The Ring of k-regular Sequences, II

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Abstract

In this paper, we continue our study of k-regular sequences begun in 1992. We prove some new results, give many new examples from the literature, and state some open problems.

Key words: k-regular sequences, finite automata

1 Introduction

A sequence $(a(n))_{n \geq 0}$ over a finite alphabet Δ is said to be k-*automatic* if there exists a finite automaton with output $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a(n) = \tau(\delta(q_0, (n)_k))$ for all $n \geq 0$. Here

- Q is a finite nonempty set of states;
- $\Sigma_k = \{0, 1, \ldots, k-1\}$;

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- $\delta : Q \times \Sigma_k \to Q$ is the transition function;
- q_0 is the initial state;
- $(n)_k$ is the canonical base-k representation of n, starting with the most significant digit; and
- $\tau : Q \to \Delta$ is the output mapping.

For example, the famous Thue-Morse sequence

$$(t(n))_{n\geq 0} = \texttt{0110100110010110}\cdots$$

counts the number of 1's (mod 2) in the base-2 representation of n, and is generated by the automaton in Figure 1.

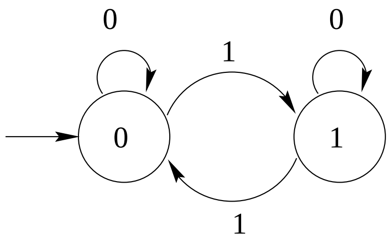

Fig. 1. Automaton generating the Thue-Morse sequence

Cobham [16] was the first to systematically study k-automatic sequences. They were then popularized and further studied by Mendès France, the authors, and others. They are extremely useful, with a well-developed theory; see, for example, [4]. However, they are somewhat restricted because of the requirement that they be defined over a finite alphabet. In [2], we gave a generalization that preserved the flavor of automatic sequences, but is defined over an infinite alphabet.

Our approach was through the k-kernel. The *k-kernel* of a sequence $(a(n))_{n\geq 0}$ is the set of subsequences

$$\{(a(k^e n + r))_{n\geq 0} \ : \ e \geq 0, \ 0 \leq r < k^e\}.$$

Then we have the following theorem of Eilenberg [19, Prop. V.3.3]:

Theorem 1 *A sequence* $\mathbf{a} = (a(n))_{n\geq 0}$ *is k-automatic if and only if the k-kernel of* $\mathbf{a}$ *is finite.*

Example 1. The Thue-Morse sequence. Consider the sequence $(t(n))_{n\geq 0}$. Then clearly

$$t(2^e n + r) \equiv t(n) + t(r) \pmod 2$$

so every sequence in the k-kernel is either $(t(n))_{n\geq 0}$ or $(t(2n+1))_{n\geq 0}$. This celebrated sequence occurs in many different areas of mathematics [3].

We now are ready to generalize k-automatic sequences. Instead of demanding that the k-kernel be finite, we instead ask that the set of sequences generated by the k-kernel be *finitely generated*. If it is, we call such a sequence *k-regular*.

More precisely, let R be a Noetherian ring. A sequence is (R, k)-regular if the R-module generated by its k-kernel is finitely generated. We usually take $R = \mathbb{Z}$, and in this case, we omit mention of R.

Example 2. Sums of digits. Consider the sequence $(s_2(n))_{n \geq 0}$, where $s_2(n)$ is the sum of the bits in the base-2 representation of n. Then

$$s_2(2^e n + r) = s_2(n) + s_2(r),$$

so every sequence in the 2-kernel is a $\mathbb{Z}$-linear combination of the sequence $(s_2(n))_{n \geq 0}$ and the constant sequence 1. Thus the 2-kernel is finitely generated, and so $(s_2(n))_{n \geq 0}$ is a 2-regular sequence.

The k-regular sequences satisfy a variety of useful properties. We recall a few results from [2]:

Theorem 2 *A sequence is k-regular and takes finitely many values if and only if it is k-automatic.*

Theorem 3 *If $(a(n))_{n \geq 0}$ and $(b(n))_{n \geq 0}$ are k-regular sequences, then so are $(a(n) + b(n))_{n \geq 0}$, $(a(n)b(n))_{n \geq 0}$, and $(ca(n))_{n \geq 0}$ for any c.*

Theorem 4 *Let $c, d \geq 0$ be integers. If $(a(n))_{n \geq 0}$ is k-regular, then so is the subsequence $(a(cn + d))_{n \geq 0}$.*

Theorem 5 *The sequence $(a(n))_{n \geq 0}$ is k-regular if and only if it is k^e-regular for any $e \geq 1$.*

2 Miscellaneous new theorems on k-regular sequences

In this section we prove a selection of new theorems about k-regular sequences.

Some sequences $\mathbf{u} = (u_n)_{n \geq 0}$ are known to satisfy recurrence relations that link subsequences such as $(u_{d^j n + i})_{n \geq 0}$ to subsequences of the same kind and to shifted sequences $(u_{n+p})_{n \geq 0}$, in a linear fashion. An example is given by the recurrence relations for the block-complexity of fixed points of uniform primitive morphisms [32]. We prove below that such sequences are d-regular. (Note that the proof of [32, Théorème 4] contains a particular case of our theorem below.) Another application of the theorem below is the computation of the palindrome complexity of fixed points of certain uniform primitive morphisms [1].

Theorem 6 *Let* $\mathbf{u} = (u_n)_{n\geq 0}$ *be a sequence with values in a Noetherian ring* R*. Suppose there exist integers* $d \geq 2$*,* $t, r, n_0 \geq 0$ *such that each sequence* $(u_{d^{t+1}n+e})_{n\geq n_0}$ *for* $0 \leq e < d^{t+1}$ *is a linear combination of the sequences* $(u_{d^j n+i})_{n\geq n_0}$ *with* $0 \leq j \leq t$*,* $0 \leq i < d^j$ *and the sequences* $(u_{n+p})_{n\geq n_0}$ *with* $0 \leq p \leq r$*. Then the sequence* u *is* d*-regular.*

Proof. First for $\ell \geq 0$ define the sequences $(\lambda_n^\ell)_{n\geq 0}$ by $\lambda_n^\ell = 1$ if $n = \ell$, and $\lambda_n^\ell = 0$ otherwise. It is clear that, for $0 \leq e < d^{t+1}$, the sequences $(u_{d^{t+1}n+e})_{n\geq 0}$ are linear combinations of the sequences $(u_{d^j n+i})_{n\geq 0}$ with $0 \leq j \leq t$, $0 \leq i < d^j$, the sequences $(u_{n+p})_{n\geq 0}$ with $0 \leq p \leq r$, and the sequences $(\lambda_n^\ell)_{n\geq 0}$ for $0 \leq \ell < n_0$.

Define the sequences $(v_n^{j,i,q})_{n\geq 0}$ by, for each $n \geq 0$, $v_n^{j,i,q} := u_{d^j(n+q)+i}$, and let $\mathcal{F}$ be the R-module generated by the finite set of sequences

$$\{(v_n^{j,i,q})_{n\geq 0},\ 0 \leq j \leq t,\ 0 \leq i < d^j,\ 0 \leq q \leq Q\} \cup \{(\lambda_n^\ell)_{n\geq 0},\ 0 \leq n < n_0\},$$

where Q is fixed such that $Q \geq \frac{d(1+r)}{d-1}$.

Since the R-module $\mathcal{F}$ is finitely generated and contains the sequence $\mathbf{u}$, for proving the d-regularity of $\mathbf{u}$ it suffices to prove that $\mathcal{F}$ is stable by the maps $(w_n)_{n\geq 0} \to (w_{dn+k})_{n\geq 0}$, where $0 \leq k < d$. It even suffices to take $(w_n)_{n\geq 0}$ to be one of the generators of $\mathcal{F}$.

Let us begin with the sequences $(v_{dn+k}^{j,i,q})_{n\geq 0}$, where $0 \leq j \leq t$, $0 \leq i < d^j$, $0 \leq q \leq Q$, and $0 \leq k < d$. Let $d^j(k+q) + i = d^{j+1}x + y$, where $0 \leq y < d^{j+1}$.

Note that $d^{j+1}x \leq d^j(k+q) + i < d^j(d-1+Q) + d^j = d^{j+1} + d^j Q$. Hence $x \leq 1 + Q/d$.

