

giacpy-cubics

We would like to prove here the following result:

let S be the cubic equation defined below and consider the 9 by 9 matrices $MQ(x,y)$ defined below. Let w_0, w_1, w_2 be 3 linearly dependent vectors of QQ^6 , satisfying the cubic equation $S(w)=0$, then the determinant $Qij=\det(MQ(wi,wj))$ satisfy an explicit relation:

$$R01*Q01+R02*Q02+R12*Q12 = 0$$

```
R.<u_1,u_2,u_3,v_1,v_2,v_3,u_10,u_20,u_30,v_10,v_20,v_30,u_11,u_21,u_31,v_11,v_21,v_31>=PolynomialRing(QQ,18);
```

```
w0=vector([u_10,u_20,u_30,v_10,v_20,u_10*u_20*u_30/(v_10*v_20)]);
w1=vector([u_11,u_21,u_31,v_11,v_21,u_11*u_21*u_31/(v_11*v_21)]);
```

```
def MQ(x,y):
    return matrix([[0,0,0,u_1,u_2,u_3,v_1,v_2,v_3],[0,0,0,x[0],x[1],x[2],x[3],x[4],x[5]],[0,0,0,y[0],y[1],y[2],y[3],y[4],y[5]],
[u_1,x[0],y[0],x[0]*y[0],0,0,0,0,0],[u_2,x[1],y[1],0,x[1]*y[1],0,0,0,0],[u_3,x[2],y[2],0,0,x[2]*y[2],0,0,0],[v_1,x[3],y[3],0,0,0,
(-(x[3]))*y[3],0,0],[v_2,x[4],y[4],0,0,0,0,(-(x[4]))*y[4],0],[v_3,x[5],y[5],0,0,0,0,(-(x[5]))*y[5]]])
```

```
def S(w):
    return W[0]*W[1]*W[2]-W[3]*W[4]*W[5]
```

```
MQ(w0,w1)
```

$$\begin{pmatrix} 0 & 0 & 0 & u_1 & u_2 & u_3 & v_1 & v_2 & v_3 \\ 0 & 0 & 0 & u_{10} & u_{20} & u_{30} & v_{10} & v_{20} & \frac{u_1 u_2 u_3}{v_1 v_2} \\ 0 & 0 & 0 & u_{11} & u_{21} & u_{31} & v_{11} & v_{21} & \frac{u_1 u_2 u_3}{v_1 v_2} \\ u_1 & u_{10} & u_{11} & u_1 u_{11} & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_{20} & u_{21} & 0 & u_2 u_{21} & 0 & 0 & 0 & 0 \\ u_3 & u_{30} & u_{31} & 0 & 0 & u_3 u_{31} & 0 & 0 & 0 \\ v_1 & v_{10} & v_{11} & 0 & 0 & 0 & -v_1 v_{11} & 0 & 0 \\ v_2 & v_{20} & v_{21} & 0 & 0 & 0 & 0 & -v_2 v_{21} & 0 \\ v_3 & \frac{u_1 u_2 u_3}{v_1 v_2} & \frac{u_1 u_2 u_3}{v_1 v_2} & 0 & 0 & 0 & 0 & 0 & -\frac{u_1 u_2 u_3 u_1 u_2 u_3}{v_1 v_2 v_1 v_2} \end{pmatrix}$$

```
#time D01sage=MQ(w0,w1).det()
```

```
from giacpy import *
```

```
// Giac share
root-directory:/home/fred/sagefred/git-trac-command/local/share/giac\
/
// Using keyword file
/home/fred/sagefred/git-trac-command/local/share/giac/doc/fr/keyword\
s
// Giac share
root-directory:/home/fred/sagefred/git-trac-command/local/share/giac\
/
Help file
/home/fred/sagefred/git-trac-command/local/share/giac/doc/fr/aide_ca\
s not found
Added 0 synonyms
```

Computing those symbolic determinant with the minors method is much faster

```
time Q01=(libgiac(MQ(w0,w1))).det_minor()
```

```
Time: CPU 0.02 s, Wall: 0.07 s
```

```
a0=(-u_10*u_20*u_30*v_11^2*v_21^2+u_10*v_11*v_21*u_21*u_31*v_10*v_20+u_20*v_11*v_21*u_31*v_10*v_20*u_11+u_30*v_11*v_21*u_21*v_10*v_20*u_11-
v_11*u_21*u_31*v_10*v_20^2*u_11-v_21*u_21*u_31*v_10^2*v_20*u_11)/(v_11*v_21*v_10*v_20)
```

```
a1=(u_10*u_20*u_30*v_10*v_11^2+u_10*u_20*u_30*v_11^2*v_20*v_21-u_10*u_20*u_31*v_10*v_11*v_20*v_21-
u_10*u_21*u_30*v_10*v_11*v_20*v_21-u_11*u_20*u_30*v_10*v_11*v_20*v_21+u_11*u_21*u_31*v_10^2*v_20^2)/(v_10*v_11*v_20*v_21)
```

```
w2=a0*w0+a1*w1
```

```
w2
```

