Université Sorbonne Paris Nord, Université de Paris et Sorbonne Université

Mémoire de stage

Skein categories, quantum groups and associators

Juan Ramón Gómez García

sous la direction de

Adrien Brochier

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Introduction

The revolutionary work on knot invariants presented by Jones in the 1980s supposed a revolution in knot theory, revealing a new class of invariants called quantum invariants, which revealed the existence of interactions never before explored between low-dimensional topology and certain hitherto exotic algebraic structures such as Hopf algebras. These connections have since given rise to a large number of interesting results and constructions in different branches of mathematics and physics, such as quantum mechanics, statistical mechanics, quantum group theory or the theory of Lie algebras. In this report we review three constructions that have arisen in this context: the category of ribbon graphs, the oriented-framed skein category and the category of chord diagrams.

The category $\mathrm{Rib}_{\mathcal{V}}$ of ribbon graphs over a ribbon category $\mathcal{V}$ and, in particular, the subcategory of tangles, provides an appropriate algebraic structure for a family of low-dimensional topological objects, the tangles, that generalise knots providing a suitable environment for a systematic study of topological invariants. It is a category whose morphisms are isotopy classes of a certain type of surfaces and which is universal among ribbon categories. Specifically, for any ribbon category $\mathcal{V}$, there is a uniquely determined functor

$$F : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V},$$

which connects the topological structure of ribbon graphs with the algebraic properties of the categories with the same name. The existence of this connection has interesting immediate consequences since it allows us to obtain topological properties from the algebra by constructing invariants but also, in the other way round, it allows us to deduce algebraic results from the topology of the surfaces.

The second of the categories we will be interested in, the oriented-framed skein category $\mathcal{OS}(z,t)$, is directly connected to quantum groups. Quantum groups were introduced between 1983 and 1985 by Drinfeld and Jimbo, giving a precise mathematical formulation to the work of some physicists on the Yang Baxter equation. Roughly speaking, they can be described as uniparametric deformations of the enveloping algebra of semisimple Lie algebras. These algebras can be described in terms of generators and relations. By modifying these relations in an appropriate way through the introduction of a parameter q, it is possible to obtain new objects, which include the classical objects as a particular case (by taking $q = 1$ we recover the initial enveloping algebra) and which possess interesting properties. One of these properties is the fact that their category of representations has a ribbon category structure and can therefore be used in the study of topological invariants. Although their combinatorial description is complicated, quantum groups admit a graphical representation through diagrams that can be organised into a certain category, which turns out to be equivalent to the quotient of the linearisation of the tangle category by certain skein relations.

The third object to be described is the category of chord diagrams, which has its origin in the Vassiliev invariants of knots. These invariants were introduced around 1989 by Vassiliev and brought out once again the deep connection between the theory of invariants of knots, Lie theory and quantum theory. Vassiliev's idea was to use singular knots with a finite number of self-intersections to describe certain subspaces of linear combinations of knots. Using this description, a Vassiliev invariant of degree n can be defined non-constructively as an invariant which cancels out in the subspace represented by the singular knots with $> m$ self-intersections. The introduction of singular knots produces a filtration in the space of linear combinations of isotopy classes of knots which is compatible with the filtration of the vector space of all Vassiliev invariants and whose study is sufficient for the understanding of all finite type invariants. It quickly follows that the invariant of degree m of a knot with m double points depend only on the combinatorics of the self-intersection points of singular knots. This combinatorics can be easily described in terms of diagrams on the circle constructed by adding a series of chords connecting two points whose image is a self-intersection point. In particular, each Vassiliev invariant produces a function on the space of linear combinations of chords diagrams which descends to the quotient $\mathcal{A}$ of this vector space modulo the so-called one-term and four-term relations. The main result of Vassiliev's theory states that the Vassiliev vector space of invariants is isomorphic, as a filtered vector space, to the graded dual of $\mathcal{A}$, where the graduation comes from the number of chords. The proof of this result consists in the construction of a certain invariant, the Kontsevich integral, which is universal in the sense that it captures all finite type invariants and also all quantum invariants. The chord diagrams can be organised into a category $\mathcal{CD}$, which will be the object of study in the third section, that admits a structure of symmetric infinitesimal category. The theory developed by Drinfeld shows that any such category can be deformed into a larger category $\mathcal{CD}^{\Phi}$ which can be equipped with a ribbon structure using the so-called Drinfeld associators. The universal property of the tangle category then establishes the existence of a functor

$$F : \mathrm{Rib}_{\mathcal{CD}^{\Phi}} \to \mathcal{CD}^{\Phi}$$

which turns out to be a generalisation of the Kontsevich invariant.

It follows from all this discussion that there are two ways to produce ribbon categories. The first, through the category of representations of quantum groups, is explicit but unnatural. The second construction, based on the existence of Drinfeld associators, is less explicit but more general and conceptually clearer. It turns out that there is indeed a connection between both. Using again the theory of Drinfeld associators, it is possible to obtain a ribbon category $\mathrm{Rep}^{\Phi}G$ from the category of representations of a certain class of group G. In the same way that the category of representations of the associated quantum group, $\mathrm{Rep}_q G$ is closely related to a quotient of the (ribbon) category of tangles, it is possible to construct a functor $\bar{F} : \mathcal{CD}^{\Phi} \to \mathrm{Rep}^{\Phi}G$ with the property that any invariant constructed using the universal property of the category of tangles over $\mathrm{Rep}^{\Phi}G$ is obtained by specialising an invariant $F : \mathrm{Rib}_{\mathcal{CD}^{\Phi}} \to \mathcal{CD}^{\Phi}$. Finally, a fundamental theorem in quantum algebra states the existence of an equivalence of ribbon categories

$$\mathrm{Rep}_q G \simeq \mathrm{Rep}^{\Phi}G,$$

which establishes the link between the three categories we are going to study.

1 The category of ribbon graphs

The discovery of the Jones polynomial in 1984 radically transformed the study of knot invariants, as it revealed the existence of previously unexplored relationships between objects of low-dimensional topology and certain abstract algebraic structures, relatively exotic until then. However, the set of knots in $\mathbb{R}^3$ is too narrow to define interesting structures to exploit these interconnections. One way to solve this difficulty is to consider the knots as part of a larger, well-structured object on which sophisticated algebraic techniques can be properly applied. In particular, knots are a specific case of a wider family of topological objects, the ribbon graphs, which can be organised into a category defined from their elementary topological properties. It turns out that this category provides the natural environment in which to explore the connections between low-dimensional topology and certain interesting algebraic structures: it possesses a universal property that intimately connects it to the ribbon categories. This connection can be exploited in two directions: on the one hand, it produces a machinery for generating topological invariants; on the other hand, it allows one to use surface topology to produce algebraic results, for example through the graph calculus on monoidal categories that we will discuss in more detail later.

1.1 Ribbon categories

First of all, we introduce the algebraic background on which the rest of the constructions presented in this report will be based. This is provided by the strict monoidal categories, which constitute the most elementary natural example of "categories with additional structure". Roughly speaking, they are categories with a tensor product. When, in addition, they are equipped with braiding, twist and duality, they give rise to particularly interesting objects in the construction of certain topological invariants: the ribbon categories. The definition of these categories has been inspired, to a large extent, by the Hopf algebras, which we briefly discuss in the appendix.

1.1.1 Strict monoidal categories. A *tensor product* in a category $\mathcal{V}$ is a covariant (bi)functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. In particular, we have the following identities

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g'),$$

$$\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W},$$

whenever they make sense.

Definition 1.1.1. A *strict monoidal category* is a category $\mathcal{V}$ endowed with a tensor product and a distinguished object $\mathbf{1} \in \mathcal{V}$, called *the unit object*, such that

$$(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad V \otimes \mathbf{1} = V, \quad \mathbf{1} \otimes V = V,$$

$$(f \otimes g) \otimes h = f \otimes (g \otimes h), \quad f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f,$$

for all objects U, V, W and all morphisms f, g, h in $\mathcal{V}$.

Strict monoidal categories are a particular case of (not necessarily strict) monoidal categories. The general definition and main properties of these categories are given in the appendix.

A first example of strict monoidal category is provided by the category of tangles, that we will treat in further detail in the next section:

Example 1.1.2 (A first definition of a category of tangles)**.** An *unoriented tangle* is the image of a piecewise smooth embedding $f : \left(\mathbb{S}^1\right)^{\sqcup n} \sqcup [0,1]^{\sqcup m} \to \mathbb{R}^2 \times [0,1]$ such that boundary point maps to boundary points and interior points maps to interior points. Points lying in $\mathbb{R}^2 \times \{0\}$ are the *inputs* of the tangle and those lying in $\mathbb{R}^2 \times \{1\}$ are the *outputs*.

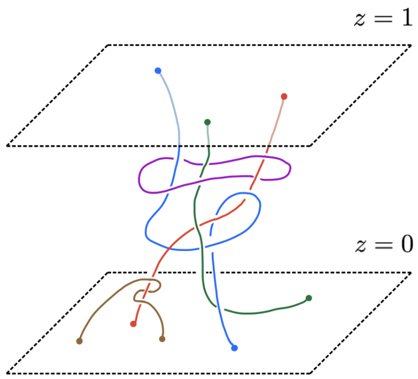


Figure 1: A tangle with 5 inputs and 3 outputs.

For any $p, q \geqslant 0$, let $\tilde{T}_{p,q}$ be the set of all tangles which have p inputs and q outputs, all of them lying in $\mathbb{R} \times \{0\} \times \{\pm 1\}$. Let $T_{p,q}$ be the set of isotopy classes of elements of $\tilde{T}_{p,q}$ such that during the isotopy the inputs and outputs stay in $\mathbb{R} \times \{0\} \times \{\pm 1\}$ and are not allowed to meet each other. We can define a canonical multiplication map $T_{p,q} \times T_{q,r} \to T_{p,r}$ by concatenating two representatives of the isotopy classes such that the outputs of the first one coincides with the inputs of the second one, and rescaling.

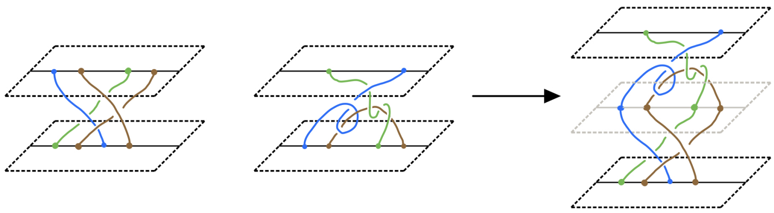


Figure 2: Concatenation of two tangles $t_1 \in T_{4,4}$ and $t_2 \in T_{4,2}$.

We define now the *category of unoriented tangles up to isotopy* as the category $\mathcal{T}$ whose objects are nonnegative integers and, for every $p, q \in \mathbb{N}$, $\mathrm{Hom}_{\mathcal{T}}(p, q) = T_{p,q}$. The composition is given by the concatenation of tangles that we have defined above and, for each $p \in \mathbb{N}$, the identity $\mathrm{id}_p \in T_{p,p}$ is the tangle consisting of p vertical segments. This category admits the following structure of monoidal category: for each $p, q \in \mathbb{N}$, set $p \otimes q \coloneqq p + q$ and, for each pair of tangles $t_1 \in T_{p,q}$, $t_2 \in T_{r,s}$, $t_1 \otimes t_2$ is the class of isotopy of the union of two representatives of t_1 and t_2 in such a way that every input/output of the first tangle is to the left of every input/output of the second one, as illustrated in the following picture. We will discuss the details of this construction in later sections.

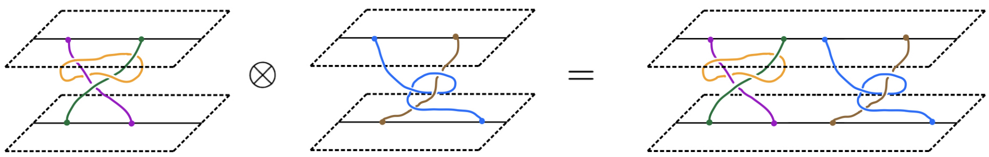


Figure 3: Tensor product of two tangles.

1.1.2 Presentation of a strict monoidal category by generators and relations. One of the most efficient ways to describe a group is by generators and relations. A similar formalism works for strict monoidal categories, as we will see in this subsection.

Let $\mathcal{V}$ be a strict monoidal category and $\mathcal{F}$ a family of morphisms in $\mathcal{V}$. We define *words in the alphabet* $\mathcal{F}$ inductively in the following way:

- A word of rank 1 is a symbol $f \in \mathcal{F}$ or a symbol id_V where V is an object of $\mathcal{V}$. For a word a of rank 1, we denote $\langle a \rangle$ the morphism in $\mathcal{V}$ represented by a. The only subword of such a word is the word itself.
- Suppose that words of rank $\leqslant n$ has been already defined and that for every such a word a, we have a morphism $\langle a \rangle$ in $\mathcal{V}$ and a collection of subwords. Then, the words of rank $\leqslant n+1$ are defined to be the symbols of the form $a \circ b$, with a, b of rank $\leqslant n$ and $\mathrm{source}(\langle b \rangle) = \mathrm{target}(\langle a \rangle)$, and $a \otimes b$, where a, b are words of rank $\leqslant n$. We set $\langle a \circ b \rangle := \langle a \rangle \circ \langle b \rangle$ and $\langle a \otimes b \rangle := \langle a \rangle \otimes \langle b \rangle$. By a subword of $a \circ b$ or $a \otimes b$ we mean the word itself or a subword of a or b.

Let $\{c_j, d_j\}_{j \in J}$ be a family of words in the alphabet $\mathcal{F}$ such that $\langle c_j \rangle = \langle d_j \rangle$, for all $j \in J$. We say that a and b are equivalent in the alphabet $\mathcal{F}$ if there exists a finite sequence $a = a_0, a_1 \ldots, a_k = b$ such that, for each $i = 0, \ldots k-1$, a_{i+1} is obtained from a_i replacing a subword with another subword such that either for some $j \in J$ these subwords are c_j and d_j, or the are the two sides of one of the following relations:

$$(f \circ g) \circ h \sim f \circ (g \circ h), \tag{M1}$$

$$f \circ \mathrm{id}_{s(f)} \sim f, \quad \mathrm{id}_{t(f)} \circ f \sim f, \tag{M2}$$

$$\mathrm{id}_V \circ \mathrm{id}_V \sim \mathrm{id}_V, \tag{M3}$$

$$(f \otimes g) \otimes h \sim f \otimes (g \otimes h), \tag{M4}$$

$$\mathrm{id}_{\mathbf{1}} \otimes f \sim f, \quad f \otimes \mathrm{id}_{\mathbf{1}} \sim f, \tag{M5}$$

$$\mathrm{id}_V \otimes \mathrm{id}_W \sim \mathrm{id}_{V \otimes W}, \tag{M6}$$

$$(f \otimes g) \circ (f' \otimes g') \sim (f \circ f') \otimes (g \circ g'), \tag{M7}$$

where V, W are objects of $\mathcal{V}$, f, f', g, g', h are elements of $\mathcal{F}$ and $s(f) := \mathrm{source}(f)$, $t(f) := \mathrm{target}(f)$.

We write $a \sim b$ if the words a and b are equivalent and we say that $\langle \mathcal{F} \ : \ c_j = d_j, j \in J \rangle$ is a *presentation of* $\mathcal{V}$ *by generators and relations* if

(i) $\mathcal{F}$ generates $\mathcal{V}$, i.e., each morphism of $\mathcal{V}$ is equal to $\langle a \rangle$ for some word a in the alphabet $\mathcal{F}$,
(ii) for any words a and b in the alphabet $\mathcal{F}$, the equality $\langle a \rangle = \langle b \rangle$ holds if and only if a is equivalent to b modulo the relations $\{c_j = d_j\}_{j \in J}$.

The following three lemmas will be useful in latter sections. We suppose that $\mathcal{F}$ is a collection of morphisms in a strict monoidal category $\mathcal{V}$. By equivalence of words, we mean equivalence modulo de empty set of relations.

Lemma 1.1.3. If $f_1, f_2, \ldots, f_n \in \mathcal{F}$ and the composition $f_1 \circ f_2 \circ \cdots \circ f_n$ is defined, then for any objects $V, W \in \mathcal{V}$, we have

$$(\mathrm{id}_V \otimes f_1 \otimes \mathrm{id}_W) \circ \cdots \circ (\mathrm{id}_V \otimes f_n \otimes \mathrm{id}_W) \sim \mathrm{id}_V \otimes (f_1 \circ \cdots \circ f_n) \otimes \mathrm{id}_W.$$

Proof. For $n = 2$, we have

$$\begin{aligned}(\mathrm{id}_V \otimes f_1 \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes f_2 \otimes \mathrm{id}_W) &\sim [(\mathrm{id}_V \otimes f_1) \circ (\mathrm{id}_V \otimes f_2)] \otimes (\mathrm{id}_W \circ \mathrm{id}_W) \\ &\sim (\mathrm{id}_V \circ \mathrm{id}_V) \otimes (f_1 \circ f_2) \otimes (\mathrm{id}_W \circ \mathrm{id}_W) \sim \mathrm{id}_V \otimes (f_1 \circ f_2) \otimes \mathrm{id}_W.\end{aligned}$$

The general case then directly follows by induction. ■

Lemma 1.1.4. If $f : X \to U$ and $g : V \to W$ are morphisms from $\mathcal{F}$ then the words $(f \otimes \mathrm{id}_W) \circ (\mathrm{id}_X \otimes g)$ and $(\mathrm{id}_U \otimes g) \circ (f \otimes \mathrm{id}_V)$ are equivalent.

Proof. We have

$$\begin{aligned}(f \otimes \mathrm{id}_W) \circ (\mathrm{id}_X \otimes g) &\sim (f \circ \mathrm{id}_X) \otimes (\mathrm{id}_W \circ g) \sim f \otimes g \\ &\sim (\mathrm{id}_U \circ f) \otimes (g \circ \mathrm{id}_V) \sim (\mathrm{id}_U \otimes g) \circ (f \otimes \mathrm{id}_V).\end{aligned}$$

■

Lemma 1.1.5. Every word in the alphabet $\mathcal{F}$ is equivalent either to a word id_V where V is an object of $\mathcal{V}$, or to a word of the form

$$(\mathrm{id}_{V_1} \otimes f_1 \otimes \mathrm{id}_{W_1}) \circ (\mathrm{id}_{V_2} \otimes f_2 \otimes \mathrm{id}_{W_2}) \circ \cdots \circ (\mathrm{id}_{V_k} \otimes f_k \otimes \mathrm{id}_{W_k}),$$

where $k \geqslant 1$, $f_i \in \mathcal{F}$ and $V_1, \ldots, V_k, W_1, \ldots, W_k$ are objects of $\mathcal{V}$.

Proof. By induction on the rank. The result is obvious for words of rank 1: if $f \in \mathcal{F}$, then $f \sim \mathrm{id}_{\mathbf{1}} \otimes f \otimes \mathrm{id}_{\mathbf{1}}$. Assume that the result is true for words of rank $\leqslant n$ and let a be a word of rank $\leqslant n+1$. Then either $a = b \otimes c$ or $a = b \circ c$, where b and c are certain words of rank $\leqslant n$. By the induction hypothesis, we may assume that b and c are words of the form of the lemma.

The case $a = b \circ c$ is obvious. Suppose that $a = b \otimes c$ and neither b nor c are of the form id_V. Let $b = b_1 \circ \cdots \circ b_k$ and $c = c_1 \circ \cdots \circ c_l$, where all b_i and c_j are words of the form $\mathrm{id}_V \otimes f \otimes \mathrm{id}_W$. Set $S = \mathrm{source}(b_k)$ and $T = \mathrm{target}(c_1)$. Then

$$a = b \otimes c \sim b \circ (\mathrm{id}_S)^l \otimes (\mathrm{id}_T)^k \circ c \sim (b_1 \otimes \mathrm{id}_T) \circ \cdots \circ (b_k \otimes \mathrm{id}_T) \circ (\mathrm{id}_S \otimes c_1) \circ \cdots \circ (\mathrm{id}_S \otimes c_l).$$

Thus, a has the desired form. The cases where $b = \mathrm{id}_V$ and/or $c = \mathrm{id}_W$ are treated similarly. ■

1.1.3 Strict braided monoidal categories.

A *braiding* in a monoidal category $\mathcal{V}$ is a natural family of isomorphisms

$$c = \{c_{V,W} : V \otimes W \to W \otimes V \ : \ V, W \in \mathcal{V}\}$$

such that

$$\begin{aligned} c_{U,V \otimes W} &= (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W), \\ c_{U \otimes V,W} &= (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}), \end{aligned}$$

for all objects $U, V, W \in \mathcal{V}$. In particular, taking $V = W = \mathbf{1}$ in the first identity, one has

$$c_{U,\mathbf{1}} = (\mathrm{id}_{\mathbf{1}} \otimes c_{U,\mathbf{1}\otimes\mathbf{1}}) \circ (c_{U,\mathbf{1}}, \otimes \mathrm{id}_{\mathbf{1}}) = c_{U,\mathbf{1}} \circ c_{U,\mathbf{1}}.$$

Hence, $c_{U,\mathbf{1}} = \mathrm{id}_U$, and applying the same argument to the second equality, we also get $c_{\mathbf{1},U} = \mathrm{id}_U$, for all object U in $\mathcal{V}$. Any braiding satisfies the *Yang-Baxter identity*:

$$(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}),$$

for all $U, V, W \in \mathcal{V}$. This is proven algebraically in the appendix for an arbitrary monoidal category.

Definition 1.1.6. A *strict braided monoidal category* is a strict monoidal category $\mathcal{V}$ endowed with a braiding.

If the braiding verifies

$$c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W},$$

for all objects V, W, we say that it is *symmetric*. Finally, it is straightforward to prove (cf. appendix) that the *reverse braiding*

$$\bar{c}_{V,W} \coloneqq c^{-1}_{W,V}$$

is also a braiding.

1.1.4 Duality in strict monoidal categories.

We now introduce the notion of duality in a strict monoidal category. To motivate this definition consider the category $\mathbf{Vec}_{\Bbbk}^{f}$ of finite dimensional vector spaces. Each $V \in \mathbf{Vec}_{\Bbbk}^{f}$ has a dual $V^* \in \mathbf{Vec}_{\Bbbk}^{f}$. Fix a basis $\{v_i\}_i$ and let $\{v^i\}_i$ be its dual basis. We have natural isomorphisms

$$\mathrm{ev}_V : V^* \otimes V \to \Bbbk, \quad v^i \otimes v_i \mapsto \langle v^i, v_i \rangle$$

$$\delta_V : \Bbbk \to V \otimes V^*, \quad 1 \mapsto \sum_i v_i \otimes v^i,$$

where $\langle\ ,\ \rangle$ is the duality pairing. It is then straightforward to check that the following compositions are equal to the identity:

$$V \xrightarrow{\delta_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V,$$

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes \delta_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^*}} V^*.$$

We can generalize theses objects and maps in the following way:

Definition 1.1.7. Let $\mathcal{V}$ be a strict monoidal category and $V \in \mathcal{V}$ an object. A *right dual* to V is an object $V^* \in \mathcal{V}$ with two morphisms

$$b_V : \mathbf{1} \to V \otimes V^*,$$

$$d_V : V^* \otimes V \to \mathbf{1},$$

such that the compositions

$$V \xrightarrow{b_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes d_V} V,$$

$$V^* \xrightarrow{\mathrm{id}_{V^*} \otimes b_V} V^* \otimes V \otimes V^* \xrightarrow{d_V \otimes \mathrm{id}_{V^*}} V^*.$$

are equal to id_V and id_{V^*}, respectively. These requirements are called *rigidity axioms.*

1.1.5 Twists.

A *twist* in a strict monoidal category $\mathcal{V}$ is a natural family of isomorphisms

$$\theta = \{\theta_V : V \to V \ : \ V \in \mathcal{V}\},$$

such that, for any two objects $V, W \in \mathcal{V}$, we have

$$\theta_{V \otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W).$$

Using the naturality of the braiding, we may also write

$$\theta_{V \otimes W} = c_{W,V} \circ (\theta_V \otimes \theta_W) \circ c_{V,W} = (\theta_V \otimes \theta_W) \circ c_{V,W} \circ c_{W,V}.$$

We have seen that in a strict monoidal category $c_{\mathbf{1},\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, so we have

$$\theta_{\mathbf{1}} = \theta_{\mathbf{1} \otimes \mathbf{1}} = \theta_{\mathbf{1}} \otimes \theta_{\mathbf{1}} = (\theta_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}}) \circ (\mathrm{id}_{\mathbf{1}} \otimes \theta_{\mathbf{1}}) = \theta_{\mathbf{1}} \circ \theta_{\mathbf{1}},$$

Thus $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$ by the invertivility of $\theta_{\mathbf{1}}$.

1.1.6 Definition of ribbon category.

We are now ready to give the definition of a ribbon category. One of the most important features about this constructions is that it allows a consistent theory of traces and dimensions, as we will see in latter sections.

Definition 1.1.8. A *ribbon category* is a monoidal category $\mathcal{V}$ equipped with a braiding c, a twist c and a duality $(*, b, d)$ verifying the following compatibility axiom:

$$(\theta_V \otimes \mathrm{id}_{V^*}) \circ b_V = (\mathrm{id}_V \otimes \theta_{V^*}) \circ b_V,$$

for all $V \in \mathcal{V}$.

1.2 Ribbon graphs and operator invariants

Roughly speaking, ribbon graphs are compact surfaces in $\mathbb{R}^3$ that can be obtained by gluing together three elementary pieces: bands, annuli and coupons. Labelling theses pieces by objects and morphisms of a ribbon category $\mathcal{V}$, ribbon graphs can be turned into a category $\mathrm{Rib}_{\mathcal{V}}$ which topologically encodes the algebraic properties of $\mathcal{V}$.

1.2.1 Ribbon graphs and their diagrams.

The first step in the construction of $\mathrm{Rib}_{\mathcal{V}}$ consists in giving formal definitions of the elementary pieces that we mentioned above and using then to construct ribbon graphs. A *band* is a homeomorphic image of the square $[0,1] \times [0,1]$ in $\mathbb{R}^3$. The image of the segment $\{1/2\} \times [0,1]$ is called the *core* of the band and the images of $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are its *basis*. We will say that the band is *directed* if its core carries an orientation. An *annulus* is a homeomorphic image of the cylinder $\mathbb{S}^1 \times [0,1]$ in $\mathbb{R}^3$. The *core* of the annulus is the image of $\mathbb{S}^1 \times \{1/2\}$. The annulus is said to be *directed* if its core is oriented. A *coupon* is a band with a distinguished base, called the *bottom base* of the coupon. The opposite based is said to be the *top* one.

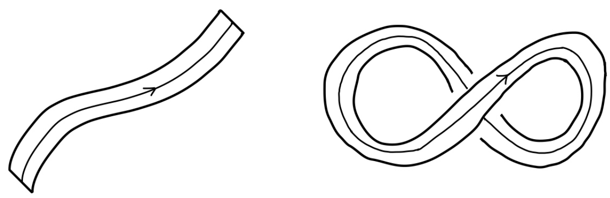


Figure 4: Directed band and directed annulus.

Definition 1.2.1. Let k, l be two non-negative integers. A (k,l)-*ribbon graph* is an oriented surface Ω embedded in $\mathbb{R}^2 \times [0,1]$ and decomposed into an union of bands, annuli and coupons such that

(i) Ω intersects the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ along the segments

$$\{[i - 1/10, i + 1/10] \times \{0\} \times \{0\} \; : \; i = 1, \ldots, k\}$$

and

$$\{[j - 1/10, j + 1/10] \times \{0\} \times \{1\} \; : \; j = 1, \ldots, l\},$$

which are the bases of certain bands of Ω. In the points of theses segments, the orientation of Ω is determined by the pair of vectors $(1,0,0)$ and $(0,0,1)$ tangent to Ω;

(ii) other bases of bands lie on the bases of coupons; otherwise, bands, annuli and coupons are disjoint;

(iii) the bands and annuli of Ω are directed.

We will consider ribbon graphs up to isotopy. Precisely, by isotopy of ribbon graphs, we mean an isotopy in the strip $\mathbb{R}^2 \times [0,1]$ constant on the boundary intervals and preserving the splitting into annuli, bands and coupons as well as the directions of bands and coupons and the orientation of the surface. By rotating an annulus around its core by the angle of π we get the same annulus with the opposite orientation, so the orientations of annuli are then superfluous. On the other hand, the orientation of bands and coupons determine a "preferred side" of the surface Ω.

Ribbon graphs allow a more manageable representation in terms of diagrams, which can be constructed as follows. Using isotopy, we can always deform a ribbon graph in such a way that coupons and bans are parallel to the plane $\mathbb{R} \times \{0\} \times \mathbb{R}$ and the projections of the cores in this plane are in generic position, i.e., there are finitely many intersections, each intersection point has exactly two preimages and the two tangent vectors are linearly independent. In the addition, we require that top bases of coupons are represented on top of bottom bases. We will then say that the ribbon graph is in *generic position.* Projecting bands and annuli onto their cores and $\mathbb{R}^2 \times [0,1]$ onto $\mathbb{R} \times \{0\} \times \mathbb{R}$, and labelling each intersection by  or , we obtain a representation of the ribbon graph by a diagram which determines it up to isotopy.

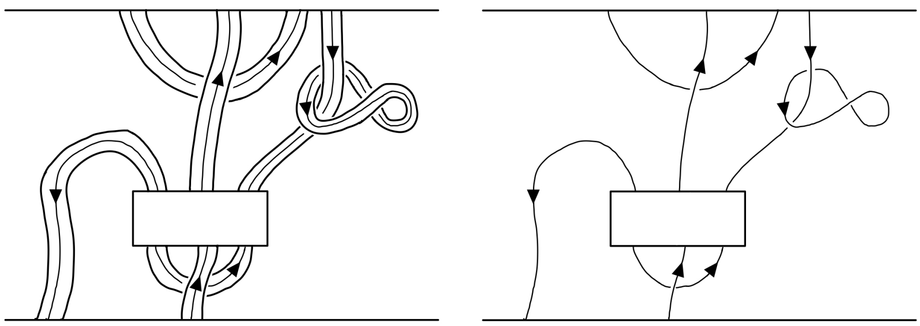


Figure 5: A ribbon graph and its diagram.

The only ambiguity comes from the fact that bands may be twisted several times along its cores. However, both positive and negative twists in a band are isotopic to curls which go parallel to the plane, as shown in Figure 6.

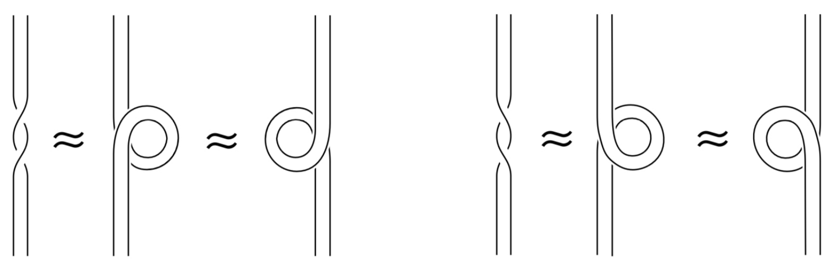


Figure 6: Twisted bands are isotopic to cursl.

1.2.2 The category of $\mathcal{V}$-coloured ribbon graphs.

Let now $\mathcal{V}$ a strict monoidal category with duality. Labelling bands and annuli by objects and coupons by morphism of $\mathcal{V}$, we will construct the category $\mathrm{Rib}_{\mathcal{V}}$ which will be key in latter sections to matching the topology of ribbon graphs with the algebra of ribbon categories. Precisely, a ribbon graph is said to be *coloured* if each band and each annulus of the graph is equipped with an object of $\mathcal{V}$. This object is called the *colour* of the band or annulus.

Let Ω be a coloured ribbon graph. Let Q be a coupon of Ω and let $V_1, \ldots, V_m$ be the colours of the bands incident to the bottom base of Q, enumerated in the order induced by the orientation of Ω restricted to Q. Let $W_1, \ldots, W_n$ be the colours of the bands intersecting the top base of Q, encountered in the order induced by the opposite orientation of Q. For each $i = 1, \ldots, m$, define ε_i by

$$\varepsilon_i = \begin{cases} +1, & \text{if the band coloured by } V_i \text{ is directed “out” of } Q, \\ -1, & \text{otherwise.} \end{cases}$$

Similarly,

$$\nu_j = \begin{cases} -1, & \text{if the band coloured by } W_j \text{ is directed “out” of } Q, \\ +1, & \text{otherwise.} \end{cases}$$

A *colour of the coupon* Q is an arbitrary morphism

$$f : V^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n},$$

where we set $V^{+1} := V$ and $V^{-1} := V^*$, for every object V of $\mathcal{V}$. A ribbon graph is *v-coloured* over $\mathcal{V}$ if it is coloured and coupons are provided with colours as above. Again, we will consider colored ribbon graphs up to isotopy. By isotopy of coloured (resp. v-coloured) ribbon graphs, we will mean colour-preserving isotopy. We are now ready to define the category $\mathrm{Rib}_{\mathcal{V}}$.

Definition 1.2.2. The *category* $Rib_{\mathcal{V}}$ *of v-colored ribbon graphs* is the category defined by the following data:

- Objects: finite sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$, where $V_1, \ldots, V_m$ are objects of $\mathcal{V}$, and $\varepsilon_1, \ldots, \varepsilon_m \in \{+1, -1\}$.
- Morphisms: given two sequences η, η' as above, a morphism $f : \eta \to \eta'$ is a v-coloured ribbon graph such that η and η' are the sequences of colours and directions of those bands meeting the bottom and top boundary intervals, respectively, with $\varepsilon = 1$ corresponding to the downwards direction and $\varepsilon = -1$ corresponding to the upwards direction of the bands.