We have $v_{dn+k}^{j,i,q} = u_{d^j(dn+k+q)+i} = u_{d^{j+1}(n+x)+y}$. We distinguish two cases:

- Case 1: $j < t$. Then $u_{d^{j+1}(n+x)+y} = v_n^{j+1,y,x}$, where $j+1 \leq t$, $0 \leq y < d^{j+1}$, and $0 \leq x \leq 1 + Q/d \leq Q$ since $Q \geq d(1+r)/(d-1) \geq d/(d-1)$. Hence $(v_{dn+k}^{j,i,q})_{n\geq 0}$ is one of the generators of $\mathcal{F}$.
- Case 2: $j = t$. Then $u_{d^{j+1}(n+x)+y} = u_{d^{t+1}(n+x)+y}$. Hence, from the remark at the beginning of this proof (replacing n by $n+x$), the sequence $(v_{dn+k}^{j,i,q})_{n\geq 0} = (v_{dn+k}^{t,i,q})_{n\geq 0}$ is a linear combination of the sequences $(u_{d^\ell(n+x)+i})_{n\geq 0}$ with $0 \leq \ell \leq t$, $0 \leq i < d^\ell$, the sequences $(u_{n+x+p})_{n\geq 0}$ with $p \leq r$, and the sequences $(\lambda_{n+x}^\ell)_{n\geq 0}$ for $0 \leq \ell < n_0$. We saw in the first case that $x \leq 1 + Q/d \leq Q$.

Hence the sequences $(u_{d^\ell(n+x)+i})_{n\geq 0}$ with $0 \leq \ell \leq t$, $0 \leq i < d^\ell$ are in the set of generators of $\mathcal{F}$.

On the other hand the sequences $(u_{n+x+p})_{n\geq 0}$ are also in this set of generators, since $x+p \leq x+r \leq 1+Q/d+r \leq Q$ from the choice of Q. Finally the sequences $(\lambda^\ell_{n+x})_{n\geq 0}$ satisfy:

· either $\ell \geq x$ and $(\lambda^\ell_{n+x})_{n\geq 0} = (\lambda^{\ell-x}_n)_{n\geq 0}$. Then $\ell - x \leq \ell < n_0$ and the sequence $(\lambda^\ell_{n+x})_{n\geq 0}$ belongs to the set of generators of $\mathcal{F}$;

· or $\ell < x$ and $(\lambda^\ell_{n+x})_{n\geq 0}$ is the constant sequence 0.

Hence all the sequences $(v^{j,i,q}_{dn+k})_{n\geq 0}$, where $0 \leq j \leq t$, $0 \leq i < d^j$, $0 \leq q \leq Q$, and $0 \leq k < d$, belong to the R-module $\mathcal{F}$.

Let us look now at the sequences $(\lambda^\ell_{dn+k})_{n\geq 0}$ for $0 \leq \ell < n_0$, and $0 \leq k < d$.

- If $\ell \neq k \bmod d$, then clearly the sequence $(\lambda^\ell_{dn+k})_{n\geq 0}$ is the constant sequence 0.
- If $\ell = k \bmod d$, say $\ell = dz + k$ with $z \in \mathbb{Z}$, then:
  · either $z \geq 0$, hence $(\lambda^\ell_{dn+k})_{n\geq 0} = (\lambda^z_n)_{n\geq 0}$, which belongs to the R-module $\mathcal{F}$ since $z \leq \ell < n_0$;
  · or $z < 0$, hence the sequence $(\lambda^\ell_{dn+k})_{n\geq 0}$ is the constant sequence 0.

Hence all the sequences $(\lambda^\ell_{dn+k})_{n\geq 0}$ for $0 \leq \ell < n_0$ and $0 \leq k < d$ belong to the R-module $\mathcal{F}$. ■

A celebrated theorem of Cobham says that if a sequence is both k- and ℓ-automatic, with k, ℓ multiplicatively independent, then it is ultimately periodic. The analogous conjecture about k-regular sequences is still open [2, p. 195]. But we do have the following result:

Theorem 7 *Let k and ℓ be integers ≥ 2. Let $\mathbf{x}$ be a sequence that is both k-regular and ℓ-regular. Then $\mathbf{x}$ is $k\ell$-regular.*

Proof. Suppose $\mathbf{x} = (x_n)_{n\geq 0}$ is both k-regular and ℓ-regular, for $k, \ell \geq 2$. Since $\mathbf{x}$ is k-regular, we know there exist sequences $(x^{(1)}_n)_{n\geq 0}, \ldots, (x^{(d)}_n)_{n\geq 0}$, each of the form $(x_{k^\alpha n+\beta})_{n\geq 0}$ for some $\alpha \geq 0$ and $\beta < k^\alpha$, such that $(x^{(1)}_n)_{n\geq 0} = (x_n)_{n\geq 0}$, and any sequence $(x_{k^\gamma n+\delta})_{n\geq 0}$ with $\gamma \geq 0$ and $\delta < k^\gamma$ is a $\mathbb{Z}$-linear combination of the sequences $(x^{(i)n}_{n\geq 0})$, for $i = 1, 2, \ldots, d$.

Since the sequence $(x_n)_{n\geq 0}$ is ℓ-regular, it follows from Theorem 2.6 of [2] that the sequences $(x^{(i)}_{n\geq 0})$ are also ℓ-regular, for each of them is of the form $(x_{k^\alpha n+\beta})_{n\geq 0}$. Hence for each $i = 1, 2, \ldots, d$ there exist sequences $(x^{(i,1)}_n)_{n\geq 0}$, $(x^{(i,2)}_n)_{n\geq 0}, \ldots, (x^{(i,e_i)}_n)_{n\geq 0}$, each of the form $(x^{(i)}_{\ell^\alpha n+\beta})_{n\geq 0}$ for some $\alpha \geq 0$ and

$\beta < \ell^\alpha$, such that $(x_n^{(i,1)})_{n\geq 0} = (x_n^{(i)})_{n\geq 0}$, and any sequence $(x_{\ell^\gamma n+\delta}^{(i)})_{n\geq 0}$ with $\gamma \geq 0$ and $\delta < \ell^\gamma$ is a linear combination of the sequences $(x_n^{(i,j)})_{n\geq 0}$, for $j = 1, 2, \ldots, e_i$.

Now let $\alpha \geq 0$ and $\beta < (k\ell)^\alpha$, and consider the sequence $(x_{(k\ell)^\alpha n+\beta})_{n\geq 0}$. Let $\beta = k^\alpha q + r$, with $q \geq 0$ and $0 \leq r < k^\alpha$. Then $k^\alpha q \leq k^\alpha q + r = \beta < (k\ell)^\alpha$. Hence $q < \ell^\alpha$.

We then have $x_{(k\ell)^\alpha n+\beta} = x_{k^\alpha(\ell^\alpha n+q)+r}$. The sequence $(x_{k^\alpha n+r})_{n\geq 0}$ is a $\mathbb{Z}$-linear combination of the sequences $(x_n^{(i)})_{n\geq 0}$, with $i = 1, 2, \ldots, d$. Hence the sequence $(x_{(k\ell)^\alpha n+\beta})_{n\geq 0}$ is the same linear combination of the sequences $(x_{\ell^\alpha n+q}^{(i)})$, and hence, since $q < \ell^\alpha$, a linear combination of the sequences $(x_n^{(i,j)})_{n\geq 0}$, with $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, e_j$. ■

Theorem 8 *Let α, β be real numbers, and let k be an integer ≥ 2. The sequence $(\lfloor n\alpha + \beta \rfloor)_{n\geq 0}$ is k-regular if and only if α is rational.*

Proof. Suppose $\mathbf{a} := (\lfloor n\alpha + \beta \rfloor)_{n\geq 0}$ is k-regular. Then by Theorem 3 the sequence $\mathbf{b} := (\lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor - \lfloor \alpha \rfloor)_{n\geq 0}$ is also k-regular. But $\mathbf{b}$ takes its values in $\{0, 1\}$, and hence by Theorem 2 is k-automatic. But then, by a classic theorem about automatic sequences, the limiting frequency of the symbol 1, if it exists, must be rational [16]. But it is easy to see that 1 occurs with frequency α mod 1. Hence α mod 1 is rational, and so α is rational.

On the other hand, if α is rational, then it is easy to see that $\mathbf{c} := (\lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor)_{n\geq 0}$ is periodic and hence k-regular. Then by [2, Theorem 3.1], the running sum sequence $\mathbf{d} = (\sum_{0\leq i<n} c_i)_{n\geq 0}$ of $\mathbf{c} = (c_n)_{n\geq 0}$ is also k-regular. But $\mathbf{d} = \mathbf{a}$. ■

In our previous paper [2] we remarked that if p is a prime number, the sequence $(\nu_p(n!))_{n\geq 0}$ is p-regular. Here $\nu_q(m)$ denotes the exponent of the highest power of q dividing m. Note that

$$\nu_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

In our last theorem of this section, we generalize this result as follows:

Theorem 9 *Let k be an integer ≥ 2, and let $(c_n)_{n\geq 0}$ be a k-regular sequence with $c(0) = 0$. Then the sequence $(b_n)_{n\geq 0}$ defined by*

$$b_n = c_n + c_{\lfloor n/k \rfloor} + c_{\lfloor n/k^2 \rfloor} + \cdots$$

is also k-regular.

