On representations of mapping class groups in Integral TQFT

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In this talk, I discussed the theory of integral TQFT which I have developed in joint work with Patrick Gilmer [6, 7]. In usual Reshetikhin-Turaev TQFT, the mapping class group of a compact orientable surface Σ is represented on a finite-dimensional vector space, say $V_p(\Sigma)$, over a cyclotomic field, say $\mathbb{Q}(\zeta_p)$ (here ζ_p is a primitive pth root of unity). For integral TQFT, the vector space should be replaced by a free lattice $\mathcal{S}_p(\Sigma)$ over $\mathbb{Z}(\zeta_p)$ = the ring of algebraic integers in $\mathbb{Q}(\zeta_p)$. In particular, it means that mapping classes are now represented by matrices with integral coefficients.

In [7], we have shown how to construct such an integral TQFT refinement for the Reshetikhin-Turaev $SO(3)$ TQFT at $q = \zeta_p$, p an odd prime, starting from the skein-theoretical approach to this TQFT as in [2]. The integral lattice $\mathcal{S}_p(\Sigma)$ is contained in the vector space $V_p(\Sigma)$ and has a natural definition in terms of the vector-valued quantum $SO(3)$-invariants for 3-manifolds with boundary (see below). (If $p \equiv 1 \bmod 4$, the coefficient ring considered in [7] is actually a quadratic extension of $\mathbb{Z}(\zeta_p)$, but for simplicity of exposition I will ignore this and similar details in this talk.)

The mapping class group representation on the lattice $\mathcal{S}_p(\Sigma)$ preserves a natural non-degenerate hermitian form with values in $\mathbb{Z}(\zeta_p)$. One may ask whether the image of the mapping class group under this representation coincides with the automorphism group of this form. Note that the analogous statement for the U(1)-TQFT is the well-known fact that the image of the mapping class group acting in homology is the symplectic group $Sp(2g, \mathbb{Z})$, that is, the group of automorphisms of the integral homology lattice of Σ which preserve the intersection form.

Another question about the image of the mapping class group concerns its group theoretic structure. It is known that Dehn twists are represented by matrices of order p; are there any other relations in the image that don't already hold in the mapping class group? For the torus without boundary, there must be more relations, because the image is known to be a finite group (Gilmer [4]). But for the torus with one boundary component, I can show that there are no other relations. One may wonder whether this is a general fact for hyperbolic surfaces, and if so, what is its geometric meaning?

Bases of the vector space $V_p(\Sigma)$ are well understood in terms of admissible colorings of uni-trivalent graphs. But the $\mathbb{Z}(\zeta_p)$-span of such a *graph basis* is almost never invariant under the mapping class group, and hence cannot be equal to the integral lattice $\mathcal{S}_p(\Sigma)$. In [7], we show that $\mathcal{S}_p(\Sigma)$ admits what we call *graph-like* bases associated to a special kind of uni-trivalent graph which we call a *lollipop tree*. Roughly speaking, a graph-like basis is obtained from the usual graph basis associated to the lollipop tree by the composition of two operations: a certain triangular base change, and some rescaling depending on the colors. For precise definitions, see [7].

Integral TQFT contains more topological information than the usual TQFT over a field. For example, it allows to study embedding questions as follows. Consider the following problem. Given an orientable compact connected 3-manifold N with boundary $\partial N = \Sigma$, does it embed into the 3-sphere? This translates in TQFT to a condition on the vector $v = v_p(N)$ in $V_p(\Sigma)$ associated to N: since

$$N \cup (S^3 - N) = S^3$$

there must be a vector v' (namely $v' = v_p(S^3 - N)$) such that

$$\langle v, v' \rangle = 1$$

(since the quantum invariant of S^3 is 1 in the normalization which is relevant here). In usual TQFT, this condition just requires v to be non-zero (since the form $\langle\ ,\ \rangle$ is non-degenerate). But in integral TQFT, both v and v' must lie in the integral lattice $\mathcal{S}_p(\Sigma)$. This puts lots of restrictions on v, and they may be used to show in some cases that N does not embed into S^3. An example is given at the end of our paper [7]. More examples can be found in Gilmer [5].

To understand how this works in practice, one needs to know that the integral lattice $\mathcal{S}_p(\Sigma)$ is exactly the span, over $\mathbb{Z}(\zeta_p)$, of the vectors $v_p(N')$ where N' has boundary Σ and no closed components. The numbers $\langle v, v' \rangle$ where $v' \in \mathcal{S}_p(\Sigma)$ span an ideal in $\mathbb{Z}(\zeta_p)$ which we call the FKB-ideal since Frohman and Kania-Bartoszynska were the first to consider this kind of quantum obstruction to embedding one manifold into another [3]. Clearly, if N embeds into S^3, then there is a v' in $\mathcal{S}_p(\Sigma)$ such that $\langle v, v' \rangle = 1$, so the FKB-ideal is trivial (i.e., contains 1). But to decide effectively whether such a v' exist, we need a basis (or at least, a finite generating set) of $\mathcal{S}_p(\Sigma)$. Frohman and Kania-Bartoszynska could not compute the ideal except in rather trivial situations. But our integral TQFT-bases from [7] make their idea into an effective tool. I like to think that this shows at the same time that integral TQFT, which is defined over a ring of algebraic integers, represents the actual topological information much more closely than the usual TQFTs defined over a field.

