The exponential-logarithmic equivalence classes of surreal numbers.

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Introduction

Surreal numbers :

- form a **proper class totally ordered** that contains "all numbers great and small", e.g. real numbers and ordinal numbers (Conway 76) ;
- form a **divisible ordered abelian group** that is a universal domain for ordered abelian groups (Conway 76, Ehrlich 2001) ;

Introduction

Surreal numbers :

- form a **real closed field** that is a universal domain for real fields (Conway 76, Ehrlich 2001) ;
- form a **field of generalized power series** (Conway 76) ;
- form a **real exponential field** that is a non standard model of the theory of $\mathbb{R}_{an,\exp}$ (Gonshor 86, van den Dries-Ehrlich 2001).

Introduction

Our result : describe a natural and explicit system Φ_0 of **representatives of the exponential-logarithmic equivalence classes** of surreal numbers.

Our conjectures :

- **Conjecture 1** Surreal numbers form an **exponential-logarithmic transseries field** :

$$\mathbf{NO} = \mathrm{ELT}(\Phi_0, \sigma_0)$$

for some automorphism $\sigma_0 = \log_{|\Phi_0}$ of Φ_0.

$\rightarrow$ Φ_0 is the chain of **initial fundamental monomials** (S. Kuhlmann's context of *EL-series*) ;
$\rightarrow$ Φ_0 is the chain of **log-atomic elements** (van der Hoeven's context of **transseries**).

Introduction

- **Conjecture 2** Surreal numbers carry a **Hardy type derivation** (i.e. a derivation with same valuative properties as the derivation in Hardy fields or H-fields).

  $\rightsquigarrow$ We expect $(\mathbf{NO}, \exp, d)$ to be a **universal domain for real differential exponential fields**.

Introduction

My aims today are :

- to give a short survey on **NO** ;
- to give a description of our results on what should be Φ_0 ;
- to give a short survey on ELT fields.

Definitions and results due to Conway.

Denote by **ON** the *(proper) class* of all ordinals numbers.

- A **surreal number** $a \in$ **NO** is defined to be a map

  $$a : \alpha \to \{\ominus, \oplus\} \quad \text{for some } \alpha \in \textbf{ON}$$

  A surreal number is usually identified to its image : a well ordered sequence of $\ominus$'s and $\oplus$'s

  $$\oplus \oplus \ominus \oplus \ominus \ominus \cdots$$

- The support $\alpha \in$ **ON** of a is called its **length** $l(a)$.

Definitions and results due to Conway.

- Surreal numbers are **ordered lexicographically** with :

$$\ominus < \emptyset < \oplus.$$

- Surreal numbers carry a partial ordering called simplicity :

  a is simpler than *b*, write $a <_s b$ iff *a* is a proper initial subsequence of *b*

Definitions and results due to Conway.

Key property : any subsets $F < G$ in **NO** define a **unique shortest element** such that $F < a < G$. One denotes

$$a = \langle F | G \rangle,$$

also unique up to **cofinal** representations.

- **NO** is a **continuum** containing **ON** ;
- any surreal number has a **canonical representation** $a = \langle a^L | a^R \rangle$ where a^L and a^R are simpler than a ;
- one can define functions on **NO** by **transfinite recursion** : from the shortest to the largest surreal numbers.

Field structure and normal form.

- There is a field structure on **NO** which makes it a **real closed fields** containing $\mathbb{R}$ $\rightarrow$ universal domain for RCF.
- The ω-map : for any $a \in$ **NO**, define

$$\omega^a := \langle 0, n\omega^{a^L} | \omega^{a^R}/2^n \rangle,$$

then ω^a is the shortest representative of an **Archimedean equivalence class** of **NO**.

$\rightarrow$ **Conway normal form** of the surreal numbers i.e. generalized series field structure of **NO** :

$$\mathbf{NO} = \mathbb{R}((\omega^{\mathbf{NO}}))$$

Generalized series field.

- $\omega^{\omega^{\mathbf{NO}}}$ is a system of representatives of the multiplicative equivalence classes of **NO**.

  $\rightarrow$ **Hahn series field** structure of **NO** :

  $$\mathbf{NO} = \mathbb{R}((H(\omega^{\omega^{\mathbf{NO}}})))$$

  where $H(\omega^{\omega^{\mathbf{NO}}})$ is the **Hahn group** generated by $\omega^{\omega^{\mathbf{NO}}}$.

Kruskal-Gonshor results.

- The ω-map is not an exponential map, since it has fixed points : the **generalized epsilon numbers** s.t. $a = \omega^a$.
- **NO** admits an exponential
$$\exp : (\mathbf{NO}, +) \to (\mathbf{NO}_{>0}, .)$$
and a logarithm $\log = \exp^{-1}$.

$$\mathbf{NO} \models T_{\mathrm{an},\exp}$$

Kruskal-Gonshor results.