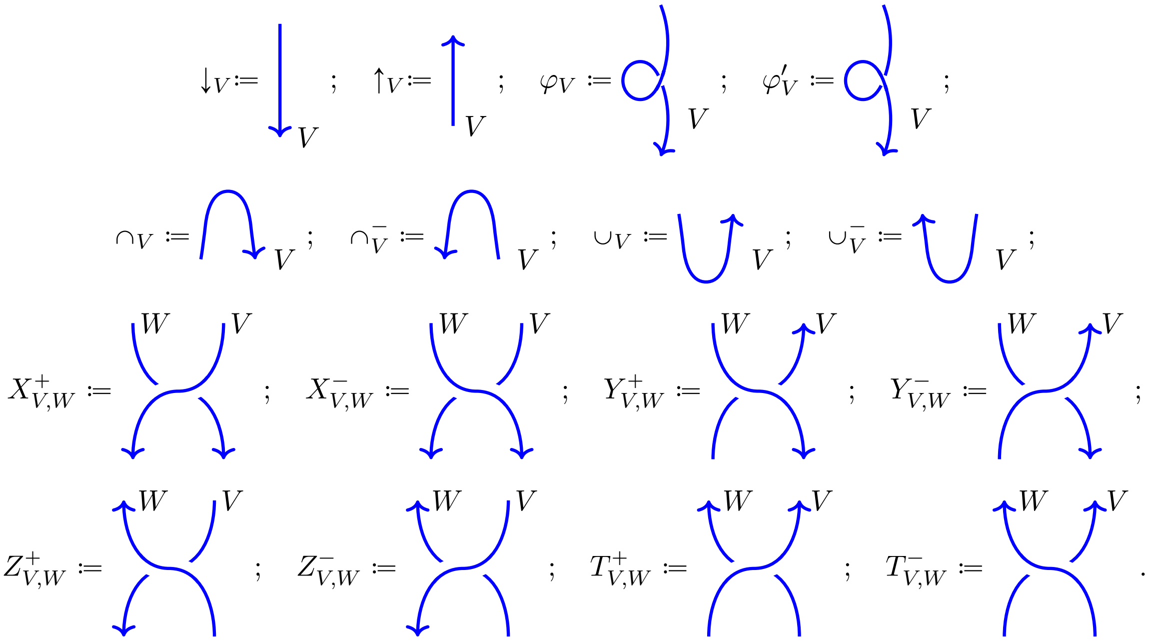
Composition of morphisms is obtained is obtained by stacking one ribbon graph on the other, i.e., $f \circ g = \begin{matrix} f \\ g \end{matrix}$, and the identity morphisms are represented by ribbon graphs which have no annuli and no coupons, and consists of untwisted vertical bands.

This category is endowed with an structure of strict monoidal category by defining the tensor product on objects by juxtaposition and the tensor product of two morphisms f and g by placing a v-coloured graph representing f to the left of a v-coloured ribbon graph representing g.

1.2.3 Presentation of $\mathrm{Rib}_{\mathcal{V}}$ by generators and relations.

An important subcategory of $\mathrm{Rib}_{\mathcal{V}}$ is provided by *ribbon tangles*. A ribbon tangle is a ribbon graph containing no coupons. The *category of v-coloured ribbon tangles* is thus the subcategory of $\mathrm{Rib}_{\mathcal{V}}$ with the same objects and whose morphisms are ribbon graphs containing no coupons. Diagrams representing morphisms in this category are called *tangle diagrams*. These diagrams consist basically of oriented strands connecting different points of $\mathbb{R} \times \{0\} \times \{0, 1\}$.

The isotopy invariance allows us to deform the diagrams to obtain other equivalent diagrams, for example by bending or stretching one of the strands. There are, however, a number of pathological situations that cannot be solved using isotopy. First of all, remember that each strand represents a band in the ribbon graph, so loops cannot be eliminated by simply stretching the strand (this would produce a twist in the corresponding band). On the other hand, if the two ends of a strand are at the same height, it is inevitable that the strand will have a “maximum” or a “minimum”. Finally, two strands cannot intersect, so crossings between them cannot be resolved by an isotopy that leaves the endpoints fixed. Around one of these pathological points, the tangle diagrams "look like" one of the morphisms in the following list:



We will see that, in effect, any tangle diagram can be obtained by composition and tensor product of morphisms from this list.

Given a tangle diagram D, we call *height function* of D the restriction π_D of the projection π : $\mathbb{R} \times [0,1] \to [0,1]$, $(x,y) \mapsto y$, to D. The extremal points of π_D and the crossing points of D are the *singular points* of the diagram. A *generic* tangle diagram is a tangle diagram such that:

(i) the set of the extremal points of π_D and the set of crossing points of D are disjoint;
(ii) there is a finite number of singular points;
(iii) the heights of two different singular points are different;
(iv) π_D is non-degenerate in all extremal points.

With this definition we exclude points of the following nature:

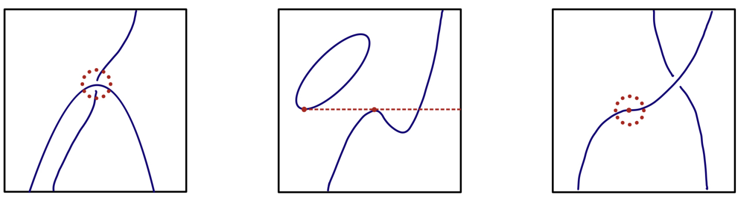


Figure 7: Examples of "pathological" point in tangle diagrams.

Since we are considering diagrams up to isotopy, it is clear that every tangle diagram is equivalent to a generic one, obtained applying a small deformation on it. Let D be such a generic diagram. The number of singular points is finite and they lie at different heights, so we can take a partition $0 = x_0, x_1, \ldots, x_r = 1$ of the unit interval such that there is at most one singular point in $\mathbb{R} \times [x_{i-1}, x_i]$, for each $i = 1, \ldots r$. Cutting the strip $\mathbb{R} \times [0,1]$ along the lines $\mathbb{R} \times \{x_i\}$, $i = 0, \ldots r$, we can decompose the tangle diagram into a composition of tangle diagrams containing at most one singular point, as the following picture suggests:

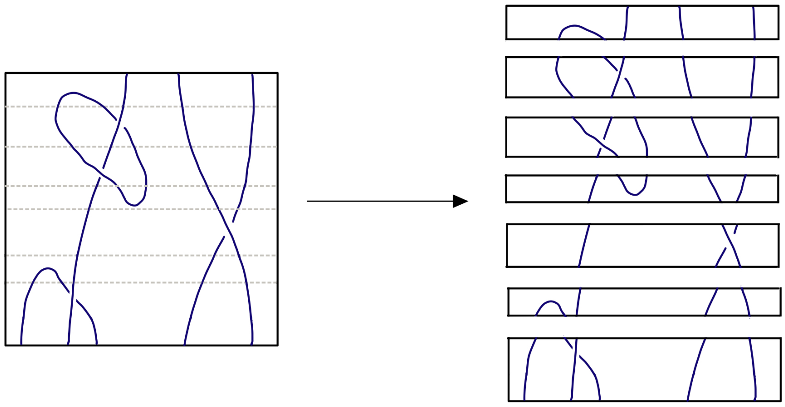


Figure 8: Decomposition of a generic tangle diagram.

Taking into account that generic points are either crossings or extremal points, it is now straightforward to see that every block in the previous decomposition is obtained as a finite tensor product of some of the morphisms that we listed at the beginning of the paragraph (except φ_V and φ'_V).

The general case of ribbon graphs, eventually with some coupons, can be treated in a similar way. Isolating coupons as we have just done with singular points, we can decompose any diagram representing a ribbon graph into simpler pieces containing at most one coupon or one singular point. An *elementary* ribbon graph is a ribbon graph consisting of one coupon and a set of unlinked untwisted vertical bands incident to this coupon.

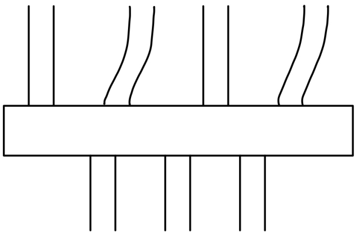


Figure 9: Elementary ribbon graph.

The discussion above shows that the list of elementary morphisms together with elementary ribbon graphs provides with a full set of generators. Moreover, some of them are superfluous, since they can obtained by composition and tensorization of the rest. Indeed, we have the following equalities:

$$\cup_V^- = (\uparrow_V \otimes \varphi'_V) \circ Z^+_{V,V} \circ \cup_V,$$

$$\cap_V^- = \cap_V \circ Z^-_{V,V} \circ (\varphi_V \otimes \uparrow_V),$$

$$Y^\nu_{V,W} = (\cap_V \otimes \downarrow_W \otimes \uparrow_V) \circ (\uparrow_V \otimes X^\nu_{W,V} \otimes \uparrow_V) \circ (\uparrow_V \otimes \downarrow_W \otimes \cup_V),$$

$$T^\nu_{V,W} = (\cap_V \otimes \uparrow_W \otimes \uparrow_V) \circ (\uparrow_V \otimes Y^\nu_{W,V} \otimes \uparrow_V) \circ (\uparrow_V \otimes \uparrow_W \otimes \cup_V),$$

where $\nu \in \{-1, +1\}$. The right-hand side of the above relations is represented in Figure 10 for $\nu = 1$. A similar graphical argument works for the other cases.

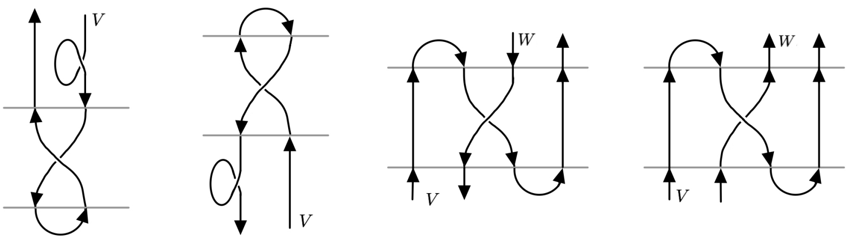


Figure 10: Graphical representation of the decomposition of $\cup_V^-$, $\cap_V^-$, $Y^+_{V,W}$ and $T^+_{V,W}$.

All in all, we have proven the following lemma:

Lemma 1.2.3. The coloured ribbon tangles

$$X^\nu_{V,W}, Z^\nu_{V,W}, \varphi_V, \varphi'_V, \cup_V, \cap_V$$

where V, W runs over objects of $\mathcal{V}$ and ν runs over $\{+1, -1\}$ generate the category of ribbon tangles. The same ribbon tangles together with all elementary v-coloured ribbon graphs generate $\mathrm{Rib}_{\mathcal{V}}$. ■

Recall from paragraph 1.1.2 that a presentation by generators and relations for a monoidal category $\mathcal{V}$ consists of a collection $\mathcal{F}$ of morphisms together with a family of relations such that any morphisms of $\mathcal{V}$ can be written as a word in the elements of $\mathcal{F}$ (cf. subsection 1.1.2 for a precise definition of word) and two words represent the same morphism if and only if they are connected by a finite string of words obtained by substituting one letter by another one which is equivalent modulo the relations.

We center again on the subcategory of ribbon tangles. We claim that the following is a list of fundamental relations for the tangles from lemma 1.2.3:

$$(\uparrow_W \otimes X^+_{U,V}) \circ (X^+_{U,W} \otimes \downarrow_V) \circ (\downarrow_U \otimes X^+_{V,W}) = (X^+_{V,W} \otimes \downarrow_U) \circ (\downarrow_V \otimes X^+_{U,W}) \circ (X^+_{U,V} \otimes \downarrow_W), \tag{R1}$$

$$\downarrow_V = (\downarrow_V \otimes \cap_V) \circ (\cup_V \otimes \downarrow_V), \tag{R2}$$

$$\uparrow_V = (\cap_V \otimes \uparrow_V) \circ (\uparrow_V \otimes \cup_V), \tag{R3}$$

$$X^-_{V,W} = (X^+_{W,V})^{-1}, \tag{R4}$$

$$\varphi'_V = (\varphi_V)^{-1}, \tag{R5}$$

$$X^\nu_{V,W} \circ (\downarrow_V \otimes \varphi_W) = (\varphi_W \otimes \downarrow_V) \circ X^\nu_{V,W}, \tag{R6}$$

$$Z^\nu_{V,W} = \Big[(\cap_W \otimes \downarrow_V \otimes \uparrow_W) \circ (\uparrow_W \otimes X^{-\nu}_{V,W} \otimes \uparrow_W) \circ (\uparrow_W \otimes \downarrow_V \otimes \cup_W)\Big]^{-1}, \tag{R7}$$

$$(\varphi_V)^2 = (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V). \tag{R8}$$

In the proof of lemma 1.2.3, we gave an explicit method to assign a word to a given ribbon tangle. Indeed, if D is a tangle diagram with no singular points, we can associate to D the word $a(D) = \mathrm{id}_V$, for some object of $\mathcal{V}$. On the other hand, if D has $n \geqslant 1$ singular points, we take $n-1$ horizontal lines in $\mathbb{R} \times [0, 1]$ in such a way that there is exactly one singular point between two consecutive lines. This singular point is a crossing or an extremal point of the hight function, so the graph locally looks like one of the preferred diagrams that we presented at the beginning of this section. Strands not containing this point are isotopic to vertical lines. Hence, the part of the diagram lying between two adjacent horizontal lines is a morphism of $\mathrm{Rib}_{\mathcal{V}}$ of the form $\mathrm{id}_V \otimes f \otimes \mathrm{id}_W$, where V, W are objects and f is represented by one of the aforementioned morphisms. Writing f in terms of the generators presented in lemma 1.2.3, we get a word representing the part of the diagram lying between two lines. We assign to the diagram D the composition $a(D)$ of these n words written down from the left to the right in the order of decreasing height. It is clear that the corresponding morphism $\langle a(D)\rangle$ in $\mathrm{Rib}_{\mathcal{V}}$ is the isotopy class of D.

In order to prove that the set of relations listed above is a fundamental set of relations, we shall show that two words in the generators from lemma 1.2.3 representing isotopic ribbon tangles are equivalent modulo these relations. The theory of ribbon tangles widely generalizes the theory of framed knots and links. The following well-known theorem of knot theory characterizes isotopy of framed links in terms of transformations of link diagrams:

Theorem 1.2.4 (Reidemeister's theorem)**.** Two link diagrams in $\mathbb{R}^2$ represent isotopic framed links in $\mathbb{R}^3$ if and only if they may be related by an ambient isotopy of the plane and a finite sequence of moves $\Omega_0, \Omega_2, \Omega_3$ of the same type as those represented in picture 11. ■

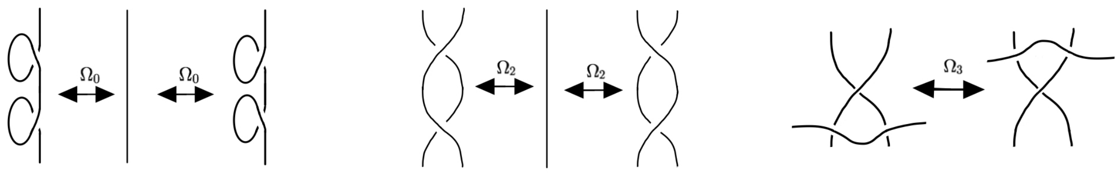


Figure 11: Reidemeister moves.

The proof of Reidemeister's theorem involves examining how a certain deformation on an arc of a link changes its diagram. This argument is entirely local to arcs of the link. In particular, it does not depend on the fact that those arcs are part of a circle or an interval, so the theorem extends to the setting of tangles. The same works when we introduce colours and directions, so the Reidemeister's theorem gives a method to check whether two ribbon tangles are isotopic or not by inspecting their diagrams.

The proof that (R1) - (R8) is a fundamental set of relations comes now to show that, under transformations of a tangle diagram consisting of Reidemeister moves and/or ambient isotopies of $\mathbb{R}\times[0,1]$ constant on the boundary of the strip, the word associated with the diagram is replaced by an equivalent one.

The Reidemeister move Ω_2. Considering directions, we have the following eight oriented versions of the second Reidemeister move Ω_2:

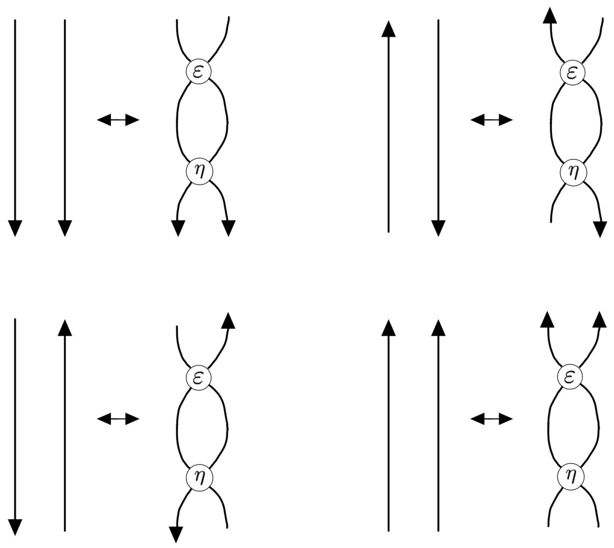


Figure 12

where, in each diagram, we let ε and η be different elements of $\left\{ \quad , \quad \right\}$.

Given two diagrams D and D' such that D' is obtained from D by an application of one of these moves, we shall show that $a(D) \sim a(D')$. Note that Reidemeister moves of type Ω_2 involve only two consecutive singular points, so they modify only two consecutive blocks in the decomposition of $\mathbb{R} \times [0,1]$ into horizontal bands that we use to compute words. These two blocks are of the form $\mathrm{id}_V \otimes f \otimes \mathrm{id}_W$ and $\mathrm{id}_V \otimes g \otimes \mathrm{id}_W$, where $f, g \in \{X^{\nu}_{U,Q}, Y^{\nu}_{U,Q}, Z^{\nu}_{U,Q}, T^{\nu}_{U,Q} \ : \ \nu = 0, 1\}$, for some objects U, V, W and Q. We have then to show that

$$(\mathrm{id}_V \otimes f \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes g \otimes \mathrm{id}_W) \sim \mathrm{id}_V \otimes \mathrm{id}_U \otimes \mathrm{id}_Q \otimes \mathrm{id}_W.$$

However, lemma 1.1.3 allows us to write

$$(\mathrm{id}_V \otimes f \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes g \otimes \mathrm{id}_W) \sim \mathrm{id}_V \otimes (f \circ g) \otimes \mathrm{id}_W,$$

which reduces the problem to prove that $f \circ g \sim \mathrm{id}_U \otimes \mathrm{id}_Q$, i.e., we can suppose, without loss of generality, that D and D' are the simple diagrams represented in Figure 12.

We examine the first three cases where $\varepsilon =$ and $\eta =$. The rest can be treated in an analogous way. For the first graph, we directly have from (R4) that

$$X^{+}_{V,W} \circ X^{-}_{W,V} \sim X^{+}_{V,W} \circ (X^{+}_{V,W})^{-1} = \mathrm{id}_{W \otimes V} = \downarrow_W \downarrow_V .$$

In the second case, the resulting graph for our choice of ω and η is $Z^{-}_{V,W} \circ Y^{+}_{W,V}$. It follows from the proof of lemma 1.2.3 that the related word is

$$Z^{-}_{V,W} \circ (\cap_W \otimes \downarrow_V \otimes \uparrow_W) \circ (\uparrow_W \otimes X^{+}_{V,W} \otimes \uparrow_W) \circ (\uparrow_W \otimes \downarrow_V \otimes \cup_W),$$

which is equivalent to $\uparrow_W \downarrow V$ by (R7). The third diagram is $Y^{-}_{V,W} \circ Z^{+}_{W,V}$ and the same argument works. The ideas for the fourth diagram are the same, but the computations are tedious since they involve longer words. A pictorial argument for this case is given in [8, section I.4.4.].

We give some ideas of the proof for Reidmeister moves of type Ω_3 and ambient isotopies. For further details and a complete proof refer again to [8, sections I.4.5. to I.4.6.].

The Reidemeister move Ω_3. The third Reidemeister move involves three strands and captures the idea that when one of them goes over or under the other two, we can move it to both sides of the crossings without changing the isotopy type of the ribbon tangle represented by the diagram. This move is represented schematically in Figure 13 where each symbol A, B, C stands for one of the crossings $X^{\nu}_{V,W}, Y^{\nu}_{V,W}, Z^{\nu}_{V,W}, T^{\nu}_{V,W}$, where $\nu \in \{-1, +1\}$ and V, W are objects of $\mathcal{V}$.

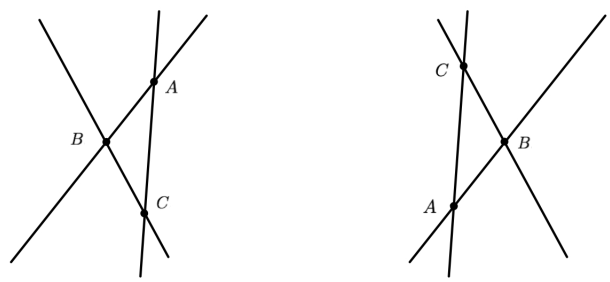


Figure 13

We will say that a triple (A, B, C) is *compatible* if one of the strands goes over or under the other two and the directions and colours of A, B, C are induced by a choice of colour and orientation for each strand. Example of triples which are not compatible are given if figure 14.

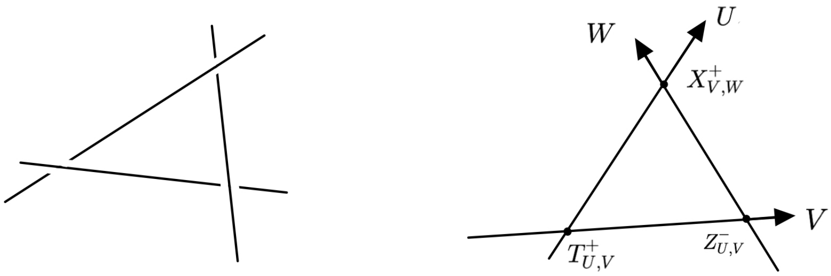


Figure 14: Non compatible triples.

For each compatible triple (A, B, C), we may replace the crossing points in Figure 13 by the corresponding graphical representation of A, B, C, giving rise to two tangle diagrams $D(A, B, C)$ and $D'(A, B, C)$. We will say that a compatible triple is *good* if $a(D(A, B, C))$ is equivalent to $a(D'(A, B, C))$. Hence, proving that a Reidemeister move of type Ω_3 produces an equivalent word comes to prove that all compatible triples are good. We will only address here the case where A, B and C are of type X^ν to give an idea of how that can be proven.

Consider first the involution $A \mapsto \overline{A}$ in the set of crossings $\{X^\nu_{V,W}, Y^\nu_{V,W}, Z^\nu_{V,W}, T^\nu_{V,W}\}$ defined by symmetry on the vertical axis, as shown in the following picture:

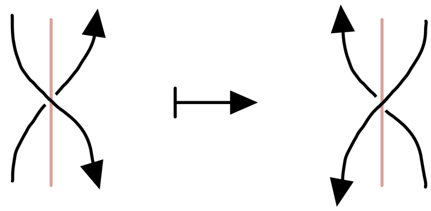


Figure 15: Symmetry on crossings.

Note that the diagrams of the form $A \circ \overline{A}$ and $\overline{A} \circ A$ are exactly those concerned by the second Reidemeister move. Composing $D(A, B, C)$ and $D'(A, B, C)$ from above by $\overline{C} \otimes |$ and from below by $| \otimes \overline{C}$ and applying a move of type Ω_2, we get $D'(B, A, \overline{C})$ and $D(B, A, \overline{C})$, respectively, as shown in Figure 16.

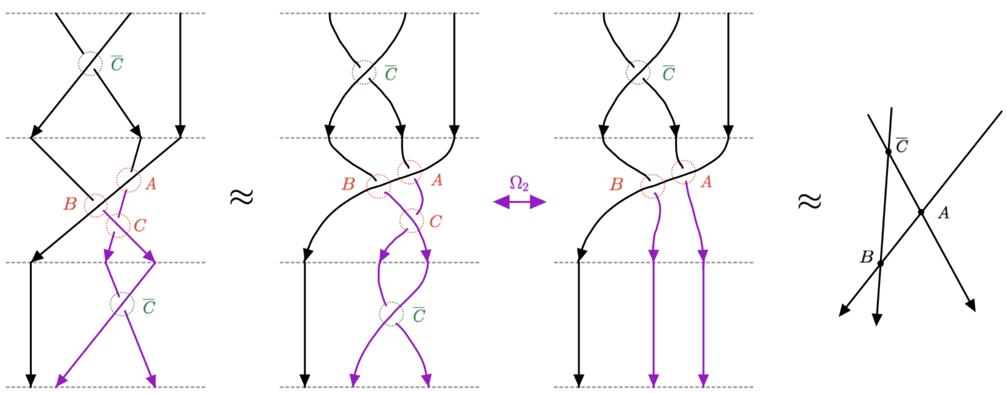


Figure 16: $D(A, B, C) \sim D'(B, A, \overline{C})$.

The preceding argument shows that if (A, B, C) is a compatible triple, then $(B, A, \overline{C})$ is compatible too. Moreover, if (A, B, C) is good, we have

$$\begin{aligned} a\left(D'(B, A, \overline{C})\right) &\sim a\left(\overline{C} \otimes |\right) \circ a\left(D(A, B, C)\right) \circ a\left(| \otimes \overline{C}\right) \\ &\sim a\left(\overline{C} \otimes |\right) \circ a\left(D'(A, B, C)\right) \circ a\left(| \otimes \overline{C}\right) = a\left(D(B, A, \overline{C})\right), \end{aligned}$$

i.e., the triple $(B, A, \overline{C})$ is also good. The same argument can be applied to show that the compatibility and goodness of $(B, A, \overline{C})$ imply those of (A, B, C), and we can prove similarly that (A, B, C) is good if and only if $(\overline{A}, C, B)$.

Now consider the case where A, B, C are crossings of type X^ν, with $\nu = \pm 1$. Replacing if necessary (A, B, C) by $(\overline{A}, C, B)$ or $(B, A, \overline{C})$ we may always suppose that the signs ν of the first and the third element of the triple coincide. This forces the second element of the triple to have also the same sign. If this common sign is $\nu = +1$, we have

$$\begin{aligned} a\left(D(A, B, C)\right) &= (\downarrow_U \otimes X^+_{W,V}) \circ (X^+_{W,U} \otimes \downarrow_V) \circ (\downarrow_W \otimes X^+_{V,U}) \\ &\sim (X^+_{V,U} \otimes \downarrow_W) \circ (\downarrow_V \otimes X^+_{W,U}) \circ (X^+_{W,V} \otimes \downarrow_U) = a\left(D'(A, B, C)\right), \end{aligned}$$

where the equivalence between the two words follows from (R1). Otherwise, $\nu = -1$, and we get

$$\begin{aligned} a(D(A, B, C)) &= (\downarrow_U \otimes X^-_{W,V}) \circ (X^-_{W,U} \otimes \downarrow_V) \circ (\downarrow_W \otimes X^-_{V,U}) \\ &\sim (\downarrow_U \otimes X^+_{V,W})^{-1} \circ (X^+_{U,W} \otimes \downarrow_V)^{-1} \circ (\downarrow_W \otimes X^+_{U,V})^{-1} \\ &\sim \left[(\downarrow_W \otimes X^+_{U,V}) \circ (X^+_{U,W} \otimes \downarrow_V) \circ (\downarrow_U \otimes X^+_{V,W})\right]^{-1} \\ &\sim \left[(X^+_{V,W} \otimes \downarrow_U) \circ (\downarrow_V \otimes X^+_{U,W}) \circ (X^+_{U,V} \otimes \downarrow_W)\right]^{-1} \\ &\sim (X^-_{V,U} \otimes \downarrow_W) \circ (\downarrow_V \otimes X^-_{W,U}) \circ (X^-_{W,V} \otimes \downarrow_U) = a(D'(A, B, C)), \end{aligned}$$

where the first and the fourth equivalence are a consequence of (R4), the second one is trivial and the third one is given by (R1).

The rest of the cases are trickier, but the details are given in [8].

Ambient isotopies. As mentioned before, we consider ambient isotopies of $\mathbb{R} \times [0, 1]$ constant on the boundary. We can think of such an isotopy as a continuous deformation of the tangle diagram keeping the endpoints of every strand fixed. We have to show that two generic diagrams connected by such a transformations are represented by equivalent words. These deformations of tangle diagrams will affect singular points so, at a given instant of time, a situation may arise in which two singular points are "instantaneously" at the same height, a crossing "instantaneously" coincides with an extremal point of the height function or new singular points are generated or annihilated when bending or stretching a strand. Thus, isotopic generic diagrams may be obtained from each other by a finite sequence of the following transformations:

(A1) an isotopy in the class of generic diagrams;
(A2) an isotopy interchanging the order of two singular points with respect to the height function;
(A3) birth or annihilation of a pair of local extrema;
(A4) isotopies shown in Figure 17.

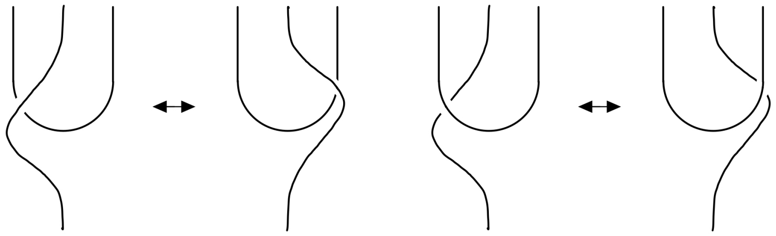


Figure 17

Transformations of type (A1) do not modify the configuration of the set of singular points so they do not change the word at all. The result for transformations of type (A2) is a direct consequence of lemma 1.1.4. The argument for transformations of type (A3) depends on the orientation of the strands. The simplest case is represented in Figure 18, which is nothing but the graphical analogous of (R3).

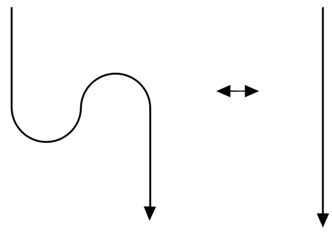


Figure 18

The rest of transformations of type (A3) and transformations of type (A4) involve Reidemeister moves and some sophisticated transformations of diagrams which are better explained graphically. The details are given in [8, section I.4.6.]

The Reidemeister move Ω_0. We finally treat the first transformation depicted in Figure 11. In this case, we have to show that

$$a(\varphi_V) \circ a(\varphi'_V) = a(\varphi \circ \varphi'_V) \sim \mathrm{id}_V,$$

$$a(\varphi'_V) \circ a(\varphi_V) = a(\varphi \circ \varphi'_V) \sim \mathrm{id}_V.$$

We have

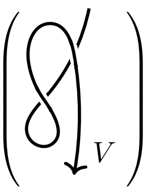
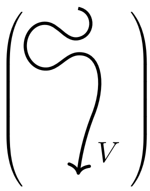
$$\begin{aligned}
a(\varphi_V) &= (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ (a(\cup^-_V) \otimes \downarrow_V) \\
&= (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ \left(\left[(\uparrow_V \otimes \varphi'_V) \circ Z^+_{V,V} \circ \cup_V \right] \otimes \downarrow_V \right) \\
&\sim (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ (\uparrow_V \otimes \varphi'_V \otimes \downarrow_V) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V) \\
&\sim (\cap_V \otimes \downarrow_V) \circ \left(\uparrow_V \otimes \left(X^+_{V,V} \circ (\varphi'_V \otimes \downarrow_V) \right) \right) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V) \\
&\sim (\cap_V \otimes \downarrow_V) \circ \left(\uparrow_V \otimes \left((\downarrow_V \otimes \varphi'_V) \circ X^+_{V,V} \right) \right) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V) \\
&\sim (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes \downarrow_V \otimes \varphi'_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V) \\
&\sim \varphi'_V \circ (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^+_{V,V}) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V) \\
&\sim \varphi'_V \circ \varphi_V \circ \varphi_V \\
&\sim \varphi_V,
\end{aligned}$$

where the first equality holds by definition of a; the second one by the decomposition of $\cup^-_V$ given in the proof of lemma 1.2.3; the first equivalence follows from (M3) and (M7); the second one from (M7); the third one is a combination of (R4), (R5) and (R6); the fourth one is a consequence of (M3) and (M7); the fifth one can be obtained from (M5) and (M7); and the two last are a consequence of (R8) and (R5), respectively.

Applying the same arguments one shows that

$$a(\varphi'_V) \sim \varphi'_V \circ (\cap_V \otimes \downarrow_V) \circ (\uparrow_V \otimes X^-_{V,V}) \circ (Z^+_{V,V} \otimes \downarrow_V) \circ (\cup_V \otimes \downarrow_V).$$

Representing graphically the right-hand side of the last equation and applying the Reidemeister move Ω_2 and an ambient isotopy, we get

$$a\left(\varphi'_V\right) \sim \varphi'_V \circ a$$

$$\sim \varphi'_V \circ a$$

$$= \varphi'_V \circ \mathrm{id}_V \sim \varphi'_V.$$

The claim now follows straightforward from the previous computations and (R5).

