Some thoughts

What I have understood about what you're interested in

Let Λ be Lazard's ring and F^{univ} be the universal group law over Λ. Let

$$\mathcal{FG} = \left[\underline{\mathrm{Aut}}(F^{univ})\backslash \mathrm{Spec}(\Lambda)\right]$$

be the stack of one dimensional formal group laws. Let $\mathrm{Qcoh} - \mathcal{FG}$ be quasicoherent modules over this stack. This means simply

- for each couple (R, F) where R is a ring and F a formal group law one gives oneself an R-module $M_{(R,F)}$
- moreover for each $f : (R, F) \longrightarrow (R', F')$ that is to say a morphism $f : R \longrightarrow R'$ together with an isomorphism $F \otimes_R R' \xrightarrow{\sim} F'$ you give yourself an isomorphism $\gamma_f : M_{(R,F)} \otimes_{R,f} R' \xrightarrow{\sim} M_{(R',F')}$. Moreover you ask for a cocyle condition for $\gamma_{f_1 \circ f_2}$...

Then if I understand cobordism gives you a cohomology theory

$$X \longmapsto M(X) \in \mathrm{Qcoh} - \mathcal{FG}$$

where let's say X is a CW complex (maybe more generaly a spectrum). And with some finiteness hypothesis on X, like X a finite CW complex, $M(X)$ is a coherent sheaf.

Now there is the formal stack $\mathcal{D}iv$ of height n one dimensional formal p-divisible groups over $\mathrm{Spf}(\mathbb{Z}_p)$ (I take $\mathbb{Z}_p$ but all of this is the same with formal $\mathcal{O}$-modules with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$). This stack is uniformized in the following way. Let $\mathbb{H}$ be a one dimensional formal p-divisible group of height n over $\mathbb{F}_p$ and $\mathfrak{X}$ be its deformation space. Then

$$\mathcal{D}iv = [\underline{\mathrm{Aut}}(\mathbb{H})\backslash \mathfrak{X}]$$

where $\underline{\mathrm{Aut}}(\mathbb{H})$ is a localy constant étale pro-group scheme that become $\mathcal{O}_D^\times$ over $\mathrm{Spf}(W(\mathbb{F}_{p^n}))$ (with D the invariant $\frac{1}{n}$ division algebra over $\mathbb{Q}_p$).

One can define coherent modules over $\mathcal{D}iv$ as the data

- for each noetherian scheme S on which p is localy nilpotent together with a height n one dimensional formal p-div. group H over S one gives oneself $\mathcal{M}_{(S,H)}$ a coherent sheaf on S
- for each couples (S', H') and (S, H) together with a morphism $f : S' \longrightarrow S$ and an isomorphism $f^*H \xrightarrow{\sim} H'$ one has an isomorphism $f^*\mathcal{M}_{(S,H)} \xrightarrow{\sim} \mathcal{M}_{(S',H')}$...all of this satisfying some cocycle conditions...

Now of course there is a pullback/specialisation functor

$$\mathrm{Coh} - \mathcal{FG} \longrightarrow \mathrm{Coh} - \mathcal{D}iv$$

(seing $\mathcal{D}iv$ as a formal completion of $\mathcal{FG}$) and this gives you a cohomological theory with values in $\mathrm{Coh} - \mathcal{D}iv$.

It is difficult to define the generic fiber of the stack $\mathcal{D}iv$ as a stack in the category of rigid spaces but anyway what one wants to do is define the sheaf of coherents modules on it and that's easy to do by defining it as equivariant coherent sheaves on Lubin-Tate spaces as rigid spaces (open ball + action of $\mathcal{O}_D^\times$). Here one should add a continuity condition in the definition of an equivariant coherent sheaves on rigid L.T. spaces, condition always satisfied if such a sheaf comes by taking the generic fiber from $\mathrm{Coh} - \mathcal{D}iv$.

Now as I understand you're interested in invertible objects in this category of equivariant coherent objects on the rigid L.T. space, that is to say equivariant line bundles. I suspect this is due to the fact you want to apply this functor to invertible spectrums or something like that ?

The link with the isomorphism between L.T. and Drinfeld towers

There is a striking link between those equivariant objects on L.T. space and the isomorphism between L.T. and Drinfeld towers.

The fact is the following (all of this will appear in a book in Progress in Math. in a few months with the proof of the existence of the isomorphism) :

- If X is a rigid space with a "continuous" action of a localy profinite group (one can define precisely what it means, this has been done by Berkovich) then there is a good category of smooth equivariant rigide étale sheaves on $X_{\text{ét}}$. Here the main point is this smoothness condition, you may want to call it a discrete sheaf : nothing else than the stabilizator of a section is open
- In your paper with Gross you considered the period space that is the projective space $\mathbb{P}^{n-1}$ over $\mathbb{Q}_{p^n}$ with an action of $D^\times$. In fact this space as a natural descent data that descends it to a Severi-Brauer variety (or rather the associated rigid space) over $\mathbb{Q}_p$. Let's note P for this rigid Severi-Brauer variety with its action of $D^\times$
- Now you can consider smooth $D^\times$-equivariant sheaves on $P_{\text{ét}}$ and smooth $\mathrm{GL}_n(\mathbb{Q}_p)$-equivariant sheaves on Drinfeld's $\Omega_{\text{ét}}$. You can consider smooth $O_D^\times$-equivariant sheaves on Lubin-Tate open ball and smooth $\mathrm{GL}_n(\mathbb{Z}_p)$-equivariant sheaves on $\Omega_{\text{ét}}$ too.

