MOTIVES AND AUTOMORPHIC FORMS : THE (POTENTIALLY) ABELIAN CASE

by

Laurent Fargues

Abstract. — This article is a review about the link between automorphic representations and potentially abelian motives. We also treat Shimura-Taniyama theory in details with complete proofs from a "modern point of view", give a description of the Taniyama group, the conjugation theory of CM abelian varieties and motives and its translation into the language of zero-dimensional Shimura varieties.

Introduction

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an irreducible continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, that is to say an irreducible representation of $\mathrm{Gal}(F|\mathbb{Q})$ for some $F|\mathbb{Q}$ finite Galois. According to Artin conjecture the L-function of ρ should have an holomorphic extension to the complex plane and satisfy a functional equation. In fact one conjectures more precisely there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_\mathbb{Q})$ such that $L(s,\Pi) = L(s,\rho)$. For example if $n = 2$, one conjectures $L(s,\rho) = L(s,f)$ where f is a weight one modular cusp form that is holomorphic if $\det\rho(c) = -1$ and a non-holomorphic Mass form with associated eigenvalue of the Hyperbolic Laplacian $\frac{1}{4}$ if $\det\rho(c) = 1$, where c is complex conjugation.

One knows thanks to Brauer's theorem we can write this L-function as a product/quotient of Dirichlet's L-functions. Thus we know $L(s,\rho)$ is "virtually" automorphic and has a meromorphic extension to $\mathbb{C}$ with a functional equation.

The L-function of ρ is the same as the one of the associated simple Artin motive over $\mathbb{Q}$ with $\overline{\mathbb{Q}}$-coefficients. One of the main purpose of this review is to explain the generalization of the preceding conjecture and fact in the framework of CM motives.

I am first giving in section 1 one example where one can attach to an automorphic representation something that looks like (and in fact is but we will know it later) the system of ℓ-adic realizations of a Motive. This is the case of algebraic Hecke characters.

In the other direction I explain in section 2 how to associate an automorphic representation to the motive $h^1(A)$ where A is an absolutely simple abelian variety having complex multiplication over its field of definition. I give complete proofs in "modern language", using a little bit of cristalline cohomology. For sake of completeness I do the same in section 3 for the Shimura Taniyama reciprocity law.

Then in section 4 I explain very quickly the general conjectures that should link automorphic representations and motives in the purpose to explain where are Taniyama and global Weil groups coming from.

In sections 5 and 6 I explain in details the connected Serre group, the Serre extension and its link with algebraic Hecke characters.

Section 7 is devoted to the general theorem about the Taniyama group. In this section we only give statements about the Taniyama group without details. One of the main statements I wanted to reach in this review is explained; the fact that the L-function of a CM motive is "virtually automorphic" and thus has a meromorphic extension to the complex plane and satisfies a functional equation. For sake of completeness we give a statement of the Fontaine-Mazur conjecture (actually a theorem) in the abelian case in section 8.

In section 9 we give much more details about the Taniyama group, its structure, Langlands construction, Deligne's theorem that it coincides with the CM motivic Galois group and try to explain why to understand this group, or rather extension, is equivalent to understand the conjugation theory of CM motives. Here I have tried to clarify the arguments in [**4**], for example about the construction of Langlands cocycle where it is hidden in complex computations in Milne and Shih's article. We present a sketch of the proof of Deligne's theorem in a little bit different language. In this section we assume the reader is familiar with the language of Tannakian categories.

We give in section 10 an application of the preceding to the conjugation of CM abelian varieties by a general automorphism of $\mathbb{C}$ not fixing the reflex field. This is the most general reciprocity law for CM abelian varieties.

Finally in section 11 we translate the results of the preceding sections in terms of the Shimura varieties that are moduli spaces of CM motives with additional structures. This means we give the general reciprocity law for CM motives equipped with additional structures.

Here is a list of references about this subject :

- The original article [**11**] of Weil defining algebraic Hecke characters and their properties
- The book [**9**] where the Serre group and the Serre extension are defined.
- The lecture notes 900 [**4**] (available on J.S. Milne's web page) where the theory of Hodge cycles is defined and studied, the Taniyama group is constructed and it is proved that this group is the Galois group of CM motives
- "Abelian Varieties with Complex Multiplication (for Pedestrians)" available on J.S. Milne's web page, http://www.jmilne.org/math/
- One of the lacks of this survey is that we don't explain a lot about the global Weil group. We strongly recommend Tate's article [**10**] (available on the AMS website in the online books section) and for the detailed construction of this group the last chapter of [**2**].

Notations

If F is a number field we note $\mathbf{A}_F$ for the adeles of F, $\mathbf{A}_{F,f}$ its finite part, $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$ its infinite part. A subscript like $\mathbf{A}_{F,S}$ indicates we take only places in S, a superscript like $\mathbf{A}_F^S$ means we take the places outside S. We sometimes note C_F for the idele class group $\mathbf{A}_F^\times/F^\times$. We normalize Artin reciprocity law $\mathbf{A}_F^\times/F^\times \longrightarrow \mathrm{Gal}(F^{ab}|F)$ in such a way a uniformizing element is sent to a geometric Frobenius, the inverse of the arithmetic Frobenius $x\mapsto x^q$.

If T is a torus over F and $\overline{F}$ a fixed algebraic closure of F we note $X^*(T)=\mathrm{Hom}(T_{\overline{F}},\mathbb{G}_{m\overline{F}})$ its group of characters and $X_*(T)=\mathrm{Hom}(\mathbb{G}_{m\overline{F}},T_{\overline{F}})$ its group of cocharacters as Galois modules.

If $E|F$ is a finite field extension and G an algebraic group over E we note $\mathrm{Res}_{E/F}G$ for Weil's scalar restriction defined by $\forall A$ an F-algebra $(\mathrm{Res}_{E/F}G)(A)=G(A\otimes_F E)$. For example if T is a torus over E this corresponds at the level of characters to take Galois induction, $X^*(\mathrm{Res}_{E/F}T) = \mathrm{Ind}_{E/F}X^*(T)$. As people are used to do in automorphic forms, when the base field F is clear (like $\mathbb{Q}$ for example), we sometimes note $E^\times$ for the torus $\mathrm{Res}_{E|F}\mathbb{G}_m$. This is harmless since $E^\times$ is Zariski dense in $\mathrm{Res}_{E/F}\mathbb{G}_m$.

We note $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ for Deligne's torus. Its characters are $\mathbb{C}^\times \ni z \longmapsto z^p\overline{z}^q$ where $p,q\in\mathbb{Z}$. We normalize the correspondence between $\mathbb{R}$-Hodge structures and representations of $\mathbb{S}$ in such a way $\mathbb{S}$ acts through $z^p\overline{z}^q$ on $H^{p,q}$. We have $\mathbb{S}_\mathbb{C} = \mathbb{C}^\times \times \mathbb{C}^\times$ on which complex conjugation is $(z_1, z_2) \mapsto (\overline{z}_2, \overline{z}_1)$. If $h : \mathbb{S} \to \mathrm{GL}(V)$ is a $\mathbb{R}$-Hodge structure then we note $\mu_h : \mathbb{G}_{m\mathbb{C}} \to \mathrm{GL}(V_\mathbb{C})$, the Hodge cocharacter defining the Hodge filtration, for the composite of h over $\mathbb{C}$ with $\mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}_\mathbb{C}$ defined by $z \mapsto (z, 1)$. The weight w_h, defined over $\mathbb{R}$, is the composite of h with the inclusion $\mathbb{R}^\times \to \mathbb{C}^\times$. We recall by definition the weight of a $\mathbb{Q}$-Hodge structure is always defined over $\mathbb{Q}$.