We summarize the all the information presented above in the following lemma:

Lemma 1.2.5. Relations (R1)-(R8) form a complete set of relations between the generators of the category of coloured ribbon tangles presented in lemma 1.2.3. ■

To generalize this result to the category $\mathrm{Rib}_{\mathcal{V}}$, we need to introduce new relations involving elementary ribbon graphs. The idea is the same as in the case of ribbon tangles: we describe an isotopy of ribbon graph through a finite sequence of transformations connecting their respective ribbon graphs and we show that these transformations translates into an equivalence between the associated words modulo the desired relations.

Let Ω and Ω' be two ribbon graphs and let D and D' be two diagrams representing them. Coupons are then parallel to a vertical plane in such representations. In the course of the isotopy, they move as solid rectangles so, in particular, they describe a loop in $SO(3)$. We may visualize this by thinking of every coupon as having three ortogonal axis attached to its center; the position of these axis will change as we deform Ω into Ω', so that at each time they represent the element of $SO(3)$ (the rotation taking the initial reference system into the reference system at time t). Since the coupons are in generic position in both D and D', the final configuration of the axis coincides with the initial one so, indeed, we have a loop in $SO(3)$ with base point the identity. All such loops may be deformed into $SO(2)$ (cf. remark 1.2.7), which implies that we may deform the initial isotopy into a another isotopy between Ω and Ω' that keeps all coupons parallel to the plane. Moreover, since the fundamental group of $SO(2)$ is generated by a 2π-rotation, we may relate Ω and Ω' by a composition of isotopies of the following two types: (i) isotopies keeping the bases of all coupons horizontal and (ii) isotopies rotating a coupon in the plane by the angle of 2π.

Now suppose that D and D' are two diagrams representing graphs connected by an isotopy of the first type. In the regions where no coupon is present, diagrams are modified as in the case of tangle diagrams, while in a neighbourhood of a coupon the only possible changes are to push a strand of diagram under or over the coupon, as shown in Figure 19. Hence, isotopies of type (i) may be presented as compositions of the following transformations:

1. ambient isotopies of the strip $\mathbb{R} \times [0, 1]$ constant at the boundary,
2. Reidemeister moves Ω_0, Ω_2 and Ω_3,
3. pushing a strand of the diagram over or under a coupon.

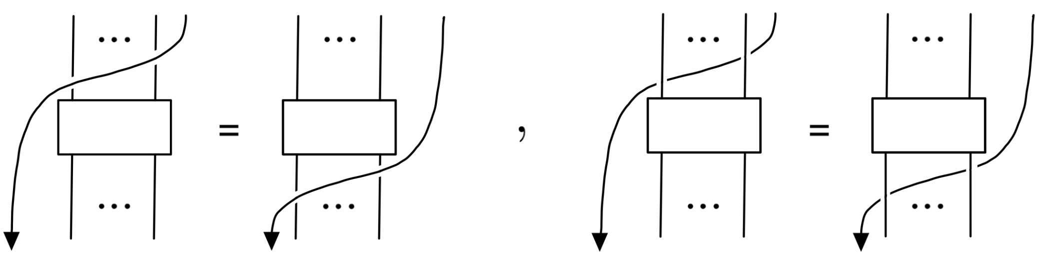


Figure 19

Note that in Figure 19 we have represented only the case where the strand in question is oriented downwards. This is general enough, since the case where it is oriented upwards can be obtained by combining ambient isotopies, Reidemeister moves and the case represented above. This can be done as in Figure 20.

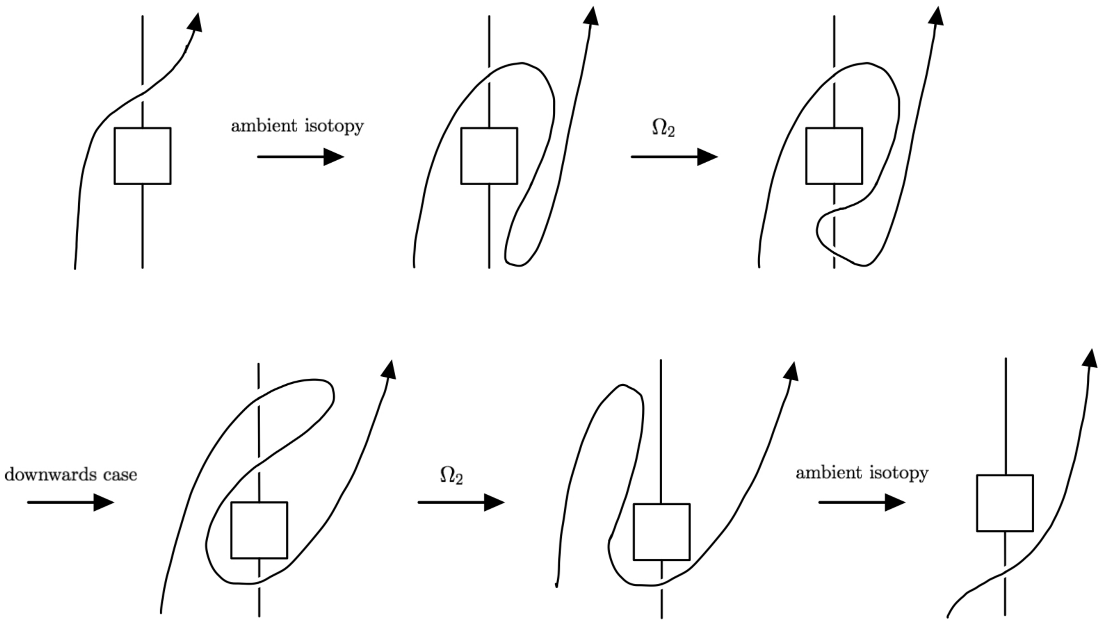


Figure 20

On the other hand, suppose that D and D' are connected by an isotopy rotating a coupon by the angle of 2π, as shown in the following diagram:

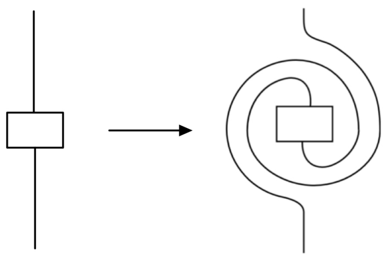


Figure 21

But we can push the strands incident to the top (bottom) face over (under) the coupon as follows:

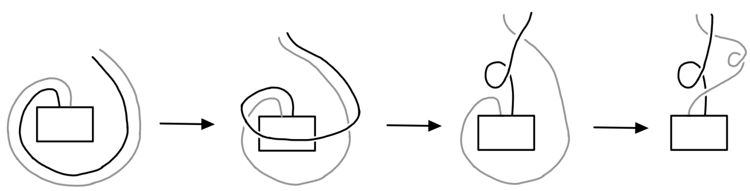


Figure 22

As the picture suggests, he result of this movement is an elementary coupon with a full right-hand twist of the bunch of the top bands and a left-hand twist of the bunch of the bottom bands. All in all, isotopies rotating a coupon may be presented at the level of diagrams as a composition of isotopies keeping the bases horizontal and movements relating the following two graphs:

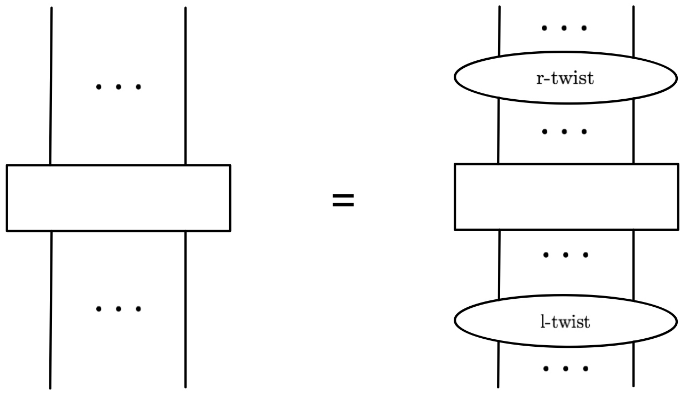


Figure 23

This discussion together with the arguments given for the case of tangle diagrams imply the following result:

Lemma 1.2.6. Relations (R1)-(R8) together with the relations presented in Figure 20 and Figure 23 form a complete set of relations between the generators of the category $\mathrm{Rib}_{\mathcal{V}}$ listed in lemma 1.2.3. ■

Remark 1.2.7. In the previous description of isotopies of ribbon graphs, we claimed that any loop in $SO(3)$ with basepoint the identity may be deformed into a loop in $SO(2)$. Indeed, any element of $SO(3)$ is represented by a point of the ball $B_\pi(0) = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = \pi^2\}$. The direction of the vector pointing from the origin gives the axis of the rotation and the length of this vector gives its angle. In particular, the identity map is represented by the origin. Moreover, antipodal points gives raise to the same element of $SO(3)$ so we have an homeomorphism

$$SO(3) \cong {}^{B_\pi(0)}/_{\sim} \cong \mathbb{R}P^3,$$

where $\sim$ is the equivalence relation whose classes are pairs of antipodal points. On the other hand, $SO(2)$ can be identified with the subset of $SO(3)$ obtained by fixing one direction, i.e., $SO(2)$ the projection of one diameter on $B_\pi(0)$, and it is clear that every loop based at the origin can be deformed into a loop contained in this diameter.

1.2.4 The Reshetikhin-Turaev functor. In the first subsection, we introduced the concept of ribbon categories based on the elementary "blocks" that make up its structure: braiding, twist and duality. On the other hand, all the work we have done so far in the second subsection has resulted in a decomposition of the elements of the ribbon graph category through their presentation by generators and relations. Let us now see how we can relate these two decompositions, through the construction of a functor $F : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ that matches the topology of ribbon graphs with the algebraic properties of monoidal categories.

Theorem 1.2.8. Let $\mathcal{V}$ be a strict monoidal category with braiding c, twist θ, and compatible duality $(*, b, d)$. There exists a unique covariant functor $F : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ preserving tensor product and satisfying the following conditions:

(i) $F((V,+1)) = V$ and $F((V,-1)) = V^*$;

(ii) for any objects $V, W \in \mathcal{V}$, we have

$$F\left(X^+_{V,W}\right) = c_{V,W}, \quad F(\varphi_V) = \theta_V, \quad F(\cup_V) = b_V, \quad F(\cap_V) = d_V;$$

(iii) for any elementary v-coloured ribbon graph Γ, $F(\Gamma) = f$, where f is the colour of the only coupon of Γ.

We call the morphism $F(\Omega)$ the *operator invariant* of Ω.

Proof. We start noting that, since F is a covariant functor, we have

$$F(\downarrow_V) = \mathrm{id}_V, \quad F(\uparrow_V) = \mathrm{id}_{V^*}, \quad F\begin{pmatrix}\Omega \\ \Omega'\end{pmatrix} = F(\Omega) \circ F(\Omega'),$$

for any two composable v-coloured ribbon graphs Ω and Ω'. On the other hand, we recall that a complete set of generators for $\mathrm{Rib}_{\mathcal{V}}$ is given by the morphisms

$$X^{\nu}_{V,W},\ Z^{\nu}_{V,W},\ \varphi_V,\ \varphi'_V,\ \cup_V,\ \cap_V,$$

where $\nu \in \{+1,-1\}$ and V, W runs over the objects of $\mathcal{V}$, together with elementary v-coloured ribbon graphs. If a functor F as in the theorem exists, then its image over the generators is uniquely determined by conditions (ii) and (iii), relations (R4), (R5) and (R7), and the fact that F preserves the tensor product. This implies the uniqueness.

To prove the existence of F we have to show that any two words representing a given morphism are sent to the same object by F, which is equivalent to prove that F preserves the relations from lemma 1.2.6.

Applying F to both sides of (R1) we recover the Yang-Baxter equation for strict monoidal categories that we introduced in paragraph 1.1.3. The images of (R2) and (R3) are nothing but the rigidity axioms of duality (cf. definition 1.1.7) and that of (R6) follows from the naturality braiding. We impose relations (R4), (R5) and (R7) to get the images of φ'_V, $X^-_{V,W}$,. $Z^+_{V,W}$ and $Z^-_{V,W}$, for any objects V, W of $\mathcal{V}$.

In order to check relation (R8), let us express $F(Z^+_{V,W})$ and $F(Z^-_{V,W})$ in a more suitable way. Consider the following commutative diagram:

$$\begin{tikzcd}
V = V \otimes \mathbf{1} \arrow[r, "\mathrm{id}_V \otimes b_W"] \arrow[d, "\mathrm{id}_V"'] & V \otimes W \otimes W^* \arrow[r, "c_{V,W} \otimes \mathrm{id}_{W^*}"] \arrow[d, "c_{V,W\otimes W^*}"'] & W \otimes V \otimes W^* \arrow[d, "\mathrm{id}_W \otimes c_{V,W^*}"'] \\
V = \mathbf{1} \otimes V \arrow[r, "b_W \otimes \mathrm{id}_V"] & W \otimes W^* \otimes V \arrow[r, equal] & W \otimes W^* \otimes V.
\end{tikzcd}$$

Commutativity of the left square follows from the naturality of the braiding and the fact that $c_{V,\mathbf{1}} = \mathrm{id}_V$. Commutativity of the right square follows from the definition of braiding. Thus,

$$b_W \otimes \mathrm{id}_V = (\mathrm{id}_W \otimes c_{V,W^*}) \circ (c_{V,W} \otimes \mathrm{id}_{W^*}) \circ (\mathrm{id}_V \otimes b_W).$$

Multiplying the left-hand side of this equation by id_{W^*} and composing with $d_W \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_V$, we have

$$\begin{aligned}(d_W \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_V) &\circ (\mathrm{id}_{W^*} \otimes b_W \otimes \mathrm{id}_V) \\ &= [(d_W \otimes \mathrm{id}_{W^*}) \circ (\mathrm{id}_{W^*} \otimes b_W)] \otimes \mathrm{id}_V = \mathrm{id}_{W^*} \otimes \mathrm{id}_V,\end{aligned}$$

by the properties of the tensor product and the rigidity axioms of duality. Doing the same thing on the right-hand side, we get

$$\begin{aligned}
&(d_W \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_V) \circ \big[\mathrm{id}_{W^*} \otimes (\mathrm{id}_W \otimes c_{V,W^*}) \circ (c_{V,W} \otimes \mathrm{id}_{W^*}) \circ (\mathrm{id}_V \otimes b_W)\big] \\
&\quad = (d_W \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_V) \circ (\mathrm{id}_{W^*} \otimes \mathrm{id}_W \otimes c_{V,W^*}) \circ (\mathrm{id}_{W^*} \otimes c_{V,W} \otimes \mathrm{id}_{W^*}) \circ (\mathrm{id}_{W^*} \otimes \mathrm{id}_V \otimes b_W) \\
&\quad = c_{V,W^*} \circ [(d_W \otimes \mathrm{id}_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{W^*} \otimes c_{V,W} \otimes \mathrm{id}_{W^*}) \circ (\mathrm{id}_{W^*} \otimes \mathrm{id}_V \otimes b_W)].
\end{aligned}$$

A similar argument applied to the diagram

$$\begin{tikzcd}
W^* \otimes V \otimes W \arrow[d, "\mathrm{id}_{W^*}\otimes c_{V,W}"'] & V \otimes W^* \otimes W \arrow[l, "c_{V,W^*}\otimes \mathrm{id}_W"'] \arrow[r, "\mathrm{id}_V \otimes d_W"] \arrow[d, "c_{V,W^*\otimes W}"'] & V \otimes \mathbf{1} = V \arrow[d, "\mathrm{id}_V"'] \\
W^* \otimes W \otimes V \arrow[r, equal] & W^* \otimes W \otimes V \arrow[r, "d_W \otimes \mathrm{id}_V"] & \mathbf{1} \otimes V = V
\end{tikzcd}$$

shows that the expression in brackets in the last equation is c^{-1}_{V,W^*}. But the inverse of that bracket is exactly the image by F of the right-hand side of (R8) with $\nu = -1$, which coincides with $F(Z^-_{V,W})$ by construction. All in all, we have that $F(Z^-_{V,W}) = c_{V,W^*}$. Applying this to the reverse braiding, we get $F(Z^+_{V,W}) = c^{-1}_{W^*,V}$.

We are now ready to check (R8). Let V be an object of $\mathcal{V}$. We have to show that

$$\theta_V^2 = (d_V \otimes \mathrm{id}_V) \circ (\mathrm{id}_{V^*} \otimes c_{V,V}) \circ (c^{-1}_{V^*,V} \otimes \mathrm{id}_V) \circ (b_V \otimes \mathrm{id}_V)$$

holds. Using the naturality of the twist and the fact that $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, we have

$$b_V = b_V \circ \theta_{\mathbf{1}} = \theta_{V\otimes V^*} \circ b_V = c_{V^*,V} \circ c_{V,V^*} \circ (\theta_V \otimes \theta_{V^*}) \circ b_V.$$

The compatibility axiom of θ implies that $(\theta \otimes \theta_{V^*}) \circ b_V = (\theta_V^2 \otimes \mathrm{id}_{V^*}) \circ b_V$, so the preceding equality reads

$$\big(\theta_V^2 \otimes \mathrm{id}_{V^*}\big) \circ b_V = c^{-1}_{V,V^*} \circ c^{-1}_{V^*,V} \circ b_V.$$

Multiplying the left-hand side from the right by id_V and composing with $\mathrm{id}_V \otimes d_V$ we have

$$\begin{aligned}
(\mathrm{id}_V \otimes d_V) &\circ \big((\theta_V^2 \otimes \mathrm{id}_{V^*}) \circ b_V\big) \otimes \mathrm{id}_V \\
&= (\mathrm{id}_V \otimes d_V) \circ (\theta_V^2 \otimes \mathrm{id}_{V^*} \otimes \mathrm{id}_V) \circ (b_V \otimes \mathrm{id}_V) \\
&= (\mathrm{id}_V \circ \theta_V^2) \otimes (d_V \circ (\mathrm{id}_{V^*} \otimes \mathrm{id}_V)) \circ (b_V \otimes \mathrm{id}_V) \\
&= (\theta_V^2 \circ \mathrm{id}_V) \otimes (\mathrm{id}_{\mathbf{1}} \circ d_V) \circ (b_V \otimes \mathrm{id}_V) \\
&= \theta_V^2 \circ (\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \theta_V^2.
\end{aligned}$$

Doing the same on the right-hand side and using the explicit computation of c^{-1}_{V,V^*} that we found in the previous paragraph, one gets the right-hand side of the relation (R8). Explicit computations are omitted since they are quite tedious but similar to what we have just done.

Let's now verify the relation in $\mathcal{V}$ obtained from the relation in Figure 19. First, note that we may suppose that all the bands intersecting the coupon are directed downwards. Indeed, if one of these bands is directed upwards we may replace its color by the corresponding dual and change the direction of the core. We get like this a new ribbon graph whose image by F is the same as the initial one, so it gives rise to the same relation in $\mathcal{V}$.

Every crossing in Figure 19 corresponds to a ribbon generator $X^\nu_{V,W}$. We treat first the case $\nu = 1$. Let $f : U_1 \otimes \cdots \otimes U_n \to V_1 \otimes \cdots \otimes V_m$ be the colour of the coupon and let W be the color of the long band. Applying F to the ribbon graph results in the following equality:

$$\begin{aligned}
&(\mathrm{id}_{V_1} \otimes \cdots \otimes \mathrm{id}_{V_{n-1}} \otimes c_{W,V_n}) \circ \cdots \circ (c_{W,V_1} \otimes \mathrm{id}_{V_2} \otimes \cdots \otimes \mathrm{id}_{V_n}) \circ (\mathrm{id}_W \otimes f) \\
&\quad = (f \otimes \mathrm{id}_W) \circ (\mathrm{id}_{U_1} \otimes \cdots \otimes \mathrm{id}_{U_{m-1}} \otimes c_{W,U_m}) \circ \cdots \circ (c_{W,U_1} \otimes \mathrm{id}_{U_2} \otimes \cdots \otimes \mathrm{id}_{U_m}).
\end{aligned}$$

Now the axioms for the braiding in strict monoidal categories exactly say that the following two v-coloured graphs correspond to the same morphism in $\mathcal{V}$:

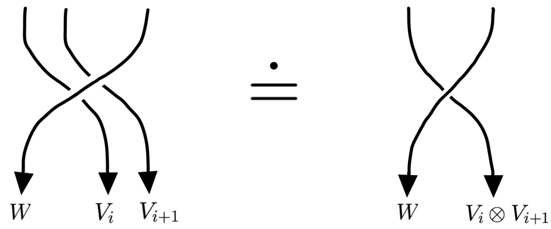


Figure 24

Applying it repeatedly to both sides of the relation in Figure 19, one finds that the corresponding relation in $\mathcal{V}$ is equivalent to

$$c_{W,V_1\otimes\cdots\otimes V_n} \circ (\mathrm{id}_W \otimes f) = (f \otimes \mathrm{id}_W) \circ c_{W,U_1\otimes\cdots\otimes U_m},$$

which holds by the naturality of c.

The case $\nu = 1$ is proven by the same argument applied to the reverse braiding.

Finally, we check the relation in $\mathcal{V}$ arising from Figure 23. Denote by $r(V_1, \ldots, V_n)$ the full right-hand twist appearing in that relation and let us prove by induction that

$$F\left(r(V_1, \ldots, V_n)\right) = \theta_{V_1\otimes\cdots\otimes V_n}.$$

If $n = 1$, then $r(V_1) = \varphi_{V_1}$ and $F(r(V_1)) = F(\varphi_{V_1}) = \theta_{V_1}$. Assume the result for $n \geqslant 2$. Set $V = V_1 \otimes \cdots \otimes V_{n-1}$ and $W = V_n$ and note that

$$r(V_1, \ldots, V_n) = X^+_{W,V} \circ X^+_{V,W} \circ \left(r(V_1, \ldots, V_{n-1})) \otimes r(V_n)\right).$$

Applying F to this equality, we get

$$\begin{aligned} F\left(r(V_1, \ldots, V_n)\right) &= c_{W,V} \circ c_{V,W} \circ \left(F(r(V_1, \ldots, V_{n-1})) \otimes F(r(V_n))\right) \\ &= c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W) \\ &= \theta_{V\otimes W} = \theta_{V_1\otimes\cdots\otimes V_n}, \end{aligned}$$

where the second equality holds by the induction hypothesis and the third one by the axioms of the twist. Applying the same argument to the left-hand twist $l(U_1, \ldots, U_m))$, we get that

$$F\left(l(U_1, \ldots, U_m)\right) = \theta^{-1}_{U_1\otimes\cdots\otimes U_m}.$$

Supposing as above that all the bands are directed downwards, the identity in $\mathcal{V}$ corresponding to the relation in Figure 23 has the form

$$\theta_{V_1\otimes\cdots\otimes\cdot V_n} \circ f \circ \theta^{-1}_{U_1\otimes\cdots\otimes U_m} = f.$$

This identity follows directly from the naturality of twist in $\mathcal{V}$. ■

Corollary 1.2.9. The functor F has the following properties:

$$F(X^-_{V,W}) = c^{-1}_{W,V}, \quad F(Y^+_{V,W}) = c^{-1}_{W,V*}, \quad F(Y^-_{V,W}) = c_{V*,W}),$$

$$F(Z^+_{V,W}) = c^{-1}_{W*,V}, F(Z^-_{V,W} = c_{V,W*},$$

$$F(T^+_{V,W}) = c_{V*,W*}, F(T^-_{V,W}) = c^{-1}_{W*,V*}, F(\varphi'_V) = \theta^{-1}_V,$$

where V, W are objects of $\mathcal{V}$.

Proof. Some cases have already been treated in the proof of the previous theorem. The rest are similar. ■

The functor Reshetikhin-Turaev functor may be regarded from several viewpoints. Firstly, it yields to isotopy invariants of v-coloured ribbon graphs. By definition, the morphisms of $\mathrm{Rib}_{\mathcal{V}}$ are isotopy classes of ribbon graphs so two graphs Ω and Ω' representing the same isotopy class will verify $F(\Omega) = F(\Omega')$. Secondly, it elucidates the important the role of ribbon categories in the construction of isotopy invariants. Finally, it renders rigorous the graphical calculus in ribbon categories that we explain below.

1.3 From topology to algebra: graphical calculus in ribbon categories

The formalism introduced in the previous section provides a very powerful tool for producing identities in ribbon categories. The idea should now be clear: we can represent each morphism in a strict monoidal category $\mathcal{V}$ by the diagram of a ribbon graph which is in its preimage by F. Applying transformations on such a diagram which do not alter the isotopy class of the corresponding graph, and using F to descend back to the category $\mathcal{V}$, we obtain a new algebraic expression which is associated to the same isotopy class and which must therefore be equal to the initial expression. We will apply this technique to prove some properties about the quantum trace and dimension, that we will define later in this section.

1.3.1 Basic rules of graphical calculus. Let us develop this ideas further. Let $\mathcal{V}$ be a ribbon category. We use the symbol $\doteq$ to indicate that two diagrams represent ribbon graphs produce the same arrow in $\mathcal{V}$. We can represent a morphism $f : U \to V$ in $\mathcal{V}$ by the diagram

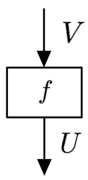


Figure 25

Using the composition in $\mathrm{Rib}_{\mathcal{V}}$, we get the following equivalent representations for the composition of two morphisms $f : U \to W$ and $g : W \to V$ in $\mathcal{V}$

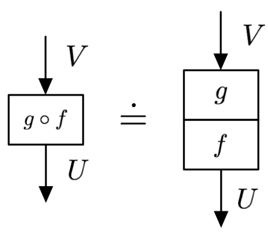


Figure 26

A morphism $f : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n$ is represented by

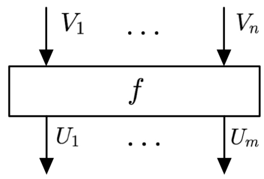


Figure 27

and the tensor product of two morphisms $f \otimes g$ by

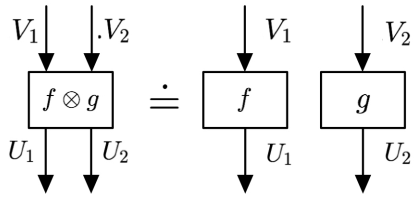


Figure 28

The identity arrow of V will be represented by a vertical arrow $\downarrow_V$ directed downwards. Using the isotopy invariance, one gets

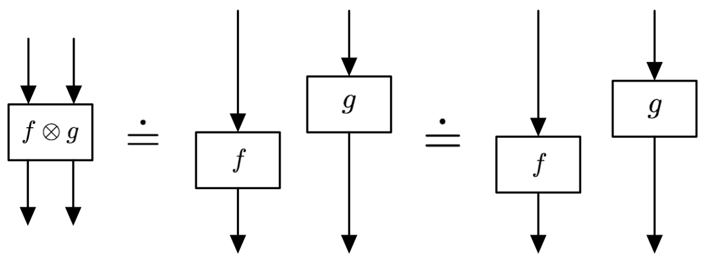


Figure 29

which is nothing but a graphical expression of the identity

$$f \otimes g = (f \circ \mathrm{id}) \otimes (\mathrm{id} \circ g) = (\mathrm{id} \circ f) \otimes (g \circ \mathrm{id}).$$

The braiding and each inverse are represented by

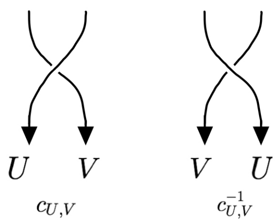


Figure 30

for each pair of objects $U, V \in \mathcal{V}$, so we obviously have

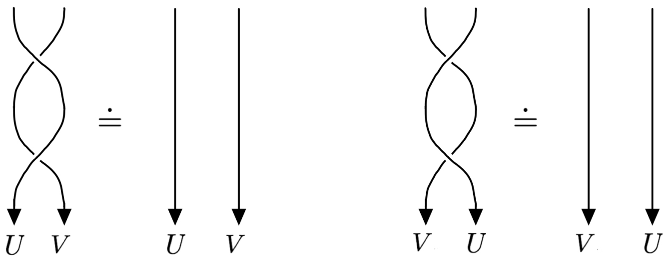


Figure 31

and we can expressed pictorially the naturality of the braiding as

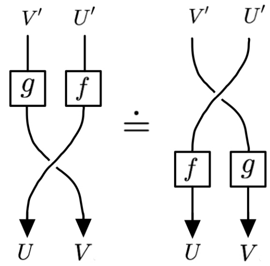


Figure 32

where $f : U \to U'$ and $g : V \to V'$ are morphisms in $\mathcal{V}$. As for duality, the identity map of V will represented by a vertical arrow $\uparrow_V$ pointing upwards and coloured by V. The duality maps are given by

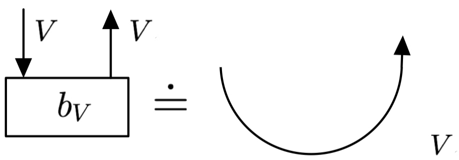


Figure 33

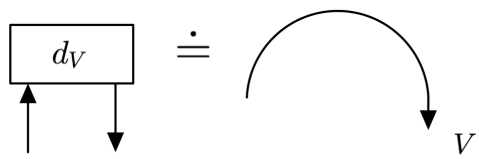


Figure 34

Finally, the rigidity axioms are represented by

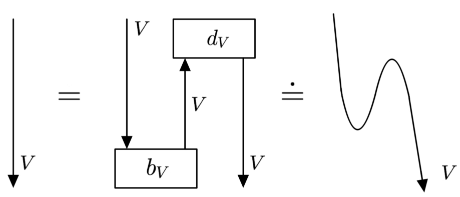


Figure 35

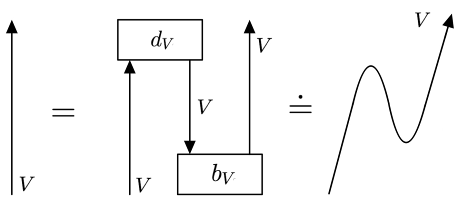


Figure 36

The following result, that we prove using graphical calculus, allows us to identify canonically V and V^{**}:

Corollary 1.3.1. For any object V of $\mathcal{V}$, the object V^{**} is canonically isomorphic to V.

Proof. Let $\Omega \coloneqq (\cap_V^- \otimes \uparrow_{V^*}) \circ (\downarrow_V \otimes b_{V^*})$ and $\Omega' \coloneqq (d_{V^*} \otimes \mathrm{id}_V) \circ (\uparrow_{V^*} \otimes \cup_V^-)$ be the v-coloured ribbon graphs corresponding to the following diagrams:

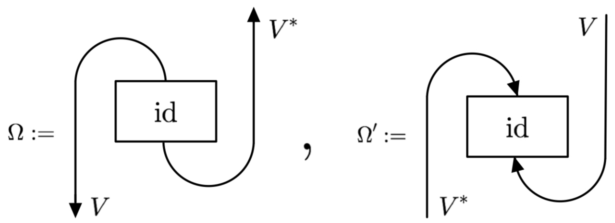


Figure 37

Note that $F(\Omega) \in \mathrm{Hom}_{\mathcal{V}}(V, V^{**})$ and $F(\Omega') \in \mathrm{Hom}_{\mathcal{V}}(V^{**}, V)$. The following graphical argument shows that $F(\Omega') \circ F(\Omega) = \mathrm{id}_V$:

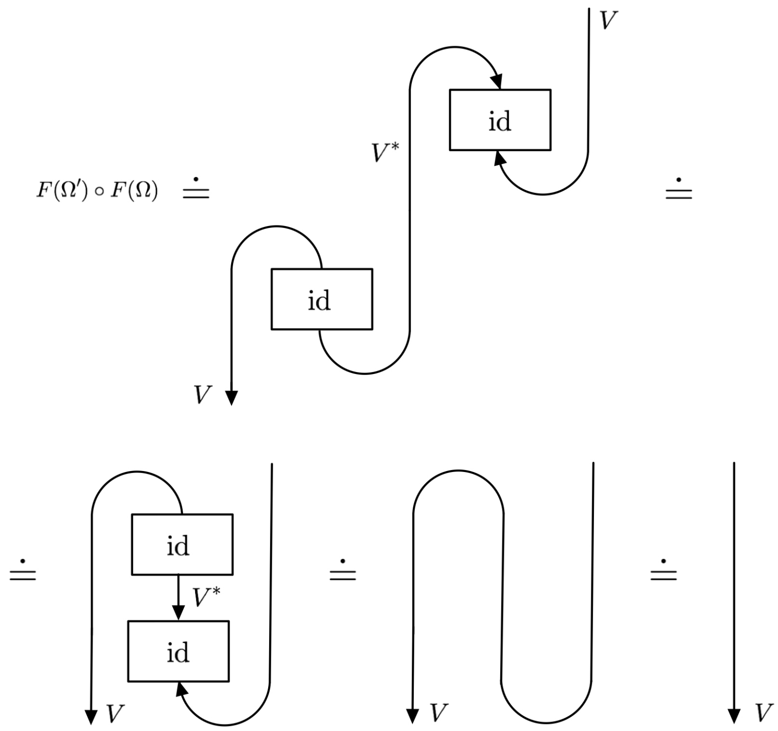


Figure 38

The first and the last equalities come from the functoriality of F; in the second one, we deform the ribbon graph by isotopy; for the third one, we change the direction of the band between the two coupons and recolour it by V^*, and we delete the coupons coloured by the identity. A similar argument shows that $F(\Omega) \circ F(\Omega') = \mathrm{id}_{V^*}$, so $F(\Omega)$ is an isomorphism between V and V^{**}. ■

Using this identification and the definition of $\mathrm{Rib}_{\mathcal{V}}$ and the functor F, we can write

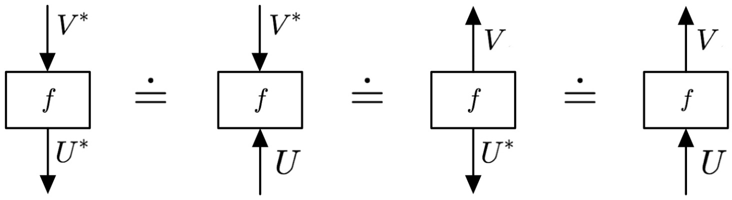


Figure 39

for every morphism $f : U^* \to V^*$.