Proof. We have $b_n = c_n + b_{\lfloor n/k \rfloor}$. Since $(c_n)_{n\geq 0}$ is k-regular, the $\mathbb{Z}$-module generated by its k-kernel is finitely generated, say by $T = \{(c_{k^{e_i}n+f_i})_{n\geq 0} \ : \ 1 \leq i \leq t\}$. Let $e = \max_{1\leq i\leq t} e_i$. Let $\mathcal{F}$ be the R-module generated by

$$T \ \cup \ \{(b_{k^d n+a})_{n\geq 0} \ : \ d \leq e \text{ and } 0 \leq a < k^d\}.$$

Clearly $(b_n)_{n\geq 0} \in \mathcal{F}$. Further $\mathcal{F}$ is stable under each of the maps $(w_n)_{n\geq 0} \to (w_{kn+a})_{n\geq 0}$, $0 \leq a < k$. ■

The result about the k-regularity of $\nu_p(n!)$ follows by setting $c_n = n$. Then $\nu_p(n!) = b_n - n$.

3 k-regular sequences and arithmetic fractals

In this section we explore a connection between k-regular sequences and the "arithmetic fractals" of Morton and Mourant [30,31].

Let G be an abelian group, written additively, and let $\mathbf{a} = (a(n))_{n\geq 0}$ be a sequence of elements of G. For $n, q \geq 0$ we define a subword of $\mathbf{a}$ of length k^q as follows:

$$X_n^q = (a(nk^q), a(nk^q+1), \ldots, a(nk^q+k^q-1)).$$

For a subword $v = b_1 b_2 \cdots b_t$ over the set G, and $c \in G$, we define $v - c = d_1 d_2 \cdots d_t$, where $d_i = b_i - c$ for $1 \leq i \leq t$. Morton and Mourant defined the group $\Gamma_k(G)$ to be the set of all sequences $\mathbf{a}$ for which the sequence of blocks $(X_n^q - a(n))_{n\geq 0}$ is purely periodic, for all $q \geq 0$. In other words, a sequence $\mathbf{a}$ is in $\Gamma_k(G)$ if there exists an integer $M \geq 1$ such that for all $q \geq 0$ we have $X_m^q - a(m) = X_n^q - a(n)$ if $m \equiv n \pmod{M}$.

We now prove a simple characterization of $\Gamma_k(\mathbb{Z})$.

Theorem 10 *Let $\mathbf{a} = (a(n))_{n\geq 0}$ be a sequence of integers. Then $\mathbf{a} \in \Gamma_k(\mathbb{Z})$ if and only if the sequence $(a(n) - a(\lfloor n/k \rfloor))_{n\geq 0}$ is purely periodic.*

Proof. Suppose $\mathbf{a} \in \Gamma_k(\mathbb{Z})$. Then by taking $q = 1$ in the definition of $\Gamma_k(\mathbb{Z})$, we see there exists $M \geq 1$ such that $X_m^1 - a(m) = X_n^1 - a(m)$ if $m \equiv n \pmod{M}$. In other words, if $m \equiv n \pmod{M}$, then

$$a(km+i) - a(m) = a(kn+i) - a(n) \tag{1}$$

for all i, $0 \leq i < k$. We now show that $(a(n) - a(\lfloor n/k \rfloor))_{n\geq 0}$ is purely periodic with period Mk. Suppose $m' \equiv n' \pmod{Mk}$. Then $m' \equiv n' \pmod{k}$, so that we can write $m' = mk + i$, $n' = nk + i$ for some m, n with $0 \leq m, n < M$.

Then by Eq. (1), we have

$$a(m') - a(\lfloor m'/k \rfloor) = a(n') - a(\lfloor n'/k \rfloor),$$

which is the desired conclusion.

For the other direction, suppose that $(a(n) - a(\lfloor n/k \rfloor))_{n \geq 0}$ is purely periodic. In other words, suppose that if $m \equiv n \pmod{M}$, then

$$a(m) - a(\lfloor m/k \rfloor) = a(n) - a(\lfloor n/k \rfloor). \tag{2}$$

Now suppose $m \equiv n \pmod{M}$. Fix a $q \geq 0$ and i_0 with $0 \leq i_0 < k^q$. Define

$$\begin{aligned} m_0 &:= mk^q + i_0 \\ n_0 &:= nk^q + i_0 \end{aligned}$$

Then $m_0 \equiv n_0 \pmod{M}$, so by Eq. (2) we have

$$a(m_0) - a(\lfloor m_0/k \rfloor) = a(n_0) - a(\lfloor n_0/k \rfloor). \tag{3}$$

But $\lfloor m_0/k \rfloor = mk^{q-1} + \lfloor i_0/k \rfloor$ and similarly $\lfloor n_0/k \rfloor = nk^{q-1} + \lfloor i_0/k \rfloor$. Hence, defining

$$\begin{aligned} m_1 &:= mk^{q-1} + i_1 \\ n_1 &:= mk^{q-1} + i_1 \end{aligned}$$

where $i_1 := \lfloor i_0/k \rfloor$ we get

$$a(m_0) - a(m_1) = a(n_0) - a(n_1). \tag{4}$$

Note that $0 \leq i_1 < k^{q-1}$ and $m_1 \equiv n_1 \pmod{M}$, so we can repeat this procedure. This gives us a system of equalities

$$a(m_j) - a(m_{j+1}) = a(n_j) - a(n_{j+1}) \tag{5}$$

for $0 \leq j < q$ where

$$\begin{aligned} m_j &:= mk^{q-j} + i_j \\ n_j &:= nk^{q-j} + i_j \end{aligned}$$

for $0 \leq j \leq q$ and

$$i_{j+1} := \lfloor i_j/k \rfloor$$

for $0 \leq j < q$. Furthermore $0 \leq i_j < k^{q-j}$. Now add all of the equalities (5) together. Telescoping cancellation gives

$$a(m_0) - a(m_q) = a(n_0) - a(n_q).$$

Since $m_q = m$ and $n_q = n$, we get and hence

$$a(mk^q + i_0) - a(m) = a(nk^q + i_0) - a(n),$$

provided $m \equiv n \pmod{M}$. It follows that $X_m^q - a(m) = X_n^q - a(n)$ if $m \equiv n \pmod{M}$. Hence $\mathbf{a} \in \Gamma_k(\mathbb{Z})$. ∎

Now we consider the more general case of $\Gamma_k(G)$, where G is an abelian group. We will show

Theorem 11 *Let* $\mathbf{a}$ *be a sequence in* $\Gamma_k(G)$*, i.e., if* $m \equiv n \pmod{M}$ *then* $X_m^q - a(m) = X_n^q - a(n)$ *for all* $q \geq 0$*. Then there exist a* k*-uniform morphism* $\varphi : (G \times \mathbb{Z}/M\mathbb{Z}) \to (G \times \mathbb{Z}/M\mathbb{Z})^k$ *and a coding* $\tau : G \times \mathbb{Z}/M\mathbb{Z} \to G$ *such that* $\mathbf{a} = \tau(\varphi^\omega([a(0), 0]))$*.*

Proof. Define

$$\varphi([g,i]) = [g_0, ki][g_1, ki+1] \cdots [g_{k-1}, ki+k-1]$$

where (of course) the second entries are computed mod M, and $g + X_i^1 - a(i) = (g_0, g_1, \ldots, g_{k-1})$. Note that this map is well-defined, since if $i' \equiv i \pmod{M}$ then

$$\varphi([g,i']) = [g_0', ki'] \cdots [g_{k-1}', ki'+k-1]$$

where $(g_0', \ldots, g_{k-1}') = g + X_{i'}^1 - a(i')$ and by definition we have $X_{i'}^1 - a(i') = X_i^1 - a(i)$ if $i' \equiv i \pmod{M}$. Also define $\tau([g,i]) = g$. We now claim that

$$\varphi([a(i), i]) = [a(ki), ki] \cdots [a(ki+k-1), ki+k-1]. \tag{6}$$

To see this, note that

$$\varphi([a(i), i]) = [g_0, ki] \cdots [g_{k-1}, ki+k-1]$$

where

$$\begin{aligned}(g_0, \ldots, g_{k-1}) &= a(i) + X_i^1 - a(i) \\ &= X_i^1 \\ &= (a(ki), \ldots, a(ki+k-1)).\end{aligned}$$

Now a simple induction on j, together with Eq. (6), proves that

$$\varphi^j([a(0), 0]) = [a(0), 0][a(1), 1] \cdots [a(k^j - 1), k^j - 1].$$

From this the desired result follows. ∎

As a corollary, we get a result of Morton and Mourant [31]:

Corollary 12 *If G is finite, the sequence* $\mathbf{a}$ *is k-automatic.*

Proof. From Theorem 11, the sequence $\mathbf{a}$ is the image, under a coding, of a fixed point of a k-uniform morphism over a finite alphabet (with $M|G|$ letters). Hence, by a well-known theorem [16], $\mathbf{a}$ is k-automatic. ■

4 New examples of k-regular sequences

In our 1992 paper [2] we gave about 30 examples of k-regular sequences from the literature. Now we give about twenty more. For more information about the sequences we discuss, the reader can consult Sloane and Plouffe's *Encyclopedia of Integer Sequences* [45], or the newer on-line version of this reference work [44]. Sequences from the former work are given in the form M`xxxx` and from the latter work in the form A`xxxxxx`.