$$\left(\frac{v_1^2 u_2^2 u_3^2 u_4^2 u_5^2 - u_1 v_1 u_2^2 u_3^2 u_4^2 u_5^2 v_1 - u_1 u_2^2 u_3^2 u_4^2 u_5^2 v_2 - u_2 u_3 u_4^2 u_5^2 v_1 v_2 + u_1^2 v_1 u_2^2 u_3^2 u_4^2 u_5^2 v_1 v_2 + u_1 u_2 u_3 u_4^2 u_5^2 v_1 v_2 + u_1 u_2 u_3 u_4^2 u_5^2 v_1 v_2 - u_1^2 u_2 u_3 u_4^2 u_5^2 v_2^2}{v_1 v_2 v_1 v_2}, \frac{v_1^2 u_2^2 u_3^2 u_4^2 u_5^2 - u_2 v_1 u_2^2 u_3^2 u_4^2 u_5^2 - u_2 v_1 u_2^2 u_3^2 u_4^2 u_5^2 v_2}{v_1 v_2 v_1 v_2} \right)$$

```
S(w0),S(w1),S(w2)
```

```
(0,0,0)
```

```
MQ(w0,w2)
```

$$\begin{pmatrix}
 0 & 0 \\
 0 & 0 \\
 0 & 0 \\
 u_1 & u_{10} \\
 u_2 & u_{20} \\
 u_3 & u_{30} \\
 v_1 & v_{10} \\
 v_2 & v_{20} \\
 v_3 & \frac{u_1 u_2 u_{30}}{v_1 v_{20}}
 \end{pmatrix}
 \begin{pmatrix}
 \frac{v_1^3 \theta_2^2 \theta_1^2 u_2 u_{31} - u_1 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_1 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} - u_2 \theta_3 \theta_1 \theta_2 \theta_1^2 v_1 v_{21} + u_1^2 \theta_1 \theta_2 \theta_2 u_3 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_1 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 \frac{v_1^3 \theta_2^2 \theta_1 u_2 u_{31} - u_2 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_2 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} - u_1 \theta_3 \theta_1 \theta_2 \theta_2^2 v_1 v_{21} + u_1^2 \theta_1 \theta_2 \theta_2 u_3 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_2 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 v_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 \frac{v_1^3 \theta_2^2 \theta_1 u_2 u_{31} - u_3 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_3 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} + u_3^2 \theta_1 \theta_2 \theta_1 u_2 v_1 v_{21} - u_1 \theta_2 \theta_1 \theta_2 \theta_3^2 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_3 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 v_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 -\frac{v_1^3 \theta_1 u_2 u_{31} + u_3 \theta_1^2 \theta_1 u_2 v_{11} + u_2 \theta_1^2 \theta_1 u_3 v_{11} + u_1 \theta_1^2 \theta_2 u_3 v_{11} - u_2 \theta_3 \theta_1 \theta_1 v_{11}^2 - u_1 \theta_3 \theta_1 \theta_2 v_{11}^2 - u_1 \theta_2 \theta_1 \theta_1}{v_1 \theta_1} \\
 -\frac{v_2^3 \theta_1 u_2 u_{31} + u_3 \theta_1^2 \theta_1 u_2 v_{21} + u_2 \theta_1^2 \theta_1 u_3 v_{21} + u_1 \theta_1^2 \theta_2 u_3 v_{21} - u_2 \theta_3 \theta_2 \theta_1 v_{21}^2 - u_1 \theta_3 \theta_2 \theta_2 v_{21}^2 - u_1 \theta_2 \theta_2 \theta_2}{v_2 \theta_2} \\
 \frac{v_1^3 \theta_2^2 \theta_1^2 u_2^2 u_{31} - u_2 \theta_3 \theta_1^2 \theta_2^2 \theta_1^2 u_2 u_3 v_{11} v_{21} - u_1 \theta_3 \theta_1 \theta_2^2 \theta_1 u_2^2 u_3 v_{11} v_{21} - u_1 \theta_2 \theta_2 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 \theta_2 u_3 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 \theta_2 u_3 v_{11}^2}{v_1^2 \theta_2^2 \theta_1^2 v_{21}^2}
 \end{pmatrix}$$

```
time Q02=libgiac.det_minor(MQ(w0,w2))
```