In the article $\overline{\mathbb{Q}}$ means the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We stress the reader to be careful with which fields are embedded fields, in $\mathbb{C}$, and which one are "abstract" fields.

1. Algebraic Hecke characters

In this section we sometimes pronounce the word "motive", but only as a motivation. In fact all results are proved by elementary algebraic number theory methods.

1.1. Definition. — Let $F|\mathbb{Q}$ finite. Consider the torus

$$T = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$$

We have the decomposition

$$T_{\overline{\mathbb{Q}}} = \prod_{\tau : F \hookrightarrow \overline{\mathbb{Q}}} \mathbb{G}_{m\overline{\mathbb{Q}}}$$

$$X^*(T) = \{ \sum_{\tau : F \hookrightarrow \overline{\mathbb{Q}}} a_\tau[\tau] \mid a_\tau \in \mathbb{Z}\}$$

on which $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts through by $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ $\sigma.[\tau] = [\sigma \circ \tau]$. If $\chi = \sum_\tau a_\tau[\tau]$

$$\begin{array}{ccc} \prod_\tau \mathbb{G}_{m\overline{\mathbb{Q}}} = T_{\overline{\mathbb{Q}}} & \xrightarrow{\chi} & \mathbb{G}_{m\overline{\mathbb{Q}}} \\ (x_\tau)_\tau & \longmapsto & \prod_\tau x_\tau^{a_\tau} \end{array}$$

and $\forall x \in F^\times = T(\mathbb{Q})$ $\rho(x) = \prod_\tau \tau(x)^{a_\tau}$.

Moreover one has $T(\mathbb{R}) = F_\infty^\times$, for a prime p $T(\mathbb{Q}_p) = \prod_{v|p} F_v^\times$ and $T(\mathbf{A}_\mathbb{Q}) = \mathbf{A}_F^\times$.

Let

$$\chi : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

be a Hecke character, $\chi = \bigotimes_v \chi_v = \chi_\infty \otimes \chi_f$. This means χ is continuous that is to say χ_∞ is continuous on $F_\infty^\times$ and χ_f is trivial on an open-compact subgroup of $\mathbf{A}_{F,f}^\times$. In particular χ_v is unramified for almost all v.

Definition 1.1. — Let $\rho \in X^*(T)$. The Hecke character χ is algebraic of weight ρ if

$$\chi_{|(F_\infty^\times)^0} = \rho^{-1} : T(\mathbb{R})^0 \hookrightarrow T(\mathbb{R}) \hookrightarrow T(\mathbb{C}) \xrightarrow{\rho^{-1}} \mathbb{C}^\times$$

Algebricity mean nothing else than for each archimedian place v :

– If v is real after fixing an isomorphism $F_v \simeq \mathbb{R}$ then χ_v is of the form $x \mapsto \mathrm{signe}(x)^\epsilon |x|^n$ for $\epsilon \in \{0, 1\}$ and $n \in \mathbb{Z}$
– If v is complex after fixing an isomorphism $F_v \simeq \mathbb{C}$ then χ_v is of the form $z \mapsto z^p\overline{z}^q$ for some $p, q \in \mathbb{Z}$

Example 1.2. — – χ is algebraic of weight $0 \iff \chi_{|(F_\infty^\times)^0} = 1 \iff \chi : \mathbf{A}_F^\times / \overline{(F_\infty^\times)^0 F^\times} \longrightarrow \mathbb{C}^\times$. And thus via Artin reciprocity map $\mathbf{A}_F^\times / \overline{(F_\infty^\times)^0 F^\times} \xrightarrow{\sim} \mathrm{Gal}(F^{ab}|F)$ weight zero algebraic Hecke characters correspond to finite order characters $\mathrm{Gal}(F^{ab}|F) \longrightarrow \mathbb{C}^\times$ and thus to simple Artin motives over F with $\overline{\mathbb{Q}}$-coefficients that "split" over an abelian extension.

- The idele norm $\|.\| : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbb{R}_+^\times$ is algebraic of weight $\rho = N_{F/\mathbb{Q}}^{-1} = -\sum_\tau [\tau]$. It corresponds to Tate motive $\overline{\mathbb{Q}}(1)$.
- As one already sees on the preceding example not all Hecke character correspond to a motive: for $t \in \mathbb{C}$ the character $\|.\|^t$ is algebraic iff $t \in \mathbb{Z}$ iff the L-function $L(s, \|.\|^t) = \zeta_\mathbb{Q}(s+t)$ is the L-function of a motive. The motive $\overline{\mathbb{Q}}(t)$ exists iff $t \in \mathbb{Z}$.
- More generally if $\chi : \mathrm{Gal}(F^{ab}|F) \longrightarrow \mathbb{C}^\times$ is of finite order, $M(\chi)$ is the associated simple Artin motive over F and $k \in \mathbb{Z}$ then $\chi\|.\|^k$ corresponds to the motive $M(\chi)(k)$ (Tate twist).

1.2. The associated ℓ-adic compatible system. —

Lemma 1.3. — *Let χ be algebraic of weight ρ. Then $\exists E \subset \overline{\mathbb{Q}}$ a number field s.t. $Im(\chi_f) \subset E^\times$.*

Proof. Let $K_f \subset \mathbf{A}_{F,f}^\times$ be compact open s.t. $\chi_{f|K_f} = 1$. Then the statement is a consequence of the two facts

$$|F^\times \backslash \mathbf{A}_{F,f}^\times / K_f| < +\infty$$
$$\forall x \in F^\times \quad \chi_f(x) = \pm\rho(x) \in \text{a fixed number field} \subset \mathbb{C}$$

□

Let E be a number field like in the preceding lemma (take the smallest one for example). The character ρ is defined over E ($\forall x \in F^\times = T(\mathbb{Q})$ $\rho(x) = \pm\chi_f(x) \in E^\times$ and $T(\mathbb{Q})$ is Zariski dense in T) , $\rho : T_E \longrightarrow \mathbb{G}_{mE}$ thus defining

$$\rho^E : T \longrightarrow \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

Let now ℓ be a prime number. One has

$$\begin{tikzcd} T(\mathbb{Q}_\ell) = \prod_{v|\ell} F_v^\times \arrow[r, hook] \arrow[d, "\rho^E_{\mathbb{Q}_\ell} = (\rho_\lambda)_{\lambda|\ell}"] & T(\mathbf{A}_\mathbb{Q}) = \mathbf{A}_F^\times \\ (E \otimes_\mathbb{Q} \mathbb{Q}_\ell)^\times = \prod_{\lambda|\ell} E_\lambda^\times & \end{tikzcd}$$