1.3.2 More results on duality. We use the rules of graphical calculus that we have just described to prove some more properties of duality that will be useful in latter sections. The following result shows that duality is compatible with the tensor product:

Corollary 1.3.2. For any objects V, W of $\mathcal{V}$, the objects $W^* \otimes V^*$ and $(V \otimes W)^*$ are canonically isomorphic.

Proof. It suffices to note that the operator invariants associated to the following v-coloured ribbon graphs are mutually inverse:

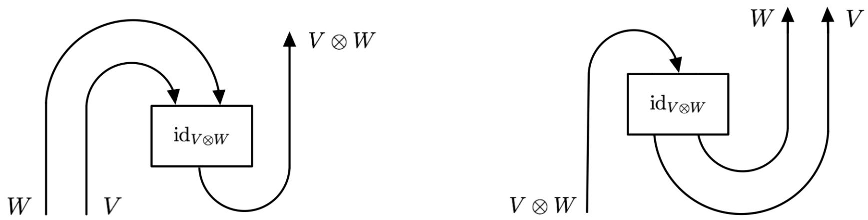


Figure 40

■

If $\mathcal{V}$ is a strict monoidal category, then $\mathrm{End}_{\mathcal{V}}(\mathbf{1})$ is a commutative monoid. Indeed, for any $f, g \in \mathrm{End}_{\mathcal{V}}(\mathbf{1})$,

$$f \circ g = (f \otimes \mathrm{id}_{\mathbf{1}}) \circ (\mathrm{id}_{\mathbf{1}} \otimes g) = f \otimes g = (\mathrm{id}_{\mathbf{1}} \otimes g) \circ (f \otimes \mathrm{id}_{\mathbf{1}}) = g \circ f.$$

We then have:

Corollary 1.3.3. The morphisms $b_V : \mathbf{1} \to \mathbf{1}^*$ and $d_V : \mathbf{1}^* \to \mathbf{1}$ are mutually inverse isomorphisms.

Proof. By the previous corollary and the properties of the unit object, we have

$$\mathbf{1}^* = \mathbf{1}^* \otimes \mathbf{1} \cong \mathbf{1}^* \otimes \mathbf{1}^{**} \cong (\mathbf{1}^* \otimes \mathbf{1})^* = \mathbf{1}^{**} \cong \mathbf{1}.$$

On the other hand, by the axioms of duality,

$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V,$$

for every object V of $\mathcal{V}$. In particular, taking $V = \mathbf{1}$, we have $d_{\mathbf{1}} \circ b_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$. Finally, this equality and the previous remark imply that, for any isomorphism $g : \mathbf{1} \to \mathbf{1}^*$,

$$\left(g^{-1} \circ b_{\mathbf{1}}\right) \circ (d_{\mathbf{1}} \circ g) = (d_{\mathbf{1}} \circ g) \circ \left(g^{-1} \circ b_{\mathbf{1}}\right) = d_{\mathbf{1}} \circ b_{\mathbf{1}} = d_{\mathbf{1}}.$$

Composing with g to the left and with g^{-1} to the right, we get $b_{\mathbf{1}} \circ d_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}^*}$. ■

Let $f : U \to V$ be a morphism in a monoidal category with duality. The *transpose* $f^* : V^* \to U^*$ is defined by

$$f^* = (d_V \otimes \mathrm{id}_{U^*}) \circ (\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*}) \circ (\mathrm{id}_{V^*} \otimes b_U).$$

It can be depicted as follows:

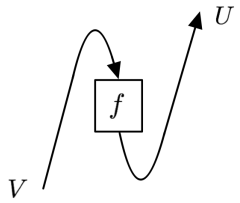


Figure 41

It follows from the definition that the transpose of an isomorphism is an isomorphism.

Proposition 1.3.4. If $f : V \to W$ and $g : U \to V$ are morphisms in a monoidal category with duality, then we have $(f \circ g)^* = g^* \circ f^*$ and $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$ for any object V.

Proof. Using the graphical calculus we have

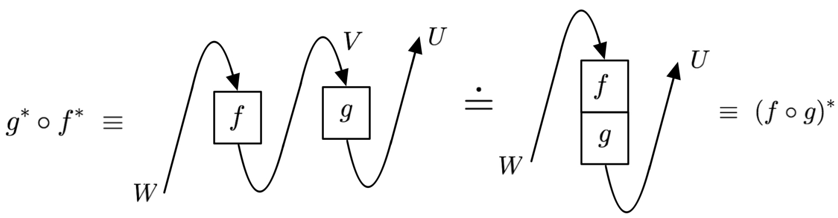


Figure 42

Then,

$$(\mathrm{id}_V)^* \circ f^* = (f \circ \mathrm{id}_V)^* = f^*$$

and

$$g^* \circ (\mathrm{id}_V)^* = (\mathrm{id}_V \circ g)^* = g^*,$$

i.e., $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$. ■

1.3.3 Quantum trace and dimension

By analogy with vector spaces, we now introduce the concepts of trace and dimension in ribbon categories. Recall that if V is a finite dimensional vector space with basis $\{v_i\}_{i \in I}$ and if $\{v^i\}_{i \in I}$ is the dual basis, the trace of an endomorphism $f : V \to V$ is given by

$$\begin{aligned} \mathrm{tr}(f) &= \sum_i v^i\,(f(v_i)) = \sum_i \mathrm{ev}_V\left(v^i \otimes f(v_i)\right) = \sum_i \mathrm{ev}_V \circ (\mathrm{id}_{V^*} \otimes f)(v^i \otimes v_i) \\ &= \mathrm{ev}_V \circ (\mathrm{id}_{V^*} \otimes f)\left(\sum_i v^i \otimes v_i\right) = \mathrm{ev}_V \circ (f \otimes \mathrm{id}_{V^*}) \circ \tau_{V,V^*} \circ \delta_V(1), \end{aligned}$$

where we use the same notations as in paragraph 1.1.7 and we set $\tau_{V,V^*}(v^i \otimes v_i) = v_i \otimes v^i$. This motivates the following definition:

Definition 1.3.5. For any object V and any endomorphism $f : V \to V$ in a ribbon category $\mathcal{V}$, we define the *quantum trace* of f as the element

$$\mathrm{tr}(f) = d_V \circ c_{V,V^*} \circ ((\theta_V \circ f) \otimes \mathrm{id}_{V^*}) \circ b_V$$

of the monoid $\mathrm{End}_{\mathcal{V}}(\mathbf{1})$.

It follows directly from theorem 1.2.8 that the quantum trace is the operator inavariant of the ribbon graphs represented in the following figure:

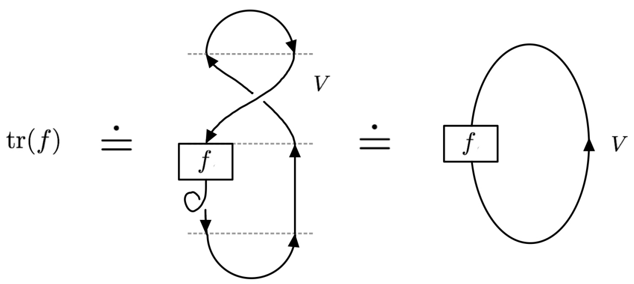


Figure 43

Note that if $V = W_1 \otimes \cdots \otimes W_n$ and $f = F(\Omega)$, for some v-coloured ribbon graph Ω, we may replace the band coloured by f in the previous diagram by a bunch of n bands coloured by $W_1, \ldots, W_n$, respectively, and the coupon coloured by f by the ribbon graph Ω. This substitutions do not change the associated operator invariant, so we have

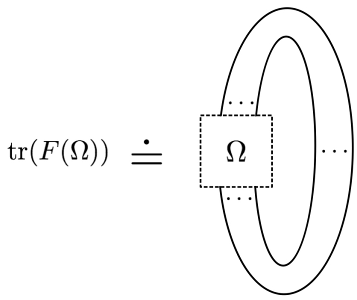


Figure 44

The ribbon graph on the right-hand side of the last equality is the v- coloured ribbon $(0,0)$-graph obtained by closing the free ends of Ω. In particular, taking $\Omega = X^+_{W,V} \circ X^+_{V,W}$, we get

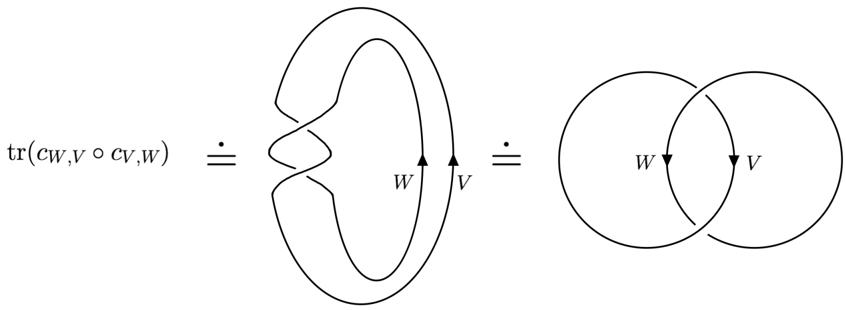


Figure 45

We collect the main properties of the trace in the following lemma:

Lemma 1.3.6. 1. For any morphisms $f : V \to W$, $g : W \to V$, we have $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$.
2. For any endomorphisms f, g of objects of $\mathcal{V}$, we have $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \circ \mathrm{tr}(g)$.
3. For any morphism $k : \mathbf{1} \to \mathbf{1}$, we have $\mathrm{tr}(k) = k$.

Proof. By the isotopy invariance of F, we have

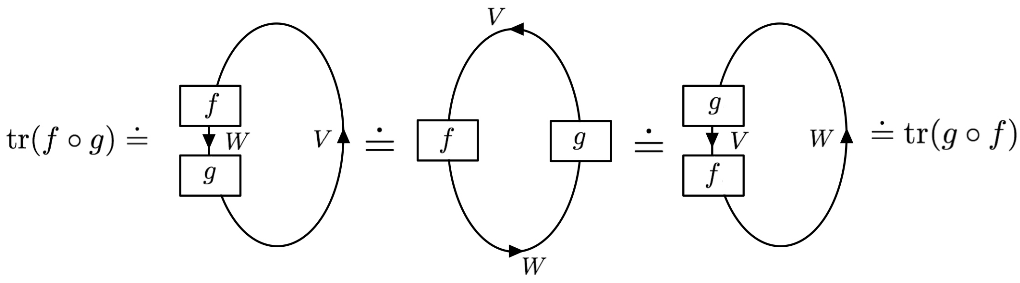


Figure 46

which proves the first assertion. The second one follows from the following argument:

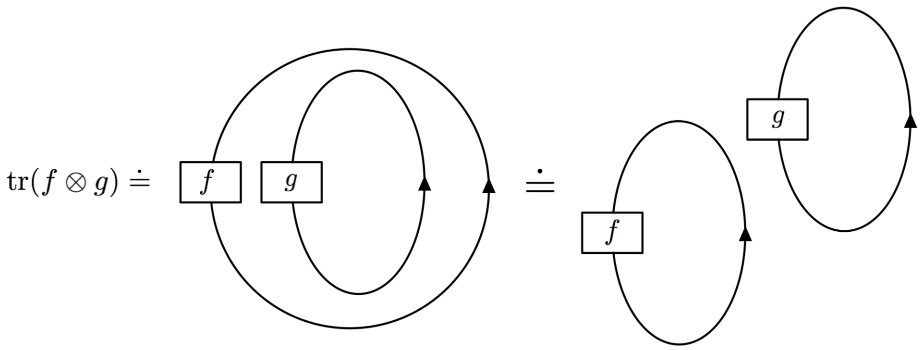


Figure 47

Finally, we prove the last one algebraically. Indeed, for any $k \in \mathrm{End}_{\mathcal{V}}(\mathbf{1})$, we have

$$\begin{aligned}
\mathrm{tr}(k) &= d_{\mathbf{1}} \circ (k \circ \mathrm{id}_{\mathbf{1}}) \circ b_{\mathbf{1}} \\
&= (\mathrm{id}_{\mathbf{1}} \otimes d_{\mathbf{1}}) \circ (k \otimes \mathrm{id}_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}^*}) \circ (\mathrm{id}_{\mathbf{1}} \otimes b_{\mathbf{1}}) \\
&= k \otimes (d_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}^*}) \circ b_{\mathbf{1}}) \\
&= k \otimes d_{\mathbf{1}} \circ b_{\mathbf{1}} = k \otimes \mathrm{id}_{\mathbf{1}} = k,
\end{aligned}$$

where the first equality holds by the definition of trace and the fact that $\theta_{\mathbf{1}} = c_{\mathbf{1},V} = c_{V,\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, the fourth is the result of corollary 1.3.3 and the rest are clear. ∎

Coming back to the case of vector spaces, the following equality holds for any finite dimensional vector space V:

$$\mathrm{tr}(\mathrm{id}_V) = \dim(V).$$

This generalizes now in a natural way to the case of ribbon categories.

Definition 1.3.7. The *quantum dimension* of an object $\mathcal{V}$ of a ribbon category $\mathcal{V}$ is the morphis

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) \in \mathrm{End}_{\mathcal{V}}(V).$$

It follows straightforward from the discussion above that $\dim(V)$ is the operator invariant of the following v-coloured ribbon graph

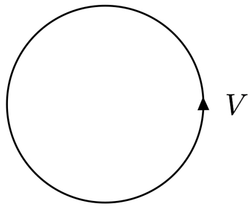
$\dim(V) \doteq$ 

Figure 48

This graph is isotopic to the same graph with the opposite direction, which means that, for any object V of $\mathcal{V}$, we have

$$\dim(V) = \dim(V^*).$$

The main properties of the dimension are collected in the following result, which is a direct corollary of lemma 1.3.6:

Corollary 1.3.8. 1. Isomorphic objects have equal dimensions.

2. For any objects V, W, we have $\dim(V \otimes W) = \dim(V) \circ \dim(V)$.
3. $\dim(\mathbf{1}) = \mathrm{id}_{\mathbf{1}}$.

Proof. If $g : V \to W$ is an isomorphism, then we have

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) = \mathrm{tr}(g^{-1} \circ g) = \mathrm{tr}(g \circ g^{-1}) = \mathrm{tr}(\mathrm{id}_W) = \dim(W).$$

The other two points are two particular cases of points 2 and 3 from lemma 1.3.6. ■

2 The oriented skein category and its connection to Rep $U_q(\mathfrak{gl}_n)$

The formalism developed in the previous section highlights the existence of connections between objects of a topological nature and algebraic objects with a given structure. The axiomatisation of ribbon categories is largely motivated by the discovery of Hopf algebras, which are bialgebras with additional structure that allows their category of representations to be endowed with a well-defined tensor product. In particular, this category of representations is a linear category, in the sense that each set of morphisms has a linear structure inherited from the structure of the underlying algebra. Thus, giving a linear structure to the category of tangles previously described will allow us to further study the connections of this category with the representations of an important class of Hopf algebras, the quantum groups, defined from deformations in classical algebraic structures defined by generators and relations.

2.1 Definition and first properties

2.1.1 The category of framed tangles. Let $\mathcal{V}_0$ be the trivial monoidal category generated by just one object $*$ and one morphism $\mathrm{id}_* : * \to *$. We define the *category $\mathcal{FOT}$ of framed oriented tangles* to be the subcategory of $\mathrm{Rib}_{\mathcal{V}_0}$ whose objects are finite strings on $+ := (*, +)$ and $- := (*, -)$ and whose morphisms are isotopy classes of ribbon graphs containing no coupons.

In order to simplify the exposition, we adapt to this particular case the notation that we used to describe $\mathrm{Rib}_{\mathcal{V}}$. The tensor product in $\mathcal{FOT}$ is given by concatenation so we will usually omit the sign $\otimes$ and just write objects as words in $\langle +, - \rangle$, e.g., $++-+- := + \otimes + \otimes - \otimes + \otimes -$. We will use bold type to refer to these words. If $\mathbf{a}, \mathbf{b} \in \langle +, - \rangle$, a ribbon graph representing a morphism $f : \mathbf{a} \to \mathbf{b}$ will be called an $(\mathbf{a}, \mathbf{b})$-*ribbon*. Note that $\mathrm{id}_+ = \downarrow$ and $\mathrm{id}_- = \uparrow$.

Lemma 1.2.6 gives a description of $\mathcal{FOT}$ in terms of generators and relations. We restate this result omitting the colouring of morphisms, as there is only one possibility:

Lemma 2.1.1. The category $\mathcal{FOT}$ is generated by the objects $+, -$ and the morphisms

$$X^+,\ X^-,\ Z^+,\ Z^-,\ \cap,\ \cup,\ \varphi,\ \varphi',$$

subject to the relations

$$(\uparrow \otimes X^+) \circ (X^+ \otimes \downarrow) \circ (\downarrow \otimes X^+) = (X^+ \otimes \downarrow) \circ (\downarrow \otimes X^+) \circ (X^+ \otimes \downarrow), \tag{R1$'$}$$

$$\downarrow = (\downarrow \otimes \cap) \circ (\cup \otimes \downarrow), \tag{R2$'$}$$

$$\uparrow = (\cap \otimes \uparrow) \circ (\uparrow \otimes \cup), \tag{R3$'$}$$

$$X^- = (X^+)^{-1}, \tag{R4$'$}$$

$$\varphi' = \varphi^{-1}, \tag{R5$'$}$$

$$X^\nu \circ (\downarrow \otimes \varphi) = (\varphi \otimes \downarrow) \circ X^\nu, \quad \nu \in \{+1, -1\}, \tag{R6$'$}$$

$$Z^\nu = \left[(\cap \otimes \downarrow \otimes \uparrow) \circ (\uparrow \otimes X^{-\nu} \otimes \uparrow) \circ (\uparrow \otimes \downarrow \otimes \cup)\right]^{-1}, \quad \nu \in \{+1, -1\}, \tag{R7$'$}$$

$$\varphi^2 = (\cap \otimes \downarrow) \circ (\uparrow \otimes X^+) \circ (Z^+ \otimes \downarrow) \circ (\cup \otimes \downarrow). \tag{R8$'$}$$

■

The category $\mathcal{FOT}$ may be endowed with the structure of a ribbon category in a natural way. Indeed, to define the braiding $c_{\mathbf{a},\mathbf{b}} : \mathbf{a} \otimes \mathbf{b} \to \mathbf{b} \otimes \mathbf{a}$, where $\mathbf{a} = \mathbf{a}_1 \cdots \mathbf{a}_k$ and $\mathbf{b} = \mathbf{b}_1 \cdots \mathbf{b}_l$, we take a bunch of k strands oriented as $\mathbf{a}_1, \ldots, \mathbf{a}_k$, respectively, and place it from above across a bunch of l vertical strands oriented as $\mathbf{a}_1, \ldots, \mathbf{a}_l$, respectively (see figure 49). The right dual of $\mathbf{a} = \mathbf{a}_1 \cdots \mathbf{a}_k$ is $\mathbf{a}^* := \mathbf{a}_k^* \cdots \mathbf{a}_1^*$, where $+^* = -$, with duality maps $d_{\mathbf{a}} : \mathbf{a}^* \otimes \mathbf{a} \to \mathbf{1}$ and $b_{\mathbf{a}} : \mathbf{1} \to \mathbf{a} \otimes \mathbf{a}^*$ defined as in the second and the third diagrams in figure 49. Finally, the twist is represented by the last diagram in the same figure. It is straightforward to check that this construction verify all the axioms in the definition of ribbon category.

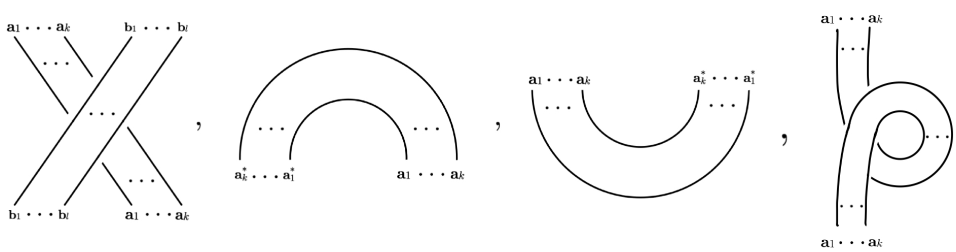
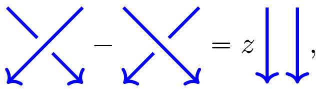


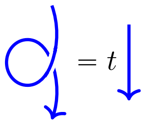
Figure 49: $c_{\mathbf{a},\mathbf{b}}$, $d_{\mathbf{a}}$ and $b_{\mathbf{a}}$.

2.1.2 The oriented skein category $\mathcal{OS}(z,t)$. Let $\Bbbk$ be a commutative ground ring and fix parameters $z, t \in \Bbbk^\times$. The $\Bbbk$-*linearisation* of $\mathcal{FOT}$ is the category whose objects are the same as $\mathcal{FOT}$ and whose morphisms are $\Bbbk$-linear combinations of morphisms in $\mathcal{FOT}$.

Definition 2.1.2. The *extended oriented skein category* $\widehat{\mathcal{OS}}(z,t)$ is the quotient of the $\Bbbk$-linearisation of $\mathcal{FOT}$ by the following two relations:



(S)



(T)

This relations are called the *Conway skein relation* and the *twist relation*, respectively.

The relations in the above definition are local in ribbon graphs, i.e., two morphisms are equivalent if one can be obtained from the other by substituting locally in the graph one side of the equalities for the other and keeping the rest of the diagrams unchanged. Algebraically, they can be written as

$$X^+ - X^- = z(\downarrow \otimes \downarrow), \tag{S}$$

$$\varphi = t \downarrow . \tag{T}$$

Relation (R5)$'$ implies that φ is invertible of inverse φ', so (T) is equivalent to

$$\varphi' = t^{-1} \downarrow . \tag{T$'$}$$

We then have

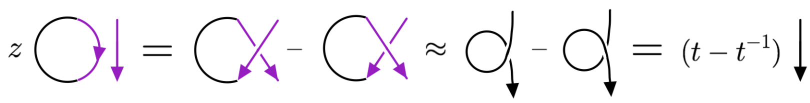
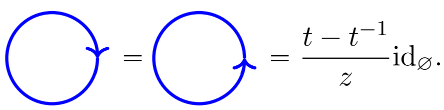


Figure 50

This motivates the following definition:

Definition 2.1.3. The *oriented skein category* $\mathcal{OS}(z,t)$ is the quotient of the $\Bbbk$-linearisation of $\mathcal{FOT}$ by the relations (S), (T) and



(D)

Note that (D) fixes the quantum dimension of the unique object $* \in \mathcal{V}_0$ of the underlying trivial monoidal category. On the other hand, the ribbon structure of $\mathcal{FOT}$ induces a ribbon structure on the oriented skein category $\mathcal{OS}(z,t)$ and using lemma 2.1.1 we can give the following efficient monoidal representation:

Theorem 2.1.4. The oriented skein category $\mathcal{OS}(z,t)$ is isomorphic to the strict $\Bbbk$-linear monoidal category $\mathcal{C}$ generated by objects E and F and morphisms

$$S : E \otimes E \to E \otimes E, \quad T : E \otimes F \to F \otimes E, \quad C : \mathbf{1} \to E \otimes F, \quad D : F \otimes E \to \mathbf{1},$$

subject to the following relations:

$$S^2 = zS + \mathrm{id}_E \otimes \mathrm{id}_E; \tag{OS1}$$

$$(S \otimes \mathrm{id}_E) \circ (\mathrm{id}_E \otimes S) \circ (S \otimes \mathrm{id}_E) = (\mathrm{id}_E \otimes S) \circ (S \otimes \mathrm{id}_E) \circ (\mathrm{id}_E \otimes S); \tag{OS2}$$

$$(\mathrm{id}_E \otimes D) \circ (C \otimes \mathrm{id}_E) = \mathrm{id}_E; \tag{OS3}$$

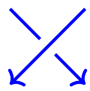
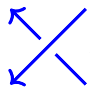
$$(D \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes C) = \mathrm{id}_F; \tag{OS4}$$

$$T^{-1} = (D \otimes \mathrm{id}_E \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes S \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C); \tag{OS5}$$

$$tD \circ T \circ C = \frac{t - t^{-1}}{z}\mathrm{id}_{\mathbf{1}}. \tag{OS6}$$

Remark 2.1.5. The description of a $\Bbbk$-linear monoidal category in terms of generators and relations is analogous to the case of a monoidal category (cf. paragraph 1.1.2) with the particularity that morphisms are now expressed as $\Bbbk$-linear combinations of finite tensor products and compositions of generators, and relations also involve $\Bbbk$-linear combinations of morphisms.

We will now prove that an explicit $\Bbbk$-linear monoidal functor giving an isomorphism between the category $\mathcal{C}$ of the theorem and $\mathcal{OS}(z,t)$ is given by

$E \mapsto +,\quad F \mapsto -,\quad S \mapsto$ , $\quad T \mapsto$ , $\quad C \mapsto$ , $\quad D \mapsto$ .

If we denote this functor by Φ, we can describe it algebraically by

$$\Phi(E) = +, \quad \Phi(F) = -, \quad \Phi(S) = X^+, \quad \Phi(T) = Z^-, \quad \Phi(C) = \cup, \quad \Phi(D) = \cap.$$

To check that Φ is well-defined we have to verify that the relations (OS1)-(OS6) hold in $\mathcal{OS}(z,t)$. Indeed, by the Conway skein relation, we have

$$(X^+)^2 = X^+ \circ (z(\downarrow \otimes \downarrow) + X^-) = zX^+ + \downarrow \otimes \downarrow,$$

so (OS1) holds in $\mathcal{OS}(z,t)$. Applying Φ to (OS2), (OS3), (OS4) and (OS5) we recover directly the relations (R1)$'$, (R2)$'$, (R3)$'$ and (R7)$'$. Finally, we check (OS6) graphically:

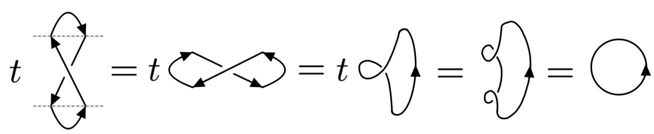


Figure 51

The leftmost element is exactly $\Phi\,(tD \circ T \circ C)$. The first, the second and the last equalities follow from the isotopy invariance of morphisms and the third one is the twist relation (T). Finally, the rightmost element is $\dim(*)$, so we have the result by (D).

Proof of theorem 2.1.4. We construct an strict $\Bbbk$-linear monoidal functor $\Psi : \mathcal{OS}(z,t) \to \mathcal{C}$ which is the inverse of Φ. Note that we already have a presentation of $\mathcal{OS}(z,t)$ by generators and relations: this is given by the generators and relations of $\mathcal{FOT}$ (cf. lemma 2.1.1) together with the relations (S), (T) and (D). Therefore, to define a $\Bbbk$-linear monoidal functor $\Psi : \mathcal{OS}(z,t) \to \mathcal{C}$ it suffices to choose the images of these generators and take the unique strict $\Bbbk$-linear monoidal extension to $\mathcal{OS}(z,t)$. The values of Ψ on objects are forced by those of Φ, so we have

$$\Psi(+) = E \quad \text{and} \quad \Psi(-) = F.$$

By the same reason, there is only one possible choice for each of the generators X^+, Z^-, $\cup$ and $\cap$. This is given by

$$\Psi\left(X^+\right) = S, \quad \Psi\left(Z^-\right) = T, \quad \Psi(\cup) = C \quad \text{and} \quad \Psi(\cap) = D.$$

On the other hand, the skein relation (S) forces

$$\Psi\left(X^-\right) = \Psi\left(X^+\right) - z\Psi(\downarrow \otimes \downarrow) = S - z\left(\mathrm{id}_E \otimes \mathrm{id}_E\right),$$

and the twist relations (S) and (S)$'$ yield to

$$\Psi\left(\varphi\right) = t\,\mathrm{id}_E \quad \text{and} \quad \Psi\left(\varphi'\right) = t^{-1}\,\mathrm{id}_E.$$

It remains to define the value of $\Psi\left(Z^-\right)$. In $\mathcal{OS}(z,t)$ we have

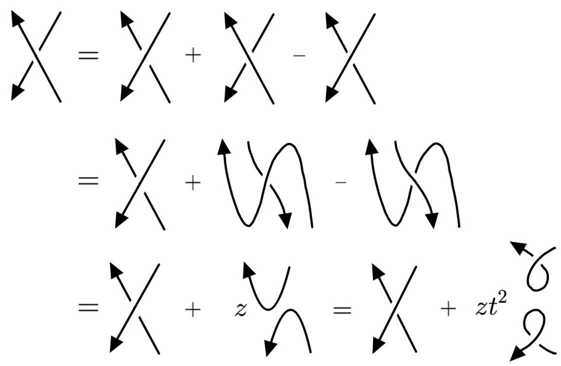


Figure 52

where the first equality is trivial, the second one is a consequence of the isotopy invariance of morphisms, in the third one we apply the skein relation (S) and in the last one we use the twist relation (T) twice. The rightmost term is $Z^- + zt^2\left(Z^- \circ \cup \circ \cap \circ Z^-\right)$, which forces

$$\Psi\left(Z^+\right) = T + zt^2\left(T \circ C \circ D \circ T\right).$$

Let us verify that Ψ is well defined by checking that (R1)$'$-(R8)$'$ hold in $\mathcal{C}$. First we note that (OS1) implies

$$(S - z(\mathrm{id}_E \otimes \mathrm{id}_E)) \circ S = S \circ (S - z(\mathrm{id}_E \otimes \mathrm{id}_E)) = S^2 - zS = \mathrm{id}_E \otimes \mathrm{id}_E,$$

so (R4)$'$ holds. Applying Ψ to (R1)$'$, (R2)$'$ and (R3)$'$ we get (OS2), (OS3) and (OS4). (R5)$'$ is clear by construction of Ψ and to check (R6)$'$ we just note that $\Psi(\downarrow \otimes \varphi) = t\,\mathrm{id}_E \otimes \mathrm{id}_E = \Psi(\varphi \otimes \downarrow)$, so

$$\Psi\left(X^\nu\right) \circ \Psi(\downarrow \otimes \varphi) = t\Psi\left(X^\nu\right) = \Psi(\varphi \otimes \downarrow) \circ \Psi\left(X^\nu\right).$$

For $\nu = -1$, the relation (R7)$'$ is just the version in $\mathcal{OS}(z,t)$ of (OS5), so there is nothing to check. Consider the case where $\nu = 1$. Applying Ψ to the expression in square brackets in (R7)$'$, we get

$$\begin{aligned}
&\Psi\left((\cap \otimes \downarrow \otimes \uparrow) \circ (\uparrow \otimes X^- \otimes \uparrow) \circ (\uparrow \otimes \downarrow \otimes \cup)\right) \\
&\quad = (D \otimes \mathrm{id}_E \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes (S - z(\mathrm{id}_E \otimes \mathrm{id}_E)) \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C) \\
&\quad = (D \otimes \mathrm{id}_E \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes S \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C) \\
&\qquad\quad - z(D \otimes \mathrm{id}_E \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes \mathrm{id}_E \otimes \mathrm{id}_F) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C) \\
&\quad = T^{-1} - z(D \otimes C) = T^{-1} - z(\mathrm{id}_{\mathbf{1}} \otimes C) \circ (D \otimes \mathrm{id}_{\mathbf{1}}) = T^{-1} - z(C \circ D),
\end{aligned}$$

where the first equality in the last line comes from (OS5) and the rest are clear. We have to prove that this is the two-sided inverse of $\Psi(Z^+) = T + zt^2(T \circ C \circ D \circ T)$. Indeed,

$$\begin{aligned}
&\left(T^{-1} - z(C \circ D)\right) \circ \left(T + zt^2(T \circ C \circ D \circ T)\right) \\
&\quad = \mathrm{id}_E \otimes \mathrm{id}_F + zt^2(C \circ D \circ T) - z(C \circ D \circ T) - z^2t^2(C \circ D \circ T \circ C \circ D \circ T) \\
&\quad = \mathrm{id}_E \otimes \mathrm{id}_F + zt^2(C \circ D \circ T) - z(C \circ D \circ T) - zt^2(C \circ D \circ T) + z(C \circ D \circ T) \\
&\quad = \mathrm{id}_E \otimes \mathrm{id}_F,
\end{aligned}$$

where the second equality is a consequence of (OS6) applied to the last term. The other computation is analogous. We check now (R8)$'$. To do so, we claim that

$$(D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S) = (\mathrm{id}_E \otimes D) \circ \left(T^{-1} \otimes \mathrm{id}_E\right). \qquad (\star)$$

Applying Ψ to the right-hand side of the equality one gets $t^2\,\mathrm{id}_E$. Doing the same to the left-hand side and applying $(\star)$ and (OS8), we have

$$\begin{aligned}
&(D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S) \circ \left[\left(T + zt^2(T \circ C \circ D \circ T)\right) \otimes \mathrm{id}_E\right] \circ (C \otimes \mathrm{id}_E) \\
&\quad = (\mathrm{id}_E \otimes D) \circ \left(T^{-1} \otimes \mathrm{id}_E\right) \circ \left[\left(T + zt^2(T \circ C \circ D \circ T)\right) \otimes \mathrm{id}_E\right] \circ (C \otimes \mathrm{id}_E) \\
&\quad = (\mathrm{id}_E \otimes D) \circ \left[\left(C + zt^2(C \circ D \circ T \circ C)\right) \otimes \mathrm{id}_E\right] \\
&\quad = (\mathrm{id}_E \otimes D) \circ \left(1 + t\left(t - t^{-1}\right)\right)(C \otimes \mathrm{id}_E) = t^2\,\mathrm{id}_E.
\end{aligned}$$

The equality $(\star)$ follows easily from (OS5) and the properties of the tensor product. Indeed,

$$\begin{aligned}
&(\mathrm{id}_E \otimes D) \circ \left(T^{-1} \otimes \mathrm{id}_E\right) \\
&\quad = (\mathrm{id}_E \otimes D) \circ \left[\left((D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S)\right) \otimes \mathrm{id}_F \otimes \mathrm{id}_E\right] \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C \otimes \mathrm{id}_E) \\
&\quad = \left[\left((D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S)\right) \otimes D\right] \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C \otimes \mathrm{id}_E) \\
&\quad = (D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes \mathrm{id}_E \otimes D) \circ (\mathrm{id}_F \otimes \mathrm{id}_E \otimes C \otimes \mathrm{id}_E) \\
&\quad = (D \otimes \mathrm{id}_E) \circ (\mathrm{id}_F \otimes S).
\end{aligned}$$

Finally, (D) translates into (OS8) by Ψ.

