The heat kernel and frequency localized functions on the Heisenberg group

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Abstract. The goal of this paper is to study the action of the heat operator on the Heisenberg group $\mathbb{H}^d$, and in particular to characterize Besov spaces of negative index on $\mathbb{H}^d$ in terms of the heat kernel. That characterization can be extended to positive indexes using Bernstein inequalities. As a corollary we obtain a proof of refined Sobolev inequalities in $\dot{W}^{s,p}$ spaces.

Keywords. Heat kernel, Besov space, Heisenberg group, frequency localization.

1. Introduction

This paper is concerned mainly with a characterization of Besov spaces on the Heisenberg group using the heat kernel. In [1], a Littlewood-Paley decomposition on the Heisenberg group is constructed, and Besov spaces are defined using that decomposition. It is classical that in $\mathbb{R}^d$ there is an equivalent definition, for negative regularity indexes, in terms of the heat kernel. This characterization in $\mathbb{R}^d$ can be extended to positive regularity indexes thanks to Bernstein's inequalities which express that derivatives act almost as homotheties on distributions, the Fourier transform of which is supported in a ring of $\mathbb{R}^d$ centered at zero.

The aim of this text is to present a similar characterization of Besov spaces on $\mathbb{H}^d$ using the heat flow. One of the main steps of the procedure in $\mathbb{R}^d$ consists in studying frequency localized functions and the action of derivatives, and more generally Fourier multipliers, on such functions (the correponding inequalities for derivatives are known as Bernstein inequalities). In the Heisenberg group there is a priori no simple notion of frequency localization, since the Fourier transform is a family of operators on a Hilbert space; however frequencies may be understood by studying the action of the Laplacian on a Hilbertian basis of that space, which allows to define a notion of frequency localization (see Definition 18 below). One can then try to investigate the action of the semi-group of the heat equation on the Heisenberg group on such frequency localized functions. That is achieved in this paper; we also prove a similar characterization of Besov spaces in terms of the heat flow, as in the classical $\mathbb{R}^d$ case. This allows to prove refined Sobolev inequalities, for $\dot{W}^{s,p}$ spaces. Finally we are able by similar techniques to recover the fact that the heat semi-group is the convolution by a function in the Schwartz class (as in previous works by Gaveau in [6] and Hulanicki in [8]).

Let us mention that by a different method, Furioli, Melzi and Veneruso obtained in [5] a characterization of Besov spaces in terms of the heat kernel for Lie groups of polynomial growth.

1.1. The Heisenberg group $\mathbb{H}^d$

In this introductory section, let us recall some basic facts on the Heisenberg group $\mathbb{H}^d$. The Heisenberg group $\mathbb{H}^d$ is the Lie group with underlying $\mathbb{C}^d \times \mathbb{R}$ endowed with the following product law:

$$\forall\,((z,s),(z',s')) \in \mathbb{H}^d \times \mathbb{H}^d, \quad (z,s)\cdot(z',s') = (z+z', s+s'+2\mathrm{Im}(z\cdot\overline{z}')),$$

where $z\cdot\overline{z}' = \sum_{j=1}^d z_j \overline{z}'_j$. It follows that $\mathbb{H}^d$ is a non commutative group, the identity of which is $(0,0)$; the inverse of the element (z,s) is given by $(z,s)^{-1} = (-z,-s)$. The Lie algebra of left invariant vector fields on the Heisenberg group $\mathbb{H}^d$ is spanned by the vector fields

$$Z_j = \partial_{z_j} + i\overline{z}_j\partial_s, \quad \overline{Z}_j = \partial_{\overline{z}_j} - iz_j\partial_s \quad \text{and} \quad S = \partial_s = \frac{1}{2i}[\overline{Z}_j, Z_j],$$

with $j \in \{1,\dots,d\}$. In all that follows, we shall denote by $\mathcal{Z}$ the family of vector fields defined by Z_j for $j \in \{1,\dots,d\}$ and $Z_j = \overline{Z}_{j-d}$ for $j \in \{d+1,\dots,2d\}$ and for any multi-index $\alpha \in \{1,\dots,2d\}^k$, we will write

$$\mathcal{Z}^\alpha \stackrel{\mathrm{def}}{=} Z_{\alpha_1}\dots Z_{\alpha_k}. \tag{1.1}$$

The space $\mathbb{H}^d$ is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system (x,y,s) is simply the Lebesgue measure $dxdyds$.

Let us point out that on the Heisenberg group $\mathbb{H}^d$, there is a notion of dilation defined for $a > 0$ by $\delta_a(z,s) = (az, a^2 s)$. The homogeneous dimension of $\mathbb{H}^d$ is therefore $N \stackrel{\mathrm{def}}{=} 2d+2$, noticing that the Jacobian of the dilation δ_a is a^N.

The Schwartz space $\mathcal{S}(\mathbb{H}^d)$ on the Heisenberg group is defined as follows.

Definition 1. *The Schwartz space $\mathcal{S}(\mathbb{H}^d)$ is the set of smooth functions u on $\mathbb{H}^d$ such that, for any $k \in \mathbb{N}$, we have*

$$\|u\|_{k,\mathcal{S}} \stackrel{def}{=} \sup_{\substack{|\alpha|\le k \\ (z,s)\in\mathbb{H}^d}} \left|\mathcal{Z}^\alpha\left((|z|^2 - is))^{2k} u(z,s)\right)\right| < \infty.$$

Remark 2. *The Schwartz space on the Heisenberg group $\mathcal{S}(\mathbb{H}^d)$ coincides with the classical Schwartz space $\mathcal{S}(\mathbb{R}^{2d+1})$. The weight in (z,s) appearing in the definition above is related to the fact that the Heisenberg distance to the origin is defined by $\rho(z,s) \stackrel{def}{=} \left(|z|^4 + s^2\right)^{\frac14}$.*

Finally, let us present the Laplacian-Kohn operator, which is central in the study of partial differential equations on $\mathbb{H}^d$, and is defined by

$$\Delta_{\mathbb{H}^d} \stackrel{\mathrm{def}}{=} 2\sum_{j=1}^d (Z_j\overline{Z}_j + \overline{Z}_j Z_j).$$

Powers of that operator allow to construct positive order Sobolev spaces: for example we define the homogeneous space $\dot{W}^{s,p}(\mathbb{H}^d)$, for $0 < s < N/p$, as the completion of $\mathcal{S}(\mathbb{H}^d)$ for the norm

$$\|f\|_{\dot{W}^{s,p}(\mathbb{H}^d)} \stackrel{\mathrm{def}}{=} \left\|(-\Delta_{\mathbb{H}^d})^{\frac{s}{2}} f\right\|_{L^p(\mathbb{H}^d)}.$$

1.2. Statement of the results

In [1] and [3] a dyadic unity partition is built on the Heisenberg group $\mathbb{H}^d$, similar to the one defined in the classical $\mathbb{R}^d$ case. A significant application of this decomposition is the definition of Besov spaces on the Heisenberg group in the same way as in the classical case (see [1],[3]). In Section 2, we shall give a full account of this theory.