Let's note $\rho^E_{\mathbb{Q}_\ell} = (\rho_\lambda)_{\lambda|\ell}$, $\rho_\lambda : T(\mathbb{Q}_\ell) \longrightarrow E_\lambda^\times$. For such a λ define

$$\begin{array}{rcl} \chi_\lambda : \mathbf{A}_F^\times = F_\infty^\times \times (F \otimes \mathbb{Q}_\ell)^\times \times (\mathbf{A}_F^{\infty\ell})^\times & \longrightarrow & E_\lambda^\times \\ (x_\infty, x_l, x^l) & \longmapsto & \underbrace{\left(\frac{\chi_\infty}{\rho^{-1}}\right)(x_\infty)}_{\in\{\pm1\}} . \underbrace{\rho_\lambda^{-1}(x_l)}_{\in E_\lambda^\times} . \underbrace{\chi_f(x_\ell x^\ell)}_{\in E^\times} \end{array}$$

The idea being to take the ρ^{-1} defining the Hodge structure at the archimedean factors of the motive we would like to construct and to move it at the prime ℓ to an ℓ-adic Hodge-Tate structure in the sens of Fontaine.

One verifies immediately that

$$\chi_\lambda : \mathbf{A}_F^\times/F^\times \longrightarrow E_\lambda^\times$$

and the following lemma.

Lemma 1.4. — *χ_λ is continuous for the λ-adic topology.*

Thus, $E_\lambda^\times$ being totally discontinuous there is a factorisation

$$\chi_\lambda : \pi_0(\mathbf{A}_F^\times/F^\times) \longrightarrow E_\lambda^\times$$

and via Artin reciprocity map : $\pi_0(\mathbf{A}_F^\times/F^\times) \xrightarrow{\sim} \mathrm{Gal}(F^{ab}|F)$ it induces a λ-adic character still denoted χ_λ

$$\chi_\lambda : \mathrm{Gal}(F^{ab}|F) \longrightarrow E_\lambda^\times$$

Lemma 1.5. — *Let S be a finite set of finite places of F such that χ is unramified outside S ($\forall v \notin S.\infty\ \chi_v : F_v^\times/\mathcal{O}_{F_v}^\times \longrightarrow \mathbb{C}^\times$). Then $\forall \lambda|\ell\ \ \chi_\lambda$ is unramified outside $S.\ell$. Moreover*

$$\forall v \notin S.\ell\ \chi_\lambda(Frob_v) \in E^\times$$

(rationality of the Frobenius) and the system $(\chi_\lambda)_\lambda$ satisfies the following compatibility condition

$$\forall \lambda|\ell\ \forall \lambda'|\ell'\ \forall v \notin \infty.S.\ell.\ell'\ \ \chi_\lambda(Frob_v) = \chi_{\lambda'}(Frob_v)$$

Proof. Immediate. □

Thus the compatible system $(\chi_\lambda)_\lambda$ satisfies all the properties of the compatible system attached to the étale ℓ-adic cohomology when ℓ varies of a motive over F with coefficients in E, of rank 1 over E. For example the rationality of the Frobenius at a place of good reduction should follow from the fact that the characteristic polynomial of Frobenius being rational over $\mathbb{Q}$ the conjugacy class of Frobenius is defined over $\mathbb{Q}$ thus the Frobenius is rational since it lives in an abelian world (the motive we want to construct are abelian).

One verifies moreover

Lemma 1.6. — *For all ℓ, v a place of F, $v|\ell$ and $\lambda|\ell$ the Galois representation $\chi_{\lambda|Gal(F_v^{ab}|F_v)}$ is potentially cristalline. It is cristalline iff χ is unramified at v.*

Proof. — If $L|\mathbb{Q}_p$ and $M|\mathbb{Q}_p$ are finite a character $\mathrm{Gal}(L^{ab}|L) \longrightarrow M^\times$ is cristalline iff it is of the form $\prod_\sigma \sigma \circ \chi^{b_\sigma}$ where σ goes through embeddings from L into M, χ is the Lubin-Tate character associated to $L|\mathbb{Q}_p$ and $b_\sigma \in \mathbb{Z}$.

The lemma follows then from the fact that the associated character $F_v^\times \longrightarrow E_\lambda^\times$ restricted to an open subgroup of $\mathcal{O}_{F_v}^\times$ is given by $z \longmapsto \prod_\sigma \sigma(z)^{b_\sigma}$ where σ goes through embeddings $F_v \hookrightarrow E_\lambda$ and $b_\sigma \in \mathbb{Z}$. □

Remark 1.7. — Any motive should have potentially semi-stable reduction at any place. In fact this is known to be true at the level of their ℓ-adic realizations (Grothendieck ℓ-adic monodromy theorem) and their p-adic one (Faltings and Tusji's theorems in the framework of Fontaine's theory asserting that the p-adic Galois representations coming from smooth projective algebraic varieties are potentially semi-stable in the sens of Fontaine). But here the Motive we are interested in will be abelian, that is to say their Mumford-Tate group will be a torus. Thus the monodromy operator (in any theory, p-adic or ℓ-adic) that is a nilpotent operator has to be trivial (no unipotent in a torus). Thus for our CM motives potentially semi-stable will be equivalent to potentially cristalline.

The Fontaine-Mazur conjecture characterizes p-adic global Galois representations coming from motives by saying they are the one that are unramified at almost all places an potentially semi-stable at places dividing p (see [**5**]).

Corollary 1.8. — *The system $(\chi_\lambda)_\lambda$ satisfies all hypotheses of the Fontaine-Mazur conjecture.*

One can even go further. Since $C_F \simeq C_F^1 \times \mathbb{R}_+^\times$ and C_F^1 (norm 1 idele classes) is compact any (continuous) character $C_F \longrightarrow \mathbb{R}_+^\times$ is of the form $\|.\|^s$ for some $s \in \mathbb{R}$. The composite $\mathbf{A}_F^\times/F^\times \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{|.|} \mathbb{R}_+^\times$ is thus equal to $\|.\|^{-w/2}$ for some $w \in \mathbb{R}$ and one verifies easily $w \in \mathbb{Z}$. This should be the weight of the associated motive. In fact one can prove by purely elementary methods (use the next classification proposition 1.12) :

Lemma 1.9. — *The compatible system $(\chi_\lambda)_\lambda$ is pure of weight w : $\forall \lambda|\ell, v \notin \infty.S.\ell\ \forall \sigma : E \hookrightarrow \mathbb{C}\ \ |\chi_\lambda(Frob_v)|_\sigma = q_v^{w/2}$. The $\chi_\lambda(Frob_v)$ are Weil numbers of weight w.*

In fact one will see later in section 7 that χ corresponds to a simple motive $M(\chi)$ (motive over F with coefficients in E in the sens of Deligne's Hodge cycles).