To conclude, we have to prove that Ψ is the two-side inverse functor of Φ. By construction, $\Psi \circ \Phi = \mathrm{Id}_{\mathcal{C}}$ and it is straightforward to verify that, in fact, $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{OS}(z,t)}$. ∎

2.2 Connection to Rep $U_q(\mathfrak{gl}_n)$

2.2.1 Definition of $U_q(\mathfrak{gl}_n)$ and natural representations. Let $\Bbbk$ be a field of characteristic 0 and $q \in \Bbbk^\times$. Fix $n \in \mathbb{N}$ and assume that q is not a root of unity. By definition, the algebra $U_q(\mathfrak{gl}_n)$ is generated by elements e_i, f_i, d_j, d_j^{-1}, $i = 1, \ldots, n-1$, $j = 1, 2, \ldots, n$, subject to the relations

$$\begin{gathered}
d_i d_j = d_j d_i, \quad d_i d_i^{-1} = d_i^{-1} d_i = 1, \\
d_i e_j d_i^{-1} = q^{\delta_{ij} - \delta_{i,j+1}} e_j, \quad d_i f_j d_i^{-1} = q^{-\delta_{i,j} + \delta_{i,j+1}} f_j, \\
e_i f_j - f_j e_i = \delta_{ij} \frac{d_i d_{i+1}^{-1} - d_i^{-1} d_{i+1}}{q - q^{-1}}, \\
e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i, \quad |i - j| \leqslant 2, \\
e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0, \\
f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0.
\end{gathered}$$

This algebra becomes a Hopf algebra (cf. appendix) with the comultiplication $\Delta : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ defined on generators by

$$\Delta(e_i) = d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \quad \Delta(d_i^{\pm1}) = d_i^{\pm1} \otimes d_i^{\pm1}.$$

The antipode $S : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ is defined by

$$S(e_i) = -d_i d_{i+1}^{-1} e_i, \quad S(f_i) = -f_i d_i^{-1} d_{i+1}, \quad S(d_i) = d_i^{-1},$$

and the counit $\varepsilon : U_q(\mathfrak{gl}_n) \to \Bbbk$ is given by

$$\varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0, \quad \varepsilon(d_i) = 1.$$

We define the *natural representations* V^+ on basis $\{v_i^+ \ : \ 1 \leqslant i \leqslant n\}$ and V^- on basis $\{v_i^- \ : \ 1 \leqslant i \leqslant n\}$ to be the $U_q(\mathfrak{gl}_n)$-modules determined by the following actions of the generators of $U_q(\mathfrak{gl}_n)$, respectively:

$$\begin{array}{lll} e_i v_j^+ = \delta_{i+1,j} v_i^+, & f_i v_j^+ = \delta_{ij} v_{i+1}^+, & d_i v_j^+ = q^{\delta_{ij}} v_j^+, \\ e_i v_j^- = \delta_{i,j} v_{i+1}^-, & f_i v_j^- = \delta_{i+1,j} v_i^-, & d_i v_j^- = q^{-\delta_{ij}} v_j^-. \end{array}$$

Set Rep $U_q(\mathfrak{gl}_n)$ for the category of finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ that are isomorphic to finite direct sums of summands of the modules obtained by tensor products of V^+ and V^-.

2.2.2 A connection between $\mathcal{OS}(z,t)$ and Rep $U_g(\mathfrak{gl}_n)$.

Set $z = q - q^{-1}$. The category Rep $U_q(\mathfrak{gl}_n)$ admits the structure on a ribbon Ab-category (cf. appendix A.2), so theorem 1.2.8 ensures the existence of a non-trivial functor $F : \mathcal{FOT} \to$ Rep $U_q(\mathfrak{gl}_n)$. On the other hand, we have seen that the oriented skein category $\mathcal{OS}(z,t)$ is also a ribbon Ab-category obtained from the linearisation of $\mathcal{FOT}$. The questions then arises as to the existence of a functor $\Gamma : \mathcal{OS}(z,t) \to$ Rep $U_q(\mathfrak{gl}_n)$ transporting the ribbon Ab-structure of $\mathcal{OS}(z,t)$ to Rep $U_q(\mathfrak{gl}_n)$. We construct such a functor using theorem 2.1.4.

Let E, F, S, T, C, D be the set of generators from theorem 2.1.4. We will freely identify this generators with their images in $\mathcal{OS}(z,t)$ given by the functor Φ of the previous paragraph. Set $\Gamma(+) = V^+$ and $\Gamma(-) = V^-$. The objects $+$ and $-$ are dual in $\mathcal{OS}(z,t)$, so let us identify V^- with the dual of V^+. To do so, we have to fix a non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle \ : \ V^+ \otimes V^- \to \Bbbk.$$

Recall from section A.2 that the action of $U_q(\mathfrak{gl}_n)$ as a Hopf algebra in $(V^+)^*$ is defined by

$$(u\varphi)(v^+) = \varphi(S(u)v^+)$$

for any $u \in U_q(\mathfrak{gl}_n), \varphi \in (V^+)^*, v^+ \in V^+$. Hence, in order to get a well-defined morphism

$$V^- \xrightarrow{\sim} (V^+)^*, \quad v^- \mapsto \langle \cdot, v^- \rangle$$

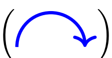
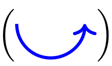
of $U_q(\mathfrak{gl}_n)$-modules, the pairing has to be compatible with the U_q-module structure of $(V^+)^*$. Precisely, if we set $\varphi_{v^-} : V^+ \to \Bbbk, \quad v^+ \mapsto \varphi_{v^-}(v^+) := \langle v^+, v^- \rangle$, we must have

$$\langle S(u)v^+, v^- \rangle = \varphi_{v^-}(S(u)v^+) = (u\varphi_{v^-})(v^+) = \varphi_{uv^-}(v^+) = \langle v^+, uv^- \rangle,$$

for all $u \in U_q(\mathfrak{gl}_n)$, $v^+ \in V^+$ and $v^- \in V^-$. There is a unique (up to scalars) non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle$ satisfying this condition, given by $\langle v_i, v_j \rangle := (-1)^i q^i \delta_{ij}$. The associated evaluation and coevaluation maps are

$$\begin{array}{rccc} \mathrm{ev} : & V^- \otimes V^+ & \to & \Bbbk, \\ & v_i^- \otimes v_j^+ & \mapsto & (-1)^i q^i \delta_{ij}, \end{array} \qquad \begin{array}{rccc} \mathrm{coev} : & \Bbbk & \to & V^+ \otimes V^-, \\ & 1 & \mapsto & \sum_{j=1}^n (-1)^j q^{-j} v_j^+ \otimes v_j^-. \end{array}$$

Then, defining

$\Gamma\Big($  $\Big) = \mathrm{ev}$ and $\Gamma\Big($  $\Big) = \mathrm{coev}$,

the relations (OS3) and (OS4) from theorem 2.1.4 are automatically satisfied.

Next we choose a candidate for the image of S. This has to be an isomorphism which is a solution of the Yang-Baxter equation (OS2) and satisfies (OS1). For our choice of $z = q - q^{-1}$, this is satisfied by the isomorphism $R : V^+ \otimes V^+ \to V^+ \otimes V^+$ given by

$$R(v_i^+ \otimes v_j^+) := \begin{cases} v_j^+ \otimes v_i^+, & \text{if } i < j, \\ q v_j^+ \otimes v_i^+, & \text{if } i = j, \\ v_j^+ \otimes v_i^+ + (q - q^{-1}) v_i^+ \otimes v_j^+, & \text{if } i > j. \end{cases}$$

The value of $\Gamma(T)$ can be obtained by applying Γ to (OS5). Using this, the expression of $\cap^-$ and $\cup^-$ in terms of the generators of $\mathcal{FOT}$ and the relations (T) and (OS6), one can check that the images of $\cap^-$ and $\cup^-$ by Γ are

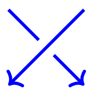
$$\Gamma(\cap^-) = \mathrm{ev}' \quad \text{and} \quad \Gamma(\cup^-) = \mathrm{coev}',$$

where

$$\begin{array}{rccc} \mathrm{ev}' : & V^+ \otimes V^- & \to & \Bbbk, \\ & v_i^+ \otimes v_j^- & \mapsto & (-1)^i q^{n+1-i} \delta_{ij}, \end{array} \qquad \begin{array}{rccc} \mathrm{coev}' : & \Bbbk & \to & V^- \otimes V^+, \\ & 1 & \mapsto & \sum_{j=1}^n (-1)^j q^{j-n-1} v_j^- \otimes v_j^+. \end{array}$$

Equally, one verifies that dimension relation (D) imposes $t = q^n$. More details on this are given in [1, section 3]. We have proved the following result:

Theorem 2.2.1. Let $\Bbbk$ be a field of characteristic zero and $q \in \Bbbk^\times$. Set $z = q - q^{-1}$ and $t = q^n$, for $n \in \mathbb{N}$. There exists a $\Bbbk$-linear monoidal functor $\Gamma : \mathcal{OS}(z, t) \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ sending $+ \mapsto V^+$, $- \mapsto V^-$ and

 $\mapsto R$,  $\mapsto \mathrm{ev}$,  $\mapsto \mathrm{coev}$,  $\mapsto \mathrm{ev}'$,  $\mapsto \mathrm{coev}'$.

■

2.2.3 Some consequences. We can use the last result to prove some properties of the oriented skein category $\mathcal{OS}(z, t)$. We consider here an arbitrary commutative ground ring $\Bbbk$ and $z, t \in \Bbbk^\times$.

Let $\mathbf{a}, \mathbf{b} \in \langle +, - \rangle$ be objects of $\mathcal{OS}(z, t)$ such that x (resp. x') letters of $\mathbf{a}$ and y (resp. y') letters of $\mathbf{b}$ are equal to $+$ (resp. $-$). It is clear that the set $\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathbf{a}, \mathbf{b})$ is zero unless $r := x + y' = x' + y$, so let us assume that this is the case. Our first aim is to prove that the $\Bbbk$-module $\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathbf{a}, \mathbf{b})$ is isomorphic to $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$. Intuitively, we will construct an isomorphism that stretches all the strands pointing upwards and folds them by passing each of their ends through one side of the graph. Formally, let $b : +^{\otimes y} \to \mathbf{b} \otimes +^{\otimes y'}$ be the unique morphism that consists of y' nested rightwards cups on top of y vertical downwards strands. Let $a : +^{\otimes x'} \otimes \mathbf{a} \to +^{\otimes x}$ the unique morphism consisting of x' nested leftwards cups on top of x vertical downwards strands. These morphisms correspond to the top and bottom blocks in picture 53, respectively. Then, the linear map

$$\begin{array}{rccc} \theta : & \mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathbf{a}, \mathbf{b}) & \to & \mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r}) \\ & f & \mapsto & (\downarrow^{\otimes x'} \otimes b) \circ (\downarrow^{\otimes x'} \otimes f \otimes \downarrow^{\otimes y'}) \circ (a \otimes \downarrow^{\otimes x'}). \end{array}$$

has an obvious two-sided inverse, obtained by stretching and folding in the opposite direction the ends of the strands pointing downwards.

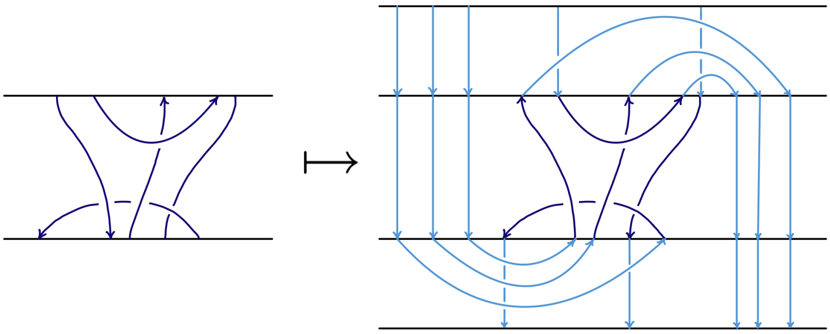


Figure 53

Fix now $r \in \mathbb{N}$ and let us study in more detail the set $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$. The generators of these set are isotopy classes of ribbon diagrams whose strands go from $\mathbb{R} \times \{0\} \times \{1\}$ to $\mathbb{R} \times \{0\} \times \{0\}$. Take a representative of such a generator. If any of its strands is twisted, we may apply the relation (T) to write it as a ribbon diagram containing no twists multiplied by a certain power of t. By the isotopy invariance of morphisms, we can stretch all these strands so that we obtain a ribbon diagram where the z coordinate of every strand is strictly decreasing. This is better explained by the following picture:

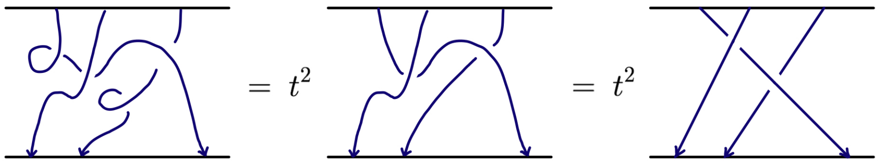


Figure 54

Moreover, by recursively applying the skein relation (S) to every negative crossing, we can change this morphism by a $\mathbb{k}$-linear combination of morphisms of $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$ whose crossings are all positive. We can describe these morphisms using the symmetric group $\mathfrak{S}_r$ in the following way: to permutation $\pi \in \mathfrak{S}_r$ of $\{1, \ldots, r\}$, we associate the ribbon diagram $\xi_\pi \in \mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$ with a vertical strictly descending strand going from $(i, 0, 1)$ to $(\pi(i), 0, 0)$, for each $i \in \{1, ..., r\}$ in such a way that all crossings are positive. The previous discussion shows that, in fact, the set $\{\xi_\pi\}_{\pi \in \mathfrak{S}_r}$ provides a set of $r!$ generators for $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$ as a $\mathbb{k}$-module. Recall that every permutation can be written as a composition of transpositions so we would like to transport this decomposition to $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$ in order to get a more reduced set of generators of this space as a $\mathbb{k}$-algebra. Nevertheless, the relations defining $\mathfrak{S}_r$ are too restrictive, since $\sigma^2 = \mathrm{id}_{\mathfrak{S}_r}$, for any transposition $\sigma \in \mathfrak{S}_r$ and this relation does not hold in $\mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r})$. Indeed, for such a transposition, ξ_σ has the form $\downarrow \otimes \cdots \otimes \downarrow \otimes X^+ \otimes \downarrow \otimes \cdots \otimes \downarrow$, so the relation (OS1) from theorem 2.1.4 implies that $\xi_\sigma^2 = z\xi_\sigma + \downarrow^{\otimes r}$. For this condition to be verified we should rather work in a suitable deformation of the group algebra $\mathbb{k}[\mathfrak{S}_r]$. This is provided by the *Iwahori-Hecke algebra.*

Definition 2.2.2. The *Iwahori-Hecke algebra* $H_r(\mathbb{k}; z)$ is the unitary associative $\mathbb{k}$-algebra with $r-1$ generators $\sigma_1, \ldots, \sigma_{r-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geqslant 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-2,$$

$$\sigma_i^2 = z\sigma_i + 1, \quad i = 1, \ldots, n-1.$$

When the context is clear we will simply denote this algebra by H_r. It is a well-known fact that it has rank $r!$, with basis $\{w\}_{w \in \mathfrak{S}_r}$ defined by letting w be the word in generators σ_i arising from a reduced expression for w. It is now clear from the discussion above that we have a surjective morphism of algebras

$$\begin{array}{rccl} \iota_r : & H_r & \to & \mathrm{Hom}_{\mathcal{OS}(z,t)}(+^{\otimes r}, +^{\otimes r}) \\ & \sigma_i & \mapsto & \xi_{\sigma_i}. \end{array}$$

In fact, we have more than that:

Lemma 2.2.3. The map ι is an isomorphism of $\Bbbk$-algebras.

Proof. In the course of the proof we will vary the ground ring $\Bbbk$, so we will denote $\mathcal{OS}(z,t)$ by $\mathcal{OS}(z,t)_{\Bbbk}$. Consider first the case where $\Bbbk = \mathbb{Z}\left[z, z^{-1}, t, t^{-1}\right]$. For any $n \geqslant r$, let $\Gamma : \mathcal{OS}(q - q^{-1}, q^n)_{\mathbb{Q}(q)} \to \mathrm{Rep}\, U_q(\mathfrak{gl}_n)$ be the functor from theorem 2.2.1. Let $\omega : \mathcal{OS}(z,t)_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \to \mathcal{OS}(q - q^{-1}, q^n)_{\mathbb{Q}(q)}$ the obvious strict $\mathbb{Z}$-linear monoidal functor sending $z \mapsto q - q^{-1}$, $t \mapsto q^n$. Now take a linear relation

$$\sum_{\pi \in \mathfrak{S}_r} \lambda_\pi(z,t)\xi_\pi = 0,$$

with $\lambda_\pi(z,t) \in \mathbb{Z}\left[z, z^{-1}, t, t^{-1}\right]$. Applying ω and Γ to the last expression, one gets

$$\sum_{\pi \in \mathfrak{S}_r} \lambda_\pi(q - q^{-1}, q^n)\Gamma(\xi_\pi) = 0.$$

It follows from theorem 2.2.1 and the definition of ξ_π, that $\Gamma(\xi_\pi)$ is a composition of morphisms of the form $\mathrm{id}_{V^+} \otimes \cdots \otimes \mathrm{id}_{V^+} \otimes R \otimes \mathrm{id}_{V^+} \otimes \cdots \otimes \mathrm{id}_{V^+}$ (see figure 55). Thus, an easy induction on the length of an irreducible decomposition of π on the generators gives

$$\Gamma(\xi_\pi)(v_1 \otimes \cdots \otimes v_n) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(n)}.$$

Since $n \geqslant r$, this elements are linearly independent in V^+, which implies that $\lambda_\pi(q - q^{-1}, q^n) = 0$ for all $\pi \in \mathfrak{S}_r$ and, since this is true for infinitely many values of n, we get that $\lambda_\pi(z,t) = 0$. Hence, $\{\xi_\pi\}_{\pi \in \mathfrak{S}_r}$ is a basis of $\mathrm{Hom}_{\mathcal{OS}(z,t)_{\mathbb{Z}[z,z^{-1},t,t^{-1}]}}(+^{\otimes r}, +^{\otimes r})$, so this has rank $r!$. Then, ι_r is an isomorphism.

The general case where $\Bbbk$ is an arbitrary commutative ring and $\bar{z}, \bar{t} \in \Bbbk^\times$ follows from the following observation. Viewing $\Bbbk$ as a $\mathbb{Z}\left[z, z^{-1}, t, t^{-1}\right]$-module so z and t act via $\bar{z}$ and $\bar{t}$, there is an obvious strict $\Bbbk$-linear monoidal functor

$$\mathcal{OS}(\bar{z}, \bar{t})_{\Bbbk} \to \mathcal{OS}(z,t)_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \otimes_{\mathbb{Z}[z,z^{-1},t,t^{-1}]} \Bbbk$$

sending each generator ξ_π to the same generator tensored by $1_{\Bbbk}$. Since the set $\{\xi_\pi \otimes 1_{\Bbbk}\}_{\pi \in \mathfrak{S}_r}$ is linearly independent, so is $\{\xi_\pi\}_{\pi \in \mathfrak{S}_r}$ in $\mathcal{OS}(z,t)_{\Bbbk}$, which proves the result.

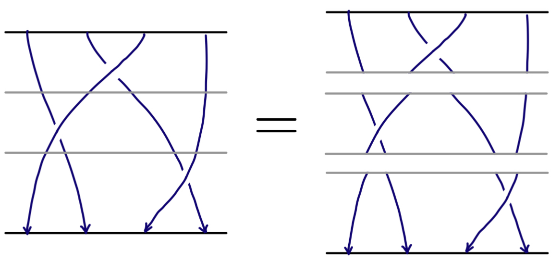


Figure 55

■

Let $\mathbf{a}, \mathbf{b} \in \langle +, - \rangle$ and $r := x + y' = x' + y$ as above. Using the isomorphism θ constructed at the beginning of the paragraph, we have:

Theorem 2.2.4. The morphism space $\mathrm{Hom}_{\mathcal{OS}(z,t)}(\mathbf{a}, \mathbf{b})$ is a free $\Bbbk$-module of rank $r!$. ■

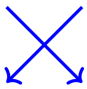
3 Chord diagrams and the Kontsevich's theorem

In this section, we describe the category of string diagrams. This category admits an infinitesimal symmetric category structure, so it can be deformed, using Drinfeld's theory of associates which we will discuss briefly, giving rise to a ribbon category whose structure is related to the category of tangles. The origin of the string diagrams is to be found in the combinatorial description of the finite degree invariants, or Vassiliev invariants, which we describe below.

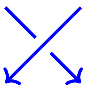
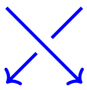
3.1 Vassiliev invariants of links

3.1.1 Vassiliev invariants of unframed knots. Let us start by recalling that an *oriented knot* is an oriented embedding of the circle $\mathbb{S}^1$ in $\mathbb{R}^3$, i.e., it is a three-dimensional copy of the circle containing no self-intersections. If we allow self-intersections to occur in a finite number, we have a weaker notion of knot, which we will refer to as *singular knot*. Precisely, a singular knot is an immersion of an oriented copy of $\mathbb{S}^1$ in $\mathbb{R}^3$, where we impose all the self-intersections to be double points with transversal branches and we assume that they are in finite number. There is a natural notion of isotopy of singular knots, taking double points to double points. For a fixed field $\Bbbk$, we denote by $\mathcal{K}$ (resp. $\mathcal{S}$) the vector space of $\Bbbk$-linear combinations of isotopy classes of oriented knots (resp. regular knots). Clearly, $\mathcal{K} \subset \mathcal{S}$.

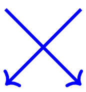
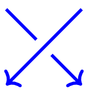
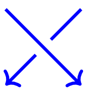
We can describe singular oriented knots using planar diagrams. Double points will be represented by

 .

There are two ways to desingularise a double point, yielding two singular oriented knots with one double point less. We will represent these two options by

$K_+ =$  and $K_- =$  .

Let us define $\mathcal{D}$ as the quotient of $\mathcal{S}$ by the ideal generated by the relations of the form

 $=$  $-$  , (V)

where the three knots involved are identical outside the neighbourhood represented in the diagram. By recursively applying the relation (V) to every double point, we can associate an element of $\mathcal{K}$ to each singular knot $K \in \mathcal{S}$. This extends by linearity to a map

$$\mathcal{S} \to \mathcal{K}$$

which induces a isomorphism

$$\mathcal{D} \xrightarrow{\sim} \mathcal{K}.$$

Hence, the vector space $\mathcal{D}$ is nothing but a different representation of $\mathcal{D}$ which allows us to talk about knots with double points in $\mathcal{K}$. In particular, if $\mathcal{K}_m$ denotes the vector space generated by isotopy classes of oriented knots with $> m$ points, we get a filtration

$$\mathcal{K} \supset \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots .$$

Note that applying (V) to a double point of a knot $K \in \mathcal{K}_m$, we have a linear combination of two knots in $\mathcal{K}_{m-1}$ so, indeed, $\mathcal{K}_m$ is linearly generated by the isotopy classes of oriented knots with exactly $m+1$ double points.

Any invariant of oriented knots P of oriented knots taking values in $\Bbbk$ gives rise to a $\Bbbk$-linear map $\theta : \mathcal{K} \to \Bbbk$ and, the other way round, any such linear map induces an invariant of oriented knots. These invariants are called *Vassiliev invariants* and it follows that they are in bijection with $\mathrm{Hom}_{\Bbbk}(\mathcal{K}, \Bbbk)$. The set $\mathcal{V}$ of all Vassiliev invariants inherits an structure of $\Bbbk$ linear space from $\Bbbk$.

Definition 3.1.1. A knot invariant θ is a *Vassiliev invariant of degree* m if the induced map $\theta : \mathcal{K} \to \Bbbk$ factors through $\mathcal{K}_m$, that is,

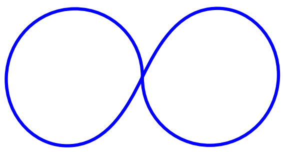
$$\theta_{|\mathcal{K}_m} = 0.$$

We denote by $\mathcal{V}_m$ the vector space of all Vassiliev invariant of degree m.

If θ is a Vassiliev invariant of degree m, then it is also a Vassiliev invariant of degree n for each $n > m$, since $\mathcal{K}_n \subset \mathcal{K}_m$. Hence, the filtration of $\mathcal{K}$ induces a filtration

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}.$$

The space $\mathcal{V}_0$ of Vassiliev invariants of degree 0 is easy to describe. If $\theta \in \mathcal{V}_0$, then its value is zero for every knot containing a double point. In particular, applying θ to the Vassiliev relation (V), we get $\theta(K_+) = \theta(K_-)$, so changing a single crossing does not alter the value of θ. Since every knot can be turned into the trivial knot by a finite sequence of crossing changes, it follows that a Vassiliev invariant of degree 0 is a constant function on the set of knots. By exactly the same argument, Vassiliev invariants are constant over the sets of knots with one single point that differs in a finite number of crossing changes. However, all theses knots can be turned into



but the positive and the negative desingularisations of this knot are both isotopic, so (V) implies that the value of the invariant on this knot is 0. All in all, every Vassiliev invariant of degree 1 is a Vassiliev invariant of degree 0.

3.1.2 Emergence of chord diagrams.

There is a natural way to extend any Vassiliev invariant $\theta : \mathcal{K} \to \Bbbk$ to an invariant $\bar{\theta} : \mathcal{S} \to \Bbbk$ of singular knots: if K is such a knot, then we define $\bar{\theta}(K)$ to be the value of θ on the knot obtained by desingularising K using (V). It is clear that if θ has degree m, then $\bar{\theta}$ is zero on the subspace $\mathcal{S}_{m+1}$ of $\mathcal{S}$ spanned by the isotopy classes of singular knots with exactly $m+1$ double points. Conversely, by a previous observation, the canonical projection $\mathcal{S} \to \mathcal{D} \cong \mathcal{K}$ maps $\mathcal{S}_{m+1}$ onto $\mathcal{K}_m$, so any map $\bar{\theta} : \mathcal{S} \to V$ verifying $\theta_{|\mathcal{S}_{m+1}} = 0$ factors through $\mathcal{K}$ giving rise to a Vassiliev invariant $\theta : \mathcal{K} \to V$. To clarify the exposition, we will use a bar to distinguish singular knots $\bar{K} \in \mathcal{S}$ from knots $K \in \mathcal{K}$.

If $\bar{K}_+$ and $\bar{K}_-$ are two singular knots in $\mathcal{S}_m$ which differ by a single crossing change and isotopy, then there exists a knot $\bar{K} \in \mathcal{S}_{m+1}$ whose projection in $\bar{K}$ equals the projection of $\bar{K}_+ - \bar{K}_-$, hence

$$0 = \bar{\theta}(\bar{K}) = \bar{\theta}(\bar{K}_+) - \bar{\theta}(\bar{K}_-),$$

for every Vassiliev invariant θ of degree m. Applying the same argument recursively, one shows that, indeed, $\bar{\theta}(\bar{K}) = \bar{\theta}(\bar{K})$, for all $\bar{K}, \bar{K}' \in \mathcal{S}_m$ differing by a finite number of crossing changes. Moreover, up to isotopy and a finite number of crossing changes, every singular knot is determined by the cyclic order of its double points, so we have shown the following.

Proposition 3.1.2. Vassiliev invariants depend only upon the cyclic order of the singular points of a singular knot. ■

All the combinatorial information about double points can be encoded in a more convenient way as diagram over the circle, that we construct in the following way. Let $K : \mathbb{S}^1 \to \mathbb{R}^3$ be a singular knot with m double points. For any double point, consider the segment connecting its two preimages and label arbitrarily the m segments obtained by the elements of a finite set of m elements, for instance, $\{1, \ldots, m\}$. The diagram so obtained is called *chord diagram of degree* m and it is defined up to orientation preserving diffeomorphism of the circle. Fixing a basepoint $\star$ in $\mathbb{S}^1$ whose image is not a double point, one can associate a finite sequence of labels to each chord diagram by going across the circle from $\star$, following the orientation and recording the labels of the preimages of the double points in the order that we encounter them. For instance, the sequence corresponding to the diagram in figure 56 is $(1, 4, 2, 3, 4, 3, 1, 2)$. If we choose a different basepoint, the corresponding sequence is a circular permutation of the initial one, while a different labelling of the chords turns into a relabelling of the element of the sequence. Since the combinatorics of chords diagrams do not depend on these choices, we say that two diagrams are *equivalent* if, possibly after relabelling the chords, they determine the same circular sequences.

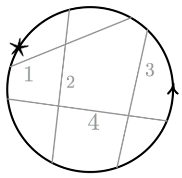


Figure 56

If $\mathcal{C}$ denote the vector space of $\Bbbk$-linear combinations of equivalence classes of chord diagrams and $\mathcal{C}_m$ is the subspace generated by equivalence classes of chord diagrams of degree m, we have constructed a map

$$\phi_m : \mathcal{S}_m \to \mathcal{C}_m.$$

Pairwise identification of the extremal points of the chords of a diagram produces a (not uniquely defined) singular knot whose associated chord diagram is the starting one, so ϕ_m is indeed surjective. Moreover, the discussion above shows that, if θ is a Vassiliev invariant of degree m and $\bar{K}, \bar{K}' \in \mathcal{S}_m$, then $\phi_m(\bar{K}) = \phi_m(\bar{K}')$ implies $\bar{\theta}(\bar{K}) = \bar{\theta}(\bar{K}')$, i.e., the value of a degree m Vassiliev invariant of a singular knot with exactly m singular points depends only on the chord diagram of the singular knot. Thus, we can define a unique map $\omega_\theta : \mathcal{C}_m \to \Bbbk$ such that

$$\begin{tikzcd} \mathcal{S}_m \arrow[r, "\phi_m"] \arrow[dr, "\bar{\theta}"'] & \mathcal{C}_m \arrow[d, "\omega_\theta"] \\ & \Bbbk \end{tikzcd}$$

is a commutative diagram. We call ω_θ the *symbol* of θ and it induces a linear map

$$\begin{array}{rccl} \alpha_m : & \mathcal{V}_m & \to & \mathcal{C}_m^* = \mathrm{Hom}_{\Bbbk}(\mathcal{C}_m, \Bbbk) \\ & \theta & \mapsto & \omega_\theta. \end{array}$$

One easily shows that $\ker(\alpha_m) = \mathcal{V}_{m-1}$, so α_m factors through $\mathcal{V}_m/\mathcal{V}_{m-1}$ yielding an injective morphism

$$\begin{array}{rccl} \bar{\alpha}_m : & \mathcal{V}_m/\mathcal{V}_{m-1} & \to & \mathcal{C}_m^* \\ & [\theta] & \mapsto & \omega_\theta. \end{array}$$

3.1.3 The Vassiliev-Kontsevich theorem. It turns out that every element ω in the image of $\bar{\alpha}_m$ satisfies the *one-term relations* (1T)

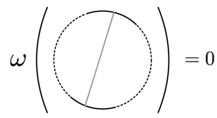


Figure 57

and the *four-term relations* (4T)

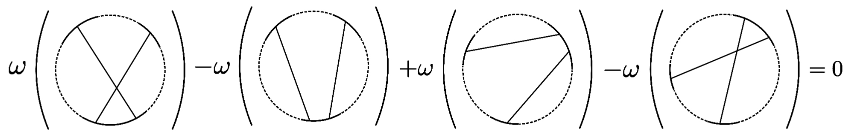


Figure 58

where the bold arcs represents regions where there are no ends of chords except for those that are represented. There may be chords with ends on the dotted sections and these chords may intersect the ones shown in the figures. A function on chord diagrams satisfying these two relations is called a *weight system* and the Kontsevich's theorem states that they are equivalent to Vassiliev invariants:

Theorem 3.1.3 (Kontsevich's theorem)**.** The map

$$\bar{\alpha}_m : \mathcal{V}_m/\mathcal{V}_{m-1} \to \mathcal{W}_m,$$

where $\mathcal{W}_m$ is the vector space of $\Bbbk$-linear combinations of weight systems of degree m, is a vector space isomorphism. ■

We will prove a more general result in the following subsections. The importance of this theorem stems for the fact that we have the following isomorphisms

$$\begin{tikzcd} \mathcal{V}_m \arrow[r, phantom, "\cong"] \arrow[d, dashed, "\beta_m"'] & \mathcal{V}_m/\mathcal{V}_{m-1} \arrow[r, phantom, "\oplus"] \arrow[d, "\bar{\alpha}_m"'] & \mathcal{V}_{m-1}/\mathcal{V}_{m-2} \arrow[r, phantom, "\oplus"] \arrow[d, "\bar{\alpha}_{m-1}"'] & \cdots \arrow[r, phantom, "\oplus"] & \mathcal{V}_1/\mathcal{V}_0 \arrow[r, phantom, "\oplus"] \arrow[d, "\bar{\alpha}_1"'] & \mathcal{V}_0 \arrow[d, "\bar{\alpha}_0"'] \\ \mathcal{W}_{\leqslant m} \arrow[r, phantom, "\cong"] & \mathcal{W}_m \arrow[r, phantom, "\oplus"] & \mathcal{W}_{m-1} \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\oplus"] & \mathcal{W}_1 \arrow[r, phantom, "\oplus"] & \mathcal{W}_0 \end{tikzcd}$$

which establish the equivalence of Vassiliev invariants and weight systems.