The main result of this paper describes the action of the semi-group associated with the heat equation on the Heisenberg group, on a frequency localized function. We refer to Definition 18 below for the notion of a frequency localized function, which requires the definition of the Fourier transform on $\mathbb{H}^d$, and is therefore slightly technical.

Lemma 3. *Let (r_1, r_2) be two positive real numbers, and define $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0, r_1, r_2)$ the ring centered at the origin, of small and large radius respectively r_1 and r_2. Two positive constants c and C exist such that, for any real number $p \in [1,\infty]$, any couple (t,β) of positive real numbers and any function u frequency localized in the ring $\beta\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, we have*

$$\|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p(\mathbb{H}^d)} \leq C e^{-ct\beta^2} \|u\|_{L^p(\mathbb{H}^d)}. \tag{1.2}$$

That lemma is the key argument in the proof of the following theorem which is well known in $\mathbb{R}^d$ and proved by a different method in [5] for Lie groups of polynomial growth. The definition of Besov spaces is provided in the next section.

Theorem 4. *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. A constant C exists which satisfies the following property. For $u \in \dot{B}^{-2s}_{p,r}(\mathbb{H}^d)$, we have*

$$C^{-1}\|u\|_{\dot{B}^{-2s}_{p,r}(\mathbb{H}^d)} \leq \Big\| \|t^s e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p(\mathbb{H}^d)} \Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})} \leq C\|u\|_{\dot{B}^{-2s}_{p,r}(\mathbb{H}^d)}. \tag{1.3}$$

Remark 5. *Thanks to Bernstein's inequalities (see Proposition 20 below), we have*

$$\|u\|_{\dot{B}^{\sigma}_{p,r}(\mathbb{H}^d)} \equiv \sup_{|\alpha|=k} \big\|(-\Delta_{\mathbb{H}^d})^{\frac{\alpha}{2}} u\big\|_{\dot{B}^{\sigma-k}_{p,r}(\mathbb{H}^d)}.$$

We deduce that the caracterization of Besov spaces on the Heisenberg group in terms of the heat kernel can be extended to any positive regularity index.

This characterization is useful for instance to prove refined Sobolev inequalities. In this paper we will prove the following result.

Theorem 6. *Let $p \in [1,\infty]$ and $0 < s < N/p$ be given. There exists a positive constant C such that for any function f in $\dot{W}^{s,p}(\mathbb{H}^d)$ we have*

$$\|f\|_{L^q(\mathbb{H}^d)} \leq C\|f\|^{1-\frac{sp}{N}}_{\dot{W}^{s,p}(\mathbb{H}^d)} \|f\|^{\frac{sp}{N}}_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}},$$

with $q = pN/(N-ps)$.

Remark 7. *This is a refined Sobolev inequality since it is easy to see that $\dot{W}^{s,p}(\mathbb{H}^d)$ is continuously embedded in $\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}$, so that Theorem 6 is a refined version of the classical inequality*

$$\|f\|_{L^q(\mathbb{H}^d)} \leq C\|f\|_{\dot{W}^{s,p}(\mathbb{H}^d)}.$$

The above continuous embedding is simply due to the following estimate, applied to $u = (-\Delta_{\mathbb{H}^d})^{\frac{s}{2}} f$:

$$\|u\|_{\dot{B}^{-\frac{N}{p}}_{\infty,\infty}} = \sup_{t>0} t^{\frac{N}{2p}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^\infty(\mathbb{H}^d)} \leq C\|u\|_{L^p(\mathbb{H}^d)}.$$

Note that in the special case when $p = 2$, such an inequality was proved in [3], using the method developped in the classical case in [7].

It turns out that the techniques involved in the proof of Lemma 3 enable us to recover the following theorem, which was proved (by different methods) by Gaveau in [6] and Hulanicki in [8].

Theorem 8. *There exists a function $h \in \mathcal{S}(\mathbb{H}^d)$ such that, if u denotes the solution of the free heat equation on the Heisenberg group*

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta_{\mathbb{H}^d} u & = & 0 \quad in \quad \mathbb{R}^+ \times \mathbb{H}^d, \\ u_{|t=0} & = & u_0, \end{array} \right. \tag{1.4}$$

then we have

$$u(t,\cdot) = u_0 \star h_t$$

where $\star$ denotes the convolution on the Heisenberg group defined in Section 2 below, while h_t is defined by

$$h_t(x,y,s) = \frac{1}{t^{d+1}} h(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{s}{t}).$$

The rest of this paper is devoted to the proof of Theorems 4 to 8, as well as Lemma 3.

The structure of the paper is the following. First, in Section 2, we present a short review of Littlewood-Paley theory on the Heisenberg group, giving the notation and results that will be used in the proofs, as well as the main references of the theory. Section 3 is devoted to the proof of Theorem 4, assuming Lemma 3, and finally the proof of Lemma 3 can be found in Section 4. In Section 4 we also give the proofs of Theorems 6 and 8.