Remark 1.10. — If $\sigma \in \mathrm{Aut}(\mathbb{C})$ then $\sigma \circ \chi_f : \mathbf{A}_F^\times/F^\times \longrightarrow \sigma(E)^\times$ is the finite part of an algebraic Hecke character ${}^\sigma\chi$ of weight ρ^σ. This will correspond at the level of motives to twist the coefficients of the motive (here E) by an automorphism. This is in contrast with the fact that in general if $\Pi = \Pi_\infty \otimes \Pi_f$ is an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ then the smooth $\mathrm{GL}_n(\mathbf{A}_f)$ representation ${}^\sigma\Pi_f$ obtained by twisting the coefficients by σ is not the finite part of an automorphic representation, even for $n = 1$ (take $|.|^{1/2}$ where $|.|$ is the idele norm, fix a prime p and take an automorphism of $\mathbb{C}$ sending $\sqrt{p}$ to $-\sqrt{p}$).

1.3. Classification. — Let's begin with a remark.

Remark 1.11. — If χ_1, χ_2 are algebraic Hecke characters of weight ρ then $\chi_1\chi_2^{-1}$ has weight 0 and is thus of finite order. Thus to determine all algebraic Hecke character of a given weight it suffices to find one and twist it by all finite order Hecke characters.

From a motivic point of view this will mean all simple CM motives (with $\overline{\mathbb{Q}}$-coefficients) with a fixed Hodge type differ from a twist by a rank one abelian Artin motive.

The classification is the following proposition (see [**9**]).

Proposition 1.12. —
- *Suppose F contains no CM field. Then if χ is algebraic of weight ρ one has $\rho = N_{F/\mathbb{Q}}^{-k}$ with $k \in \mathbb{Z}$ and thus $\chi = \|.\|^k.\chi'$ where χ' is of finite order. This corresponds to motives $M(k)$ where M is a simple Artin motive with $\overline{\mathbb{Q}}$-coefficients split by an abelian extension and $M(k)$ its Tate twist.*
- *Suppose F contains a CM field and let $K \subset F$ be the biggest CM subfield of F. Then any algebraic Hecke character χ of F is obtained by base change from K to F of an algebraic Hecke character of K*
$$\exists \chi' \;\; \chi = \chi' \circ N_{F/K}$$
- *If F is CM and χ has weight $\rho = \sum_{\tau : F \hookrightarrow \mathbb{C}} a_\tau[\tau]$ then $\exists w \in \mathbb{Z}$ (the weight of χ) s.t.*
$$\forall \tau \; a_\tau + a_{c\tau} = w$$
("Hodge decomposition").

The proof follows from the fact that if $\chi = \chi_\infty.\chi_f$ is algebraic, if $K_f \subset \mathbf{A}_{F,f}^\times$ is compact open s.t. $\chi_{f|K_f} = 1$ then
$$\forall x \in F^\times \cap K_f \;\; \chi_f(x) = \rho(x) = 1$$
But $K_f \cap F^\times \subset \mathcal{O}_F^\times$ is a congruence subgroup, of finite index in the arithmetic group $\mathcal{O}_F^\times$. Now one uses Dirichlet's unit theorem
$$\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}[\{v \mid v|\infty\}]^0$$
where v means a place of F and the superscript 0 the hyperplane of sum 0 coefficients. Once one has remarked this the remaining part is easy since if $\rho = \sum_{\tau : F \hookrightarrow \overline{\mathbb{Q}}} a_\tau[\tau]$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ then looking at the composite $F^\times \xrightarrow{\rho} \mathbb{C}^\times \xrightarrow{\log|.|_\sigma} \mathbb{R}$ one has via the preceding Dirichlet's isomorphism
$$\log|\rho_{|\mathcal{O}_F^\times}|_\sigma \otimes Id \;\text{ is induced by }\; v \longmapsto \sum_{\substack{\tau : F \hookrightarrow \mathbb{C} \\ \sigma\tau \text{ induces } v}} a_\tau$$
that has to be zero on the hyperplane of sum zero coefficients for all σ.

We will see later in fact that this classification gives exactly all possible Hodge weights for CM motives : the possible ρ correspond exactly to the Hodge weights of CM motives ! This remark is the key remark to make the link between algebraic Hecke characters and CM motives.

2. The L-function of an abelian variety with complex multiplication over its field of definition

2.1. CM fields. — Let F be a number field (abstract field, not embedded). Recall the following are equivalent :

- F has a non-trivial involution $*$ that satisfies $\forall x \in F \ \mathrm{tr}_{F/\mathbb{Q}}(xx^*) \geq 0$.
- F has a non-trivial involution $*$ that induces complex conjugation in any embedding of F into $\mathbb{C}$: $\forall \tau : F \hookrightarrow \mathbb{C} \ \ c \circ \tau = \tau \circ *$ where c is complex conjugation.
- F is a totally imaginary quadratic extension of a totally real field.

Such a field F is said to be CM. The involution $*$ is unique.

If F is CM then a CM type of F is a subset $\Phi \subset \mathrm{Hom}(F, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \coprod c\Phi$.

2.2. Abelian varieties with complex multiplication over $\mathbb{C}$. — Recall. Let $A/\mathbb{C}$ be an abelian variety such that $E = \mathrm{End}(A)_\mathbb{Q}$ is a field with $[E : \mathbb{Q}] = 2 \dim A$. Then E is a CM field, complex conjugation being given by the Rosatti involution on $\mathrm{End}(A)_\mathbb{Q}$ associated to a polarization.

If $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_\mathbb{Q}$, $\iota = Id$, then the action of E on $\mathrm{Lie}(A)$ is given by its eigenspaces decomposition

$$\mathrm{Lie}\, A = \bigoplus_{\tau \in \Phi} \mathrm{Lie}(A)_\tau$$

where $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ is a CM type and E acts on $\mathrm{Lie}(A)$ through τ. In fact the Betti cohomology $H^1_B(A, \mathbb{Q})$ is one dimensional over E, $H^1_B(A, \mathbb{Q}) \simeq E$ which induces an isomorphism

$$H^1_{dR}(A/\mathbb{C}) \simeq H^1_B(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} \simeq E \otimes_\mathbb{Q} \mathbb{C} = \prod_{\tau : E \hookrightarrow \mathbb{C}} \mathbb{C}$$

and now Hodge decomposition

$$H^1_{dR}(A/\mathbb{C}) = (\mathrm{Lie} A)^\vee \oplus \overline{(\mathrm{Lie} A)^\vee}$$

splits the embeddings of E in two peaces : Φ and its complex conjugates.

Then there exists a lattice $\Lambda \subset E$ such that

$$A(\mathbb{C}) \simeq \mathbb{C}^\Phi / \Phi(\Lambda)$$

and $\mathrm{End}(A) = \mathrm{Stab}_E(\Lambda)$, an order in $\mathcal{O}_E$. The correspondence $(A, \iota) \longmapsto (E, \Phi)$ induces a bijection between couples (A, ι) up to isogeny and couples (E, Φ) with Φ a CM-type of the CM field E.

Recall too that any such (A, ι) is defined over $\overline{\mathbb{Q}}$ and thus over a number field. In fact this follows from the fact that after introducing polarizations the moduli space of $(A, \iota, \lambda, \eta)$ where λ is a polarization, η a level structure (with sufficiently big level to rigidify the situation), is a zero dimensional variety (a Shimura variety) and thus all its points are defined over a number field [(1)].

