Definable Henselian Valuations

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A valuation ring has a unique maximal ideal $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$, we call the quotient $Kv := \mathcal{O}_v/\mathfrak{m}_v$ the residue field of (K, v). We usually denote $vK := \Gamma_v$.

Example

For a field F, consider the polynomial ring $F[t]$. Then there is a natural valuation v on $F[t]$ via

$$v\left(\sum_{i=0}^{n} a_i t^i\right) = \min\{0 \leq i \leq n \mid a_i \neq 0\}$$

where $n \in \mathbb{N}$, $a_i \in F$.

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Furthermore, v extends to the power series field $F((t))$ by setting

$$v\left(\sum_{i=m}^{\infty} a_i t^i\right) = \min\{m \leq i \leq \infty \mid a_i \neq 0\}.$$

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- Let (K, v) be a valued field, consider $f \in \mathcal{O}_v[X]$ and $a \in \mathcal{O}_v$. Then $f(a) = 0$ implies $\overline{f}(\overline{a}) = 0$.

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- Let (K, v) be a valued field, consider $f \in \mathcal{O}_v[X]$ and $a \in \mathcal{O}_v$. Then $f(a) = 0$ implies $\overline{f}(\overline{a}) = 0$.
- Language of valued fields: $\mathcal{L}_{ring} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a unary relation symbol.

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If (K, v) satisfies one of the conditions in the theorem, the valuation v is called henselian. The field K is called henselian, if there exists some non-trivial henselian valuation on K.

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On the other hand, $(F((t)), v)$ is henselian as it is complete.

The Ax-Kochen/Ersov principle

Ax-Kochen/Ersov Theorem

Let (K, v) and (L, w) be henselian valued fields with $\mathrm{char}(Kv) = 0$. Then

$$(K, v) \equiv (L, w) \Longleftrightarrow Kv \equiv Lw \text{ and } vK \equiv wL.$$

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- There are also versions for positive characteristic.

Definable Valuations

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Note that separably and real closed fields do not admit any non-trivial definable valuations.

From now on, all fields K are neither real nor separably closed.

Some Facts about Definable Valuations

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- Admitting a non-trivial henselian valuation is not an elementary property in $\mathcal{L}_{ring}$ (Prestel and Ziegler, see [7]).
- Not every henselian valued field admits a $\emptyset$-definable non-trivial henselian valuation.

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- Which henselian valuations are (existentially/universally) definable?
- Which henselian valuations are $\emptyset$-definable?
- Does every (not real nor separably closed) NIP field with small absolute Galois group admit a non-trivial definable valuation?

An Application of Definable Valuations

For a field K, the absolute Galois group of K is the group

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where L ranges over all finite Galois extensions of K.

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where L ranges over all finite Galois extensions of K.

Theorem (Neukirch-Efrat-Koenigsmann-Pop)

Let K be a field. If G_K and $G_{\mathbb{Q}_p}$ are isomorphic as profinite groups, then $K \equiv \mathbb{Q}_p$.

Value group vs. residue field

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- Conditions on the value group vK are discussed in work of Koenigsmann ([6]) and Hong ([2]).
- For the remainder of this talk, we will focus on conditions on the residue field Kv.

Main ingredient

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For a valued field (K, v), the following are equivalent:

1. v extends uniquely to $K(p)$.
2. For each $f \in \mathcal{O}_v[X]$ which splits in $K(p)$ and each $a \in \mathcal{O}_v$ with $\overline{f}(\overline{a}) = 0$ and $\overline{f}'(\overline{a}) \neq 0$, there exists $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $\overline{\alpha} = \overline{a}$.

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Proposition

Let (K, v) be a henselian valued field such that $Kv \neq Kv(p)$. If $p = 2$, assume that Kv is not euclidean. If Kv is not p-henselian, then v is $\emptyset$-definable.

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Corollary

Let (K, v) be henselian such that Kv is finite. Then v is $\emptyset$-definable.

Hilbertian fields

Definition

Let K be a field and let T and X be variables. Then K is called hilbertian if for every polynomial $f \in K[T, X]$ which is separable, irreducible and monic when considered as a polynomial in $K(T)[X]$ there is some $a \in K$ such that $f(a, X)$ is irreducible in $K[X]$.

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Idea: (for $\mathrm{char}(K) \neq p$) Consider $f(T, X) = X^p - mT - 1$ for $m \in \mathfrak{m}_v$. Find $a \in \mathcal{O}_v$ such that $f(a, X)$ is irreducible. Then $f(a, X)$ splits in $K(p)$, has a simple zero in Kv but not zero in K.

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Idea: Use a p-henselian analogue of Frey-Prestel: PAC + henselian implies separably closed.

Corollary

Let (K, v) be a henselian valued field such that Kv is PRC and not real closed. Then v is $\emptyset$-definable.

Simple fields

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Proposition

Let (K, v) be a non-trivially henselian valued field such that Kv is simple and not separably closed. Then v is $\emptyset$-definable.

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Idea of the proof: Any p-henselian field admits a V-topology with a uniformly definable base of neighbourhoods of zero. This can be used to construct a definable family of strictly decreasing sets (wrt inclusion).

Help from the absolute Galois Group

Definition

Let K be a field. We say that G_K is universal if for every finite group G there are finite Galois extensions $L \subseteq M$ of K such that $\mathrm{Gal}(M/L) \cong G$.

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- pro-soluble absolute Galois groups are non-universal.
- G_K is non-universal if K is NIP of positive characteristic.
- G_K is universal if K is hilbertian.

Observation

Let (K, v) be henselian. Then G_K is non-universal if and only if G_{Kv} is non-universal.

Non-universal absolute Galois group

Theorem

Let (K, v) be a henselian valued field such that Kv is not separably nor real closed and admits no henselian valuation. If G_K is non-universal then v is $\emptyset$-definable.

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Let (K, v) be henselian such that $\mathrm{char}(K) > 0$ and K is not separably closed. If K is NIP then K admits a non-trivial $\emptyset$-definable henselian valuation.

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Thank you for your attention.

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