2. Elements of Littlewood-Paley theory on the Heisenberg group

2.1. The Fourier transform on the Heisenberg group

To introduce the Littlewood-Paley theory on the Heisenberg group, we need to recall the definition of the Fourier transform in that framework. We refer for instance to [10], [11] or [12] for more details. The Heisenberg group being non commutative, the Fourier transform on $\mathbb{H}^d$ is defined using irreducible unitary representations of $\mathbb{H}^d$. As explained for instance in [12] Chapter 2, all irreducible representations of $\mathbb{H}^d$ are unitarily equivalent to one of two representations: the Bargmann representation or the L^2-representation. The representations on $L^2(\mathbb{R}^d)$ can be deduced from Bargmann representations thanks to interlacing operators. We can consult J. Faraut and K. Harzallah [4] for more details. We shall choose here the Bargmann representations described by $(u^\lambda, \mathcal{H}_\lambda)$, with $\lambda \in \mathbb{R} \setminus \{0\}$, where $\mathcal{H}_\lambda$ are the spaces defined by

$$\mathcal{H}_\lambda = \{F \text{ holomorphic on } \mathbb{C}^d, \|F\|_{\mathcal{H}_\lambda} < \infty\},$$

while we define

$$\|F\|^2_{\mathcal{H}_\lambda} \stackrel{\text{def}}{=} \left(\frac{2|\lambda|}{\pi}\right)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2} |F(\xi)|^2 d\xi, \tag{2.1}$$

and u^λ is the map from $\mathbb{H}^d$ into the group of unitary operators of $\mathcal{H}_\lambda$ defined by

$$\begin{array}{ll} u^\lambda_{z,s} F(\xi) = F(\xi - \overline{z}) e^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)} & \text{for} \quad \lambda > 0, \\ u^\lambda_{z,s} F(\xi) = F(\xi - z) e^{i\lambda s - 2\lambda(\xi\cdot \overline{z} - |z|^2/2)} & \text{for} \quad \lambda < 0. \end{array}$$

Let us notice that $\mathcal{H}_\lambda$ equipped with the norm (2.1) is a Hilbert space and that the monomials

$$F_{\alpha,\lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\, \xi)^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^d,$$

constitute an orthonormal basis.

If f belongs to $L^1(\mathbb{H}^d)$, its Fourier transform is given by

$$\mathcal{F}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(z,s) u^\lambda_{z,s} dz ds.$$

Note that the function $\mathcal{F}(f)$ takes its values in the bounded operators on $\mathcal{H}_\lambda$. As in the $\mathbb{R}^d$ case, one has a Plancherel Theorem and an inversion formula. More precisely, let $\mathcal{A}$ denote the Hilbert space of one-parameter families $A = \{A(\lambda)\}_{\lambda\in\mathbb{R}\setminus\{0\}}$ of operators on $\mathcal{H}_\lambda$ which are Hilbert-Schmidt for almost every $\lambda \in \mathbb{R}$ with norm

$$\|A\| = \left(\frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty} \|A(\lambda)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda\right)^{\frac{1}{2}} < \infty$$

where $\|A(\lambda)\|_{HS(\mathcal{H}_\lambda)}$ denotes the Hilbert-Schmidt norm of the operator $A(\lambda)$. Then the Fourier transform can be extended to an isometry from $L^2(\mathbb{H}^d)$ onto $\mathcal{A}$ and we have the Plancherel formula:

$$\|f\|^2_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}}\sum_{\alpha\in\mathbb{N}^d}\int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda}|\lambda|^d d\lambda.$$

On the other hand, if

$$\sum_{\alpha\in\mathbb{N}^d}\int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda}|\lambda|^d d\lambda < \infty \tag{2.2}$$

then we have for almost every w,

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)\right)|\lambda|^d d\lambda, \tag{2.3}$$

where

$$\operatorname{tr}\left(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)\right) = \sum_{\alpha\in\mathbb{N}^d}(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda}$$

denotes the trace of the operator $u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)$.

Remark 9. *The above hypothesis* (2.2) *is satisfied in* $\mathcal{S}(\mathbb{H}^d)$*, where* $\mathcal{S}(\mathbb{H}^d)$ *is defined in Definition* 1*. This follows from Proposition* 10 *which is proved for the sake of completeness, directly below its statement.*

Let us moreover point out that we have the following useful formulas, for any $k \in \{1, ..., d\}$.

Denoting by $1_k = (0, \ldots, 1, \ldots)$ the vector whose k - component is one and all the others are zero, one has

$$\mathcal{F}(Z_k f)(\lambda)F_{\alpha,\lambda} = -\sqrt{2|\lambda|}\sqrt{\alpha_k+1}\mathcal{F}(f)(\lambda)F_{\alpha+1_k,\lambda} \tag{2.4}$$

if $\lambda > 0$, and similarly

$$\mathcal{F}(Z_k f)(\lambda)F_{\alpha,\lambda} = \sqrt{2|\lambda|}\sqrt{\alpha_k}\mathcal{F}(f)(\lambda)F_{\alpha-1_k,\lambda} \tag{2.5}$$

if $\lambda < 0$. Furthermore,

$$\mathcal{F}(\overline{Z_k} f)(\lambda)F_{\alpha,\lambda} = \sqrt{2|\lambda|}\sqrt{\alpha_k}\mathcal{F}(f)(\lambda)F_{\alpha-1_k,\lambda} \tag{2.6}$$

if $\lambda > 0$, and

$$\mathcal{F}(\overline{Z_k} f)(\lambda)F_{\alpha,\lambda} = -\sqrt{2|\lambda|}\sqrt{\alpha_k+1}\mathcal{F}(f)(\lambda)F_{\alpha+1_k,\lambda} \tag{2.7}$$

if $\lambda < 0$. Therefore, we have easily, for any $\rho \in \mathbb{R}$,

$$\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)F_{\alpha,\lambda} = (4|\lambda|(2|\alpha|+d))^\rho\,\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} \tag{2.8}$$

and

$$\mathcal{F}\left(e^{t\Delta_{\mathbb{H}^d}} f\right)(\lambda)F_{\alpha,\lambda} = e^{-t(4|\lambda|(2|\alpha|+d))}\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}.$$

Using those formulas, we can prove the following proposition, which justifies Remark 9 stated above. The proof of this proposition is new to our knowledge.