All the theory is the same over $\overline{\mathbb{Q}}$ and $\mathbb{C}$.

An abelian variety A over $\mathbb{C}$ whose Mumford-Tate group is a torus is isogenous to a direct sum of copies of the preceding type of abelian varieties (the Mumford-Tate group of (A, ι) as before

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[(1)] A more elementary way to see it is the following. Let (A, ι) over $\mathbb{C}$ with CM type Φ. A standard argument shows we can suppose we have an irreducible algebraic variety X over a Galois closure $\widetilde{E}$ of E, an abelian scheme $\mathcal{A}$ over X equipped with an action ι of an order in $\mathcal{O}_E$, an embedding in $\mathbb{C}$ of the field of rational functions of X such that via this embedding at the generic point $(\mathcal{A}, \iota) \mapsto (A, \iota)$. We have a decomposition of $\mathcal{O}_X \otimes_\mathbb{Q} E$-module $\mathrm{Lie}(\mathcal{A}) = \bigoplus_{\tau \in \Phi'} \mathrm{Lie}(\mathcal{A})_\tau$ where $\Phi' \subset \mathrm{Hom}(E, \widetilde{E})$ and via the composite of embeddings $\widetilde{E} \hookrightarrow$ rational functions on $X \hookrightarrow \mathbb{C}$ one has $\Phi' \mapsto \Phi$. Then one can chose a closed point $x \in X(\overline{\mathbb{Q}})$ where $\overline{\mathbb{Q}} \subset \mathbb{C}$ such that via x $\Phi' \mapsto \Phi$. The specialization of $(\mathcal{A}, \iota)$ at x gives an abelian variety over $\overline{\mathbb{Q}}$ with a CM action of E and CM type Φ, thus isogenous over $\mathbb{C}$ to (A, ι). The kernel of an isogeny being a finite group-scheme the isogeny is defined over $\overline{\mathbb{Q}}$.

is a subtorus of $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$) and is defined over a number field. The preceding couples (A,ι), $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_{\mathbb{Q}}$ are the simple objects of the category of abelian varieties over $\mathbb{C}$ (or $\overline{\mathbb{Q}}$) up-to isogeny whose Mumford-Tate group is a torus.

2.3. The reflex field and the relfex norm. —

2.3.1. The relfex field. — Let (E,Φ) be a CM field equipped with a CM type. Here E is an **abstract field**, not embedded into $\mathbb{C}$. Recall the reflex field of (E,Φ) is an **embedded field** $K \subset \overline{\mathbb{Q}}$ defined by

$$\mathrm{Gal}(\overline{\mathbb{Q}}|K) = \{\sigma \in \Gamma_{\mathbb{Q}} \mid \sigma\Phi = \Phi\}$$

($\sigma\Phi = \Phi$ means σ permutes the embeddings in Φ). Another useful definition is

$$K \text{ is the number field generated by the } \mathrm{tr}_\Phi(x)\ x \in E$$

where $\mathrm{tr}_\Phi = \sum_{\tau\in\Phi}\tau : E \longrightarrow \overline{\mathbb{Q}}$, $\mathrm{tr}_\Phi + c\mathrm{tr}_\Phi = \mathrm{tr}_{E/\mathbb{Q}}$ (use linear independence of embeddings). The field K is stable under complex conjugation and is a CM field. In fact

$$\forall x \in E\ \ \mathrm{tr}_\Phi(x)^c = \mathrm{tr}_{E/\mathbb{Q}}(x) - \mathrm{tr}_\Phi(x)$$

and thus K is stable under conjugation. Moreover

$$\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\ \ c\sigma\mathrm{tr}_\Phi(x) = \mathrm{tr}_{c\sigma\Phi}(x) = \mathrm{tr}_{\sigma c\Phi}(x) = \sigma c\mathrm{tr}_\Phi(x)$$

since $\forall \tau : E \hookrightarrow \mathbb{C}\ \ c\sigma\tau = \sigma\tau* = \sigma c\tau$ where $*$ is the involution of E inducing complex conjugation in any embedding. Thus c induces complex conjugation in any embedding $\sigma : K \hookrightarrow \mathbb{C}$.

There is a partial norm

$$N_\Phi : E^\times \longrightarrow K^\times$$

defined by $N_\Phi = \prod_{\tau\in\Phi}\tau$. In fact one checks this morphism is defined by evaluation at the $\mathbb{Q}$-points of a morphism of tori

$$N_\Phi : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$$

2.3.2. The relfex type. — Let's note $\sigma_0 : K \hookrightarrow \mathbb{C}$ for the canonical embedding of K (the one defining K). The group $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on the set of sets of embeddings of E in $\mathbb{C}$ and $\Gamma_K = \mathrm{Stab}(\Phi)$. Thus

$$\begin{array}{ccc} \{\sigma\Phi \mid \sigma \in \Gamma_{\mathbb{Q}}\} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C}) \\ \sigma\Phi & \longmapsto & \sigma\sigma_0 \end{array}$$

For $\tau_0 : E \hookrightarrow \mathbb{C}$ one can define a CM type of K by setting

$$\Phi'_{\tau_0} = \{\sigma\sigma_0 \mid \tau_0 \in \sigma\Phi\}$$

using the preceding correspondence between embeddings of K and the orbit of Φ under $\Gamma_{\mathbb{Q}}$. This CM type depends on the choice of τ_0 !

2.3.3. The double reflex. — Let (F,Φ''_{τ_0}) be the reflex of (K,Φ'_{τ_0}) where Φ''_{τ_0} is induced by the canonical embedding of K in $\mathbb{C}$ (this means we don't have to fix an analog of τ_0 as we did before, it is already fixed). Then

$$F \subset \tau_0(E)$$

Moreover $(\tau_0^{-1}(F), \Phi''_{\tau_0} \circ \tau_0)$ does not depend on the choice of τ_0 !

We thus obtain a couple (F',Φ'') where $F' \subset E$ and one checks that Φ is induced from Φ''. Moreover (F',Φ'') is the smallest couple formed by a CM field with a CM type such that (E,Φ)

is induced from (F', Φ'').

$$\begin{array}{c} (E,\Phi) \\ \Big| \Big)\text{Induced : } \Phi=\{\tau:E\hookrightarrow\mathbb{C} \mid \tau_{|F'}\in\Phi''\} \\ (F',\Phi'') \end{array}$$

2.3.4. The reflex norm. — We have the partial norm

$$\tau_0^{-1} \circ N_{\Phi'_{\tau_0}} : K^\times \longrightarrow F'^\times$$

that, once more, does not depend on the choice of τ_0. We note

$$N_{\Phi'} : K^\times \longrightarrow E^\times$$

the composite with $F'^\times \hookrightarrow E^\times$ and call it the reflex norm.