I would like to close this short report with two more results about TQFT representations of mapping class groups.

The first one concerns the relationship between TQFT and the Nielsen-Thurston classification of mapping classes of surfaces. In my paper [1] with J. E. Andersen and K. Ueno, we make the following

Conjecture. *Let Σ be a compact orientable surface with negative Euler characteristic and let ρ_k be the TQFT representation of the mapping class group of Σ at level k* (say for the Reshetikhin-Turaev TQFT associated to some quantum group). *Then a mapping class φ has a pseudo-Anosov piece if and only if there exists $k_0 = k_0(\varphi)$ such that the matrix $\rho_k(\varphi)$ has infinite order for all $k \geq k_0$.*

Note that it is easy to see that if φ has no pseudo-Anosov piece, then the matrix $\rho_k(\varphi)$ has finite order for all k (although φ itself may have infinite order as a mapping class). For more discussion of this conjecture, see [1].

In [1], we prove the conjecture in the $SU(n)$-case for the mapping class group $M(0,4)$ (i.e. when Σ is a four-holed sphere). In the $SU(2)$-case, we can even show that the stretching factor of a pseudo-Anosov mapping class φ is the limit, as $k \to \infty$, of the maximal eigenvalue of the TQFT-matrix $\rho_k(\varphi)$. As already mentioned in [1], I also know how to prove this for $M(1,1)$ (i.e. Σ is now a torus with one boundary component), but the proof in this case involves integral bases [8].

The second result about TQFT representations I would like to mention is unpublished work of mine from 2005 [9]. It affirms the existence of a limit representation (at least on the Torelli group) as the order of the quantum parameter $q = \zeta_p$ goes to infinity. For this result integral TQFT is crucial and I consider again the integral $SO(3)$-TQFT lattices constructed with Gilmer in [7].

Theorem. *There exist ordered bases of the integral lattices $\mathcal{S}_p(\Sigma)$ (p an odd prime), such that for every mapping class φ in the Torelli subgroup of the mapping class group of Σ, and for every (i,j), the matrix entries $(\rho_p(\varphi)_{ij}$ converge in Ohtsuki's sense as $p \to \infty$.*

Note that since the rank of $\mathcal{S}_p(\Sigma)$ goes to infinity as $p \to \infty$, for every (i,j) the matrix entry $(\rho_p(\varphi)_{ij}$ is defined for all big enough p. This matrix entry lies in $\mathbb{Z}(\zeta_p)$. The limit in Ohtsuki's sense of a sequence of algebraic integers $I_p \in \mathbb{Z}(\zeta_p)$ is defined as follows. Write

$$I_p = \sum_{n=0}^{p-2} a_{n,p}(\zeta_p - 1)^n$$

where $a_{n,p} \in \mathbb{Z}$. We say that the sequence I_p converges to a power series

$$\tau = \sum_{n=0}^{\infty} a_n h^n \in \mathbb{Q}[[h]]$$

if for every n and every prime $p >> n$, the integer $a_{n,p}$ and the rational number a_n are congruent modulo p (note that this makes sense for p bigger than the denominator of a_n).

This definition goes back to Ohtsuki. If $I_p(M)$ denotes the Reshetikhin-Turaev invariant of an integral homology sphere M, it is known by H. Murakami [11] that $I_p(M) \in \mathbb{Z}(\zeta_p)$ (a skein-theoretical proof of this result was given in my paper [10] with J. Roberts; it was the beginning of my interest in integrality questions in TQFT). Then Ohtsuki showed that $I_p(M)$ converges in the above sense to a power series $\tau(M) \in \mathbb{Q}[[h]]$ called the Ohtsuki series of M [12].

My theorem stated above generalizes Ohtsuki's result to the TQFT representation of the Torelli group. If the integral homology sphere M is obtained in the usual way from a Torelli mapping class φ, we may choose the basis of the lattice $\mathcal{S}_p(\Sigma)$ such that the invariant $I_p(M)$ is one of the entries of the matrix $\rho_p(\varphi)$ (in fact, the entry in the upper left corner of the matrix). While Ohtsuki's theorem says that this matrix entry converges as $p \to \infty$, my theorem says the same thing for *all* matrix entries. Observe that the truth of a statement of this kind will depend crucially on what basis one chooses. In fact, this convergence result would not be true without using the integral TQFT bases I found in my work with Gilmer in [7].

The limit representation can be explicitly described using skein theory, and as a corollary I obtain a purely skein-theoretical construction of the Ohtsuki series $\tau(M)$. I made some more comments in my talk about this limit representation, but for lack of space I will not reproduce them here. Hopefully a written account of this matter will soon appear elsewhere.

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