Proposition 10. *For any function $f \in \mathcal{S}(\mathbb{H}^d)$, (2.2) is satisfied. More precisely, for any $\rho > \frac{N}{2}$, there exists a positive constant C such that*

$$\sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} |\lambda|^d \, d\lambda \leq C\Big(\|f\|_{L^1(\mathbb{H}^d)} + \|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)} \Big).$$

Let us prove that result. By definition of $\mathcal{S}(\mathbb{H}^d)$, for any $\rho \in \mathbb{R}$, the function $(-\Delta_{\mathbb{H}^d})^\rho f$ belongs to $\mathcal{S}(\mathbb{H}^d)$. Therefore, we can write, using (2.8),

$$\mathcal{F}(f)(\lambda) F_{\alpha,\lambda} = \mathcal{F}((-\Delta_{\mathbb{H}^d})^{-\rho} (-\Delta_{\mathbb{H}^d})^\rho f)(\lambda) F_{\alpha,\lambda} = (4|\lambda|(2|\alpha| + d))^{-\rho} \mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda) F_{\alpha,\lambda}.$$

But that implies that

$$\|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} = (4|\lambda|(2|\alpha| + d))^{-2\rho} \Big(\frac{2|\lambda|}{\pi}\Big)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2} \Big|\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda) F_{\alpha,\lambda}(\xi)\Big|^2 d\xi.$$

According to the definition of the Fourier transform on the Heisenberg group, we thus have

$$\begin{aligned} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} &= (4|\lambda|(2|\alpha| + d))^{-2\rho} \Big(\frac{2|\lambda|}{\pi}\Big)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2} \\ &\times \Big(\int_{\mathbb{H}^d} ((-\Delta_{\mathbb{H}^d})^\rho f(z,s)) u^\lambda_{z,s} F_{\alpha,\lambda} dz ds \overline{\int_{\mathbb{H}^d} ((-\Delta_{\mathbb{H}^d})^\rho f(z',s')) u^\lambda_{z',s'} F_{\alpha,\lambda} dz' ds'} \Big) d\xi. \end{aligned}$$

Fubini's theorem allows us to write

$$\begin{aligned} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} &= (4|\lambda|(2|\alpha| + d))^{-2\rho} \\ &\times \int_{\mathbb{H}^d} \int_{\mathbb{H}^d} (-\Delta_{\mathbb{H}^d})^\rho f(z,s) \overline{(-\Delta_{\mathbb{H}^d})^\rho f(z',s')} (u^\lambda_{z,s} F_{\alpha,\lambda} \,|\, u^\lambda_{z',s'} F_{\alpha,\lambda})_{\mathcal{H}_\lambda} dz ds dz' ds'. \end{aligned}$$

Since the operators $u^\lambda_{z,s}$ and $u^\lambda_{z',s'}$ are unitary on $\mathcal{H}_\lambda$ and the family $(F_{\alpha,\lambda})$ is a Hilbert basis of $\mathcal{H}_\lambda$, we deduce that

$$\|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} \leq (4|\lambda|(2|\alpha| + d))^{-\rho} \|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)}.$$

To conclude we decompose the integral on λ into two parts, corresponding to "high and low" frequencies (the parameter $|\lambda|^{\frac{1}{2}}$ may be identified as a frequency, as will be clear in the next section – it is in fact already apparent in (2.8) above). Thus denoting $\lambda_m = (2m + d)\lambda$, we write

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} |\lambda|^d d\lambda &\leq \sum_{m \in \mathbb{N}} \begin{pmatrix} m + d - 1 \\ m \end{pmatrix} \Big(\|f\|_{L^1(\mathbb{H}^d)} \int_{|\lambda_m| \leq 1} |\lambda|^d d\lambda \\ &+ (4(2m + d))^{-\rho} \|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)} \int_{|\lambda_m| \geq 1} |\lambda|^{-\rho} |\lambda|^d d\lambda \Big). \end{aligned}$$

This gives the announced result for $\rho > N/2$. The proposition is proved. □

Finally the convolution product of two functions f and g on $\mathbb{H}^d$ is defined by

$$f \star g(w) = \int_{\mathbb{H}^d} f(wv^{-1}) g(v) dv = \int_{\mathbb{H}^d} f(v) g(v^{-1}w) dv.$$

It should be emphasized that the convolution on the Heisenberg group is not commutative. Moreover if P is a left invariant vector field on $\mathbb{H}^d$, then one sees easily that

$$P(f \star g) = f \star Pg, \tag{2.9}$$

whereas in general $P(f \star g) \neq Pf \star g$. Nevertheless the usual Young inequalities are valid on the Heisenberg group, and one has moreover

$$\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda). \tag{2.10}$$

It turns out that for radial functions on the Heisenberg group, the Fourier transform becomes simplified and puts into light the quantity that will play the role of the frequency size. Let us first recall the concept of radial functions on the Heisenberg group.

Definition 11. *A function f defined on the Heisenberg group $\mathbb{H}^d$ is said to be radial if it is invariant under the action of the unitary group $U(d)$ of $\mathbb{C}^d$, which means that for any $u \in U(d)$, we have*

$$f(z,s) = f(u(z),s), \quad \forall (z,s) \in \mathbb{H}^d.$$

A radial function on the Heisenberg group can then be written under the form

$$f(z,s) = g(|z|,s).$$

It can be shown (see for instance [10]) that the Fourier transform of radial functions of $L^2(\mathbb{H}^d)$, satisfies the following formulas:

$$\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda}$$

where

$$R_m(\lambda) = \begin{pmatrix} m+d-1 \\ m \end{pmatrix}^{-1} \int e^{i\lambda s} f(z,s) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} dz ds,$$

and where $L_m^{(p)}$ are Laguerre polynomials defined by

$$L_m^{(p)}(t) = \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m+p \\ m-k \end{pmatrix} \frac{t^k}{k!}, \quad t \geq 0, \quad m,p \in \mathbb{N}.$$

Note that in that case

$$\|f\|_{L^2(\mathbb{H}^d)} = \|(R_m)\|_{L^2_d(\mathbb{N}\times\mathbb{R})} \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}} \sum_m \begin{pmatrix} m+d-1 \\ m \end{pmatrix} \int_{-\infty}^{\infty} |Q_m(\lambda)|^2 |\lambda|^d d\lambda \right)^{\frac{1}{2}}, \tag{2.11}$$

which corresponds to the Plancherel formula recalled above, in the radial case. We also have the following inversion formula: if R_m belongs to $L^2_d(\mathbb{N}\times\mathbb{R})$ defined in (2.11), then the function

$$f(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} R_m(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda \tag{2.12}$$

is a radial function in $L^2(\mathbb{H}^d)$ and satisfies

$$\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda}.$$

2.2. Littlewood-Paley theory on the Heisenberg group

Now we are ready to define the Littlewood-Paley decomposition on $\mathbb{H}^d$. We will not give any proof but refer to the construction in [1] and [3] for all the details. We simply recall that the key point in the construction of the Littlewood-Paley decomposition on $\mathbb{H}^d$ lies in the following proposition proved in [1]. Note that Proposition 12 enables one to show in particular that functions of $-\Delta_{\mathbb{H}^d}$ may be seen as convolution operators by Schwartz class functions (a result proved by Hulanicki [8] in the case of general nilpotent Lie groups).