2.3.5. Toric interpretation of the reflex norm. — Let (E, Φ) be as before. Let $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ and the $\Gamma_\mathbb{Q}$-module $X^*(T) = \mathrm{Hom}(T_{\overline{\mathbb{Q}}}, \mathbb{G}_{m\overline{\mathbb{Q}}})$. One has

$$\begin{aligned} X^*(T) &= \{ \sum_{\tau:E\hookrightarrow\mathbb{C}} a_\tau[\tau] \mid a_\tau \in \mathbb{Z}\} \\ X_*(T) &= \{\sum_\tau b_\tau[\tau] \mid b_\tau \in \mathbb{Z}\} \end{aligned}$$

where $([\tau])_\tau$, resp. $([\tau])_\tau$, are dual basis of $X^*(T)$, resp. $X_*(T)$.

The type Φ defines a cocharacter

$$\mu_\Phi = \sum_{\tau\in\Phi}[\tau] \in X_*(T)$$

The field of definition of μ_Φ is the reflex field K and thus

$$\mu_\Phi : \mathbb{G}_{mK} \longrightarrow T_K$$

and thus by applying the Weil scalar restriction a morphism

$$\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{K/\mathbb{Q}}T_K$$

For any tori S over a field k and a separable finite extension $k'|k$ there is a norm morphism $N_{k'/k} : \mathrm{Res}_{k'/k}S_{k'} \longrightarrow S$ that is an adjunction morphism, a trace morphism in étale topology : if $f : \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k)$, as étale sheaves, $S_{k'} = f^*S = f^!S$ and $\mathrm{Res}_{k'/k}S_{k'} = f_*f^*S = f_!f^!S$ (this is easily described at the level of characters/cocharacters as an adjunction since scalar extension to k' corresponds to restriction of a Galois module to a sub Galois group and Weil scalar restriction to an induction of a Galois module).

Thus coming back to our problem, by composition we have a composite morphism of $\mathbb{Q}$-tori

$$\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \xrightarrow{N_{K/\mathbb{Q}}} T$$

This morphism is $N_{\Phi'}$!

2.3.6. Hodge-theoretic interpretation. — Let (A, ι) over $\overline{\mathbb{Q}}$ where $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_\mathbb{Q}$. Let Φ be the associated CM type and K the relfex field. Then K is the definition field of the isogeny class of (A, ι). In particular if (A, ι) is defined over a number field K' on has $K \subset K'$.

The $\mathbb{Q}$-Hodge structure $H^1_{Betti}(A(\mathbb{C}), \mathbb{Q})$ corresponds to a morphism

$$h : \mathbb{S} \longrightarrow \mathrm{GL}_\mathbb{Q}(H^1_B(A, \mathbb{Q}))_\mathbb{R}$$

and thus a morphism

$$\mu_h : \mathbb{G}_{m\mathbb{C}} \longrightarrow \mathrm{GL}_\mathbb{Q}(H^1_B(A, \mathbb{Q}))_\mathbb{C}$$

defining the Hodge filtration. Now the $\mathbb{Q}$-Hodge structure has an action of E and thus in fact via ι those morphisms factor through a maximal torus

$$\begin{aligned} h &: \mathbb{S} \longrightarrow (\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{R}} \\ \mu_h &: \mathbb{G}_{m\mathbb{C}} \longrightarrow (\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{C}} \end{aligned}$$

Then one has

$$\mu_h = \mu_\Phi$$

and the Mumford-Tate group of A is the smallest subtorus of $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ through which μ_Φ factorizes. Thus

$$MT(A) = \mathrm{Im}\ N_{\Phi'}$$

Remark 2.1. — From the preceding description of the Mumford-Tate group it is clear it is always contained in the subtorus $\{x \in E^\times \mid x\overline{x} \in \mathbb{Q}^\times\}$. This reflects the fact that our abelian variety has a polarization. But of course this can be smaller, for example if the CM type (E, Φ) is induced from a strictly smaller CM type (E', Φ') since then the M.T.-group is contained in $\{x \in E'^\times \mid x\overline{x} \in \mathbb{Q}^\times\}$.

2.4. The L-function. — Let $F \subset \mathbb{C}$ be a number field and (A, ι) over F with $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_{\mathbb{Q}}$. By this we mean not only A is defined over F, but the CM action too. Let Φ be the associated CM type of E. Then one has

$$K \subset F$$

Following Taniyama and Shimura we will construct for each embedding $\sigma : E \hookrightarrow \mathbb{C}$ an algebraic Hecke characters $\chi_\sigma : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$ such that

$$L(s, A) = \prod_\sigma L(s, \chi_\sigma)$$

where $L(s, A)$ is the L-function of the motive $h^1(A)$ (this equality can be understood with the archimedean Gamma factors of the L-functions included, see [**10**] for the precise definition of those factors). Moreover the λ-adic compatible system associated to the χ_σ as in section 1 will coincide with the one associated to (A, ι) through the Galois action on étale ℓ-adic cohomology for varying ℓ. This will prove the automorphic nature of $h^1(A)$ and in particular the fact that $L(s, A)$ has an holomorphic continuation to the whole complex plane and satisfies a functional equation.

2.4.1. The Frobenius at places of good reduction. — Let's begin by defining the character at places where all the situation has good reduction.

Thus let S be a finite set of places of F containing the infinite one such that

- if $v \notin S$, F is unramified at v
- $\forall v \notin S$ A has good reduction at v
- If $\mathfrak{a} \subset \mathcal{O}_E$ is the order $\iota^{-1}(\mathrm{End}(A))$ then $\forall v \notin S, v|p$ $\mathfrak{a}$ is maximal at p and E is unramified at all places dividing p

If $v \notin S$ let's note $k(v)$ the residue field of v and $A_{k(v)}$ the reduction of the Neron model of $A \otimes_F F_v$. The extension property of the Neron model implies it is still equipped with an action of $\iota^{-1}(\mathrm{End}(A))$ and thus we have a couple $(A_{k(v)}, \iota)$ over $k(v)$.

One has

$$\mathrm{End}(A_{k(v)}, \iota)_{\mathbb{Q}} = E$$

(although $\mathrm{End}(A_{k(v)})_{\mathbb{Q}}$ is in general much bigger, but $\iota(E) \subset \mathrm{End}(A_{k(v)})_{\mathbb{Q}}$ is a maximal commutative subalgebra of this semi-simple algebra since $[E : \mathbb{Q}] = 2 \dim A_{k(v)}$).

Now one has the geometric Frobenius

$$\mathrm{Frob}_v \in \mathrm{End}(A, \iota)$$

and thus $\mathrm{Frob}_v \in E^\times$ via ι (Frob_v being an isogeny $\mathrm{Frob}_v \in \mathrm{End}(A)_{\mathbb{Q}}^\times$).

In fact for all $\ell \neq p$ where p is the residue characteristic of v, via ι $\mathrm{Frob}_v \in E^\times$ acts as the inverse of an arithmetic Frobenius at v inside $\mathrm{Gal}(\overline{\mathbb{Q}}|F)$ on

$$H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = V_\ell(A)^\vee$$

Thus from Riemann hypothesis we deduce

Lemma 2.2. — *For all λ a finite place of E not dividing p $Frob_v$ is a λ-adic unit. One has*

$$\forall \sigma : E \hookrightarrow \mathbb{C} \;\; |Frob_v|_\sigma = q_v^{1/2}$$

That is to say $\mathrm{Frob}_v \in E$ **is a weight** 1 q_v**-Weil number**.