Although we believe this catalogue is interesting in its own right, we observe that by [2, Corollary 4.6], it follows that each of these sequences can be computed in polynomial time. In some cases (e.g., Example 17), this is not at all obvious.

Example 3. Families of Separating Subsets. Consider a set S containing n elements. If a family $F = \{A_1, A_2, \ldots, A_k\}$ of subsets of S has the property that for every pair (x, y) of distinct elements of S, we can find indices $1 \leq i, j \leq k$ such that

(i) $A_i \cap A_j = \emptyset$; and
(ii) $x \in A_i$ and $y \in A_j$,

then we call F a *separating family* for S. Let $f(n)$ denote the minimum possible cardinality of F.

For example, the letters of the alphabet can be separated by only 9 subsets:

$$\begin{array}{ll}
\{a,b,c,d,e,f,g,h,i\} & \{j,k,l,m,n,o,p,q,r\} \\
\{s,t,u,v,w,x,y,z\} & \{a,b,c,j,k,l,s,t,u\} \\
\{d,e,f,m,n,o,v,w,x\} & \{g,h,i,p,q,r,y,z\} \\
\{a,d,g,j,m,p,s,v,y\} & \{b,e,h,k,n,q,t,w,z\} \\
\{c,f,i,l,o,r,u,x\} &
\end{array}$$

Cai Mao-Cheng showed (see [23, Chapter 18]) that

$$f(n) = \min_{0 \le i \le 2} f_i(n),$$

where

$$f_i(n) = 2i + 3\lceil \log_3 n/2^i \rceil.$$

The first few terms of this sequence are given in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f(n)$	0	2	3	4	5	5	6	6	6	7	7	7	8	8

It is Sloane and Plouffe's sequence $M0456$ and Sloane's sequence $A007600$.

A priori, it is not clear that f is 3-regular, since the minimum of two k-regular sequences is not necessarily k-regular. However, in this case it is possible to prove the following characterization:

Theorem 13 *Let j be an integer such that $3^j < n \le 3^{j+1}$, i.e., $j = \lceil \log_3 n \rceil - 1$. Then*

$$f(n) = \begin{cases} 3j+1, & \text{if } 3^j < n \le 4 \cdot 3^{j-1}; \\ 3j+2, & \text{if } 4 \cdot 3^{j-1} < n \le 2 \cdot 3^j; \\ 3j+3, & \text{if } 2 \cdot 3^j < n \le 3^{j+1}. \end{cases}$$

Proof. First, note that $f_i(n) \ne f_j(n)$ for $i \ne j$, since if $i \ne j$, then $f_i(n)$ and $f_j(n)$ are in different residue classes, modulo 3.

Next, note that $f_0(n) < f_1(n)$ if and only if $3\lceil \log_3 n \rceil < 2 + 3\lceil \log_3 n/2 \rceil$, if and only if $\lceil \log_3 n \rceil \le \lceil \log_3 n/2 \rceil$, if and only if there exists j such that $2 \cdot 3^j < n \le 3^{j+1}$.

Similarly, $f_0(n) < f_2(n)$ if and only if there exists j such that $4 \cdot 3^j < n \le 3^{j+1}$, and $f_1(n) < f_2(n)$ if and only if there exists j such that $4 \cdot 3^j < n \le 2 \cdot 3^{j+1}$.

It follows that $f_0(n) \le \min(f_1(n), f_2(n))$ if and only if $2 \cdot 3^j < n \le 3^{j+1}$. In this case, $f_0(n) = 3j + 3$. Similar reasoning suffices for the other two cases. ∎

From Theorem 13, it easily follows that $f(n)$ is 3-regular. In fact, if we define $g(n+1) = f(n) - 3j$, where $j = \lceil \log_3 n \rceil - 1$, then it is easy to prove that $(g(n))_{n \ge 0}$ is actually a 3-automatic sequence which is the image under τ of the fixed point of φ, where $\tau(abcde) = 32312$, and φ sends $a \to abc$, $b \to dee$, $c \to ccc$, $d \to ddd$, and $e \to eee$.

Example 4. The sequences of Mallows and Propp. C. Mallows observed that there is a unique monotone sequence $(a(n))_{n\geq 0}$ of non-negative integers such that $a(a(n)) = 2n$ for $n \neq 1$. Here are the first few terms of this sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(n)$	0	1	3	4	6	7	8	10	12	13	14	15	16	18	20	22	24

It can be shown that $a(2^i + j) = 3\cdot 2^{i-1} + j$ for $0 \leq j < 2^{i-1}$, and $a(3\cdot 2^{i-1} + j) = 2^{i+1} + 2j$ for $0 \leq j < 2^{i-1}$. This sequence is also 2-regular. It is Sloane and Plouffe's sequence $M2317$ and Sloane's sequence $A007378$. We have

$$\begin{aligned}
a(4n) &= 2a(2n);\\
a(4n+1) &= a(2n) + a(2n+1);\\
a(4n+3) &= -2a(n) + a(2n+1) + a(4n+2);\\
a(8n+2) &= 2a(2n) + a(4n+2);\\
a(8n+6) &= -4a(n) + 2a(2n+1) + 2a(4n+2).
\end{aligned}$$

Let P be a string of 0's and 1's that starts with a 1. We define the *pattern function* $e_P(n)$ to be the number of (possibly overlapping) occurrences of P in the binary representation of n. For example, $e_{101}(21) = 2$. The *pattern sequence* is the sequence $(e_P(n))_{n\geq 0}$.

We observe that the sequence $a(n+1) - a(n)$ has the following interesting representation as a sum of pattern sequences:

$$a(n+1) - a(n) = 1 + e_1(n) - e_{10}(n) + \left(\sum_{i\geq 0} e_{10^i 10}(n)\right) - \left(\sum_{i\geq 0} e_{10^i 1}(n)\right).$$

The proof is left to the reader.

J. Propp [37] introduced the sequence $(s(n))_{n\geq 0}$, defined to be the unique monotone sequence such that $s(s(n)) = 3n$. The table below gives the first few terms:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n)$	0	2	3	6	7	8	9	12	15	18	19	20	21	22	23	24	25

It is Sloane and Plouffe's sequence $M0747$ and Sloane's sequence $A003605$. Patruno [33] showed that

$$s(n) = \begin{cases} n + 3^k, & \text{if } 3^k \le n < 2 \cdot 3^k; \\ 3(n - 3^k), & \text{if } 2 \cdot 3^k \le n < 3^{k+1}. \end{cases}$$

This sequence is 3-regular, and satisfies the recurrence

$$\begin{aligned} s(3n) &= 3s(n); \\ s(9n+1) &= 6s(n) + s(3n+1); \\ s(9n+2) &= 6s(n) + s(3n+2); \\ s(9n+4) &= 2s(3n+1) + s(3n+2); \\ s(9n+5) &= s(3n+1) + s(3n+2); \\ s(9n+7) &= -6s(n) + 3s(3n+1) + 2s(3n+2); \\ s(9n+8) &= -12s(n) + 6s(3n+1) + s(3n+2). \end{aligned}$$

More generally, one can consider solutions to the equation $a(a(n)) = dn$ for a fixed integer $d \ge 4$. Elsewhere we show that the lexicographically least monotone increasing solution to $a(a(n)) = dn$ is d-regular, and its first differences are d-automatic [5].

Example 5. The Hurwitz-Radon function. Every $n \ge 1$ can be uniquely expressed as $n = 2^{4a+b}u$, where u is odd, and $0 \le b \le 3$. In [27, p. 131], T. Y. Lam discusses the function $\rho_F(n) = 8a + 2^b$ in connection with real periodicity and Clifford modules. The table below gives the first few terms of this sequence:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho_F(n)$	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	9

Also see [43]. This function is 2-regular. It is Sloane and Plouffe's sequence $M0161$ and Sloane's sequence $A003484$.