Proposition 12. *For any $Q \in \mathcal{D}(\mathbb{R}\setminus\{0\})$, the series*

$$g(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q((2m+d)\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda$$

converges in $\mathcal{S}(\mathbb{H}^d)$.

The Littlewood-Paley operators are then constructed using the following proposition (see [1] and [3]).

Proposition 13. *Define the ring* $\mathcal{C}_0 = \{\tau \in \mathbb{R},\ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\}$ *and the ball* $\mathcal{B}_0 = \{\tau \in \mathbb{R},\ |\tau| \leq \frac{4}{3}\}$. *Then there exist two radial functions* $\widetilde{R}^*$ *and* R^* *the values of which are in the interval* $[0,1]$, *belonging respectively to* $\mathcal{D}(\mathcal{B}_0)$ *and to* $\mathcal{D}(\mathcal{C}_0)$ *such that*

$$\forall \tau \in \mathbb{R}, \quad \widetilde{R}^*(\tau) + \sum_{j \geq 0} R^*(2^{-2j}\tau) = 1 \quad \text{and} \quad \forall \tau \in \mathbb{R}^*, \quad \sum_{j \in \mathbb{Z}} R^*(2^{-2j}\tau) = 1,$$

and satisfying as well the support properties

$$|p-q| \geq 1 \Rightarrow \text{supp } R^*(2^{-2q}\cdot) \cap \text{supp } R^*(2^{-2p}\cdot) = \emptyset$$

$$\text{and} \quad q \geq 1 \Rightarrow \text{supp } \widetilde{R}^* \cap \text{supp } R^*(2^{-2q}\cdot) = \emptyset.$$

Moreover, there are radial functions of $\mathcal{S}(\mathbb{H}^d)$, *denoted* ψ *and* φ *such that*

$$\mathcal{F}(\psi)(\lambda)F_{\alpha,\lambda} = \widetilde{R}^*_{|\alpha|}(\lambda)F_{\alpha,\lambda} \quad \text{and} \quad \mathcal{F}(\varphi)(\lambda)F_{\alpha,\lambda} = R^*_{|\alpha|}(\lambda)F_{\alpha,\lambda},$$

where we have noted $\widetilde{R}^*_m(\tau) = \widetilde{R}^*((2m+d)\tau)$ *and* $R^*_m(\tau) = R^*((2m+d)\tau)$.

Now as in the $\mathbb{R}^d$ case, we define Littlewood-Paley operators in the following way.

Definition 14. *The Littlewood-Paley operators* Δ_j *and* S_j, *for* $j \in \mathbb{Z}$, *are defined by*

$$\begin{aligned} \mathcal{F}(\Delta_j f)(\lambda)F_{\alpha,\lambda} &= R^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} \\ \mathcal{F}(S_j f)(\lambda)F_{\alpha,\lambda} &= \widetilde{R}^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}. \end{aligned}$$

Remark 15. *It is easy to see that*

$$\Delta_j u = u \star 2^{Nj}\varphi(\delta_{2^j}\cdot) \quad \text{and} \quad S_j u = u \star 2^{Nj}\psi(\delta_{2^j}\cdot)$$

which implies that those operators map L^p *into* L^p *for all* $p \in [1,\infty]$ *with norms which do not depend on* j.

Along the same lines as in the $\mathbb{R}^d$ case, we can define homogeneous Besov spaces on the Heisenberg group (see [1]).

Definition 16. *Let* $s \in \mathbb{R}$ *be given, as well as* p *and* r, *two real numbers in the interval* $[1,\infty]$. *The Besov space* $\dot{B}^s_{p,r}(\mathbb{H}^d)$ *is the space of tempered distributions* u *such that*

- *The series* $\sum_{-m}^{m} \Delta_q u$ *converges to* u *in* $\mathcal{S}'(\mathbb{H}^d)$.
- $\|u\|_{\dot{B}^s_{p,r}(\mathbb{H}^d)} \stackrel{def}{=} \big\|2^{qs}\|\Delta_q u\|_{L^p(\mathbb{H}^d)}\big\|_{\ell^r(\mathbb{Z})} < \infty$.

Remark 17. *Sobolev spaces* $\dot{H}^s(\mathbb{H}^d)$ *have a characterization using Littlewood-Paley operators, as well as noninteger Hölder spaces (see* [1],[3]*). More precisely, one has* $\dot{H}^s(\mathbb{H}^d) = \dot{B}^s_{2,2}(\mathbb{H}^d)$ *for any* $s \in \mathbb{R}$, *and for any* $\rho \in \mathbb{R} \setminus \mathbb{N}$, $\dot{C}^\rho(\mathbb{H}^d) = \dot{B}^\rho_{\infty,\infty}(\mathbb{H}^d)$.

2.3. Frequency localized functions and Bernstein inequalities on the Heisenberg group

Let us first define the concept of localization procedure in the frequency space in the framework of the Heisenberg group. We will only state the definition in the case of smooth functions – otherwise one proceeds by regularizing by convolution (see [1] or [3]).

Definition 18. *Let* $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0,r_1,r_2)$ *be a ring of* $\mathbb{R}$ *centered at the origin. A function* u *in* $\mathcal{S}(\mathbb{H}^d)$ *is said to be frequency localized in the ring* $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, *if*

$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+d)^{-1}2^{2j}\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}}(\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}.$$

Remark 19. *Equivalently, a frequency localized function in the sense of Definition* 18 *satisfies*

$$u = u \star \phi_j,$$

where $\phi_j = 2^{Nj}\phi(\delta_{2^j}\cdot)$*, and* ϕ *is a radial function in* $\mathcal{S}(\mathbb{H}^d)$ *such that*

$$\mathcal{F}(\phi)(\lambda)F_{\alpha,\lambda} = R((2|\alpha|+d)\lambda)F_{\alpha,\lambda},$$

with R *compactly supported in a ring of* $\mathbb{R}$ *centered at zero.*

In order to estimate the cost of applying powers of the Laplacian on a frequency localized function, we shall need the following proposition, which ensures that the action powers of the Laplacian act as homotheties on such frequency localized functions. The proof of that proposition may be found in [3].