Now the theorem is (the second part of the theorem is not in the book the arguments are exactly the same) :

Théorème 1 *There is a topos equivalence between*

- *smooth $D^\times$-equivariant sheaves on $P_{\acute{e}t}$ and smooth $GL_n(\mathbb{Q}_p)$-equivariant sheaves on $\Omega_{\acute{e}t}$*
- *smooth $\mathcal{O}_D^\times$-equivariant sheaves on L.T. open ball and smooth $GL_n(\mathbb{Z}_p)$-equivariant sheaves on $\Omega_{\acute{e}t}$*

Through those topos equivalence the smooth-equivariant cohomology complex of L.T. and Drinfeld towers are isomorphic

Of course as you see it is very tempting to extend this to equivariant coherent sheaves with a continuity condition for the action (moraly this has to be true and the only thing is to go through awfull technical details I think). The good thing being that on Ω the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ is completly transparent (by homographies) and much easier to compute (you have the Bruhat-Tits building as a skeleton of Ω that will help you classify those objects, although the building already appears in the Lubin-Tate open ball it is more hidden). Of course this correspondance should conserve line bundles.

One last thing : this correspondence for the étale site is a geometric version of the local Jacquet-Langlands correspondence. What's funny is that it seems the correspondence for coherent sheaves would suggest there is a cohomological theory different from cobordism that would give those equivariant objects on the Drinfeld space. At the end this correspondence would give an isomorphism between two cohomological theories ? ?

Hecke Sheaves and a question

Here I want to discuss something more. Let X be Lubin-Tate open ball and $(X_K)_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$ be the Lubin-Tate tower as a tower of rigid spaces with $X = X_{GL_n(\mathbb{Z}_p)}$ (I'm cheating a little bit on what is the L.T. tower but that's not very important). There are Hecke correspondences for $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$ open and $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ s.t. $g^{-1}Kg \subset \mathrm{GL}_n(\mathbb{Z}_p)$

$$\begin{array}{ccccc} & X_{K\cap gKg^{-1}} & \xrightarrow[\sim]{g} & X_{g^{-1}Kg\cap K} & \\ {}^{\pi_{K\cap gKg^{-1},K}}\swarrow & & & & \searrow^{\pi_{g^{-1}Kg\cap K,K}} \\ X_K & & & & X_K \end{array}$$

and what I call a Hecke sheaf is a compactible system of sheaves $(\mathcal{F}_K)_K$ on the étale site (this could be coherent sheaves) where each $\mathcal{F}_K$ is $\mathcal{O}_D^\times$-equivariant and moreover there are Hecke corresondences between all those $(\mathcal{F}_K)_K$:

$$g^*\pi_{g^{-1}Kg\cap K,K}\mathcal{F}_K \xrightarrow{\sim} \pi^*_{K\cap gKg^{-1},K}\mathcal{F}_K$$

all satisfying some cocyle conditions and so on... The fact is that those sheaves can be idenfifed with sheaves on the "rigid stack" $\mathcal{D}iv_{\mathbb{Q}}$ of one dimensional height n formal p-divisible groups "up to isogeny" (I don't want to give a precise definition). Moreover I prove at the same time as preceding theorem that Hecke sheaves are the same as $D^\times$-equivariant sheaves on $P_{\text{ét}}$. This means if $\breve{\pi} : X \longrightarrow P$ is the period morphism as defined in your article with Gross then all Hacke sheaves are of the form

$$(\breve{\pi} \circ \pi_{K,GL_n(\mathbb{Z}_p)})^*\mathcal{G}$$

where $\mathcal{G}$ is a $D^\times$-equiariant sheaf on the period space P (put the smoothness condition or not on the action, this has nothing to do with it).

Now the question I have is :

Starting from a finite CW complex Y or anything like that you're interested in, is the associated $D^\times$-equivariant coherent sheaf on L.T. space an Hecke sheaf ?

Because if that's the case they all come from the period space and are easy to classify ! ! I think that's not the case, you would have remarked it but one never knows.

To help you to respond my question here is a criterion to be a Hecke sheaf. Simply I am asking if R is a (complete local to fix the ideas) ring, H a height n one dimensional formal group law on R, $f : H \longrightarrow H'$ an isogeny, M_H and $M_{H'}$ the associated R-modules of finite type through cobordism (or whatever I don't understand). Then does f induces an R-morphism $M_{H'} \longrightarrow M_H$? Does this morphism coincide with multiplication by p when $H = H'$ and f is multiplication by p on the formal group law ?