- For any purely infinite surreal a and any surreal b,

$$\begin{array}{rcl} \exp(\omega^a) & = & \omega^{\omega^{g(a)}} \\ \log(\omega^{\omega^b}) & = & \omega^{h(b)} \end{array}$$

with $g = h^{-1}$ and $h(b) := \langle 0, h(b^L) | h(b^R), \omega/2^n \rangle$.

- log and exp are **strong morphisms.**

Our results : recursive definition.

- **Exponential-logarithmic equivalence relation** :
  $\forall x, y \in \mathbf{NO}_{>>1}$,

$$x \sim_{EL} y \Leftrightarrow \exists n \in \mathbb{N},\ \log_n(x) \leq y \leq \exp_n(x)\,;$$

$$x >_{EL} y \Leftrightarrow \forall n,\ \log_n(x) > y.$$

- **The κ-map** : for any $a \in \mathbf{NO}$, define

$$\kappa(a) = \kappa_a := \langle \exp^n(0), \exp^n(\kappa_{a^L}) \mid \log^n(\kappa_{a^R}) \rangle \quad (n \in \mathbb{N}).$$

Examples.

- $$\begin{array}{rcl} \kappa_0 & := & \langle \exp_k(0),\ k \in \mathbb{N} \mid \emptyset \rangle \\ & = & \langle k \in \mathbb{N} \mid \emptyset \rangle \\ & = & \omega \end{array} ;$$
- $$\begin{array}{rcl} \kappa_1 & := & \langle \exp_k(0),\ \exp_k(\omega)\ k \in \mathbb{N} \mid \emptyset \rangle \\ & = & \langle \omega_k(1),\ k \in \mathbb{N} \mid \emptyset \rangle \\ & = & \epsilon_0 \end{array} ;$$
- $$\begin{array}{rcl} \kappa_{-1} & := & \langle \exp_k(0),\ k \in \mathbb{N} \mid \log_k(\omega),\ k \in \mathbb{N} \rangle \\ & = & \langle k \in \mathbb{N} \mid \omega^{\omega^{-k}},\ k \in \mathbb{N} \rangle \\ & = & \omega^{\omega^{-\omega}} \end{array} .$$

First theorem for the κ-map.

Theorem 1.

*The map κ is well defined on **NO** and for any $a \in$ **NO** :*

• $\forall n \in \mathbb{Z}$, $\kappa_{a,n} := \log^n \kappa_a \in \omega^{\omega^{\mathbf{NO}}}$;

• κ_a *is the shortest element of an exponential equivalence class.*

About our conjecture.

Heuristic.

$$\epsilon_{\mathbf{NO}} \subsetneq \kappa_{\mathbf{NO}} \subsetneq \omega^{\omega^{\mathbf{NO}}} \subsetneq \omega^{\mathbf{NO}} \subsetneq \mathbf{NO}$$

Set :

- $\kappa_{a,n} := \log^n(\kappa_{a,n}),\ n \in \mathbb{Z}$;
- $\Phi_0 := \bigcup_{a \in \mathbf{NO}} \left(\bigcup_{n \in \mathbb{Z}} \kappa_{a,n} \right)$;
- $\sigma_0 : \Phi_0 \to \Phi_0$ by $\sigma_0 := \log_{|\Phi_0}$

Conjecture :

$$\mathbf{NO} = \mathrm{ELT}(\Phi_0, \sigma_0)$$

Our results : sign sequence.

- Denote by $\&$ the "**concatenation**" of sign sequences.
- For any $a \in \mathbf{NO}$, write its sign sequence as the following transfinite concatenation :

$$a = (a_0 \oplus) \& (b_0 \ominus) \& (a_1 \oplus) \& (b_1 \ominus) \& \cdots$$

<u>N.B</u> : a_α is possibly 0 for $\alpha = 0$ or α is a limit ordinal.

Our results : sign sequence.

- **The μ-map** : for any $a \in \mathbf{NO}$, define :

$$\mu(a) := \langle \mu(a^L) \& (\omega_n(\epsilon_c + 1)\oplus) \mid \mu(a^R) \& (n\ominus) \rangle$$

where c is the total number of $\oplus$'s in a, minus 1.

Second theorem for the κ-map.