Proposition 20 ([3])**.** *Let* p *be an element of* $[1,\infty]$ *and let* (r_1, r_2) *be two positive real numbers. Define* $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0, r_1, r_2)$ *the ring centered at the origin, of small and large radius respectively* r_1 *and* r_2*. Then for any real number* ρ*, there is a constant* C_ρ *such that if* u *is a function defined on* $\mathbb{H}^d$*, frequency localized in the ring* $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$ *then*

$$C_\rho^{-1}2^{-j\rho}\|(-\Delta_{\mathbb{H}^d})^{\frac{\rho}{2}}u\|_{L^p(\mathbb{H}^d)} \leq \|u\|_{L^p(\mathbb{H}^d)} \leq C_\rho 2^{-j\rho}\|(-\Delta_{\mathbb{H}^d})^{\frac{\rho}{2}}u\|_{L^p(\mathbb{H}^d)}.$$

3. Proof of Theorem 4

In this section we shall prove Theorem 4, assuming Lemma 3. It turns out that the proof is very similar to the $\mathbb{R}^d$ case, and we sketch it here for the convenience of the reader.

Let us start by estimating $\|t^s e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p}$. Using Lemma 3 and the fact that the operator Δ_j commutes with the operator $e^{t\Delta_{\mathbf{H}^d}}$, we can write

$$\|t^s\Delta_j e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p} \leq Ct^s 2^{2js}e^{-ct2^{2j}}2^{-2js}\|\Delta_j u\|_{L^p}.$$

Using the definition of the homogeneous Besov (semi) norm, we get

$$\|t^s e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p} \quad \leq \quad C\|u\|_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c_{r,j}$$

where $(c_{r,j})_{j\in\mathbb{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of $\ell^r(\mathbb{Z})$. In the case when $r = \infty$, the required inequality comes immediately from the following easy result: for any positive s, we have

$$\sup_{t>0}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}} < \infty. \tag{3.1}$$

In the case when $r < \infty$, using the Hölder inequality with the weight $2^{2js}e^{-ct2^{2j}}$ and Inequality (3.1) we obtain

$$\begin{aligned}\int_0^\infty t^{rs}\|e^{t\Delta_{\mathbb{H}^d}}u\|^r_{L^p}\frac{dt}{t} &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}\Big)^{r-1}\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\Big)\frac{dt}{t}\\ &\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\frac{dt}{t}\cdot\end{aligned}$$

This gives directly the result by Fubini's theorem.

In order to prove the other inequality, let us observe that for any s greater than -1, we have

$$\int_0^\infty \tau^s e^{-\tau}d\tau \stackrel{\text{def}}{=} C_s.$$

Using the fact that the Fourier transform on the Heisenberg group is injective, we deduce the following identity (which may be easily proved by taking the Fourier transform of both sides)

$$\Delta_j u = C_s^{-1} \int_0^\infty t^s (-\Delta_{\mathbb{H}^d})^{s+1} e^{t\Delta_{\mathbb{H}^d}} \Delta_j u dt.$$

Then Lemma 3, the obvious identity $e^{t\Delta_{\mathbb{H}^d}} u = e^{\frac{t}{2}\Delta_{\mathbf{H}^d}} e^{\frac{t}{2}\Delta_{\mathbb{H}^d}} u$ and the fact that the operator Δ_j commutes with the operator $e^{t\Delta_{\mathbf{H}^d}}$, lead to

$$\|\Delta_j u\|_{L^p} \leq C \int_0^\infty t^s 2^{2j(s+1)} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p} dt. \tag{3.2}$$

In the case $r = \infty$, we simply write

$$\begin{aligned}
\|\Delta_j u\|_{L^p} &\leq C\Big(\sup_{t>0} t^s \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}\Big) \int_0^\infty 2^{2j(s+1)} e^{-ct2^{2j}} dt \\
&\leq C 2^{2js} \Big(\sup_{t>0} t^s \|e^{t\Delta_{\mathbf{H}^d}} u\|_{L^p}\Big).
\end{aligned}$$

In the case $r < \infty$, Hölder's inequality with the weight $e^{-ct2^{2j}}$ gives

$$\begin{aligned}
\Big(\int_0^\infty t^s e^{-ct2^{2j}} \|e^{t\Delta_{\mathbf{H}^d}} u\|_{L^p} dt\Big)^r &\leq \Big(\int_0^\infty e^{-ct2^{2j}} dt\Big)^{r-1} \int_0^\infty t^{rs} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r dt \\
&\leq C 2^{-2j(r-1)} \int_0^\infty t^{rs} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r dt.
\end{aligned}$$

Thanks to (3.1) and Fubini's theorem, we infer from (3.2) that

$$\begin{aligned}
\sum_j 2^{-2jsr} \|\Delta_j u\|_{L^p}^r &\leq C \int_0^\infty \Big(\sum_{j\in\mathbb{Z}} t 2^{2j} e^{-ct2^{2j}}\Big) t^{rs} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r \frac{dt}{t} \\
&\leq C \int_0^\infty t^{rs} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r \frac{dt}{t}\cdot
\end{aligned}$$

The theorem is proved. □

4. Proofs of Lemma 3 and Theorems 6 and 8

Now we are left with the proof of Lemma 3, as well as Theorems 6 and 8. Lemma 3 is proved in Paragraph 4.1, while the proofs of Theorems 6 and 8 can be found in Paragraphs 4.2 and 4.3 respectively.

4.1. Proof of Lemma 3

By density, it suffices to suppose that the function u is an element of $\mathcal{S}(\mathbb{H}^d)$. Now the frequency localization of u in the ring $\beta\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$ allows us to write

$$\mathcal{F}(e^{t\Delta_{\mathbb{H}^d}} u)(\lambda) F_{\alpha,\lambda} = e^{-t\beta^2(4|\beta^{-2}\lambda|(2|\alpha|+d))} R_{|\alpha|}(\beta^{-2}\lambda) \mathcal{F}(u)(\lambda) F_{\alpha,\lambda}, \tag{4.1}$$

with $R_{|\alpha|}(\lambda) = R((2|\alpha| + d)\lambda)$ and $R \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ is equal to 1 near the ring $\mathcal{C}_{(r_1,r_2)}$. We can then assume in what follows that $\beta = 1$.