Example 6. The Stanton-Kocay-Dirksen sequence. In [46], Stanton, Kocay, and Dirksen studied the sequence $f(n)$, defined to be the cardinality of a certain set. More precisely, define the product of sets C, D to be $CD = \{cd \ : c \in C, d \in D\}$. Let $A(n) = \{1, 2, \ldots, n\}$, and $B = \{2^i \ : \ i \ge 0\}$; then we define $f(n) := |A(n) \setminus (A(\lfloor n/2 \rfloor)B)|$.

They showed that $f(0)=0$, and $f(n)=f(n-1)+(-1)^{\nu_2(n)}$ for $n\geq 1$. They also introduced the functions $F(n)=\sum_{1\leq k\leq n} f(k)$ and $R(n)=\sum_{1\leq k\leq n} h(k)$, where $h(k)$ is the alternating sum of the binary digits of i. More precisely, write $k=\sum_{i\geq 0} a_i(k)2^i$, where each a_i is either 0 or 1; then $h(k):=\sum_{i\geq 0}(-1)^i a_i$.

Here is a brief table of these sequences:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(n)$	0	1	0	1	2	3	2	3	2	3	2	3	4	5
$F(n)$	0	1	1	2	4	7	9	12	14	17	19	22	26	31
$R(n)$	0	1	0	0	1	3	3	4	3	3	1	0	0	1
$h(n)$	0	1	-1	0	1	2	0	1	-1	0	-2	-1	0	1

All of these sequences are 2-regular. For example, it is easy to see that the following relations hold:

$$\begin{aligned}
f(4n) &= f(4n+2) = 2f(n)+f(2n);\\
f(4n+1) &= f(4n+3) = 2f(n)+f(2n+1);\\
h(2n) &= -h(n);\\
h(2n+1) &= 1-h(n).
\end{aligned}$$

The sequence $R(n)$ is Sloane and Plouffe's sequence $M2274$ and Sloane's sequence $A005536$. The sequence $h(n)$ is Sloane's sequence $A065359$.

From the recursion relations for $h(n)$, the following relations follow easily:

$$\begin{aligned}
R(2n+1) &= n+1-R(n) \quad (n\geq 0);\\
R(4n) &= R(n)+3R(n-1) \quad (n\geq 1);\\
R(4n+2) &= 3R(n)+R(n-1) \quad (n\geq 1).
\end{aligned}$$

Combining these with the identity $R(2n)=n-R(n)-R(n-1)$ for $n\geq 1$ demonstrates that $(R(n))_{n\geq 0}$ is 2-regular.

Furthermore, $R(n)$ is always non-negative, although it is not immediately obvious from the preceding identities. To prove this fact, first observe that

$$R(n)=\sum_{0\leq k\leq n}\sum_{j\geq 0}(a_{2j}(k)-a_{2j+1}(k))$$

for $n\geq 0$. To prove $R(n)$ non-negative, it suffices to show that each of the sequences $\sum_{0\leq k\leq n}(a_{2j}(k)-a_{2j+1}(k))$ is non-negative.

To see this, observe that

$$(a_{2j}(i))_{i\geq 0} = (0^r, 1^r)^\omega, \tag{7}$$

where $r = 2^{2j}$, the exponents in the right-hand-side of Eq. (7) denote repetitions, and comma denotes concatenation. Similarly,

$$(a_{2j+1}(i))_{i\geq 0} = (0^{2r}, 1^{2r})^\omega.$$

It follows that

$$(a_{2j}(i) - a_{2j+1}(i))_{i\geq 0} = (0^r, 1^r, (-1)^r, 0^r)^\omega.$$

Hence

$$\left(\sum_{0\leq i\leq n} (a_{2j}(i) - a_{2j+1}(i))\right)_{n\geq 0} = (0^r, 1, 2, \ldots, r, r-1, r-2, \ldots, 2, 1, 0, 0^r)^\omega.$$

The result now follows.

Example 7. Length of subgroup chains. Let G be a finite group. A *subgroup chain of length* m is a strictly descending chain of the form

$$G = G_0 > G_1 > \cdots > G_m = 1.$$

Let $\ell(G)$ be the maximum possible chain length for G. In [7], L. Babai investigated $\ell(G)$ for $G = S_n$, the symmetric group on n letters. He conjectured that $\ell(S_n) = \lceil 3n/2 \rceil - s_2(n) - 1$, where $s_2(n)$ denotes the number of 1's in the binary expansion of n. This conjecture was proved by Cameron, Solomon, and Turull [9]. The sequence $\ell(S_n)$ is 2-regular, as it is the sum of three sequences, each of which is 2-regular.

Example 8. The "odious" numbers. Consider the set

$$B = \{1, 2, 4, 7, 8, 11, 13, 14, 16, \ldots\}$$

of integers containing an odd number of 1's in their base-2 expansion. Let $b_0 = 1$ and in general, let b_i be the i'th smallest number in this set. See [38, p. 22]; [28]. For sets $S \subseteq \mathbb{N}$ define a function $f : S \times \mathbb{N} \to \mathbb{N}$ as follows: $f_S : n \to$ number of solutions to $x + y = n$, with $x, y \in S$. Then Lambek and Moser [28] showed that the sets B and $A = \mathbb{N} \setminus B$ have the property that $f_A = f_B$, and furthermore these are the only two complementary sets with this property.

The sequence $(b_i)_{i\geq 0}$ is 2-regular, as it satisfies the following recurrence relations:

$$
\begin{aligned}
b_{4n} &= -2b_n + 3b_{2n};\\
b_{4n+1} &= -2b_n + 2b_{2n} + b_{2n+1};\\
b_{4n+2} &= \frac{2}{3}b_n + \frac{5}{3}b_{2n+1};\\
b_{4n+3} &= 6b_n - 3b_{2n} + 2b_{2n+1}.
\end{aligned}
$$

Example 9. The Josephus problem. The *Josephus problem* is as follows: the numbers from 1 to n are written in a circle. Starting our count with the number 1, every 2nd number that remains is crossed off until only one is left. For example, if $n = 7$, then we cross off successively $2, 4, 6, 1, 5, 3$ and we are left with 7. The "survivor" is denoted $J(n)$. The first few values of $J(n)$ are as follows:

$$1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, \cdots .$$

It is Sloane and Plouffe's sequence $M2216$ and Sloane's sequence $A006257$.

This problem was discussed by Graham, Knuth, and Patashnik [21, pp. 8–16] who observed that $J(2n) = 2J(n) - 1$ and $J(2n+1) = 2J(n) + 1$ for $n \geq 1$. It follows that $J(n)$ is 2–regular.

The same problem, where 2 is replaced by k and the result is the first uncrossed-off number encountered when there are only $k-1$ numbers left, does not appear to be k-regular in general. See [21, pp. 79–81].

We are grateful to P. Dumas for pointing out this example.

Example 10. Kimberling's paraphrases. Kimberling [24] introduced the sequence $(c(n))_{n\geq 1}$

$$1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 4, 8, \ldots$$

which arose in his study of numeration systems. It follows from his paper that this sequence is $(n/2^{\nu_2(n)} + 1)/2$, which is a 2-regular sequence. It is Sloane and Plouffe's sequence $M0145$ and Sloane's sequence $A003602$. It has the pleasant property that deleting the first occurrence of each positive integer in the sequence leaves the sequence unchanged. In fact, it is easily verified that $c(2n) = c(n)$ and $c(2n-1) = n$ for $n \geq 1$.

Example 11. The Arkin-Arney-Dewald-Ebel sequences. In [6], the sequence

$$1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 8, 8, 8, 8, 8, 9, 10, \ldots$$

is studied. This sequence $(F(n))_{n\geq 1}$ satisfies the recurrence $F(2n) = 2F(n)$; $F(2n+1) = F(n) + F(n+1)$. It is 2-regular, and is Sloane and Plouffe's sequence $M0277$ and Sloane' sequence $A006165$.

The same paper also discusses the sequence

$$(G(n))_{n\geq 1} = 1, 1, 3, 3, 3, 3, 5, 7, 9, 9, 9, 9, 9, \ldots$$

which satisfies the recurrence $G(3n) = 3G(n)$; $G(3n+1) = G(n+1) + 2G(n)$; $G(3n+2) = 2G(n+1) + G(n)$. This is 3-regular, and is Sloane and Plouffe's sequence $M2270$ and Sloane's sequence $A006166$.

Example 12. Cost of grid communications on the Connection Machine. In [49], A. Weitzman gave the following formula for $F(n)$, the optimal cost of grid communication between two processors of distance n on the Connection Machine:

$$F(j) = \begin{cases} 0, & \text{if } j = 0; \\ 1, & \text{if } j \text{ is a power of } 2; \\ 1 + \min(F(n-2^k), F(2^{k+1}-n)), & \text{if } 2^k < n < 2^{k+1}. \end{cases}$$

It is Sloane and Plouffe's sequence $M0103$ and Sloane's sequence $A007302$.