Time: CPU 1.30 s, Wall: 1.23 s

```
MQ(w1,w2)
```

$$\begin{pmatrix}
 0 & 0 \\
 0 & 0 \\
 0 & 0 \\
 u_1 & u_{11} \\
 u_2 & u_{21} \\
 u_3 & u_{31} \\
 v_1 & v_{11} \\
 v_2 & v_{21} \\
 v_3 & \frac{u_1 u_2 u_{31}}{v_1 v_{21}}
 \end{pmatrix}
 \begin{pmatrix}
 \frac{v_1^3 \theta_2^2 \theta_1^2 u_2 u_{31} - u_1 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_1 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} - u_2 \theta_3 \theta_1 \theta_2 \theta_1^2 v_1 v_{21} + u_1^2 \theta_1 \theta_2 \theta_2 u_3 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_1 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 \frac{v_1^3 \theta_2^2 \theta_1 u_2^2 u_{31} - u_2 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_2 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} - u_1 \theta_3 \theta_1 \theta_2 \theta_2^2 v_1 v_{21} + u_1^2 \theta_1 \theta_2 \theta_2 u_3 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_2 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 v_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 \frac{v_1^3 \theta_2^2 \theta_1 u_2 u_{31} - u_3 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} - u_3 \theta_1^2 \theta_2 \theta_1 u_2 u_3 v_{21} + u_3^2 \theta_1 \theta_2 \theta_1 u_2 v_1 v_{21} - u_1 \theta_2 \theta_1 \theta_2 \theta_3^2 v_1 v_{21} + u_1 \theta_2 \theta_3 \theta_2 \theta_3 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 v_1 v_{11}}{v_1 \theta_2 \theta_1 v_{21}} \\
 -\frac{v_1^3 \theta_1 u_2 u_{31} + u_3 \theta_1^2 \theta_1 u_2 v_{11} + u_2 \theta_1^2 \theta_1 u_3 v_{11} + u_1 \theta_1^2 \theta_2 u_3 v_{11} - u_2 \theta_3 \theta_1 \theta_1 v_{11}^2 - u_1 \theta_3 \theta_1 \theta_2 v_{11}^2 - u_1 \theta_2 \theta_1 \theta_1}{v_1 \theta_1} \\
 -\frac{v_2^3 \theta_1 u_2 u_{31} + u_3 \theta_1^2 \theta_1 u_2 v_{21} + u_2 \theta_1^2 \theta_1 u_3 v_{21} + u_1 \theta_1^2 \theta_2 u_3 v_{21} - u_2 \theta_3 \theta_2 \theta_1 v_{21}^2 - u_1 \theta_3 \theta_2 \theta_2 v_{21}^2 - u_1 \theta_2 \theta_2 \theta_2}{v_2 \theta_2} \\
 \frac{v_1^3 \theta_2^2 \theta_1^2 u_2^2 u_{31} - u_2 \theta_3 \theta_1^2 \theta_2^2 \theta_1^2 u_2 u_3 v_{11} v_{21} - u_1 \theta_3 \theta_1 \theta_2^2 \theta_1 u_2^2 u_3 v_{11} v_{21} - u_1 \theta_2 \theta_2 \theta_1 \theta_2^2 \theta_1 u_2 u_3 v_{11} v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 \theta_2 u_3 v_{11}^2 v_{21} + u_1 \theta_2 \theta_3 \theta_1 \theta_2 \theta_2 u_3 v_{11}^2}{v_1^2 \theta_2^2 \theta_1^2 v_{21}^2}
 \end{pmatrix}$$

```
time Q12=libgiac.det_minor(MQ(w1,w2))
```

Time: CPU 1.25 s, Wall: 1.18 s

```
print("the numerator of Q01 has:", Q01.numer().nops(), "terms")
print("the numerator of Q02 has:", Q02.numer().nops(), "terms")
print("the numerator of Q12 has:", Q12.numer().nops(), "terms")
```

```
('the numerator of Q01 has:', 258, 'terms')
('the numerator of Q02 has:', 31890, 'terms')
('the numerator of Q12 has:', 31890, 'terms')
```

define (normal is expand with simplifications) the following products:

```
R02=libgiac.normal(u_11*u_21*u_31*a0**2)
R12=libgiac.normal(u_10*u_20*u_30*a1**2)
R01=libgiac.normal(w2[0]*w2[1]*w2[2]*(a0*a1)**2)
```

number of terms of the numerators:

```
R01.numer().nops();R02.numer().nops();R12.numer().nops();
```

1908

21

21

```
time libgiac.normal(R01*Q01+R02*Q02+R12*Q12)
```

0

Time: CPU 5.34 s, Wall: 5.31 s