Since R belongs to $\mathcal{D}(\mathbb{R} \setminus \{0\})$, Proposition 12 ensures the existence of a radial function $g^t \in \mathcal{S}(\mathbb{H}^d)$ such that

$$\mathcal{F}(g^t)(\lambda) F_{\alpha,\lambda} = e^{-t(4|\lambda|(2|\alpha|+d))} R_{|\alpha|}(\lambda) F_{\alpha,\lambda}.$$

We deduce that

$$e^{t\Delta_{\mathbb{H}^d}} u = u \star g^t.$$

If we prove that two positive real numbers c and C exist such that, for all positive t, we have

$$\|g^t\|_{L^1(\mathbb{H}^d)} \leq Ce^{-ct}, \tag{4.2}$$

then the lemma is proved. To prove (4.2), let us first recall that thanks to Proposition 12

$$g^t(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} e^{-t(4|\lambda|(2m+d))} R((2m+d)\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda.$$

Now, we shall follow the idea of the proof of Proposition 12 established in [1] to obtain Estimate (4.2). Let us denote by $\mathcal{Q}$ the sub-space of $L^2_d(\mathbb{N}\times\mathbb{R})$ (defined in (2.11)) generated by the sequences (Q_m) of the type

$$Q_m(\lambda) = \int_{\mathbb{R}^n} Q((2m+f(\sigma))\lambda) P(\lambda) d\mu(\sigma), \tag{4.3}$$

where μ is a bounded measure compactly supported on $\mathbb{R}^n$, f is a bounded function on the support of μ, P is a polynomial function and Q a function of $\mathcal{D}(\mathbb{R}\setminus\{0\})$ under the form

$$Q(\tau) = e^{-4t|\tau|}\mathcal{P}(t\tau)R(\tau), \tag{4.4}$$

with $\mathcal{P}$ a polynomial and R a function of $\mathcal{D}(\mathbb{R}\setminus\{0\})$.

Now, let us recall the following useful formulas (proved for instance in [1] and [9]).

Lemma 21. *For any radial function $f\in\mathcal{S}(\mathbb{H}^d)$, we have for any $m\geq 1$,*

$$\begin{aligned}
\mathcal{F}((is-|z|^2)f)(m,\lambda) &= \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) - \frac{m}{\lambda}\Big(\mathcal{F}f(m,\lambda)-\mathcal{F}f(m-1,\lambda)\Big) \;\textit{for}\quad \lambda>0 \quad \textit{and}\\
\mathcal{F}((is-|z|^2)f)(m,\lambda) &= \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) + \frac{m+d}{|\lambda|}\Big(\mathcal{F}f(m,\lambda)-\mathcal{F}f(m+1,\lambda)\Big) \;\textit{for}\quad \lambda<0.
\end{aligned}$$

Moreover, we have the following classical property on Laguerre polynomials :

$$|L_m^{(p)}(y)e^{-y/2}| \leq C_p(m+1)^p, \quad \forall y\geq 0, \tag{4.5}$$

Let us start by proving that for any integer k, one has the following formula

$$(is-|z|^2)^k g^t(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q_m^{(k)}(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2}|\lambda|^d d\lambda, \tag{4.6}$$

where $(Q_m^{(k)})$ is an element of the space $\mathcal{Q}$. By induction the problem is reduced to proving that for (Q_m) element of $\mathcal{Q}$, the sequence $(Q_m^\star)$ defined as follows is still an element of $\mathcal{Q}$: for all $m\geq 1$

$$\begin{aligned}
Q_m^\star(\lambda) &= \frac{d}{d\lambda}Q_m(\lambda) - \frac{m}{\lambda}(Q_m(\lambda)-Q_{m-1}(\lambda)), \quad \lambda>0,\\
Q_m^\star(\lambda) &= \frac{d}{d\lambda}Q_m(\lambda) + \frac{m+d}{|\lambda|}(Q_m(\lambda)-Q_{m+1}(\lambda)), \quad \lambda<0.
\end{aligned}$$

Let us for instance compute $Q_m^\star(\lambda)$ for $\lambda>0$ and $m\geq 1$. Considering (4.3), the Taylor formula implies that

$$\frac{m}{\lambda}(Q_m(\lambda)-Q_{m-1}(\lambda)) = 2m\int_{\mathbb{R}^n}\int_0^1 Q'((2m+f(\sigma)-2u)\lambda)P(\lambda)dud\mu(\sigma).$$

Therefore

$$
\begin{aligned}
Q^\star_m(\lambda) &= \int_{\mathbb{R}^n} Q((2m+f(\sigma))\lambda)P'(\lambda)d\mu(\sigma)\\
&+ \int_{\mathbb{R}^n} Q'((2m+f(\sigma))\lambda)P(\lambda)f(\sigma)d\mu(\sigma)\\
&+ 2\int_{\mathbb{R}^n}\int_0^1\int_0^1 (2m+f(\sigma)-2us)\lambda Q''((2m+f(\sigma)-2us)\lambda)P(\lambda)ududsd\mu(\sigma)\\
&- 2\int_{\mathbb{R}^n}\int_0^1\int_0^1 Q''((2m+f(\sigma)-2us)\lambda)\lambda P(\lambda)ududsf(\sigma)d\mu(\sigma)\\
&+ 4\int_{\mathbb{R}^n}\int_0^1\int_0^1 Q''((2m+f(\sigma)-2us)\lambda)\lambda P(\lambda)u^2dusdsd\mu(\sigma).
\end{aligned}
$$

This proves that the sequence $(Q^\star_m)$ belongs to the space $\mathcal{Q}$.

Now let us end the proof of Lemma 3: defining

$$f^t_m(z,s) = \int e^{-i\lambda s}Q_m(\lambda)L^{(d-1)}_m(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda,$$

with (Q_m) element of $\mathcal{Q}$, and in view of (4.6) it is enough to prove that there exist two constants c and C which do not depend on m, such that

$$|f^t_m(z,s)| \leq Ce^{-ct}\frac{1}{m^2}\cdot \tag{4.7}$$

Due to the condition on the support of the function R appearing in (4.4), there exist two fixed constants which only depend on R, denoted c_1 and c_2 such that

$$f^t_m(z,s) = \int_{\mathbb{R}^n}\int_{c_1\leq|(2m+f(\sigma))\lambda)|\leq c_2} e^{-i\lambda s}Q((2m+f(\sigma))\lambda)P(\lambda)L^{(d-1)}_m(2|\lambda||z|^2)e^{-|\lambda||z|^2}d\mu(\sigma)|\lambda|^d d\lambda.$$

In view of (4.4) and (4.5), we obtain

$$|f^t_m(z,s)| \leq c_{d-1}\int_{\mathbb{R}^n}\int_{c_1\leq|(2m+f(\sigma))\lambda)|\leq c_2} e^{-ct}m^{d-1}d\mu(\sigma)|\lambda|^d d\lambda,$$

which leads easily to (4.7) and ends the proof of the lemma. □

4.2. Proof of Theorem 6

The proof of Theorem 6 presented here relies on the maximal function on the Heisenberg group; before starting the proof let us collect a few useful results on this function, starting with the definition of the maximal function (the interested reader can consult [11] for details and proofs).