Here is a brief table of this sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F(n)$	0	1	1	2	1	2	2	2	1	2	2	3	2	3	2	2	1

This sequence has the following beautiful expansion as a sum of pattern sequences (see Example 4 for definition):

$$F(n) = e_1(n) - \sum_{i\geq 0} e_{11(01)^i 1}(n).$$

There is also a connection between $F(n)$ and representation in base-2 using the digits $0, 1, \bar{1}$, where $\bar{1} = -1$. Define the weight of such a representation to be the number of non-zero terms. For example, $10\bar{1}01$ is a representation for 13 of weight 3. Then it can be shown that $F(n)$ is the minimum weight over all such representations for n.

Example 13. Sums of squares of digits. Porges [36] has discussed properties of the sequence $b_k(n)$, defined as follows: let $n = \sum_{i\geq 0} a_i k^i$, where $0 \leq a_i < k$, and set $b_k(n) = \sum_{i\geq 0} a_i^2$. We have $b_k(kn+a) = b_k(n) + a^2$ for $0 \leq a < k$, and so it follows that this sequence is k-regular for all $k \geq 2$.

Also see [47].

Example 14. The correlation of the Thue-Morse sequence. Define the *correlation* $\gamma_f(h)$ of a sequence $(f_n)_{n\geq 0}$ of complex numbers as follows:

$$\gamma_f(h) = \lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \overline{f(n)} f(n+h).$$

if this limit exists. In the case where $f_n = (-1)^{s_2(n)}$, the Thue-Morse sequence on symbols 1 and -1, Mahler [29] proved that $\gamma_f(0) = 1$, $\gamma_f(2k) = \gamma_f(k)$, and $\gamma_f(2k+1) = -\frac{1}{2}(\gamma_f(k) + \gamma_f(k+1))$. It follows that the sequence $\gamma_f(k)$ is 2-regular (with respect to $\mathbb{Q}$ and not $\mathbb{Z}$). We are grateful to Michel Mendès France for pointing out this example.

Example 15. Kuczma's sequences. Define $f(0) = 0$, $f(2n+1) = 2f(n)$ for $n \geq 0$, and $f(2n) = 2f(n)+1$ for $n \geq 1$. Then f is 2-regular, and it is easy to show [25] that for $n \geq 1$ we have $f(n) = 2^{\lfloor \log_2 n \rfloor + 1} - n - 1$. In fact, f maps n to the number whose base-2 representation is obtained by changing every 1 to 0, and vice-versa, in the binary representation of n. It is Sloane's sequence $A035327$.

Similarly, if we define define $g(n) = f(f(n))$, then we have

$$\begin{aligned}
g(2n) &= g(n);\\
g(4n+3) &= -2g(n) + 3g(2n+1);\\
g(8n+1) &= 4g(n) + g(4n+1);\\
g(16n+5) &= -4g(2n+1) + 4g(4n+1) + g(8n+5);\\
g(16n+13) &= -8g(n) + 8g(2n+1) + g(8n+5);
\end{aligned}$$

and so $(g(n))_{n\geq 0}$ is also 2-regular.

Finally, Kuczma defined $r(n)$ to be the least non-negative integer i such that $f^i(n) = 0$, where f^i denotes the i-fold composition of f with itself. In other words, $r(n)$ is the number of blocks of adjacent identical symbols in the base-2 representation of n. It is easy to see that

$$\begin{aligned}
r(4n) &= r(2n);\\
r(4n+1) &= r(2n) + 1;\\
r(4n+2) &= r(2n+1) + 1;\\
r(4n+3) &= r(2n+1);
\end{aligned}$$

and so $(r(n))_{n\geq 0}$ is also 2-regular. It is Sloane and Plouffe's sequence $M0110$ and Sloane's sequence $A005811$.

Here are the first few values of these sequences.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	0	0	1	0	3	2	1	0	7	6	5	4	3	2	1	0	15
$g(n)$	0	0	0	0	0	1	0	0	0	1	2	3	0	1	0	0	0
$r(n)$	0	1	2	1	2	3	2	1	2	3	4	3	2	3	2	1	2

The sequence $r(n)$ has the following simple expansion as a sum of pattern sequences (see Example 4 for definition):

$$r(n) = e_1(n) + e_{10}(n) - e_{11}(n).$$

Example 16. The sparse space for Grundy's game. In [22], the sequence $(r_n)_{n\geq 0}$ is discussed:

$$0, 1, 6, 7, 10, 11, 12, 13, 18, 19, 20, 21, 24, \ldots$$

These values are $\{n \ : \ s_2(\lfloor n/2 \rfloor) \equiv 0 \pmod 2\}$. This sequence is 2-regular, and is Sloane and Plouffe's sequence $M4060$ and Sloane's sequence $A006364$. The complementary sequence $(c_n)_{n\geq 0}$:

$$2, 3, 4, 5, 8, 9, 14, 15, 16, 17, 22, 23, \ldots$$

is also 2-regular.

Example 17. Counting overlap-free words over a binary alphabet. A word w is said to be *overlap-free* if it contains no subword of the form $axaxa$, where a is a single letter and x is a (possibly empty) word. For example, `alfalfa` is not overlap-free (take a = `a`, x = `lf`), but `overlap` is. Cassaigne [13] has shown that the sequence counting the number of overlap-free binary words is a 2-regular sequence.

Example 18. Bottomley's sequence. Henry Bottomley proposed a sequence, Sloane's $A055562$, [44], defined as follows: define $a(0) = 2$ and $a(n)$ to be the least integer $> a(n-1)$ such that $a(n) \neq a(k) + a(k-1)$ for all k with $1 \leq k < n$.

An easy induction shows that $a(2n) = 3n + 1 + (\lfloor \log_2 n \rfloor \bmod 2)$ for $n \geq 1$ and $a(2n+1) = 3n + 3$ for $n \geq 0$. It follows that $(a(n))_{n\geq 0}$ is 2-regular.

There is an interesting connection between this sequence and a sequence of Kimberling (Sloane's sequence A022441). Let $b(n) = a(n) + a(n-1)$ for $n \geq 1$.

Then $b(n) = 3n + 2 - (\lfloor \log_2 n \rfloor \bmod 2)$ for $n \geq 1$. Furthermore, the sequences $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 1}$ are complementary, in the sense that $\{a(n) \ : \ n \geq 0\} \ \cup \ \{b(n) \ : \ n \geq 1\} = \{2, 3, 4, \dots\}$.

Example 19. A counterexample sequence.

Many k-regular sequences have the property that their first differences (or, more generally, tth order differences for some $t \geq 0$) are k-automatic. It might be thought the class of such sequences exhausts the k-regular sequences. But this is not the case. For example, consider the sequence $\mathbf{u} = (u_n)_{n\geq 0}$ defined by

$$u_n = \begin{cases} n, & \text{if } n \text{ is a power of 2;} \\ 0, & \text{otherwise.} \end{cases}$$

This sequence is k-regular since it satisfies the recurrence

$$\begin{aligned} a_{2n} &= 2a_n \\ a_{4n+1} &= a_{2n+1} \\ a_{4n+3} &= 0. \end{aligned}$$

Hence all its tth order differences $\Delta^t \mathbf{u}$ are also k-regular. But $\Delta^t \mathbf{u}$ is unbounded for all t and hence never k-automatic.

Example 20. Chang and Tsai's recurrence.

Chang and Tsai [14] proved an interesting theorem on recurrences, which we reformulate as follows: Let $a \geq b \geq 0$ be integers. Then the solution to the recurrence

$$S_n = \min_{1\leq k\leq n/2} (aS_{n-k} + bS_k)$$

for $n \geq 2$ is

$$S_n = S_1 + (a + b - 1)S_1 \sum_{1\leq i\leq n-1} a^{e_0(i)} b^{e_1(i)-1}.$$

Here $e_P(n)$ is the pattern function introduced above in Example 4. Then $(S_n)_{n\geq 1}$ is 2-regular for all integers $a \geq b \geq 0$.

Example 21. A recurrence from automata theory. In a recent paper [20] the second author and co-authors were led to study the recurrence given by $V(1) = 1$ and

$$V(n) = s\Big(V(\lfloor n/2 \rfloor) + V(\lceil n/2 \rceil)\Big)$$

for $n \geq 2$.

They proved that if $n = 2^a + b$ with $0 \le b < 2^a$, then $V(n) = 2bs^{a+1} + (2^a - b)s^a$. It follows that $(V(n))_{n\ge0}$ is 2-regular. In fact, if $V(0) = 0$, this sequence satisfies the following relations for $n \ge 0$:

$$\begin{aligned}
V(2n) &= 2sV(n) \\
V(4n+3) &= (4s^3 - 2s^2)V(n) + (2s^3 + 3s)V(2n+1) - 2sV(4n+1) \\
V(8n+1) &= (4s^3 - 2s^2)V(n) - sV(2n+1) + (s+1)V(4n+1) \\
V(8n+5) &= (4s^4 - 2s^3)V(n) + (2s^3 + 5s^2)V(2n+1) - 2s^2V(4n+1).
\end{aligned}$$

In the case $s = 2$, this is Sloane's sequence $A073121$.