3.2 The categories $\mathcal{T}$ of singular tangles and $\mathcal{A}$ of chord diagrams

We introduce in this subsection two categories that generalize the vector space of linear combinations of isotopy classes of regular knots and the vector space of linear combinations of chord diagrams that we mentioned in the previous subsection. These two categories admit a filtration giving rise to a categorical version of the Kontsevich's theorem that we prove in the last subsection.

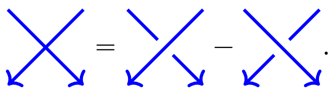
3.2.1 The category of singular tangles. Recall from section 2 that we have a category $\mathcal{FOT}$ whose objects are finite strings on $+$ and $-$ and whose morphisms are framed oriented tangles. For a given commutative ring $\Bbbk$, denote by $\mathcal{T}(\Bbbk)$ the $\Bbbk$-linearisation of $\mathcal{FOT}$. Following the ideas of the previous subsection, we will extend this category by allowing intersections between different bands of a given tangle.

Definition 3.2.1. A *singular framed oriented tangle* is defined in the same way as an ordinary framed oriented tangle (cf. definition 1.2.1 and paragraph 1.2.3) excepts that we replace the word "embedding" by "immersion", the possible singularities being a finite number of double points occurring in $\mathbb{R}^2 \times (0,1)$ and such that the framing of the two bands intersecting in any double point is the same. There is an obvious notion of isotopy of singular framed oriented tangles, carrying double points onto double points.

We are now ready to define a category $\mathcal{T}^{sing}(\Bbbk)$ which will play the role of the vector space $\mathcal{S}$ of singular knots discussed above.

Definition 3.2.2. The category $\mathcal{T}^{sing}(\Bbbk)$ of singular framed tangles is the category defined by:

- Objects: words on $+$ and $-$;
- Morphisms: $\Bbbk$-linear combinations of singular framed oriented tangles (with the usual compatibility conditions for the direction of bands) modulo the skein relation



(V)

Note that $\mathcal{T}^{sing}(\Bbbk)$ is just a different presentation of $\mathcal{T}(\Bbbk)$ (both categories are isomorphic) and the interest of introducing $\mathcal{T}^{sing}(\Bbbk)$ comes from the fact that it allows to construct a filtration of $\mathcal{T}(\Bbbk)$ by identifying certain linear combinations of tangles with singular tangles, in the same way that we did with $\mathcal{K}$ and $\mathcal{S}$. Specifically, let $\mathcal{T}_m(\Bbbk)$ be the category whose objects are the same as $\mathcal{T}(\Bbbk)$ (or $\mathcal{T}^{sing}(\Bbbk)$ or $\mathcal{T}$) and whose set of morphisms $\mathrm{Hom}_{\mathcal{T}_m}(s,s')$, for given objects s, s', is the quotient of the $\Bbbk$-module $\mathrm{Hom}_{\mathcal{T}^{sing}}(\Bbbk)(s,s')$ by the submodule generated by the singular tangles with $> m$ points. Then, there is a canonical projection $p_m : \mathcal{T}_m \to \mathcal{T}_{m-1}$ that is the identity on objects. It gives rise to a projective system of categories

$$\cdots \to \mathcal{T}_m(\Bbbk) \to \mathcal{T}_m(\Bbbk) \to \cdots \to \mathcal{T}_0(\Bbbk),$$

which is analogous to the filtration of $\mathcal{K}$ by the $\mathcal{K}_m$. We denote by $\hat{\mathcal{T}}(\Bbbk)$ its projective limit. The map that is the identity on objects and sends a singular tangle to its class in $\mathcal{T}_m(\Bbbk)$ is a functor $\mathcal{T}^{sing}(\Bbbk) \to \mathcal{T}_m(\Bbbk)$, which induces functors

$$\mathcal{T} \to \mathcal{T}^{sing}(\Bbbk) \to \hat{\mathcal{T}}(\Bbbk).$$

Finally, we note that the category of singular framed tangles has an structure of ribbon category, induced by the ribbon structure of $\mathcal{FOT}$.

3.2.2 The category of chord diagrams. We now introduce the second of the categories involved in the categorical generalisation of Kontsevich's theorem: the category of chord diagrams.

A (k, l)-*curve* Γ is a compact oriented 1-manifold (i.e. a disjoint union of a finite number of oriented intervals and circles) such that each connected component is equipped with an element of $\mathbb{Z}/2\mathbb{Z}$, called the *residue*, and such that the boundary $\partial\Gamma$ is decomposed as a disjoint union of two totally ordered sets U and V verifying that $\mathrm{card}(U) = k$ and $\mathrm{card}(V) = l$. The elements of U are called *inputs* of Γ and the elements of V are the *outputs*. By a homeomorphism of (k, l)-curves, we mean an orientation preserving homemorphism that respects the residue, the splitting of the boundary into inputs and outputs and the order of U and V.

Let Γ be a (k, l)-curve. For each $1 \leqslant i \leqslant k$, we set $\varepsilon_i = +1$ if the interval containing the ith is oriented towards this input and $\varepsilon_i = -1$ otherwise. On the other hand, for each $1 \leqslant j \leqslant l$, we set $\eta_j = -1$ if the interval containing the jth output is oriented towards it, and $\eta_j = +1$ otherwise. The sequence $(\varepsilon_1, \ldots, \varepsilon_k)$ is the *source* of Γ and $(\eta_1, \ldots, \eta_l)$ is the *output*.

Definition 3.2.3. A *chord diagram* on a (k, l)-curve Γ is a finite (possibly empty) set of pairs of points on $\Gamma\backslash\partial\Gamma$, all points being distinct. By a *homeomorphism of chord diagrams* we mean a homeomorphism of the underlying curves preserving the distinguished pairs of points.

In figures, we draw curves lying inside a horizontal strip, with the k inputs lying on the bottom boundary line with the order increasing from left to right and the l outputs are represented on the top line with the same order. We draw a dashed line called a *chord* between the two points of a distinguished pair.

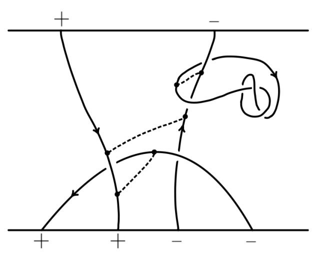


Figure 59: A chord diagram.

Consider now a chord diagram C and choose three points $a, b, c \in C$ such that none of them belongs to any distinguished pair. Let a', b', c' three points obtained by slightly pushing a, b, c following the orientation of the corresponding strand. Then, we can construct four different chord diagrams from C by adding two chords in the following way:

1. C_1 is the diagram obtained by adding the pairs of distinguished points $\{a, b\}$ and $\{a', c\}$;
2. C_2 is the diagram obtained by adding the pairs of distinguished points $\{a, c\}$ and $\{a', b\}$;
3. C_3 is the diagram obtained by adding the pairs of distinguished points $\{a, b\}$ and $\{b', c\}$;
4. C_4 is the diagram obtained by adding the pairs of distinguished points $\{a, b'\}$ and $\{b, c\}$.

For example, one possibility for the chord diagram of the figure 59 is represented in the following picture:

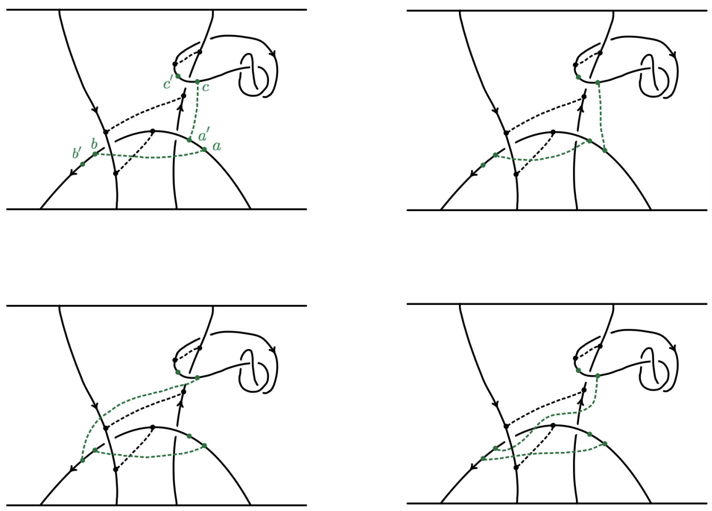


Figure 60

Definition 3.2.4. The category $\mathcal{A}(\Bbbk)$ of chords diagrams is defined by:

- Objects: words on $+$ and $-$;
- Morphisms: $\Bbbk$-linear combinations of homeomorphism classes of chord diagrams modulo the four term relation
$$C_1 - C_2 + C_3 - C_4 = 0. \tag{4T}$$

The identity morphism is represented by chordless diagrams consisting of intervals with residue zero such that, for each interval, one boundary point is an input and the other is an output with the same numbering. The composition $C' \circ C$ of two morphisms C, C' is obtained by gluing C with C' along the (unique) order-preserving homemorphism of the set of inputs of D' onto the set of outputs of D. The residue $r \in \mathbb{Z}/2\mathbb{Z}$ of each connected component D of the composition is given by

$$r = \sum_i r(\alpha_i) + \sum_j r(\beta_j) + \sum_{i<i'} r(\alpha_i, \alpha_{i'}) + \sum_{j<j'} r(\beta_j, \beta_{j'}),$$

where α_i (resp. β_j) are the components of C (resp. C') contained in D, and $r(\alpha_i, \alpha_{i'}) = 0$ if α_i and $\alpha_{i'}$ can be embedded in the horizontal strip, the order of their bottom and top endpoints being preserved, without intersecting each other, and $r(\alpha_i, \alpha_{i'}) = 1$ otherwise (see figure 61). Note that this formula only involves sums and does not mix components coming from different chords diagrams, so the composition is indeed associative. Also note that the composition of a digram with m chords and a diagram with n chords is a diagram with $m + n$ chords.

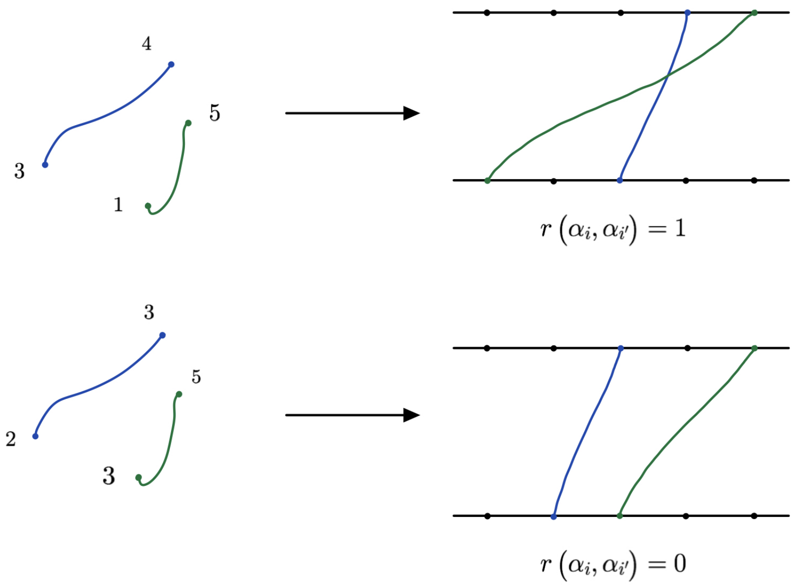


Figure 61

We can construct a projective system for $\mathcal{A}(\Bbbk)$ as we did for $\mathcal{T}(\Bbbk)$. To do so, let $\mathcal{A}_m(\Bbbk)$ be the category obtained by factoring out the morphisms of $\mathcal{A}(\Bbbk)$ by the $\Bbbk$-submodule generated by homemorphism classes of chord diagrams with $> m$ chords. There is a canonical projection $p_m : \mathcal{A}_m(\Bbbk) \to \mathcal{A}_{m-1}(\Bbbk)$ that is the identity on objects. Thus, we have

$$\cdots \to \mathcal{A}_m(\Bbbk) \to \mathcal{A}_{m-1}(\Bbbk) \to \cdots \to \mathcal{A}_0(\Bbbk).$$

The category $\mathcal{A}(\Bbbk)$ has an obvious structure of strict braided monoidal category with duality. The tensor product of two objects is just the concatenation. The tensor product of two diagrams D, D' is defined by the disjoint union in such a way that the inputs (resp. outputs) of D precede the inputs (resp. outputs) of D'. This extends by $\Bbbk$-linearity to a tensor product for arbitrary morphism in $\mathcal{A}(\Bbbk)$. The unit object is the empty word. As for the duality, the dual of a sequence $s = (\varepsilon_1, \dots, \varepsilon_k)$ is the sequence $s^* = (-\varepsilon_k, \dots, -\varepsilon_1)$. The duality maps b, d and the braiding σ are represented by the same chordless diagrams as in $\mathcal{FOT}$ (see figure 49) with residues equal to zero. Moreover, σ is a symmetry.

3.3 Infinitesimal symmetric categories.

We now study a property of some symmetric categories, namely the existence of an infinitesimal braiding, allowing to define a ribbon structure on (a deformation of) them. To do so, we sketch some ideas of the theory of associators developped by Drinfeld. We will see in the following subsection that the category $\mathcal{A}(\Bbbk)$ of chord diagrams described in the previous one is, indeed, an infinitesimal symmetric category.

3.3.1 Infinitesimal braidings. Here, we fix a field $\Bbbk$ of characteristic 0 and we suppose that all categories are $\Bbbk$-linear. Recall that a symmetric braided category is a braided monoidal category whose braiding c is a symmetry, i.e., $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$, for all objects V, W.

Remark 3.3.1. The definition of infinitesimal braiding easy generalizes to the case of general monoidal categories, not necessarily strict. Indeed, it is enough to introduce the associativity isomorphisms and their inverses in the axioms in order to make them coherent.

Definition 3.3.2. Let $\mathcal{S}$ be a strict symmetric $\Bbbk$-linear monoidal category. An *infinitesimal braiding* is a natural family of endomorphisms

$$t = \{t_{V,W} : V \otimes W \to V \otimes W\}_{V,W},$$

such that

$$c_{V,W} \circ t_{V,W} = t_{W,V} \circ c_{V,W},$$
$$t_{U,V\otimes W} = t_{U,V} \otimes \mathrm{id}_W + (c_{U,V} \otimes \mathrm{id}_W)^{-1} \circ (\mathrm{id}_V \otimes t_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W),$$

for all objects U, V, W. We call an *infinitesimal symmetric category* a symmetric category together with an infinitesimal braiding.

The interest of this definition lies in the fact that we can construct a strict ribbon category $\mathcal{S}[[\hbar]]^{str}$ out of an infinitesimal symmetric category $\mathcal{S}$. We will use this strict ribbon category will be a source of some interesting invariants via the theorem 1.2.8. Before giving this construction, we need the concept of a *Drinfeld associator*.

3.3.2 Drinfeld associators. The notion of Hopf algebra that we introduced in paragraph A.2 is a particular case of a more general algebraic structure, the *bialgebras*. A bialgebra over a field $\Bbbk$ is an algebra A equipped with two algebra morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \Bbbk$ such that

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta = \mathrm{id}_A.$$

As we noticed in remark A.2.5, the tensor product of vector spaces induces a monoidal structure in the category of representations of a bialgebra. Drinfeld noticed that, in fact, we do not need to require the comultiplication to be coassociative.

Definition 3.3.3. Let A be an algebra equipped with a comultiplication $\Delta : A \to A \otimes A$ and a counit $\varepsilon : A \to \Bbbk$. We say that A is a *quasi-bialgebra* if the tensor product of the category of vector spaces induces a monoidal structure in the category $\mathrm{Rep}(A)$ of A-modules.

Drinfeld proved that this definition is equivalent to the existence of an invertible element Φ in $A \otimes A \otimes A$ and invertible elements $l, r \in A$ such that, for all $a \in A$,

$$(\mathrm{id}_A \otimes \Delta)(\Delta(a)) = \Phi \left((\Delta \otimes \mathrm{id}_A)(\Delta(a))\right) \Phi^{-1},$$
$$(\varepsilon \otimes \mathrm{id}_A)(\Delta(a)) = l^{-1}al, \quad (\mathrm{id}_A \otimes \varepsilon)(\Delta(a)) = r^{-1}ar,$$
$$(\mathrm{id}_A \otimes \mathrm{id}_A \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\Phi) = \Phi_{234}(\mathrm{id}_A \otimes \Delta \otimes \mathrm{id}_A)(\Phi)\Phi_{123},$$
$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = r \otimes l^{-1},$$

where $\Phi_{123} = \Phi \otimes 1_A$ and $\Phi_{234} = 1_A \otimes \Phi$. When $\Phi = 1_A \otimes 1_A \otimes 1_A$ and $l = r = 1_A$ we recover the usual definition of a bialgebra.

The element Φ above is called an *associator*. The existence of such an element is not trivial. Using differential equations introduced by Knizhnik and Zamolodchikov in the context of conformal field theory, Drinfeld constructed explicitly an associator for the power series algebra of the enveloping algebra $U(\mathfrak{g})$ of a semi-simple Lie algebra $\mathfrak{g}$. This construction is complicated, but we recover here some of the notions introduced by Drinfeld that we will use in the sequel.

Definition 3.3.4. Let $n \geqslant 1$ be a integer. The *Drinfeld-Kohno algebra* is the algebra $U(\mathfrak{t}_n)$ generated by the symbols $t_{i,j}$, for all $i, j \in \{1, \dots, n\}$ distinct, and subject to the relations

(DK1) $t_{ij} = t_{ji}$;

(DK2) $[t_{ij}, t_{ik} + t_{jk}] = 0$;
(DK3) $[t_{ij}, t_{kl}] = 0$,

where i, j, k, l are distinct integers and $[f, g] = fg - gf$.

Definition 3.3.5. A formal series $\Phi(A, B)$ in two non-commuting variables A and B is called a *Drinfeld series* if it is a solution of the following system of four equations:

$$\begin{cases} \Phi(0,0) = 1, \\ \Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}) & (\text{in } U(\mathfrak{t}_4)), \\ \exp\left(\frac{1}{2}(t_{13} + t_{23})\right) = \Phi(t_{13}, t_{12}) \exp\left(\frac{1}{2}t_{13}\right) \Phi(t_{13}, t_{23})^{-1} \exp\left(\frac{1}{2}t_{23}\right) \Phi(t_{12}, t_{23}) & (\text{in } U(\mathfrak{t}_3)), \\ \exp\left(\frac{1}{2}(t_{13} + t_{12})\right) = \Phi(t_{23}, t_{13})^{-1} \exp\left(\frac{1}{2}t_{13}\right) \Phi(t_{12}, t_{13}) \exp\left(\frac{1}{2}t_{23}\right) \Phi(t_{12}, t_{23})^{-1} & (\text{in } U(\mathfrak{t}_3)). \end{cases}$$

This is the original definition given by Drinfeld but Furusho proved later on that the two last equation can be derived from the second one [3]. Using the holonomy of the Knizhnik-Zamolodchikov conection, Drinfeld first constructs a particular associator ψ_{KZ} for $\Bbbk = \mathbb{C}$ and next he deduces the existence of associators for any field of characteristic zero [2]. This associator has the form

$$\psi_{KZ}(A, B) = 1 + \frac{1}{24}[A, B] + \text{infinite sum of iterated commutators in A,B of length >2.}$$

Drinfield also showed non constructively the existence of an associator with rational coefficients.

3.3.3 Formal integration of infinitesimal symmetric categories.

Suppose that $\mathbb{Q} \subset \Bbbk$. Let Φ be a Drinfeld series. Given an infinitesimal symmetric category $\mathcal{S}$ with symmetry σ and infinitesimal braiding t, we define $\mathcal{S}[[\hbar]]$ to be the category whose objects are the same as $\mathcal{S}$ and whose set of morphisms $\mathrm{Hom}_{\mathcal{S}[[\hbar]]}(V, W)$ consists of formal series $\sum_{n \geqslant 0} f_n \hbar^n$ where $f_0, f_1, f_2, \dots$ are morphisms from V to W in $\mathcal{S}$. The composition in $\mathcal{S}[[\hbar]]$ extends the composition in $\mathcal{S}$ and the multiplication of formal series. The identity of V in $\mathcal{S}[[\hbar]]$ is the constant formal series id_V.

Theorem 3.3.6. There exists a unique structure of braided tensor category on $\mathcal{S}[[\hbar]]$ such that the tensor product on objects and the unit are the same as in $\mathcal{S}$, the tensor product of morphisms extends $\Bbbk[[\hbar]]$-linearly the tensor product in $\mathcal{S}$, the associativity isomorphism α is given by

$$\alpha_{U,V,W} = \Phi(t_{U,V}, t_{V,W}),$$

and the braiding c is given by

$$c_{V,W} = \sigma_{V,W} \circ \exp\left(\frac{1}{2}\hbar t_{V,W}\right) = \exp\left(\frac{1}{2}\hbar t_{W,V}\right) \circ \sigma_{V,W}.$$

Proof. The uniqueness is clear as the tensor product in $\mathcal{S}[[\hbar]]$ is determined by the one in $\mathcal{S}$. The pentagon axiom for α and the hexagon axioms for c then follows from the equations defining a Drinfeld series. ■

In the case when the infinitesimal symmetric category $\mathcal{S}$ has a duality $(*, b_V^0, d_V^0)$, we define the *Casimir operator* to be the natural family of morphisms defined by

$$C_V = -\left(\mathrm{id}_V \otimes (d_V^0 \circ t_{V*,V})\right) \circ (b_V^0 \otimes \mathrm{id}_V).$$

Then, it follows from the definition of infinitesimal braiding that it satisfies the equation

$$t_{V,W} = \frac{1}{2}\left(C_{V \otimes W} - C_V \otimes \mathrm{id}_W - \mathrm{id}_V \otimes C_W\right).$$

Now recall from paragraph A.1.3 that every monoidal category is monoidal equivalent to a strict monoidal category. Let $\mathcal{S}[[\hbar]]^{str}$ the strict braided tensor category associated to $\mathcal{S}[[\hbar]]$ following the procedure of the aforementioned paragraph.

Theorem 3.3.7. The strict braided tensor category $\mathcal{S}[[\hbar]]^{str}$ is a ribbon category with twist given by

$$\theta_V = \exp\left(\frac{1}{2}\hbar C_V\right)$$

and with left duality defined as follows: for any object V the dual object V^* is the same as in the category $\mathcal{S}$; the structure maps b_V and d_V are defined by

$$b_V = b_V^0 \quad \text{and} \quad d_V = d_V^0 \circ (\lambda_{V^*}^{-1} \otimes \mathrm{id}_V)$$

where λ_{V^*} is the automorphism of V^* defined by

$$\lambda_{V^*} = \left(d_V^0 \otimes \mathrm{id}_{V^*}\right) \circ \Phi\left(t_{V^*,V}, t_{V,V^*}\right) \circ \left(\mathrm{id}_{V^*} \otimes b_V^0\right).$$

Proof. The axioms for the duality follow from a computation from the definition of Drinfeld operator. The axioms for the twists are a consequence of the relation between the infinitesimal braiding and the Casimir operator given above. ■

3.4 The ribbon structure of $\mathcal{A}(\Bbbk)$ and the Kontsevhich theorem

3.4.1 Infinitesimal braiding. We provide now $\mathcal{A}(\Bbbk)$ with an infinitesimal braiding. For a nonempty sequence $s = (\varepsilon_1, \ldots, \varepsilon_n)$ and any pair (i, j) of distinct integers between 1 and n, denote by $T_s^{i,j}$ the chord diagram obtained from the identity by adding a single chord between the ith and jth intervals and let $t_s^{i,j} : s \to s$ be the morphism in $\mathcal{A}(\Bbbk)$ represented by $\varepsilon_i \varepsilon_j T_s^{i,j}$.

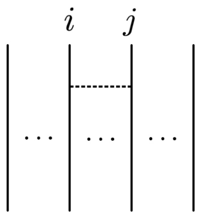
$T_s^{ij} =$ 

Figure 62

Lemma 3.4.1. For distinct integers $i, j, k, l = 1, \ldots, r$ the following relations hold in $\mathcal{A}(\Bbbk)$:

$$t_s^{i,j} = t_s^{j,i}, \qquad \left[t_s^{i,j}, t_s^{k,l}\right] = 0, \qquad \left[t_s^{i,j}, t_s^{i,k} + t_s^{j,k}\right] = 0,$$

where $[f, g] = f \circ g - g \circ f$.

Proof. The first relation holds by definition. The second one follows from the fact that the chord diagrams representing $t_s^{i,j} \circ t_s^{k,l}$ and $t_s^{k,l} \circ t_s^{i,j}$ are homeomorphic when i, j, k, l are distinct integers. Finally, the third one follows directly from the four-term relation. ■

We can use these morphisms to construct an infinitesimal braiding in the following way. For nonempty sequences $s, s' \in \langle +, - \rangle$, we define an endomorphism $t_{s,s'}$ of $s \otimes s$ by

$$t_{s,s'} = \sum_{i=1}^{r} \sum_{j=1}^{r'} t_{ss'}^{i,r+j},$$

where r and r' are the lengths of s and s'. Set

$$t_{s,\varnothing} = t_{\varnothing,s} = t_{\varnothing,\varnothing} = 0.$$

This family of morphisms defines an infinitesimal braiding in $\mathcal{A}(\Bbbk)$ (see [5] for the details), so we have:

Proposition 3.4.2. The category $\mathcal{A}(\Bbbk)$ with the infinitesimal braiding $(t_{s,s'})_{s,s'}$ is a strict infinitesimal category with duality. ■

3.4.2 The prounipotent completion $\mathcal{T}(\Bbbk)$.

Let $\mathcal{C}$ be an $\Bbbk$-linear braided category and let I be the ideal of $\mathcal{C}$ generated by the morphisms of the form

$$c_{W,V} \circ c_{V,W} - \mathrm{id}_{V \otimes W},$$

for all objects V, W. We call I the *augmentation ideal* of $\mathcal{C}$. Given an integer $m \geqslant 0$, we define I^{m+1} as the ideal generated by morphisms of the form $f_1 \star \cdots \star f_n$, where $\star$ is either the composition or the tensor product of morphisms and at least $m+1$ morphisms among the f_i's belong to I. The categories $\mathcal{C}/I^{m+1}$, $m = 0, 1, 2, \ldots$, whose objects are the same as the objects of $\mathcal{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}/I^{m+1}}(V, W) = \mathrm{Hom}_{\mathcal{C}}(V, W)/I^{m+1}(V, W),$$

where $I(V, W)$ is the set of morphisms from V to W in I^{m+1}, form a projective system. Its limit is called the *prounipotent completion of* $\mathcal{C}$.

Theorem 3.4.3. The category $\hat{\mathcal{T}}(\Bbbk)$ is isomorphic to the prounipotent completion of $\mathcal{T}(\Bbbk)$.

Proof. It is enough to show that, for all $m \geqslant 0$, we have $\mathcal{T}_m(\Bbbk) \cong \mathcal{T}(\Bbbk)/I^{m+1}$, where I is the augmentation ideal of $\mathcal{T}(\Bbbk)$.

We note that $\mathcal{T}_m(\Bbbk) = \mathcal{T}^{sing}(\Bbbk)/D_m$, where D_m is the ideal of $\mathcal{T}^{sing}(\Bbbk)$ generated by the singular tangles with $> m$ double points. But, since every morphism in $\mathcal{T}^{sing}(\Bbbk)$ is a linear combination of morphisms in $\mathcal{T}$, there exists an ideal D_m^0 in $\mathcal{T}(\Bbbk)$ such that $\mathcal{T}_m(\Bbbk) = \mathcal{T}(\Bbbk)/D_m^0$. Hence, it suffices to show that $I^{m+1} = D_m^0$.

By construction of $\mathcal{T}^{sing}$, D_m^0 is generated by morphisms whose algebraic expressions contain $> m$ occurrences of $c_{(+),(+)} - c_{(+),(+)}^{-1}$. But

$$c_{(+),(+)} - c_{(+),(+)}^{-1} = c_{(+),(+)}^{-1}(c_{(+),(+)}^2 - \mathrm{id}_{(+),(+)}),$$

which implies that $D_m^0 \subset I^{m+1}$.

On the other hand, the axioms in the definition of braiding imply that I^{m+1} is additively generated by morphisms in whose algebraic expressions the terms $c_{s',s} \circ c_{s,s'} - \mathrm{id}_{ss'}$ appear more than m times, where s and s' are length-1 sequences of $+, -$. Each such term belong to D_0^0, so $D_m^0 \subset I^{m+1}$. ■

3.4.3 The Kontsevich's theorem.

Suppose $\mathbb{Q} \subset \Bbbk$. Using the fact that $\mathcal{A}(\Bbbk)$ is an infinitesimal symmetric category, we can apply the results from the subsection 3.3 with a Drinfeld associator Φ to get a $\Bbbk$-linear strict ribbon category $\mathcal{A}(\Bbbk)[[\hbar]]^{str}$ whose objects are finite sequences of finite sequences of $+$ and $-$. We modify slightly this ribbon structure in the following way. For any object s in $\mathcal{A}(\Bbbk)[[\hbar]]^{str}$, consider the morphism $\mathrm{id}'_{\prod s}$ obtained from the identity by setting all residues equal to 1 and let

$$\gamma_s = \mathrm{id}'_{\prod s} + \sum_{i>0} 0\hbar^i.$$

The morphism γ_s commutes with all morphisms in $\mathcal{A}(\Bbbk)[[\hbar]]^{str}$ and $\gamma_s^2 = \mathrm{id}_s$. Moreover,

$$\gamma_{s \otimes s'} = \gamma_s \otimes \gamma_{s'}, \quad \gamma_{s^*} = \gamma_s^*,$$

so by composing by the twist, we get a new twist in $\mathcal{A}(\Bbbk)[[\hbar]]^{str}$. We denote the resulting ribbon category by $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$. By the universal property of the category of ribbon graphs, there exists a functor $Z : \mathcal{T} \to \mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$ preserving the tensor product ans satisfying the following conditions:

$$Z((+)) = ((+)), \quad Z((-)) = ((-)),$$
$$Z\left(c_{(+),(+)}\right) = \sigma_{(+),(+)} \circ \exp\left(\frac{1}{2}\hbar t_{(+),(+)}\right) = \exp\left(\frac{1}{2}\hbar t_{(+),(+)}\right) \circ \sigma_{(+),(+)},$$
$$Z\left(b_{(+)}\right) = b_{(+)},$$
$$Z\left(\theta_{(+)}\right) = \exp\left(\frac{1}{2}\hbar C_{(+)}\right) \circ \gamma_{(+)} = \gamma_{(+)} \circ \exp\left(\frac{1}{2}\hbar C_{(+)}\right),$$

where the morphisms appearing in the left-hand side of the equalities are the braiding, the duality morphism and the twist in the category $\mathcal{T}$; σ is the braiding in $\mathcal{A}(\Bbbk)$ and C is the Casimir operator in $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$. We extend Z by $\Bbbk$ linearity and we consider the ideal of $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$ consisting of all morphisms $\sum_{m\geqslant 0} f_m \hbar^m$ such that $f_0 = 0$ and f_1 is a linear combination of chord diagrams with at least one cord.