Theorem 2. Take a surreal

$$a = (a_0 \oplus)\&(b_0 \ominus)\&(a_1 \oplus)\&(b_1 \ominus)\&\cdots$$

and, for any α, define $c_\alpha := (\sum_{\beta \leq \alpha} a_\beta)^\flat$. Then we have :

$$\mu(a) = (\epsilon_{c_0} \oplus)\&(\omega.b_0 \ominus)\&(\epsilon_{c_1} \oplus)\&(\omega.b_1 \ominus)\&\cdots$$

and for any $n \in \mathbb{N}^*$,

$$\begin{array}{rcl} \kappa_a & = & \omega^{\omega^{\mu(a)}} \\ \kappa_{a,n} & = & \omega^{\omega^{\mu(a)\&(n\ominus)}} \\ \kappa_{a,-n} & = & \omega^{\omega^{\mu(a)\&(\omega_{n-1}(\epsilon_c+1)\oplus)}} \end{array}$$

Ideas of our proofs.

- **Theorem 1** : inspired by Gonshor's proof for the ω-map. Induction on the length of a.
- **Theorem 2** : the main ingredient is the following lemma

Lemma 1 : $\forall a \in \mathbf{NO}, \forall \beta \in \mathbf{ON}$,
$\bullet h(\mu(a)\&\beta\ominus) = \omega^{\mu(a)\&(\beta+1)\ominus}$;

$$\bullet h(\mu(a)\&\beta\oplus) = \left| \begin{array}{ll} \omega^{\mu(a)}\&(\beta-1)\oplus & \text{if } \epsilon+1 \leq \beta < \epsilon+\omega \\ \omega^{\mu(a)}\&\beta\oplus & \text{if not} \end{array} \right. .$$

Ideas of our proofs.

Proof by induction on the length of a.

For example, consider $a \in \mathbf{NO}$ with $l(a) = \alpha = \tilde{\alpha} + 1$ successor ordinal, and suppose the Lemma true for all b with $l(b) \leq \tilde{\alpha}$.
There are 2 cases :
• if $a = \tilde{a}\&\oplus$, then $a = \langle a^L | a^R \rangle = \langle \tilde{a} | \tilde{a}^R \rangle$. Moreover, $\mu(a) = \mu(\tilde{a})\&\epsilon_c\oplus$.

Ideas of our proofs.

Now, by definition,

$$\begin{aligned} h(\mu(a)) &:= \langle 0, h(\mu(\tilde{a})\&\omega_n(\epsilon_{c^L}+1)\oplus) | h(\mu(\tilde{a}^R)\&n\ominus), \omega^{\mu(a)}/2^k \rangle \\ &= \langle \omega^{\mu}(\tilde{a})\&\omega_k(\epsilon_{c^L}+1)\oplus),\ k \in \mathbb{N} | \omega^{\mu(a)}/2^k,\ k \in \mathbb{N} \rangle \\ &= \omega^{\mu(\tilde{a})}\&\epsilon_c \oplus \&\epsilon_c \omega \ominus \\ &= \omega^{\mu(\tilde{a})\&\oplus\&\ominus} \\ &= \omega^{\mu(a)\&\ominus} \end{aligned}$$

etc...

About transseries and exp-log series.

- Take $\mathbb{R}((\Gamma))$ a field of generalized series endowed with a *non surjective logarithm*
$$\log : \Gamma \to \mathbb{R}((\Gamma^{\succ 1})).$$
In particular, suppose $\Gamma = H(\Phi_0)$ for some chain Φ_0 on which $\log = \sigma_0$.
- **Key idea.** To make the log surjective and therefore define exp $\to$ apply an infinite towering exponential extension process and take the inductive limit

About transseries and exp-log series.

- **Definition** A complete subfield $\mathbb{L} \subset \mathbb{K}$ which contains Γ is called an **exp-log transseries field** if the following hold :

  ELT1. $\text{domain}(\log) = \mathbb{L}_{>0}$.

  ELT2. $\log(\Gamma) = \mathbb{L}^{\succ 1}$.

  ELT3. $\log(1+\epsilon) = \sum_{n=1}^{\infty} \epsilon^n/n \in \mathbb{L}^{\prec 1}$ for any $\epsilon \in \mathbb{L}^{\prec 1}$.

  ELT4. For any $(m_n)_n \subset \Gamma$ such that

  $$\forall n \in \mathbb{N},\ m_{n+1} \in \text{Supp } \log(m_n),$$

  then there exists a rank $N \in \mathbb{N}$ such that

  $$\forall k \in \mathbb{N},\ \log(m_{N+k}) = m_{N+k+1}.$$

To be continued.

- **Conjecture** : describe **NO** as an ELT field over $\mathbb{R}((H(\Phi_0)))$ with $\Phi_0 := \bigcup_{a \in \mathbf{NO}} (\bigcup_{n \in \mathbb{Z}} \kappa_{a,n})$.

To be continued.

- **Last theorem to find** : define a "good" derivation on $\kappa_{\mathbf{NO}}$, so that it extends to the whole **NO** as we desire :

  - strong linearity
  - generalized Leibniz rule
  - l'Hospital's rule.
  - etc

  See "Hardy type derivations on EL-series fields" (S. Kuhlmann, M.M.).