Definition 22. *Let f be in $L^1_{loc}(\mathbb{H}^d)$. The maximal function of f is defined by*

$$Mf(z,s) \stackrel{def}{=} \sup_{R>0}\frac{1}{m(B((z,s),R))}\int_{B((z,s),R)}|f(z',s')|dz'ds',$$

where $m(B((z,s),R))$ denotes the measure of the Heisenberg ball $B((z,s),R)$ of center (z,s) and radius R.

The key propoerties we will use on the maximal function are collected in the following proposition.

Proposition 23. *The maximal function satisfies the following properties.*

1. *If f is a function in $L^p(\mathbb{H}^d)$, with $1 < p \leq \infty$, then Mf belongs to $L^p(\mathbb{H}^d)$ and we have*
$$\|Mf\|_{L^p(\mathbb{H}^d)} \leq A_p \|f\|_{L^p(\mathbb{H}^d)},$$
where A_p is a constant which depends only on p and d.
2. *Let φ be a function in $L^1(\mathbb{H}^d)$ and suppose that the function $\psi(w) \stackrel{def}{=} \sup_{\rho(w') \geq \rho(w)} \varphi(w')$ belongs to $L^1(\mathbb{H}^d)$, where ρ denotes the Heisenberg distance to the origin defined in Remark 2. Then for any measurable function f, we have*
$$\Big|(f \star \varphi)(w)\Big| \leq \|\psi\|_{L^1(\mathbb{H}^d)} Mf(w).$$

Now we are ready to prove Theorem 6. By density we can suppose that f belongs to $\mathcal{S}(\mathbb{H}^d)$. Let us write
$$f = \int_0^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt$$
and decompose the integral in two parts:
$$f = \int_0^A e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt + \int_A^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt,$$
where A is a constant to be fixed later.

On the one hand, by Theorem 4, we have
$$\|e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f\|_{L^\infty} \leq \frac{C}{t^{1+\frac{1}{2}(\frac{N}{p}-s)}} \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}.$$
Therefore after integration we get
$$\int_A^\infty \|e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f\|_{L^\infty} \leq A^{\frac{1}{2}(s-\frac{N}{p})} \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbf{H}^d)}.$$
On the other hand, denoting by $g = (-\Delta_{\mathbf{H}^d})^{\frac{s}{2}} f$, we have
$$e^{t\Delta_{\mathbf{H}^d}} \Delta_{\mathbb{H}^d} f = \frac{1}{(-t)^{1-\frac{s}{2}}} e^{t\Delta_{\mathbf{H}^d}} (-t\Delta_{\mathbb{H}^d})^{1-\frac{s}{2}} g.$$
It is well-known that the heat kernel on the Heisenberg group satisfies the second assumption of Proposition 23 (the reader can consult [2], [5] or [6]), so we deduce that
$$\Big|e^{t\Delta_{\mathbb{H}^d}} (-t\Delta_{\mathbb{H}^d})^{1-\frac{s}{2}} g(x)\Big| \leq C_s M_g(x),$$
where $M_g(x)$ denotes the maximal function of the function g. This leads to
$$\Big|\int_0^A e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt\Big| \leq C A^{\frac{s}{2}} M_g(x).$$
In conclusion, we get
$$\Big|\int_0^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f(x) dt\Big| \leq C\Big(A^{\frac{s}{2}} M_g(x) + A^{\frac{1}{2}(s-\frac{N}{p})} \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}\Big),$$
and the choice of A such that $A^{\frac{N}{2p}} M_g(x) = \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}}$ ensures that
$$\Big|\int_0^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f(x) dt\Big| \leq C M_g(x)^{1-\frac{ps}{N}} \|f\|^{\frac{ps}{N}}_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}.$$
Finally taking the L^q norm with $q = \frac{pN}{N-ps}$, ends the proof of Theorem 6 thanks to Proposition 23. $\square$

4.3. Proof of Theorem 8

The proof of Theorem 8 is similar to the proof of Lemma 3 and relies on the following result.

Lemma 24. *The series*

$$h(z,s)=\frac{2^{d-1}}{\pi^{d+1}}\sum_m\int e^{-i\lambda s}e^{-4|\lambda|(2m+d)}L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda. \tag{4.8}$$

converges in $\mathcal{S}(\mathbb{H}^d)$.

Notice that Lemma 24 implies directly the theorem, as by a rescaling, it is easy to see that the heat kernel on the Heisenberg group is given by

$$h_t(x,y,s)=\frac{1}{t^{d+1}}h(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}},\frac{s}{t}).$$

□

Proof of Lemma 24 Due to the sub-ellipticity of $-\Delta_{\mathbb{H}^d}$ (see for instance [1]), it suffices to prove that for any integers k and ℓ,

$$\left\|(-\Delta_{\mathbb{H}^d})^\ell(|z|^2-is)^k h\right\|_{L^2(\mathbb{H}^d)}<\infty.$$

In order to do so, let us introduce the set $\widetilde{\mathcal{Q}}$ of sequences (Q_m) of the type

$$Q_m(\lambda)=\int_{\mathbb{R}^n}Q((2m+\theta(\sigma))\lambda)P(\lambda)d\mu(\sigma), \tag{4.9}$$

where μ is a bounded measure compactly supported on $\mathbb{R}^n$, θ is a bounded function on the support of μ, P is a polynomial function and Q a function of $\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ under the form

$$Q(\tau)=e^{-4|\tau|}\mathcal{P}(\tau), \tag{4.10}$$

where $\mathcal{P}$ is a polynomial function. As in the proof of Lemma 3 and thanks to Formula (2.8) and Lemma 21, we obtain

$$\Delta^\ell_{\mathbb{H}^d}(|z|^2-is)^k h(z,s)=\frac{2^{d-1}}{\pi^{d+1}}\sum_m\int e^{-i\lambda s}Q_m^{\ell,k}(\lambda)L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda,$$

with $(Q_m^{\ell,k})$ an element of $\widetilde{\mathcal{Q}}$ which ends the proof of the lemma thanks to (2.11). □

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