Let I be the augmentation ideal of $\mathcal{T}(\Bbbk)$. We already noted in the proof of Theorem 3.4.3 that I is generated by $c_{(+),(+)} - c^{-1}_{(+),(+)}$. The image of this element by Z is

$$\begin{aligned} Z\left(c_{(+),(+)} - c_{(+),(+)} - c^{-1}_{(+),(+)}\right) &= \sigma_{(+),(+)} \circ \left[\exp\left(\frac{1}{2}\hbar t_{(+),(+)}\right) - \exp\left(-\frac{1}{2}\hbar t_{(+),(+)}\right)\right] \\ &\equiv \hbar\sigma_{(+),(+)} \circ t_{(+),(+)} \mod \hbar^2. \end{aligned}$$

Since $\sigma_{(+),(+)} \circ t_{(+),(+)}$ has one chord, $Z(I) \subset J$ and Z induces a diagram

$$\begin{array}{ccc} \mathcal{T}(\Bbbk)/I^{m+1} & \xrightarrow{Z^m} & \mathcal{A}[[\hbar]]^{str}_\gamma/J^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{T}(\Bbbk)/I^p & \xrightarrow{Z^{p-1}} & \mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma/J^p \end{array} \qquad (\star\star)$$

which is commutative for all $p \leqslant m+1$. On the other hand, we have an inclusion

$$\kappa : \mathcal{A}(\Bbbk) \to \mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$$

which send each sequence $(\varepsilon_1, \ldots, \varepsilon_r)$ in $\mathcal{A}(\Bbbk)$ to the sequence of 1-length sequences $((\varepsilon_1), \ldots, (\varepsilon_r))$ in $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma$ and such that

$$\kappa\left(\sum_i \lambda_i D_i\right) = \sum_i \lambda_i D_i \hbar^{n(D_i)},$$

where the D_i s are chord diagrams and $n(D_i)$ is the number of chords of D_i. In particular, if D is a diagram with more than $k > m$ chords, then $\kappa(D) = D\hbar^k \in J^{m+1}$, so the functor κ induces a functor

$$\kappa_m : \mathcal{A}_m(\Bbbk) \to \mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma/J^{m+1},$$

which is injective in objects and morphisms, so we may identify $\mathcal{A}_m(\Bbbk)$ as a subcategory of the quotient category $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma/J^{m+1}$.

Lemma 3.4.4. For any integer $m \geqslant 0$, the image of the functor Z^m lies in the subcategory $\mathcal{A}_m(\Bbbk)$ of $\mathcal{A}(\Bbbk)[[\hbar]]^{str}_\gamma/J^{m+1}$.

Proof. This is clear for objects. For morphisms, note that Z^m is defined on the reductions modulo $\hbar^{m+1}$ of the morphisms $b_{(+)}$, $d_{(+)}$, $\Phi(\hbar t_{s,s'} \otimes \mathrm{id}_{s''}, \hbar \mathrm{id}_s \otimes t_{s',s''})$, $\sigma_{(+),(+)} \circ \exp\left(\frac{1}{2}\hbar t_{(+),(+)}\right)$ and $\gamma_{(+)} \circ \exp\left(\frac{1}{2}\hbar C_{(+)}\right)$, which are all in the image of κ_m, since $t_{s,s'}$ and $C_{(+)}$ are sums of chord diagrams with exactly one chord. ■

Proposition 3.4.5. For any integer $m \geqslant 0$, the functor Z^m is an isomorphism from the category $\mathcal{T}(\mathbb{k})/I^{m+1}$ to the subcategory $\mathcal{A}_m(\mathbb{k})$ of $\mathcal{A}(\mathbb{k})[[\hbar]]^{str}_{\gamma}/J^{m+1}$.

We will show that, in fact, this functor is an isomorphism. Before doing this, let us introduce some terminology. If T is a framed singular tangle, by recursively applying the Vassiliev relation to every double point, we get a nonsingular tangle T' which we represent by a planar diagram. Let $C' \subset T'$ be any component of T', we define the *residue* of C' by counting the number of self-intersections of C' and by taking the residue modulo 2, which does not depend on the choice of the diagram. By definition, the residue of any component of C of T is the residue of the corresponding connected component in any desingularization. On the other hand, generalizing the ideas that we explained in the first subsection, we can associate a singular tangle T to each chord diagram D by embedding it in $\mathbb{R}^2 \times [0,1]$ and by replacing each cord by a double point and a crossing, as in Figure 63, and such that each component of T has the same residue as the corresponding component of D. We call T a *realization* of D. If D has p chords, then any realization of D has p singular points.

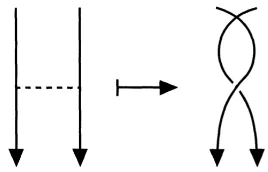


Figure 63

Proposition 3.4.6. For any integer $m \geqslant 0$, the functor Z^m is an isomorphism from the category $\mathcal{T}(\mathbb{k})/I^{m+1}$ to the subcategory $\mathcal{A}_m(\mathbb{k})$ of $\mathcal{A}(\mathbb{k})[[\hbar]]^{str}_{\gamma}/J^{m+1}$.

Proof. We have to show that Z^m induces a bijection $\mathrm{Hom}_{\mathcal{T}(\mathbb{k})/I^{m+1}}(s,s') \to \mathrm{Hom}_{\mathcal{A}_m(\mathbb{k})}(s,s')$, for any pair (s,s') of objects of $\mathcal{T}$. By $(\star\star)$, we have the following diagram of $\mathbb{k}$-modules

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{T}(\mathbb{k})/I^{m+1}}(s,s') & \xrightarrow{Z^m} & \mathrm{Hom}_{\mathcal{A}_m(\mathbb{k})}(s,s') \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{T}(\mathbb{k})/I^{p}}(s,s') & \xrightarrow{Z^{p-1}} & \mathrm{Hom}_{\mathcal{A}_{p-1}(\mathbb{k})}(s,s')
\end{array}$$

where vertical arrows are clearly surjective, for all $p \leqslant m$. Let T^p be the kernel of the left vertical morphism and A^p be the kernel of the right vertical one. We have filtrations

$$\{0\} = T^{m+1} \subset T^m \subset T^{m-1} \subset \cdots \subset T^0 = \mathrm{Hom}_{\mathcal{T}(\mathbb{k})/I^{m+1}}(s,s')$$

and

$$\{0\} = A^{m+1} \subset A^m \subset A^{m-1} \subset \cdots \subset A^0 = \mathrm{Hom}_{\mathcal{A}_m(\mathbb{k})}(s,s')$$

compatible with Z^m. It is then enough to prove that Z^m induces an isomorphism from T^p/T^{p+1} to A^p/A^{p+1} for all $p \leqslant m$.

Observe that A^p/A^{p+1} is the submodule of $A^p \subset \mathrm{Hom}_{\mathcal{A}_m(\mathbb{k})}(s,s')$ generated by all chord diagrams with exactly p chords. By a previous observation, a realization of such a diagram has p points, so it is an element of T^p, whose class module T^{p+1} is independent of the realization. It induces a well-defined surjective $\Bbbk$-linear map $Y : A^p/A^{p+1} \to T^p/T^{p+1}$, which is a right inverse for $Z^m : T^p/T^{p+1} \to A^p/A^{p+1}$ (see [5] for the details). Hence, Z^m is an isomorphism. ■

The Kontsevich's theorem now follows directly:

Theorem 3.4.7. [Kontsevich] If $\mathbb{Q} \subset \Bbbk$, then for $m = 0, 1, 2, \ldots$, there is an isomorphism of categories

$$\mathcal{T}_m(\Bbbk) \cong \mathcal{A}_m(\Bbbk)$$

such that

$$\begin{array}{ccc} \mathcal{T}_m(R) & \cong & \mathcal{A}_m(\Bbbk) \\ \downarrow & & \downarrow \\ \mathcal{T}_{m-1}(\Bbbk) & \cong & \mathcal{A}_{m-1}(\Bbbk) \end{array}$$

commutes for all $m = 0, 1, 2, \ldots$. ■

A Appendix

A.1 Monoidal categories

A.1.1 Definition and some examples. A *monoidal category* is a category $\mathcal{V}$ with the following additional data:

(i) a covariant bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the *tensor product*;

(ii) a natural family of isomorphisms

$$\alpha = \{\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) \; : \; U, V, W \in \mathcal{V}\},$$

called the *associativity isomorphisms*;

(iii) a *unit object* $\mathbf{1} \in \mathcal{V}$ and natural isomorphisms

$$\lambda = \{\mathbf{1} \otimes V \xrightarrow{\sim} V \; : \; V \in \mathcal{V}\}$$

and

$$\rho = \{V \otimes \mathbf{1} \xrightarrow{\sim} V \; : \; V \in \mathcal{V}\},$$

called *unit isomorphisms*, satisfying the following properties:

(a) Pentagon axiom. For any $V_1, \ldots, V_4 \in \mathcal{V}$, the following diagram is commutative:

$$\begin{tikzcd}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\alpha_{1,2,3} \otimes \mathrm{id}_4"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\alpha_{1,23,4}"] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\alpha_{1,2,34}"'] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\mathrm{id}_1 \otimes \alpha_{2,3,4}"] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

(b) Triangle axiom. For any $V, W \in \mathcal{V}$, the following diagram is commutative:

$$\begin{tikzcd}
(V_1 \otimes \mathbf{1}) \otimes V_2 \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & V_1 \otimes (\mathbf{1} \otimes V_2) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
 & V_1 \otimes V_2 &
\end{tikzcd}$$

Monoidal categories are ubiquitous, as the following examples suggest:

Example A.1.1. The category **Set** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set. This examples can be generalized by taking the category of sets with some structure (groups, topological spaces...).

Example A.1.2. Any additive category is a monoidal category: $\otimes$ is given by the direct sum $\oplus$ and, hence, the zero object is a unit object.

Example A.1.3. If R is a commutative unital ring, the categories R-**Mod** and R-**mod** of R-modules and R-modules of finite type are monoidal categories. The tensor product is the tensor product of modules over R and the ring R is the unit object. In particular, the category of (finite dimensional) vector spaces is a monoidal category.

Example A.1.4. Let G be a group $\Bbbk$ be a field. The category $\mathbf{Rep}_{\Bbbk}(G)$ of representations of G over $\Bbbk$ is a monoidal category. If (V, ρ_V) and (W, ρ_W) are two representations of G, their tensor product is the representation $(V \otimes W, \rho_V \otimes \rho_W)$. The unit object is given by the trivial representation $(\Bbbk, \mathrm{id}_{\Bbbk})$. Similarly, if $\mathfrak{g}$ is a Lie algebra over $\Bbbk$, the category $\mathbf{Rep}_{\Bbbk}(\mathfrak{g})$ of its representations over $\Bbbk$

can be endowed with a structure of monoidal category. The tensor product of two representations (V, ρ_V) and (W, ρ_W) is the representation $(V \otimes W, \rho_{V\otimes W})$, where

$$\rho_{V\otimes W} = \rho_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \rho_W.$$

The unit object is a one dimensional representation with the zero action of $\mathfrak{g}$.

A.1.2 Monoidal functors and equivalences of monoidal categories. Let $\mathcal{V}, \mathcal{W}$ be two monoidal categories. A *monoidal functor* is a triple (F, ϕ, φ) where $F : \mathcal{V} \to \mathcal{W}$ is a functor, $\phi : \mathbf{1}_\mathcal{W} \xrightarrow{\sim} F(\mathbf{1}_\mathcal{V})$ is an isomorphism and

$$\varphi = \{\varphi_{U,V} : F(U) \otimes F(V) \to F(U \otimes V) \ : \ U, V \in \mathcal{V}\}$$

is a family of natural isomorphisms such that the diagrams

$$\begin{tikzcd}
(F(U) \otimes F(V)) \otimes F(W) \arrow[r, "\alpha_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_{U,V}\otimes\mathrm{id}_{F(W)}"] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d, "\mathrm{id}_{F(U)}\otimes\varphi_{V,W}"] \\
F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_{U\otimes V,W}"] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_{U,V\otimes W}"] \\
F((U \otimes V) \otimes W) \arrow[r, "F(\alpha_{U,V,W})"] & F(U \otimes (V \otimes W)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1} \otimes F(U) \arrow[r, "\lambda_{F(U)}"] \arrow[d, "\phi\otimes\mathrm{id}_{F(U)}"] & F(U) \\
F(\mathbf{1}) \otimes F(U) \arrow[r, "\varphi_{\mathbf{1},U}"] & F(\mathbf{1} \otimes U) \arrow[u, "F(\lambda_U)"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
F(U) \otimes \mathbf{1} \arrow[r, "\rho_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)}\otimes\phi"] & F(U) \\
F(U) \otimes F(\mathbf{1}) \arrow[r, "\varphi_{U,\mathbf{1}}"] & F(U \otimes \mathbf{1}). \arrow[u, "F(\rho_U)"]
\end{tikzcd}$$

commute for all objects $U, V, W \in \mathcal{V}$.

A *natural monoidal transformation* $\eta : (F, \phi, \varphi) \Rightarrow (G, \phi', \varphi')$ between monoidal functors is a natural transformation $\eta : F \Rightarrow G$ such that the following diagrams are commutative, for each pair of objects $U, V \in \mathcal{V}$:

$$\begin{tikzcd}
& F(\mathbf{1}) \arrow[dd, "\eta_\mathbf{1}"] \\
\mathbf{1} \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & \\
& G(\mathbf{1})
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
F(U) \otimes F(V) \arrow[r, "\varphi_{U,V}"] \arrow[d, "\eta_U\otimes\eta_V"] & F(U \otimes V) \arrow[d, "\eta_{U\otimes V}"] \\
G(U) \otimes G(V) \arrow[r, "\varphi'_{U,V}"] & G(U \otimes V).
\end{tikzcd}$$

Finally, a *monoidal equivalence* between monoidal categories is a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ such that there exist a monoidal functor $G : \mathcal{W} \to \mathcal{V}$ and natural monoidal isomorphisms $\eta : \mathrm{id}_\mathcal{W} \overset{\sim}{\Rightarrow} FG$ and $\varepsilon : GF \overset{\sim}{\Rightarrow} \mathrm{id}_\mathcal{V}$.

Example A.1.5. A class of example of monoidal functors is given by *forgetful functors*, i.e., functors from a certain category of sets with additional structure (such as groups, vector spaces, topological spaces...) mapping each object to the underlying set and every morphism to the corresponding set-theoretical one. Such functors have an obvious monoidal structure.

A.1.3 Strictification of monoidal categories. Monoidal categories are much easier to manipulate in the particular case where the associativity and unit constraints are all identities. Recall that a monoidal category $\mathcal{V}$ is *strict* if the associativity and unit isomorphisms are all identities of the category, i.e., we have

$$(U \otimes V) \otimes W = U \otimes (V \otimes W)), \quad \mathbf{1} \otimes V = V \quad \text{and} \quad V \otimes \mathbf{1} = V,$$

for all $U, V, W \in \mathcal{V}$. In that case, the pentagon and the triangle axiom become trivial. We will see that in practice one may always assume that monoidal categories are strict. Let us begin with one example:

Example A.1.6. Let $\overline{\mathbf{Vec}_{\Bbbk}}$ the category whose objects are nonnegative integers and $\mathrm{Hom}(m, n) = \mathrm{Mat}_{m \times n}(\Bbbk)$, with composition given by the product of matrices. We define the tensor product of two objects $m, n \in N$ as $m \otimes n := mn$ and that of two morphisms $f : m \to n$, $g : p \to q$ as the Kronecker product $f \otimes g$. This endows $\overline{\mathbf{Vec}_{\Bbbk}}$ with a structure of strict monoidal category. Moreover, we have a natural inclusion $\overline{\mathbf{Vec}_{\Bbbk}} \hookrightarrow \mathbf{Vec}_{\Bbbk}$ which assigns to each $n \in \mathbb{N}$ the $\Bbbk$-vector space $\Bbbk^n$ and to each matrix $f : m \to n$ the morphism $\Bbbk^m \to \Bbbk^n$ whose matrix in the canonical basis is precisely f. This functor is clearly monoidal, fully faithful and essentially surjective, so it is an equivalence of monoidal categories.

The situation of the previous example is in fact general: starting from a arbitrary monoidal category $\mathcal{V}$, we can always construct a strict monoidal category which is monoidal equivalent to $\mathcal{V}$. This is done as follows. Let $\mathcal{S}_{\mathcal{V}}$ be the class of all finite sequences $S = (V_1, \ldots, V_k)$ of objects of $\mathcal{V}$. We call the integer k the *length* of the sequence. By convention, the length of the empty sequence is 0. If $S = (V_1, \ldots, V_k)$ and $S' = (V_{k+1}, \ldots, V_{k+n})$ are nonempty sequences of $\mathcal{S}_{\mathcal{V}}$, we set

$$S * S' := (V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+n}).$$

We also agree that $S * \varnothing = S = \varnothing * S$, for every $S \in \mathcal{S}_{\mathcal{V}}$. To any sequence $S \in \mathcal{S}_{\mathcal{V}}$, we assign an object $F(S)$ of $\mathcal{V}$ defined inductively by

$$F(\varnothing) = \mathbf{1}, \quad F((V)) = V, \quad F(S * (V)) = F(S) \otimes V.$$

We then define the *category of finite sequences of* $\mathcal{V}$ as the category $\mathcal{V}^{str}$ whose objects are the elements of $\mathcal{S}_{\mathcal{V}}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{V}^{str}}(S, S') := \mathrm{Hom}_{\mathcal{V}}(F(S), F(S')),$$

where the identities and compositions are taken from $\mathcal{V}$.

Proposition A.1.7. The categories $\mathcal{V}^{str}$ and $\mathcal{V}$ are equivalent.

Proof. The map F previously defined extends to a functor $F : \mathcal{V}^{str} \to \mathcal{V}$ which is the identity on morphisms, hence fully faithful. Moreover, $V = F((V))$ for every object $V \in \mathcal{V}$, which implies the essential surjectivity of F. Thus, F is an equivalence of categories. ∎

A quasi-inverse of the functor F above is explicitly given by the functor $G : \mathcal{V} \to \mathcal{V}^{str}$ defined by $G(V) = (V)$ on objects and by the identity on morphisms. Indeed, we have $FG = \mathrm{id}_{\mathcal{V}}$ and we have a natural isomorphism $\theta : GF \Rightarrow \mathrm{id}_{\mathcal{V}^{str}}$, where

$$\theta_S := \mathrm{id}_{F(S)} : GF(S) \to S.$$

This is well-defined because

$$\mathrm{Hom}_{\mathcal{V}^{str}}(GF(S), S) = \mathrm{Hom}_{\mathcal{V}}(FGF(S), S) = \mathrm{Hom}_{\mathcal{V}}(F(S), F(S)).$$

We now equip $\mathcal{V}^{str}$ with the structure of a strict monoidal category. The tensor product of two objects S and S' is defined to be the concatenation $S * S'$. In order to define the tensor product of two morphisms, we first construct a natural isomorphism

$$\varphi_{S,S'} : F(S) \otimes F(S') \to F(S * S')$$

for any pair objects $S, S' \in \mathcal{V}^{str}$. We do so by induction on the length of the sequence S'. First, we set

$$\varphi_{\varnothing,S} \coloneqq \lambda_{F(S)} \quad \text{and} \quad \varphi_{S,\varnothing} \coloneqq \rho_{F(S)}.$$

Next,

$$\varphi_{S,(V)} \coloneqq \mathrm{id}_{F(S)\otimes V} : F(S) \otimes V \to F(S) \otimes V = F(S * (V)).$$

Finally, if $\varphi_{S,S'}$ has been already constructed for some $S, S' \in \mathcal{V}^{str}$, $\varphi_{S,S'*(V)}$ is defined by the following commutative diagram:

$$\begin{tikzcd}
F(S) \otimes F(S' * (V)) \arrow[r, "\varphi_{S,S'*(V)}"] \arrow[d, equal] & F(S * (S' * (V))) \arrow[d, equal] \\
F(S) \otimes (F(S') \otimes V) \arrow[d, "\alpha^{-1}_{F(S),F(S'),V}"] & F((S * S') * (V)) \arrow[d, equal] \\
(F(S) \otimes F(S')) \otimes V \arrow[r, "\varphi_{S,S'}\otimes \mathrm{id}_V"] & F(S * S') \otimes V.
\end{tikzcd}$$

That is,

$$\varphi_{S,S'*(V)} = (\varphi_{S,S'} \otimes \mathrm{id}_V) \circ \alpha^{-1}_{F(S),F(S'),V}.$$

Using this natural isomorphism, we define the tensor product of two morphisms as follows. Let $f : S \to T$ and $g : S' \to T'$ be two morphisms in $\mathcal{V}^{str}$. By definition of the category of finite sequences, $f \in \mathrm{Hom}_{\mathcal{V}}(F(S), F(T))$ and $g \in \mathrm{Hom}_{\mathcal{V}}(F(S'), F(T'))$. We then define $f * g \in \mathrm{Hom}_{\mathcal{V}^{str}}(S * S', T * T') = \mathrm{Hom}_{\mathcal{V}}(F(S * S'), F(T * T'))$ by the following commutative square:

$$\begin{tikzcd}
F(S * S') \arrow[r, "\varphi^{-1}_{S,S'}"] \arrow[d, "f*g"] & F(S) \otimes F(S') \arrow[d, "f\otimes g"] \\
F(T * T') & F(T) \otimes F(T') \arrow[l, "\varphi_{T,T'}"]
\end{tikzcd}$$

It is now straightforward to prove that $\mathcal{V}^{str}$ is a strict monoidal category:

Proposition A.1.8. Equipped with $*$, the category of finite sequences $\mathcal{V}^{str}$ is a strict monoidal category. The unit element is the empty sequence.

Proof. By construction of $*$,

$$\mathrm{id}_S * \mathrm{id}_{S'} = \varphi_{S,S'} \circ (\mathrm{id}_S \otimes \mathrm{id}_{S'}) \circ \varphi^{-1}_{S,S'} = \mathrm{id}_{S*S'},$$

for every $S, S' \in \mathcal{V}^{str}$. Similarly, for $f : S \to T$, $g : T \to R$, $f' : S' \to T'$ and $g' : T' \to R'$, we have

$$\begin{aligned}
(f \circ g) * (f' \circ g') &= \varphi_{R,R'} \circ [(f \circ g) \otimes (f' \circ g')] \circ \varphi^{-1}_{S,S'} \\
&= \varphi_{R,R'} \circ (f \otimes f') \circ (g' \otimes g') \circ \varphi^{-1}_{S,S'} \\
&= \left[\varphi_{R,R'} \circ (f \otimes f') \circ \varphi^{-1}_{T,T'}\right] \circ \left[\varphi_{T,T'} \circ (g' \otimes g') \circ \varphi^{-1}_{S,S'}\right] \\
&= (f * f') \circ (g * g'),
\end{aligned}$$

which shows that $*$ is a functor. This functor is strictly associative by construction. ∎

We finish our discussion about strict monoidal categories showing that $\mathcal{V}^{str}$ is, indeed, monoidal equivalent to $\mathcal{V}$. We firs prove two technical lemmas.

Lemma A.1.9. Let $\mathcal{V}$ be an arbitrary monoidal category. The triangles

$$\begin{tikzcd}
(\mathbf{1}\otimes V)\otimes W \arrow[r, "\alpha_{\mathbf{1},V,W}"] \arrow[d, "\lambda_V\otimes \mathrm{id}_w"'] & \mathbf{1}\otimes(V\otimes W) \arrow[dl, "\lambda_{V\otimes W}"] \\
V\otimes W &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(V\otimes W)\otimes \mathbf{1} \arrow[r, "\alpha_{V,W,\mathbf{1}}"] \arrow[dr, "\rho_{V\otimes W}"'] & V\otimes(W\otimes \mathbf{1}) \arrow[d, "\mathrm{id}_V\otimes\rho_W"] \\
& V\otimes W
\end{tikzcd}$$

commute for any pair of objects $V, W \in \mathcal{V}$.

Proof. Consider the diagram

$$\begin{tikzcd}
& (U\otimes(\mathbf{1}\otimes V))\otimes W \arrow[dr, equal] & \\
((U\otimes \mathbf{1})\otimes V)\otimes W \arrow[ur, "\alpha\otimes\mathrm{id}_W"] \arrow[dr, "(\rho_U\otimes\mathrm{id}_V)\otimes\mathrm{id}_W"] \arrow[ddd, "\alpha"] & & (U\otimes(\mathbf{1}\otimes V))\otimes W \arrow[dl, "(\mathrm{id}_U\otimes\lambda_V)\otimes\mathrm{id}_W"] \arrow[ddd, "\alpha"] \\
& (U\otimes V)\otimes W \arrow[d, "\alpha"] & \\
& U\otimes(V\otimes W) & \\
(U\otimes \mathbf{1})\otimes(V\otimes W) \arrow[ur, "\rho_U\otimes\mathrm{id}_{V\otimes W}"] \arrow[dr, "\alpha"] & & U\otimes((\mathbf{1}\otimes V)\otimes W) \arrow[ul, "\mathrm{id}_U\otimes(\lambda_V\otimes\mathrm{id}_W)"'] \arrow[dl, "\mathrm{id}_U\otimes\alpha"] \\
& U\otimes(\mathbf{1}\otimes(V\otimes W)) \arrow[uu, "\mathrm{id}_U\otimes\lambda_{V\otimes W}"] &
\end{tikzcd},$$

where we dropped the subscripts of α to simplify the notation. The outer hexagon is commutative by the pentagon axiom, the two middle squares by naturality of α, and the top square and the lower left triangle by the triangle axiom. As a consequence, the lower right triangle is also commutative. Setting $U = \mathbf{1}$, we get

$$\mathrm{id}_{\mathbf{1}}\otimes(\lambda_V\otimes\mathrm{id}_W) = \mathrm{id}_{\mathbf{1}}\otimes(\lambda_{V\otimes W}\circ\alpha).$$

Applying the naturality of the unit isomorphisms to both sides of this equality, we have

$$\lambda_V\otimes\mathrm{id}_W = \lambda_{V\otimes W}\circ\alpha,$$

which shows the commutativity of the first triangle in the statement of lemma. A similar proof shows the commutativity of the second one. ∎

Lemma A.1.10. If S, S', S'' are objects on $\mathcal{V}^{str}$, we have

$$\varphi_{S,S'*S''}\circ(\mathrm{id}_S\otimes\varphi_{S',S''})\circ\alpha_{F(S),F(S'),F(S'')} = \varphi_{S*S',S''}\circ(\varphi_{S,S'}\otimes\mathrm{id}_{S''}).$$

Proof. We prove the lemma by induction on the length of S''. If $S = \varnothing$, we have

$$\begin{aligned}
&\varphi_{S,S'}\circ(\mathrm{id}_S\otimes\varphi_{S',\varnothing})\circ\alpha_{F(S),F(S'),\mathbf{1}} \\
&\qquad = \varphi_{S,S'}\circ(\mathrm{id}_S\otimes\rho_{F(S')})\circ\alpha_{F(S),F(S'),\mathbf{1}} \\
&\qquad = \varphi_{S,S'}\circ\rho_{F(S)\otimes F(S')} \\
&\qquad = \rho_{F(S*S')}\circ(\varphi_{S,S'}\otimes\mathrm{id}_{\mathbf{1}}) \\
&\qquad = \varphi_{S*S',\varnothing}\circ(\varphi_{S,S'}\otimes\mathbf{1}).
\end{aligned}$$

The first and third equality are by definition, the second one by the previous lemma, and the third one by naturality of ρ.

Let V be an object of $\mathcal{V}$ and let us prove that the equality of the lemma for S, S', S'' implies the equality for $S, S', S'' * (V)$. Indeed,

$$\begin{aligned}
&\varphi_{S,S'*S''*(V)} \circ (\mathrm{id}_S \otimes \varphi_{S',S''*(V)}) \circ \alpha_{F(S),F(S'),F(S''*(V))} \\
&\quad =(\varphi_{S,S'*S''} \otimes \mathrm{id}_V) \circ \alpha^{-1}_{F(S),F(S'*S''),V} \circ (\mathrm{id}_S \otimes \varphi_{S',S''} \otimes \mathrm{id}_V) \\
&\qquad \circ (\mathrm{id} \otimes \alpha^{-1}_{F(S'),F(S''),V}) \circ \alpha_{F(S),F(S'),F(S'')\otimes V} \\
&\quad =(\varphi_{S,S'*S''} \otimes \mathrm{id}_V) \circ (\mathrm{id}_S \otimes \varphi_{S',S''} \otimes \mathrm{id}_V) \circ \alpha^{-1}_{F(S),F(S')\otimes F(S''),V} \\
&\qquad \circ (\mathrm{id}_S \otimes \alpha^{-1}_{F(S'),F(S''),V}) \circ \alpha_{F(S),F(S'),F(S'')\otimes V} \\
&\quad =(\varphi_{S,S'*S''} \otimes \mathrm{id}_V) \circ (\mathrm{id}_S \otimes \varphi_{S',S''} \otimes \mathrm{id}_V) \\
&\qquad \circ (\alpha_{F(S),F(S'),F(S'')} \otimes \mathrm{id}_V) \alpha^{-1}_{F(S)\otimes F(S'),F(S''),V} \\
&\quad =(\varphi_{S*S',S''} \otimes \mathrm{id}_V) \circ \alpha^{-1}_{F(S*S'),F(S''),V} \circ (\varphi_{S,S'} \otimes \mathrm{id}_{S''} \otimes \mathrm{id}_V) \\
&\quad =\varphi_{S*S'',S''*(V)} \circ (\varphi_{S,S'} \otimes \mathrm{id}_{S''*(V)}).
\end{aligned}$$

The first and last equalities follows from the definition of φ, the second and fifth ones from the naturality of the associativity isomorphism, the third one by the pentagon axiom, and the fourth one by induction hypothesis. ■

Theorem A.1.11. The categories $\mathcal{V}$ and $\mathcal{V}^{str}$ are monoidal equivalent.

Proof. We have already proven that $\mathcal{V}$ and $\mathcal{V}^{str}$ are equivalent and we have shown an explicit equivalence of categories F. We claim that $(F, \mathrm{id}_{\mathbf{1}}, \varphi)$ is a monoidal functor from $\mathcal{V}^{str}$ to $\mathcal{V}$. Indeed, the preceding lemma is a reformulation of the first condition in the definition of monoidal functor, while the other two follow from the definition of $\varphi_{S,\varnothing}$ and $\varphi_{\varnothing,S}$. A direct but tedious computation shows that $(G, \mathrm{id}_{\mathbf{1}}, \mathrm{id})$ is also a monoidal functor. Finally, the natural isomorphism θ defined above is a natural tensor isomorphism. ■

Remark A.1.12. In the proof given in this subsection, we need to abandon the original category $\mathcal{V}$ in order to construct a new different category $\mathcal{V}^{str}$ which is strict. If $\mathcal{V}$ is, for example, the category of vector spaces, $\mathcal{V}^{str}$ is an strict category which is monidal equivalent to $\mathcal{V}$ but contains no vector space. In the particular case of categories of sets with additional structure, it is possible to make an alternative construction keeping the original category but redefining the tensor product, namely replacing the product of two objects by an isomorphic copy, which turns the initial category into a strict monoidal category. Details of this construction can be found in [7].

A.1.4 MacLane's coherence theorem In a monodial category, one can form n-fold products of any finite sequence of objects $V_1, \ldots, V_n$. Such a product can be obtained by adding parenthesis to the expression $V_1 \otimes \cdots \otimes V_n$ in a consistent way. For $n = 3$, the associativity axiom allows to identify the objects $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ up to a canonical isomorphism. For $n \geqslant 3$, one may use a chain of associativity isomorphism in order to identify up to isomorphism two parenthesized products of $X_1, \ldots, X_n$, $n \geqslant 3$, but one problem arises: this identification is not canonical in the sense there exist everal combinations of the associativity isomorphism giving rise to some a priori different isomorphisms between two given objects. This is solved for $n = 4$ by the pentagon axiom, which states that the two possible identifications are the same. It holds true in the general case, as the following theorem states.

Theorem A.1.13 (MacLane's Coherence Theorem)**.** Let $\mathcal{V}$ be a monoidal category and $V_1 \ldots, V_n \in \mathcal{V}$. Let P_1, P_2 be two parenthesized products of $V_1, \ldots, V_n$ (in this order) with arbitrary insertions of unit objects $\mathbf{1}$. Let $f, g : P_1 \to P_2$ be two isomorphisms obtained by composing associativity and unit isomorphism and their inverses possibly tensored with identity morphisms. Then $f = g$.

Proof. Let $(F, \phi, \varphi) : \mathcal{V} \to \mathcal{V}'$ be a monoidal equivalence between $\mathcal{V}$ and a strict monoidal category $\mathcal{V}'$. Let P_{F1}, P_{F2} the parenthesizations of $F(V_1), \ldots, F(V_n)$ corresponding to P_1 and P_2, respectively. Since $\mathcal{V}^{str}$ is strict, we have For instance, if $P_1 = (V_1 \otimes V_2) \otimes (V_3 \otimes (V_4 \otimes V_5))$, then $P_{F1} = (FV_1 \otimes FV_2) \otimes (FV_3 \otimes (FV_4 \otimes FV_5))$. Using monoidality, we will construct natural morphism ν_1, ν_2 such that the diagrams

$$\begin{tikzcd} F(P_1) \arrow[r, "F(f)"] \arrow[d, "\nu_1"] & F(P_2) \arrow[d, "\nu_2"] \\ P_{F1} \arrow[r, "\mathrm{id}"] & P_{F2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(P_1) \arrow[r, "F(g)"] \arrow[d, "\nu_1"] & F(P_2) \arrow[d, "\nu_2"] \\ P_{F1} \arrow[r, "\mathrm{id}"] & P_{F2} \end{tikzcd}$$

are commutative. Hence, $F(f) = F(g)$ and, by faithfulness of F, $f = g$.