Remark 2.3. — By definition a q-Weil number of weight $w \in \mathbb{Z}$ is an algebraic number Π that is an ℓ-adic unit for all $\ell \neq p$ and satisfies $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ $|\Pi|_\sigma = q^{w/2}$. One knows a q-Weil number up to a root of unity (here this will be a root of unity in E, thus up to a finite group) as soon as one knows its weight and its p-adic valuations. In fact if Π_1 and Π_2 have the same weight and p-adic valuations then $\Pi_1\Pi_2^{-1}$ is a unit having trivial archimedian absolute values, thus a root of unity. Thus to know $\mathrm{Frob}_v \in E^\times, v \notin S$, up to a finite set of finite group of roots of unity it remains to know its p-adic valuations. This is the following section.

2.4.2. Shimura Taniyama in modern language. — Let $v \notin S$ as before and $\mathrm{Frob}_v \in E^\times$. Now we are interested in the λ-adic valuation of Frob_v for $\lambda | p$. Those valuations are given by the slopes of the Newton polygon of the crystalline cohomology of $A_{k(v)}$.

Recall the reflex norm $N_{\Phi'} : K^\times \longrightarrow E^\times$. Since $K \subset F$ we can look at the composite

$$F^\times \xrightarrow{N_{F/K}} K^\times \xrightarrow{N_{\Phi'}} E^\times$$

Not to add too many notations we still denote it $N_{\Phi'} : F^\times \longrightarrow E^\times$. This is a morphism of tori thus inducing

$$\prod_{v|p} F_v^\times \longrightarrow \prod_{\lambda|p} E_\lambda^\times$$

for each prime number p.

Proposition 2.4 **(Shimura-Taniyama)**. — *Let $v \notin S$, $v|p$. Then $\forall \lambda | p$ a place of E*

$$v_\lambda(Frob_v) = v_\lambda(N_{\Phi'}((1, \dots, \pi_v, \dots, 1))$$

where $(1, \dots, \pi_v, \dots, 1)$ is the idele with trivial components outside v and component π_v at v.

Proof. Recall everything is unramified at p, in particular $\pi_v = p$.

Consider the cristalline cohomology of $A_{k(v)}$ over the PD thickening $\mathrm{Spec}(k(v)) \hookrightarrow \mathrm{Spec}(\mathcal{O}_{F_v})$

$$M = H^1_{cris}(A_{k(v)}/\mathrm{Spec}(\mathcal{O}_{F_v}))$$

It is a free $\mathcal{O}_{F_v}$-module equipped with a cristalline absolute Frobenius

$$\varphi : M \xrightarrow{\sim} M$$

that is σ-semi-linear, where σ is the arithmetic Frobenius of $F_v|\mathbb{Q}_p$. Moreover, as $\mathcal{O}_{F_v}$-linear automorphisms of M

$$\varphi^{[F_v:\mathbb{Q}_p]} = \mathrm{Frob}_v$$

Now thanks to Berthelot-Grothendieck comparison theorem if $\mathcal{A}_v$ is the Neron model of $A \otimes_F F_v$ over $\mathcal{O}_{F_v}$ there is a canonical isomorphism

$$M \simeq H^1_{dR}(\mathcal{A}_v)$$

and thus we have filtered φ-module $(M, \varphi, \mathrm{Fil}\, M)$ where $\mathrm{Fil}\, M \subset M$ is a direct factor and the exact sequence $0 \to \mathrm{Fil}\, M \to M \to M/\mathrm{Fil}\, M \to 0$ is the Hodge filtration

$$0 \longrightarrow \underbrace{\mathrm{Lie}(\mathcal{A}_v)^\vee}_{H^0(\mathcal{A}_v, \Omega^1_{\mathcal{A}_v})} \longrightarrow H^1_{dR}(\mathcal{A}_v) \longrightarrow \underbrace{\mathrm{Lie}(\mathcal{A}_v^\vee)}_{H^1(\mathcal{A}_v, \mathcal{O}_{\mathcal{A}_v})} \longrightarrow 0$$

The couple (M, φ) is a cristal, $pM \subset \varphi M \subset M$. Now the main fact is that the filtration $\mathrm{Fil}\, M$ reduced modulo p coincides with the kernel of $\varphi \bmod p : M/pM \to M/pM$.

In fact since $\mathrm{Fil}\, M/p\mathrm{Fil}\, M$ is the Hodge filtration of $H^1_{dR}(A_{k(v)}) = M/pM$, if $F : A_{k(v)} \to (A_{k(v)})^{(p)}$ is the absolute Frobenius

$$F^* = \varphi \bmod p : H^1_{dR}((A_{k(v)})^{(p)}) = H^1_{dR}(A_{k(v)}) \otimes_{k(v),\sigma} k(v) \longrightarrow H^1_{dR}(A_{k(v)})$$

It is clear F^* is zero on $\mathrm{Fil}\, M/p\mathrm{Fil}\, M = \mathrm{Lie}(A_{k(v)})^\vee$ since it is zero on $\mathrm{Lie}(A_{k(v)})$. We thus have an inclusion and the equality follows from the fact that we can compute the degree of an isogeny of degree a power of p on the cristalline cohomology, if $f : A_1 \longrightarrow A_2$ is an isogeny then $v_p(\deg(f)) = [H^1_{cris}(A_1/W) : f^* H^1_{cris}(A_2/W)]$, and thus since $\dim \ker(\varphi \bmod p) = \dim M/\varphi M$

$$\dim \ker(\varphi \text{ mod p}) = [M : \varphi M] = v_p(\deg F) = \dim(A_{k(v)}) = \dim \mathrm{Lie}(A_{k(v)}) = \dim \mathrm{Fil}\, M$$

Let's fix $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ an embedding inducing the place v of $F \subset \overline{\mathbb{Q}}$. It thus defines an embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$. Let L be the completion of the maximal unramified extension of F_v in $\overline{\mathbb{Q}}_p$. This defines in particular an algebraic closure $\overline{k(v)}$ of $k(v)$.

Let $(N, \varphi, \mathrm{Fil}\, N)$ be the extension of our objects to L, for example

$$N = M \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_L = H^1_{cris}(A_{\overline{k(v)}}/\mathcal{O}_L)$$

Since M has an action of an order in $\mathcal{O}_E$ that is maximal at p, thus an action of $\mathcal{O}_E \otimes \mathbb{Z}_p$, and since E is unramified at p, we can decompose

$$M = \bigoplus_{\lambda | p} M_\lambda \quad N = \bigoplus_{\lambda | p} N_\lambda$$

where λ goes through places of E dividing p. The cristalline Frobenius φ commutes with the action of E and thus leaves each M_λ and N_λ invariant. Since E is unramified at p we have a finer decomposition for N

$$N = \bigoplus_{\lambda | p} \underbrace{\bigoplus_{\substack{\tau : E \hookrightarrow L \\ \tau \text{ induces } \lambda}} N_\tau}_{N_\lambda}$$

where φ sends N_τ to $N_{\sigma\tau}$. We have the same type of decomposition for $\mathrm{Fil}\, N$. Let $d = [F_v : \mathbb{Q}_p]$, since $\varphi^d = \mathrm{Frob}_v \in E^\times$ we have

$$v_\lambda(\mathrm{Frob}_v) = d.[N_\lambda : \varphi N_\lambda] = d. \dim(\ker(\bar{\varphi} : N_\lambda/pN_\lambda \longrightarrow N_\lambda/pN_\lambda))$$