Example 22. Carlitz's sequences related to Stirling numbers.

Let $\left\{ {n \atop r} \right\}$ denote the Stirling numbers of the second kind, i.e.,

$$\left\{ {n \atop r} \right\} = \frac{1}{r!} \sum_{0 \le j \le r} (-1)^{r-j} \binom{r}{j} j^n.$$

Carlitz [10,11] defined

$$\theta_0(n) := \mathrm{Card}\{r \ : \ \left\{ {n \atop 2r} \right\} \text{ is odd and } 0 \le 2r \le n\}$$

and later [12] $\omega_0(n) := \theta_0(n+2)$. Here is a brief table of this function:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\omega_0(n)$	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5

(Carlitz's table [11] contains an error for $n = 14$.)

Carlitz showed that [10]

$$\sum_{n\ge0} \omega_0(n) X^n = \prod_{n\ge0} (1 + X^{2^n} + X^{2^{n+1}}).$$

It now follows that $(\omega_0(n))_{n\ge0}$ is 2-regular. We have the relations

$$\begin{aligned}
\omega_0(2n+1) &= \omega_0(n) \\
\omega_0(4n) &= -\omega_0(n) + 2\omega_0(2n) \\
\omega_0(4n+2) &= \omega_0(n) + \omega_0(2n).
\end{aligned}$$

In fact, $\omega_0(n)$ is just the famous Stern-Brocot sequence shifted by 1; it is Sloane and Plouffe's sequence $M0141$ and Sloane's sequence $A002487$.

More generally, for a prime p define $\theta_j(n)$ to be the number of Stirling numbers $\left\{ {n \atop k} \right\}$, $0 \leq k \leq n$, that are relatively prime to p and such that $k \equiv j \pmod{p}$. Define $\omega_j(n) = \theta_j(n+p)$. Carlitz [12] showed that there exists a polynomial f of degree $p(p-1)$ such that if $W_0(X) := \sum_{n \geq 0} \omega_0(n) X^n$, then

$$W_0(X) = \prod_{n \geq 0} f(X^{p^n}).$$

It now follows from a result of Dumas [18, Thm. 24, p. 131] that $(\omega_0(n))_{n \geq 0}$ is a p-regular sequence.

Example 23. The "infinity series" of composer Per Nørgård.

Danish composer Per Nørgård (1932–) used a particular mathematical sequence $(c_n)_{n \geq 0}$, called by some commentators the "infinity series", in many of his music compositions. Here $(c_n)_{n \geq 0}$ is defined by $c_0 = 0$, and for $n \geq 0$ we have $c_{2n} = -c_n$, and $c_{2n+1} = c_n + 1$. The first few values of this sequence are given in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
c_n	0	1	-1	2	1	0	-2	3	-1	2	0	1	2	-1	-3	4	1
m_n	G	A♭	F♯	A	A♭	G	F	B♭	F♯	A	G	A♭	A	F♯	E	B	A♭

This is Sloane's sequence $A004718$.

For example, the first 1024 notes of the second movement of his symphony *Voyage into the Golden Screen* (1968) are defined as follows: the n'th note of the composition m_n is the note offset by c_n halftones of the chromatic scale from G (sol). The sequence $(c_n)_{n \geq 0}$ is 2-regular.

For further details, see [26] and the following web pages about Per Nørgård:

http://www.pernoergaard.dk/indexeng.html

http://www.pernoergaard.dk/eng/strukturer/uendelig/uintro.html

http://www.pernoergaard.dk/eng/strukturer/uendelig/ukonstruktion.html

http://www.pernoergaard.dk/eng/strukturer/uendelig/uhierarki.html

The second movement of *Voyage into the Golden Screen* is discussed at `http://www.pernoergaard.dk/eng/udvalgte/111b.html`

with music available at

`http://www.pernoergaard.dk/ress/musexx/m1110356.mp3`

The first 128 notes of the infinity series are available at

`http://www.pernoergaard.dk/ress/musexx/mu01.mp3` .

Another 2-regular sequence appears in the so-called "rhythmic infinity system" of Nørgård [26].

Let the Fibonacci numbers $(F_n)_{n \geq 0}$ be defined as usual by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Starting with the pair $(c_0, c_1) = (F_{2n}, F_{2n+1})$, perform the following operation $n - 2$ times:

- If a number F_i appears in an even-indexed position, replace it with (F_{i-2}, F_{i-1})
- If a number F_i appears in an odd-indexed position, replace it with (F_{i-1}, F_{i-2})

Kullberg illustrates this procedure in the case $n = 5$, as follows:

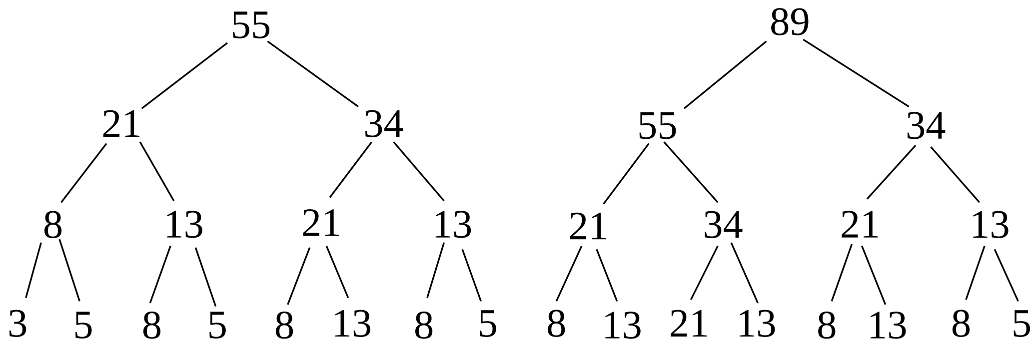

Fig. 2. Generating the rhythmic infinity series

The resulting sequence is of length 2^{n-1}. As $n \to \infty$ we get a limiting sequence $(a_i)_{i \geq 0}$:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	$\cdots$
a_i	3	5	8	5	8	13	8	5	8	13	21	13	8	13	8	5	8	13	$\cdots$

It can be shown that this sequence is 2-regular; see [42] for the details. It is Sloane's sequence $A073334$.

5 k-regular two-dimensional arrays

It is also possible to define the notion of k-regular two-dimensional array. To do so, we need a generalize the notion of k-kernel which was provided by Salon [41]. The k-kernel of a two-dimensional array $\mathbf{a} = (a(m,n))_{m,n\geq 0}$ is defined to be the set

$$\{(a(k^e m + a, k^e n + b))_{m,n\geq 0} \ : \ e \geq 0, \ 0 \leq a, b < k^e\}.$$

Again, we say $\mathbf{a}$ is k-regular if the module generated by the k-kernel is finitely generated.

One way to generate k-regular two-dimensional arrays is suggested by the following theorem:

Theorem 14 *Suppose $\mathbf{a} = (a_m)_{m\geq 0}$ and $\mathbf{b} = (b_n)_{n\geq 0}$ are k-regular sequences. Then $(a_m + b_n)_{m,n\geq 0}$ and $(a_m b_n)_{m,n\geq 0}$ are k-regular two-dimensional arrays.*

Proof. By [2, Thm. 2.2] we have that the module generated by the k-kernel of $\mathbf{a}$ (resp. $\mathbf{b}$) is generated by $\{(a_{k^i m+r})_{m\geq 0} \ : \ (i,r) \in S\}$ (resp. $\{(b_{k^j n+s})_{n\geq 0} \ : \ (j,s) \in T\}$) for some finite set S (resp. T). Then the k-kernel of $(a_m + b_n)_{m,n\geq 0}$ is a subset of

$$\{(a_{k^i m+r} + b_{k^j n+s})_{m,n\geq 0} \ : \ (i,r) \in S, \ (j,s) \in T\}.$$

Similarly, the k-kernel of $(a_m b_n)_{m,n\geq 0}$ is a subset of

$$\{(a_{k^i m+r} b_{k^j n+s})_{m,n\geq 0} \ : \ (i,r) \in S, \ (j,s) \in T\}.$$

■

We now give two examples of k-regular two-dimensional arrays.