We construct ν_1 inductively (the construction of ν_2 is identical). Write $P_1 = S \otimes T$, where S is a parenthesization of $V_1 \otimes \cdots \otimes V_i$ and T is a parenthesization of $V_{i+1} \otimes \cdots \otimes V_n$ with arbitrary intersections of unit objects, for some $0 \leqslant i \leqslant n$ (if $i = 0$, S is by convention a product of unit objects and the same holds for T if $i = n$). If $S, T \neq \mathbf{1}$, then we have a morphism

$$\varphi_{S,T}^{-1} : F(S \otimes T) = F(P_1) \to P_{F1} = F(S) \otimes F(T)$$

and we construct inductively $\nu_S : F(S) \to S_F$ and $\nu_T : F(T) \to T_F$. We then set $\nu_1 \coloneqq (\nu_1 \otimes \nu_2) \circ \varphi_{S,T}^{-1}$. On the other hand, if $P_1 = \mathbf{1} \otimes T$, we take $\nu_1 \coloneqq (\phi \otimes \nu_T) \circ \varphi_{\mathbf{1},T}^{-1}$, and similarly when $T = \mathbf{1}$.

Once we have defined ν_1 and ν_2 it remains to show that the diagrams above are commutative. We check it by induction for $F(f)$. Remember that f is a composition of associativity morphisms, unit morphisms and tensor of those with identity maps. Suppose that the outermost map in that composition is an associativity isomorphism, i.e., $f = \alpha \circ f_0$, for some f_0 (we dropped the subscripts of α for simplicity). We then have a commutative diagram

$$\begin{tikzcd} FP_1 \arrow[r, "F(f_0)"] \arrow[dddd, "\nu_1"] & F((S \otimes T) \otimes Q) \arrow[r, "F(\alpha)"] \arrow[d, "\varphi^{-1}"] & F(S \otimes (T \otimes Q)) \arrow[d, "\varphi^{-1}"] \\ & F(S \otimes T) \otimes F(Q) \arrow[d, "\varphi^{-1} \otimes \mathrm{id}"] & F(S) \otimes F(T \otimes Q) \arrow[d, "\mathrm{id} \otimes \varphi^{-1}"] \\ & (F(S) \otimes F(T)) \otimes F(Q) \arrow[r, "\alpha"] \arrow[d, "(\nu_S \otimes \nu_T) \otimes \nu_Z"] & F(S) \otimes (F(T) \otimes F(Q)) \arrow[d, "\nu_S \otimes (\nu_T \otimes \nu_Q)"] \\ & & \\ P_{F1} \arrow[r, "f_0'"] & (S_F \otimes T_f) \otimes Q_F \arrow[r, "\alpha"] & S_F \otimes (T_F \otimes Q_F), \end{tikzcd}$$

where the left square commute by induction hypothesis, the top right one by definition of monoidal functor and the bottom right one by naturality of α. The cases where the outermost map is an unit isomorphism or a tensor product are similar. ■

MacLane's coherence theorem justifies the following alternative definition of monoidal category which is usually found in some references:

Definition A.1.14. A *monoidal category* is a category $\mathcal{V}$ endowed with

(i) a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$;

(ii) a natural isomorphism

$$\alpha = \{\alpha_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) \ : \ U, V, W \in \mathcal{V}\};$$

(iii) a *unit object* $\mathbf{1} \in \mathcal{V}$ and natural isomorphism

$$\lambda = \{\mathbf{1} \otimes V \xrightarrow{\sim} V \ : \ V \in \mathcal{V}\}$$

and

$$\rho = \{V \otimes \mathbf{1} \xrightarrow{\sim} V \ : \ V \in \mathcal{V}\};$$

satisfying the following *commutativity axiom*: if P_1 and P_2 are two parenthesizations of $V_1 \otimes \cdots \otimes V_n$ with arbitrary insertions of copies of the unit object, then all the isomorphisms $\psi : P_1 \to P_2$ obtained from α, λ, ρ and the identities by composition and tensoring are equal.

A.1.5 Braided monoidal categories. A *braided monoidal category* is a pair consisting of a monoidal category $\mathcal{V}$ and a natural family of isomorphisms

$$c = \{c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V \ : \ V, W \in \mathcal{V}\},$$

called *braiding*, such that the diagrams

$$\begin{tikzcd}
 & U \otimes (V \otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\alpha_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes \mathrm{id}_W"'] & & & V \otimes (W \otimes U) \\
 & (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{id}_V \otimes c_{U,W}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[r, "c_{U\otimes V,W}"] & W \otimes (U \otimes V) \arrow[dr, "\alpha^{-1}_{W,U,V}"] & \\
U \otimes (V \otimes W) \arrow[ur, "\alpha^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U\otimes c_{V,W}"'] & & & (W \otimes U) \otimes V \\
 & U \otimes (W \otimes V) \arrow[r, "\alpha^{-1}_{U,W,V}"] & (U \otimes W) \otimes V \arrow[ur, "c_{U,W} \otimes \mathrm{id}_V"'] &
\end{tikzcd}$$

commutes for all objects $U, V, W \in \mathcal{V}$. A braided monoidal category is *symmetric* if

$$c_{W,V} \circ c_{V,W} = \mathrm{id}_{V\otimes W},$$

for all objects $V, W \in \mathcal{V}$.

Let $(\mathcal{V}, c)$ be a braided monoidal category. We define the *reverse braiding* on $\mathcal{V}$ by

$$\bar{c}_{V,W} \coloneqq c^{-1}_{W,V}.$$

Proposition A.1.15. The reverse braiding is a braiding.

Proof. The braiding axioms for c, applied to W, U, V, read

$$\alpha_{U,V,W} \circ c_{W,U\otimes V} \circ \alpha_{W,U,V} = (\mathrm{id}_U \otimes c_{W,V}) \circ \alpha_{U,W,V} \circ (c_{W,U} \otimes \mathrm{id}_V),$$

$$\alpha^{-1}_{V,W,U} \circ c_{W\otimes U,V} \circ \alpha^{-1}_{W,U,V} = (c_{W,V} \otimes \mathrm{id}_U) \circ \alpha^{-1}_{W,V,U} \circ (\mathrm{id}_W \otimes c_{U,V}).$$

Taking inverses in both sides of the last equality and applying the definition of reverse braiding, we get the braiding axioms for c'. ■

The following proposition exhibits the relationship between the brading and the unit isomorphisms in a braided monoidal category:

Proposition A.1.16. For any object V of a braided tensor category with unit $\mathbf{1}$, we have

$$\lambda_V \circ c_{V,\mathbf{1}} = \rho_V, \quad \rho_V \circ c_{\mathbf{1},V} = \lambda_V \quad \text{and} \quad c_{\mathbf{1},V} = c^{-1}_{V,\mathbf{1}}.$$

Proof. Consider the diagram

$$\begin{tikzcd}
(V\otimes \mathbf{1})\otimes W \arrow[r, "\alpha"] \arrow[dr, "\rho_V\otimes \mathrm{id}_W"] \arrow[dd, "c\otimes \mathrm{id}_W"'] & V\otimes(\mathbf{1}\otimes W) \arrow[r, "c"] \arrow[d, "\mathrm{id}_V\otimes\lambda_W"] & (\mathbf{1}\otimes W)\otimes V \arrow[d, "\lambda_W\otimes \mathrm{id}_V"] \arrow[dr, "\alpha"] & \\
& V\otimes W \arrow[r, "c"] & W\otimes V & \mathbf{1}\otimes(W\otimes V) \arrow[dl, "\mathrm{id}"] \\
(\mathbf{1}\otimes V)\otimes W \arrow[r, "\alpha"] \arrow[ur, "\lambda_V\otimes \mathrm{id}_W"] & \mathbf{1}\otimes(V\otimes W) \arrow[r, "\mathrm{id}_{\mathbf{1}}\otimes c"] \arrow[u, "\lambda_{V\otimes W}"] & \mathbf{1}\otimes(W\otimes V) \arrow[u, "\lambda_{W\otimes V}"] &
\end{tikzcd}$$

where we dropped the subscripts of α and c to simplify the notation. The outside heptagon is commutative by definition of the braiding, the top square by the naturality of c, the bottom square by the naturality of λ, the upper left triangle by the triangle axiom and the lower left and right triangles by lemma A.1.9. Thus, the right triangle commutes, i.e.,

$$\rho_V \otimes \mathrm{id}_W = (\lambda_V \circ c_{V,\mathbf{1}}) \otimes \mathrm{id}_W.$$

Taking $W = \mathbf{1}$ and applying the naturality of ρ, we get

$$\rho_V = \lambda_V \circ c_{V,\mathbf{1}},$$

which is the first equality to be proved. Replacing c by its inverse and using the second axiom of the definition of braiding, one can the second one in a similar way. Finally, the last one is a combination of the other two. ■

One of the main properties of a braided tensor category is given by the following theorem:

Theorem A.1.17. Let U, V, W be objects in a braided tensor category. Then, we have the following commutative diagram:

$$
\begin{tikzcd}[column sep=small]
 & (U\otimes V)\otimes W \arrow[dl, "c_{U,V}\otimes \mathrm{id}_W"'] \arrow[dr, "\alpha_{U,V,W}"] & \\
(V\otimes U)\otimes W \arrow[d, "\alpha_{V,U,W}"] & & U\otimes (V\otimes W) \arrow[d, "\mathrm{id}_U\otimes c_{V,W}"] \\
V\otimes (U\otimes W) \arrow[d, "\mathrm{id}_V\otimes c_{U,W}"] & & U\otimes (W\otimes V) \arrow[d, "\alpha^{-1}_{U,W,V}"] \\
V\otimes (W\otimes U) \arrow[d, "\alpha^{-1}_{V,W,U}"] & & (U\otimes W)\otimes V \arrow[d, "c_{U,W}\otimes \mathrm{id}_V"] \\
(V\otimes W)\otimes U \arrow[d, "c_{V,W}\otimes \mathrm{id}_U"] & & (W\otimes U)\otimes V \arrow[d, "\alpha_{W,U,V}"] \\
(W\otimes V)\otimes U \arrow[dr, "\alpha_{W,V,U}"] & & W\otimes (U\otimes V) \arrow[dl, "\mathrm{id}_W\otimes c_{U,V}"] \\
 & W\otimes (V\otimes U) &
\end{tikzcd}
$$

Proof. The previous dodecaedron can be obtained by gluing together the following diagrams:

$$
\begin{tikzcd}[column sep=small]
 & (V\otimes U)\otimes W \arrow[dl, "\alpha_{V,U,W}"] \arrow[ddddd, "c_{V\otimes U,W}"] \\
V\otimes (U\otimes W) \arrow[d, "\mathrm{id}_V\otimes c_{U,W}"] & \\
V\otimes (W\otimes U) \arrow[d, "\alpha^{-1}_{V,W,U}"] & \\
(V\otimes W)\otimes U \arrow[d, "c_{V,W}\otimes \mathrm{id}_U"] & \\
(W\otimes V)\otimes U \arrow[dr, "\alpha_{W,V,U}"] & \\
 & W\otimes (V\otimes U),
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
(U\otimes V)\otimes W \arrow[dr, "\alpha_{U,V,W}"] \arrow[ddddd, "c_{U\otimes V,W}"] & \\
 & U\otimes (V\otimes W) \arrow[d, "\mathrm{id}_U\otimes c_{V,W}"] \\
 & U\otimes (W\otimes V) \arrow[d, "\alpha^{-1}_{U,W,V}"] \\
 & (U\otimes W)\otimes V \arrow[d, "c_{U,W}\otimes \mathrm{id}_V"] \\
 & (W\otimes U)\otimes V \arrow[dl, "\alpha_{W,U,V}"] \\
W\otimes (U\otimes V) &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(V\otimes U)\otimes W \arrow[d, "c_{V\otimes U,W}"] & (U\otimes V)\otimes W \arrow[l, "c_{U,V}\otimes \mathrm{id}_W"'] \arrow[d, "c_{U\otimes V,W}"] \\
W\otimes (V\otimes U) & W\otimes (U\otimes V) \arrow[l, "\mathrm{id}_W\otimes c_{U,V}"]
\end{tikzcd}
.
$$

The first two diagrams are commutative by the definition of braiding and the last one by naturality of c. ■

The relation stated in the previous theorem is known as *Yang-Baxter equation*.

Many examples of monoidal categories from A.1 admit a natural braiding:

Example A.1.18. The categories **Set**, $\mathbf{Vec}_{\Bbbk}$, $\mathbf{Rep}_{\Bbbk}(G)$, $\mathbf{Rep}_{\Bbbk}(\mathfrak{g})$ and $R-\mathbf{Mod}$ admit a braiding being the transposition of factors.

A monoidal functor (F, ϕ, φ) between two monoidal categories $\mathcal{V}, \mathcal{W}$ is *braided* if the following diagram commutes

$$\begin{tikzcd} F(V) \otimes F(W) \arrow[r, "c^{\mathcal{W}}_{F(V),F(W)}"] \arrow[d, "\varphi_{V,W}"] & F(W) \otimes F(V) \arrow[d, "\varphi_{W,V}"] \\ F(V \otimes W) \arrow[r, "F(c^{\mathcal{V}}_{V,W})"] & F(W \otimes V) \end{tikzcd}$$

for all $V, W \in \mathcal{V}$.

A.2 Hopf algebras and categories of representations

Hopf algebras are bialgebras endowed with additional structure that allows a ribbon category structure to be defined in their category of representations, providing a variety of examples on which to apply the results and constructions of the previous sections.

A.2.1 Definition and some examples Let A be an algebra with unit 1_A over a field $\Bbbk$. Assume that A is provided with multiplicative $\Bbbk$-linear homomorphisms $\Delta : A \to A^{\otimes 2} := A \otimes_{\Bbbk} A$ and $\varepsilon : A \to \Bbbk$, called the *comultiplication* and *counit* respectively, and a $\Bbbk$-linear homomorphism $s : A \to A$, called the *antipode*. It is understood that $\Delta(1_A) = 1_A \otimes 1_A$ and that $\varepsilon(1_A) = 1_A$.

Definition A.2.1. We say that $(A, \Delta, \varepsilon, s)$ is a *Hopf algebra* if these homomorphisms satisfy, together with the algebra multiplication $m : A \otimes A \to A$, the following properties:

(H1) $(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta$;
(H2) $m \circ (s \otimes \mathrm{id}_A) \circ \Delta = m \circ (\mathrm{id}_A \otimes s) \circ \Delta = \varepsilon \cdot 1_A$;
(H3) $(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta = \mathrm{id}_A$.

In the first axiom, we identify $A \otimes (A \otimes A) = (A \otimes A) \otimes A$ via $(a \otimes b) \otimes c = a \otimes (b \otimes c)$. In the third one, we identify $A = A \otimes_{\Bbbk} \Bbbk = \Bbbk \otimes_{\Bbbk} A$ via $a = a \otimes 1 = 1 \otimes a$.

For every element $a \in A$, $\Delta(a)$ is a finite linear combination of pure tensors $a = \sum_i a'_i \otimes a''_i$. We formally rewrite this sum as

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

This is known as *Sweedler's sigma notation*.

It can be shown from the axioms that the antipode s is an antiautomorphism of both the algebra and the coalgebra structure in A. Precisely, this means that

$$m \circ (s \otimes s) = s \circ m \circ P_A : A^{\otimes 2} \to A,$$

and

$$P_A \circ (s \otimes s) \circ \Delta = \Delta \circ s : A \to A^{\otimes 2},$$

where P_A denotes the flip $A^{\otimes 2} \to A^{\otimes 2}$, $a \otimes b \mapsto b \otimes a$. It also follows from the axioms that $s(1_A) = 1_A$ and $\varepsilon \circ s = \varepsilon : A \to \Bbbk$.

We can define a Hopf algebra structure in the linear dual $A^* \coloneqq \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$. The multiplication, comultiplication and antipode in A^* are dual to the comultiplication, multiplication and antipode in A, respectively. The roles of the unit and counit in A^* are played by the counit of A and the homomorphism $A^* \to \Bbbk$, $y \mapsto y(1_A)$.

Example A.2.2. 1. The group algebra $\Bbbk[G]$ of a group G is a Hopf algebra. The homomorphisms Δ, ε, s are defined on the additive generators $g \in G$ by the formulas $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $s(g) = g^{-1}$.

2. Let G be a finite group. We define a Hopf algebra A as follows. As a $\Bbbk$-module, A is generated by the set $\{\delta_g\}_{g \in G}$, where δ_g is defined to be the map

$$\delta_g(h) \begin{cases} 1, & \text{if } h = g, \\ 0, & \text{otherwise.} \end{cases}$$

The multiplication is induced by the multiplication on the ground field $\Bbbk$. The comultiplication, the antipode and the counit are defined by

$$\Delta(\delta_g) = \sum_{h \in G} \delta_h \otimes \delta_{h^{-1}g}, \quad s(g) = \delta_{g^{-1}}, \varepsilon(g) = \begin{cases} 1, & g = e_G, \\ 0, & \text{otherwise,} \end{cases},$$

where e_G is the unit element of G. This Hopf algebra is dual to $\Bbbk[G]$.

3. The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is a Hopf algebra, with $\Delta(g) = g \otimes 1 + 1 \otimes g$, $\varepsilon(g) = 0$, and $s(g) = -g$.

A.2.2 Category of representations.

Let A be a Hopf algebra over a field $\Bbbk$. By an A-module of fine rank we mean a left A-module whose underlying $\Bbbk$-vector space is finite dimensional.

Definition A.2.3. We define the *category of representations of* A, $\mathrm{Rep}(A)$, as the category whose objects are A-modules of finite rank and whose morphisms are A-linear homomorphisms.

The axioms (H1) and (H3) allows to define an associative product in $\mathrm{Rep}(A)$ which turns this category into a monoidal category. For objects V, W of $\mathrm{Rep}(A)$, set $V \otimes W = V \otimes_{\Bbbk} W$, where the action of A is obtained from the obvious product action of $A^{\otimes 2}$ in $V \otimes_{\Bbbk} W$ via de comultiplication, i.e., for $a \in A$,

$$a(v \otimes w) = \sum_{(a)} a_{(1)} v \otimes a_{(2)} w.$$

The field $\Bbbk$ also carries an structure of A-module, induced by the comultiplication, i.e., $(a, k) \mapsto \varepsilon(a)k$. Axioms (H1) and (H3) imply that the canonical isomorphisms $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$, $V \cong \Bbbk \otimes A \cong A \otimes \Bbbk$ are A-linear, so the product is associative up to isomorphism in the category $\mathrm{Rep}(A)$. Finally, the tensor product of two morphisms is the standard tensor product of homomorphisms. With this definitions $\mathrm{Rep}(A)$ is a monoidal category. Moreover, for any A-modules V, W, $\mathrm{Hom}_{\mathrm{Rep}(A)}(V, W)$ is an abelian additive group with the usual addition of homomorphisms, and the composition and the tensor products of morphisms are bilinear, so $\mathrm{Rep}(A)$ is in fact a monoidal Ab-category, in the sense of the following definition:

Definition A.2.4. A category $\mathcal{C}$ is said to be an *Ab-category* if for any pair of objects V, W the set $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is an additive abelian group and the composition of morphisms is bilinear. In the case where $\mathcal{C}$ is a monoidal category, we will also require the tensor product to be bilinear.

Remark A.2.5. We did not use the axiom (H2) to construct the monoidal structure of $\mathrm{Rep}(A)$ so, indeed, it would be enough to consider a more general algebraic structure with no antipode in order to get a monoidal category of representations. Nevertheless, we need the antipode to endow $\mathrm{Rep}(A)$ with the structure of a ribbon category.

Using the antipode s, we can also provide $\mathrm{Rep}(A)$ with a duality. For any object V of $\mathrm{Rep}(A)$, set $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, where the action of A is defined by the formula $(ay)(x) = y(s(a)x)$. The duality homomorphism d_V is just the evaluation pairing $V^* \otimes V \to \Bbbk, \quad (y, x) \mapsto y(x)$. Set

$$b_V = {}^{\#}d_V : \Bbbk = \Bbbk^* \to (V^* \otimes V)^* = V^{**} \otimes V^* = V \otimes V^*,$$

where $d_V : \varphi \mapsto \varphi \circ d_V$ is the transpose of d_V and we use the standard identification $V^{**} = V$. We then have, for any $k \in \Bbbk$, $b_V(k) = kb_V(1) = k\delta_V$, where δ_V is the coevaluation map. This map expands as

$$\delta = \sum_i v_i \otimes v^i,$$

where $\{v_i\}_i$ is an arbitrary basis of V and $\{v^i\}_i$ is its dual basis.

Lemma A.2.6. The category $\mathrm{Rep}(A)$ is a monoidal Ab-category with duality.

Proof. We have to show that d_V and b_V are A-linear and satisfy the compatibility axioms of duality. Let $\{v_i\}_i$ be a $\Bbbk$-basis of V and let $\{v^i\}_i$ be its dual basis. Then we have

$$\begin{aligned}(\mathrm{id}_V \otimes d_V)\circ(b_V \otimes \mathrm{id}_v)(v_j) &= (\mathrm{id}_V \otimes d_V)(\delta_V \otimes v_j)\\ &= (\mathrm{id}_V \otimes d_V)\left(\sum_i v_i \otimes v^i \otimes v_j\right) = \sum_i v_i \otimes v^i(v_j) = v_i,\end{aligned}$$

and, similarly,

$$\begin{aligned}(d_V \otimes \mathrm{id}_{V*})\circ(\mathrm{id}_{V*} \otimes b_V)(v^j) &= (d_V \otimes \mathrm{id}_{V*})(v^j \otimes \delta_V)\\ &= (d_V \otimes \mathrm{id}_{V*})\left(\sum_i v^j \otimes v_i \otimes v^i\right) = \sum_i v^j(v^i) \otimes v^i = v^i,\end{aligned}$$

for every j, which proves the compatibility of duality.

On the other hand, for any $a \in A$ and any i, j we have

$$\begin{aligned}d_V\left(\sum_{(a)} a_{(1)}v^i \otimes a_{(2)}v_j\right) &= \sum_{(a)} a_{(1)}v^i\left(a_{(2)}v_j\right) = \sum_{(a)} v^i\left(s(a_{(1)})a_{(2)}v_j\right)\\ &= v^i\left(\sum_{(a)} s(a_{(1)})s_{(2)}v_j\right) = v^i\left([m \circ (s \otimes \mathrm{id}_A) \circ \Delta(a)]v_j\right)\\ &= v^i(\varepsilon(a)v_j) = \varepsilon(a)d_V(v^i \otimes v_j),\end{aligned}$$

where the first equality follows from the linearity and the definition of d_V, the second one from de dual A-module structure, the third one is by $\mathbb{Z}$-linearity of v^i, the fourth one is an application of (H2) and the rest are clear. This proves that d_V is A-linear.

It remains to show that b_V is too. For $a \in A$, denote the homomorphism $V \to V,\ x \mapsto ax$, by $\rho(a)$. Note that $\rho(1_A) = \mathrm{id}_A$. We have to show that $a\delta_V = \varepsilon(a)\delta_V$. We have

$$\begin{aligned}a\delta_V = \Delta(a)\delta_V &= \sum_{(a)}(a_{(1)} \otimes 1_A)(1_A \otimes a_{(2)})\delta_V = \sum_{(a)}(\rho(a_{(1)}) \otimes \mathrm{id}_{V*})(\mathrm{id}_V \otimes {}^{\#}\rho(s(a_{(2)}))(\delta_V)\\ &= \sum_{(a)}(\rho(a_{(1)}) \otimes \mathrm{id}_{V*})(\rho(a_{(2)}) \otimes \mathrm{id}_{V*})(\delta_V) = \left(\rho\left(\sum_{(a)} a_{(1)}s(a_{(2)})\right) \otimes \mathrm{id}_{V*}\right)(\delta_V),\end{aligned}$$

where the first, second and fourth equalities are given by algebra structure of $A^{\otimes 2}$ and the definition of ρ; and the third one comes from the identity

$$(\mathrm{id}_V \otimes f^*)(\delta_V) = (f \otimes \mathrm{id}_{V*})(\delta_V),$$

which is easily proven choosing a basis. Finally, (H2) ensures that $\sum_{(a)} a_{(1)} s(a_{(2)}) = \varepsilon(a) 1_A$. ■

Remark A.2.7. The category $\mathrm{Rep}(A)$ is monoidal, but not strict monoidal. However, we can apply the constructions of paragraph A.1.3 to pass from $\mathrm{Rep}(A)$ to an equivalent strict monoidal category.

A.2.3 Quasitriangular Hopf algebras and universal R-matrices.

We formulate now natural conditions on a Hopf algebra which ensures the existence of a braiding in its representation category.

Let $(A, \Delta, \varepsilon, s)$ be a Hopf algebra over a field $\Bbbk$. For $a \in A$, set

$$\Delta'(a) = P_A(\Delta(a)) = \sum_{(a)} a_{(2)} \otimes a_{(1)},$$

where P_A denotes the flip in $A^{\otimes 2}$. For any $R = \sum_j \alpha_j \otimes b_j \in A^{\otimes 2}$, set

$$R_{12} = R \otimes 1_A = \sum_j \alpha_j \otimes \beta_j \otimes 1_A \quad R_{23} = 1_A \otimes R = \sum_j 1_A \otimes \alpha_j \otimes \beta_j,$$

and

$$R_{13} = (\mathrm{id}_A \otimes P_A)(R_{12}) = (P_A \otimes \mathrm{id}_A)(R_{23}) = \sum_j \alpha_j \otimes 1_A \otimes \beta_j.$$

Note that $R_{12}, R_{13}, R_{23} \in A^{\otimes 3}$.

Definition A.2.8. Let $R \in A^{\otimes 2}$ be a invertible element. We say that (A, R) is a *quasitriangular Hopf algebra* if, for any $a \in A$, the following conditions are satisfied:

(QH1) $\Delta'(a) = R\Delta(a)R^{-1}$;
(QH2) $(\mathrm{id}_A \otimes \Delta)(R) = R_{13}R_{12}$;
(QH3) $(\Delta \otimes \mathrm{id}_A)(R) = R_{13}R_{23}$.

Note that on the right-hand sides of these formulas we use multiplications in $A^{\otimes 2}$ and $A^{\otimes 3}$ induced by the one in A. The element $R \in A^{\otimes 2}$ satisfying this conditions is called a *universal R-matrix* of A.

Remark A.2.9. A direct computation using (QH1)-(QH3) shows that the universal R-matrix of a Hopf algebra satisfies the so called Yang-Baxter equation (cf. definition **??**)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

A solution of this equation is called an *R-matrix.*

Let (A, R) be a quasitriangular Hopf algebra. For any objects V, W of $\mathrm{Rep}(A)$, we define an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ by

$$c_{V,W}(v \otimes w) = P_{V,W}\left(R(v \otimes w)\right) = \sum_j \beta_j w \otimes \alpha_j v,$$

where $R = \sum_j \alpha_j \otimes \beta_j$ and $P_{V,W} : V \otimes W \to W \otimes V$ is the flip $v \otimes w \mapsto w \otimes v$. Its inverse is defined by

$$c_{V,W}^{-1}(w \otimes v) = R^{-1}(v \otimes w).$$

Lemma A.2.10. The family of isomorphisms $c = \{c_{V,W} : V \otimes W \to W \otimes V\}_{V,W}$ is a braiding in $\mathrm{Rep}(A)$.

Proof. First we prove that $c_{V,W}$ is a morphism in $\mathrm{Rep}(A)$. Indeed, for $a \in A$ and $v \in V, w \in W$, we have

$$\begin{aligned} c_{V,W}(a(v \otimes w)) &= P_{V,W}(R\Delta(a)(v \otimes w)) \\ &= P_{V,W}(\Delta'(a)R(v \otimes w)) \\ &= \Delta(a)P_{V,W}(R(v \otimes w)) \\ &= a\, c_{V,W}(v \otimes w), \end{aligned}$$

where the firs equality is the definition of c, the second one comes from (QH1), the third one by the definition of Δ' and the last one by the definition of the A-module structure in $V \otimes W$.

Let us verify that c satisfies the compatibility axioms of braidings, i.e.,

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W),$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}).$$

We check the first equality; the second one is proven in a similar way. For any $\alpha \in U \otimes V \otimes W$, we have

$$(\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W)(\alpha) = (\mathrm{id}_V \otimes c_{U,W})(P_{12}(R_{12}\alpha)) = P_{23}(R_{23}(P_{12}(R_{12}\alpha))),$$

where P_{12} and P_{23} are the flips $u \otimes v \otimes w \mapsto v \otimes u \otimes w$ and $u \otimes w \otimes v$, respectively. Denote by $P_{1,23}$ the permutation homomorphism

$$\begin{array}{rcl} U \otimes V \otimes W & \to & V \otimes W \otimes U, \\ u \otimes v \otimes w & \mapsto & v \otimes w \otimes u. \end{array}$$

The, setting $\alpha = u \otimes v \otimes w$, we have

$$\begin{aligned} P_{23}(R_{23}(P_{12}(R_{12}\alpha))) &= P_{23}\left(R_{23}\left(\sum_j \beta_j v \otimes \alpha_j u \otimes w\right)\right) \\ &= \sum_{j,k} b_j v \otimes b_k w \otimes \alpha_k \alpha_j u = P_{1,23}(R_{13}R_{12}\alpha). \end{aligned}$$

On the other hand, it follows from the definition of $c_{U,V\otimes W}$, the action of A in $V \otimes W$ and (QH2) that

$$\begin{aligned} c_{U,V\otimes W}(\alpha) &= P_{U,V\otimes W}(Ru \otimes v \otimes w) = P_{U,V\otimes W}\left(\sum_j \alpha_j u \otimes b_j(v \otimes w)\right) \\ &= \sum_j \Delta(b_j)(v \otimes w) \otimes \alpha_j u = P_{1,23}\left((\mathrm{id}_A \otimes \Delta)(R)\alpha\right) = P_{1,23}(R_{13}R_{12}\alpha), \end{aligned}$$

which proves the assertion. ■

Remark A.2.11. In the other way round, it is easy to prove that the existence of a braiding in $\mathrm{Rep}(A)$ implies the existence of an invertible element $R \in A^{\otimes 2}$ satisfying (QH1)-(QH3) and this is not a trivial fact. There is a general method, due to Drinfeld, producing quasitriangular Hopf algebras. This method is called the *double construction* and, starting from a Hopf algebra A over a field, it produces the structure of a quasitriangular Hopf algebra in the vector space $A \otimes A^*$. Further details can be found in [6].

A.2.4 Ribbon Hopf algebras. We formulate now natural conditions on a quasitriangular Hopf algebra which ensure the existence of a twist in the representation category.

Definition A.2.12. A *ribbon Hopf algebra* is a triple (A, R, v) consisting of a quasitriangular Hopf algebra (A, R) and an invertible element v of the center of A such that

(RH1) $\Delta(v) = P_A(R)R(v \otimes v)$;
(RH2) $s(v) = v$.

The element v is called the *universal twist* of A.

For an object V of $\mathrm{Rep}(A)$, we define the twist $\theta_V : V \to V$ to be the multiplication by $v \in A$.

Lemma A.2.13. Let (A, R, v) a ribbon Hopf algebra. The family of homomorphisms $\{\theta_V : V \to V\}_V$ is a twist in the braided monoidal category $\mathrm{Rep}(A)$. This twist is compatible with duality.

Proof. Since v is a central element of A, θ_V is a homomorphism in $\mathrm{Rep}(A)$ and, since v is invertible, it is an isomorphism. For any A-modules U, W and for any $u \otimes w \in U \otimes W$ we have

$$c_{W,U} \circ c_{U,W}(u \otimes w) = P_{W,U}(RP_{U,W}(R(u \otimes w)) = \sum_{j,k} \alpha_k \beta_j u \otimes \beta_k \alpha_j w = P_A(R)R(u \otimes w),$$

where $R = \sum_j \alpha_j \otimes \beta_j$ as usual. Note that, by definition of the tensor product in $\mathrm{Rep}(A)$, $\theta_{U \otimes W}$ is the multiplication by $\Delta(u)$. Thus, (RH1) and the previous computation imply that

$$\theta_{U \otimes W}(u \otimes w) = \Delta(v)(u \otimes w) = P_A(R)R(v \otimes v)(u \otimes w) = c_{W,U} \circ c_{U,W} \circ (\theta_U \otimes \theta_W)(u \otimes w),$$

which shows that θ is a twist in $\mathrm{Rep}(A)$.

It remains to prove that it is compatible with duality, i.e., $(\theta_V \otimes \mathrm{id}_{V^*}) \circ b_V = (\mathrm{id}_V \otimes \theta_{V^*}) \circ b_V$. This is equivalente to $(v \otimes v^{-1})\delta_V = \delta_V$. (RH2) implies that for any $v \in V$, $y \in V^*$, we have $(vy)(x) = y(s(v)x) = y(vx)$. Now, for any dual basis $\{e_i\}_i$ and $\{e^i\}_i$, the basis $\{ve_i\}_i$ and $\{e^i \circ \rho(v^{-1})\}_i$, where $\rho(v^{-1})$ is the multiplication by v^{-1}, are also dual, so

$$(v \otimes v^{-1})\delta_V = \sum_i ve_i \otimes v^{-1}e^i = \delta_V,$$

by the independence of δ_V on the choice of basis. ■

We have proven the following theorem:

Theorem A.2.14. For any ribbon Hopf algebra, the category $\mathrm{Rep}(A)$ is a ribbon Ab-category. ■

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