Example 24. Let r, s be non-negative integers with base-2 representation given by $\sum_{0\leq i<t} c_i 2^i$ and $\sum_{0\leq i<t} d_i 2^i$, respectively. Define the *Nim-sum* of two integers, $r \oplus s$, to be the integer given by $\sum_{0\leq i<t}((c_i + d_i) \bmod 2)2^i$ ([50, p. 19], [17, Chapter 6], [8]). Consider the two-dimensional array $\mathbf{N} = (m \oplus n)_{m,n\geq 0}$. Here are the first few rows and columns of this array:

$\oplus$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	7	6	9	8
2	2	3	0	1	6	7	4	5	10	11
3	3	2	1	9	7	6	5	4	11	10
4	4	5	6	7	0	1	2	3	12	13
5	5	4	7	6	1	0	3	2	13	12
6	6	7	4	5	2	3	0	1	14	15
7	7	6	5	4	3	2	1	0	15	14
8	8	9	10	11	12	13	14	15	0	1
9	9	8	11	10	13	12	15	14	1	0

It is easily seen that **N** is 2-regular, as we find

$$\begin{aligned}\mathbf{N}[2i,2j] &= \mathbf{N}[2i+1,2j+1] = 2\mathbf{N}[i,j] \\ \mathbf{N}[2i+1,2j] &= \mathbf{N}[2i,2j+1] = 2\mathbf{N}[i,j]+1.\end{aligned}$$

Example 25. Let F be a field with characteristic $\neq 2$. Define $D_F(n) = \{a \in F^* \ : \ a \text{ is a sum of } n \text{ squares in } F\}$. Pfister proved that there is a binary operation on $\mathbb{N} \times \mathbb{N}$ ($\mathbb{N} = \{1, 2, 3, \dots\}$) such that $D_F(r)D_F(s) = D_F(r \circ s)$; see [43, pp. 250–252].

Here is a short table of the function $\circ$:

$\circ$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	2	4	4	6	6	8	8	10	10
3	3	4	4	4	7	8	8	8	11	12
4	4	4	4	4	8	8	8	8	12	12
5	5	6	7	8	8	8	8	7	13	14
6	6	6	8	8	8	8	8	8	14	14
7	8	8	8	8	8	8	8	8	15	16
8	8	8	8	8	8	8	8	8	16	16
9	9	10	11	12	13	14	15	16	16	16
10	10	10	12	12	14	14	16	16	16	16

It can be shown using results of Pfister [34,35] that the operation $\circ$ satisfies the following identities:

$$\begin{aligned} 2m \circ 2n &= 2(m \circ n); \\ (2m-1) \circ 2n &= 2(m \circ n); \\ 2m \circ (2n-1) &= 2(m \circ n); \\ (2m-1) \circ (2n-1) &= 2(m \circ n) - (\binom{m+n-2}{m-1} \bmod 2). \end{aligned}$$

It follows that the infinite array $(m \circ n)_{m,n \geq 1}$ is 2-regular.

6 Recognizing a k-regular sequence

Some tips for recognizing an unknown sequence are given by Sloane and Plouffe [45]. The k-regular sequences form another easy-to-recognize class.

Given a sequence $\mathbf{s} = (s_n)_{n \geq 0}$, how can we determine if it is k-regular? Evidently no finite examination of the values of $\mathbf{s}$ will suffice. But in practice the following procedure often succeeds in deducing the k-regular relations for $\mathbf{s}$ from knowledge of the first few values. The basic idea is to construct a matrix in which the rows represent truncated versions of elements of the k-kernel, together with row reduction.

(1) Initialize the matrix to have as its single row the elements s_n, $0 \leq n \leq M$, where M is an arbitrarily chosen upper limit. In practice one may start with $M = 100$.
(2) Now repeat the following step: if the subsequence $(s(k^j n + c))_{n \geq 0}$ is not linearly dependent on the previous sequences, add the $(s(k^j(kn + a) + c))_{n \geq 0}$ for $0 \leq a < k$ as rows, using as many terms as you have. Decrease the number of columns of the matrix appropriately.
(3) As elements further out in the k-kernel are examined, the number of columns of the matrix that are known in all entries decreases. If rows that are previously linearly independent suddenly become dependent with the elimination of terms further out in the sequence, then no relation can be accurately deduced; stop and retry after computing more terms.
(4) When no more linearly independent sequences can be found, you have found hypothetical relations for the sequence. These can then be verified through induction or other means.

There are some variations possible on this procedure. For example, we may initialize our matrix to have two or more rows, in which all but the last correspond to known sequences such as the constant sequence 1; the last is reserved for the sequence you wish to analyze. This frequently results in much smaller lists of relations.

This procedure has been implemented in APL, and has discovered many sequences known or conjectured to be k-regular.

As an example, let us consider a sequence due to N. Strauss [48]. Define

$$r(n) = \sum_{0 \leq i < n} \binom{2i}{i},$$

and, as above, let $f(n) = \nu_3(r(n+1))$ be the exponent of the highest power of 3 that divides $r(n+1)$. The following table gives the first few values of f:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$f(n)$	0	1	2	0	2	3	1	2	4	0	1	2	0	3	4	2	3	5	1	2

Our k-regular sequence recognizer easily produced the following conjectured relations:

$$
\begin{aligned}
f(3n+2) &= f(n)+2;\\
f(9n) &= f(9n+3) = f(3n);\\
f(9n+1) &= f(3n)+1;\\
f(9n+4) &= f(9n+7) = f(3n+1)+1;\\
f(9n+6) &= f(3n+1).
\end{aligned}
$$

With a little more work, one arrives at the conjecture

$$\nu_3(r(n)) = \nu_3(n^2\binom{2n}{n}),$$

which we proved in 1989.

A beautiful proof of this identity using 3-adic analysis was later given by Don Zagier [51]. Zagier showed that if we set

$$F(n) = \frac{\sum_{0\le k<n}\binom{2k}{k}}{n^2\binom{2n}{n}},$$

then $F(n)$ extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1+3\mathbb{Z}_3$, and can be evaluated at the negative integers as follows:

$$F(-n) = -\frac{(2n-1)!}{(n!)^2}\sum_{0\le k<n}\frac{(k!)^2}{(k-1)!}$$

for $n \ge 0$.

A heuristic k-regular sequence recognizer can produce many interesting conjectures. For example, let

$$a(n) = \sum_{0\le k\le n}\binom{n}{k}\binom{n+k}{k}.$$

Let $b(n) = \nu_3(a(n))$. Then computer experiments strongly suggest:

$$b(n) = \begin{cases} b(\lfloor n/3\rfloor) + (\lfloor n/3\rfloor \bmod 2), & \text{if } n \equiv 0,2 \pmod 3;\\ b(\lfloor n/9\rfloor)+1, & \text{if } n\equiv 1 \pmod 3.\end{cases}$$

This recurrence has been verified for $0 \le n \le 10,000$, but no proof of the conjecture is currently known.

7 Open Problems

In this section we list some open problems about k-regular sequences.

1. Prove or disprove: $(\lfloor \frac{1}{2} + \log_2 n \rfloor)_{n \geq 1}$ is not a 2-regular sequence.

Comment. Suppose $a(n) = \lfloor \frac{1}{2} + \log_2 n \rfloor$ is 2-regular. Define $b(n) := a(n + 1) - a(n)$ for $n \geq 1$. Then $(b(n))_{n \geq 0}$ would be 2-automatic, and is over the alphabet $\{0, 1\}$. The 1's in b are in positions $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, $c_4 = 11$, $c_5 = 22$, $c_6 = 45$, $c_7 = 90$, etc. Then $c_{i+1} - 2c_i$ is the i'th bit in the binary expansion of $\sqrt{2}$.

2. Suppose S and T are k–regular sequences and $T(n) \neq 0$ for all n. Prove or disprove: if $S(n)/T(n)$ is always an integer, then $S(n)/T(n)$ is k–regular.

Comment. This is an analogue of van der Poorten's Hadamard quotient theorem [39,40].

3. Prove or disprove: if a sequence $(a(n))_{n \geq 0}$ is simultaneously k- and l-regular, where k and l are multiplicatively independent, then $(a(n))_{n \geq 0}$ satisfies a linear recurrence.

Comment. This is an analogue of a famous theorem of Cobham [15] for automatic sequences.

4. Prove or disprove: if q is a polynomial taking integer values and p is a prime, then $(\nu_p(q(n)))_{n \geq 0}$ is either ultimately periodic or not p-regular.

Comment. If we understood, for example, the sequence $\nu_5(n^2 + 1)$, then we would understand the 5-adic expansion of $\sqrt{-1}$.

5. Show that if $S(n)$ is k–regular and unbounded, then it takes on infinitely many composite values.

6. Suppose $(a(n))_{n \geq 0}$ is a k-regular sequence of integers such that $a(n)$ is a perfect square for all $n \geq 0$. Prove or disprove: $(a(n)^{1/2})_{n \geq 0}$ is a k-regular sequence.

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