By definition of Φ, $\mathrm{Lie}\, A = \bigoplus_{\tau : E \hookrightarrow \overline{\mathbb{Q}}} \mathrm{Lie}(A)_\tau$. Thus

$$\mathrm{Lie}(A \otimes_{F,\nu} L) = \bigoplus_{\substack{\tau : E \hookrightarrow L \\ \tau \in \nu \circ \Phi}} \mathrm{Lie}(A \otimes_{F,\nu} L)_\tau$$

each eigenspace being one dimensional over L. Thus

$$\mathrm{Fil}\, N = \bigoplus_{\substack{\tau : E \hookrightarrow L \\ \tau \in \nu \Phi}} (\mathrm{Fil}\, N)_\tau$$

Now from the equality we proved before $\ker(\varphi \bmod p) = \mathrm{Fil}\, N/p\mathrm{Fil}\, N$ we obtain

$$\ker(\varphi \bmod p) = \bigoplus_{\substack{\tau : E \hookrightarrow L \\ \tau \in \nu \Phi}} \ker(\varphi \bmod p)_\tau$$

each factor being one dimensional over $\overline{k(v)}$. Thus

$$\begin{aligned} v_\lambda(\mathrm{Frob}_v) &= d.\dim(\ker(\bar{\varphi}: N_\lambda/pN_\lambda \longrightarrow N_\lambda/pN_\lambda)) \\ &= d.|\{\tau : E \hookrightarrow L \mid \tau \in \nu\Phi \text{ and } \tau \text{ induces } \lambda\}| \\ &= v_\lambda(N_{\Phi'}(1,\dots,\pi_v,\dots,1)) \end{aligned}$$

where last equality is verified easily (but painfully) on the concrete definition of the reflex norm given in section 2.3. □

Remark 2.5. — What we did in the preceding proof is simply to classify all weakly admissible filtered φ-modules in the sens of Fontaine that have a CM action.

2.4.3. Construction of the character on the finite ideles. — Up to enlarging S we can suppose it contains a finite place and that if $v|p$ is such a place then all places dividing p are in S. Let's define

$$\begin{aligned} \alpha : F^\times.(\prod_{v\in S, v\nmid\infty}\{1\} \times {\prod_{v\notin S}}' F_v^\times) &\longrightarrow E^\times \\ F^\times \ni x &\longmapsto N_{\Phi'}(x) \\ (y_v)_{v\notin S} &\longmapsto \prod_{v\notin S}(\mathrm{Frob}_v)^{v(y_v)} \end{aligned}$$

By weak approximation $F^\times.(\prod_{v\in S,v\nmid\infty}\{1\}\times\prod'_{v\notin S}F_v^\times)$ is dense in $\mathbf{A}_{F,f}^\times$.

Proposition 2.6. — *α extends uniquely to a continuous character*

$$\alpha : \mathbf{A}_{F,f}^\times \longrightarrow E^\times$$

where $E^\times$ is discrete (thus continuity means α is trivial on some compact open subgroup of $\mathbf{A}_{F,f}^\times$).

Proof. Let's verify first that if $x \in F^\times, (y_v)_{v\notin S} \in {\prod_{v\notin S}}' F_v^\times$ verify

$$\forall v \in S_f \ v(x) = 0 \text{ and } \forall v \notin S \ v(xy_v) = 0$$

that is to say $x.(y_v)_{v\notin S}$ is in the maximal compact subgroup of $\mathbf{A}_{F,f}^\times$ then $\alpha(x(y_v)_v)$ is a root of unity in E. Let $\beta = \alpha(x.(y_v)_{v\notin S})$. For each finite place λ of E we have thanks to proposition 2.4

$$\begin{aligned} v_\lambda(\beta) &= v_\lambda(N_{\Phi'}(x))\prod_{v\notin S} v_\lambda(N_{\Phi'}(\pi_v))^{v(y_v)} \\ &= v_\lambda\left(N_{\Phi'}\left(x.(\pi_v^{v(y_v)})_{v\notin S}\right)\right) \end{aligned}$$

where π_v means the finite idele with components 1 outside v and π_v at v, we do the computation in $\mathbf{A}_E^\times$, $(y_v)_{v\notin S}$ means the finite idele with components 1 outside v and y_v at v and S_f is the set of finite places in S. But by hypotheses $x.(\pi_v^{v(y_v)})_{v\notin S}$ is in the maximal compact subgroup of $\mathbf{A}_{F,f}^\times$, thus its image through $N_{\Phi'}$ too, thus $v_\lambda(\beta) = 0$. Thus $\beta \in \mathcal{O}_E^\times$. Now let $z \mapsto \overline{z}$ be the complex conjugation on E in any embedding of E in $\mathbb{C}$. As a morphism of tori $N_{\Phi'}\overline{N_{\Phi'}} = N_{F/\mathbb{Q}}$ thus

$$\begin{aligned} \beta\overline{\beta} &= N_{F/\mathbb{Q}}(x)\prod_{v\notin S}\left(\underbrace{\mathrm{Frob}_v\overline{\mathrm{Frob}_v}}_{q_v}\right)^{v(y_v)} \\ &= N_{F/\mathbb{Q}}(x)\prod_{v\notin S} q_v^{v(y_v)} \in \mathbb{Q}^\times \end{aligned}$$

and thus $\beta\overline{\beta}$ is a positive rational number that is a unit since $\beta \in \mathcal{O}_E^\times$, thus $\beta\overline{\beta} = 1$. From this we deduce $\forall\sigma : E \hookrightarrow \mathbb{C} \ \ |\beta|_\sigma = 1$ and thus $\beta \in \mu_\infty(E)$.

We are still not finished since now we have to prove that in a sufficiently small congruence subgroup this root of unity is 1. More precisely our α is trivial on $\prod_{v\notin S}{}' \mathcal{O}_{F_v}^\times$. We have a dense inclusion $F^\times \hookrightarrow \prod_{v\in S_f} F_v^\times$ and on this direct factor of $\mathbf{A}_{F,f}^\times$ α is equal to the character

$$\begin{array}{rcl} \Delta : F^\times & \longrightarrow & \mu_\infty(E) \\ x & \longmapsto & N_{\Phi'}(x) \prod_{v\notin S} \mathrm{Frob}_v^{-v(x)} \end{array}$$

and Δ takes values in $\mu_\infty(E)$ on $F^\times \cap \prod_{v\in S_f} \mathcal{O}_{F_v}^\times$. We have to show Δ is trivial on $F^\times \cap C$ where $C \subset \prod_{v\in S_f} \mathcal{O}_{F_v}^\times$ is an open subgroup.

Let ℓ be a prime of $\mathbb{Q}$ such that all primes of F dividing ℓ are in S. We have a decomposition

$$H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = \bigoplus_{\lambda|\ell} H_\lambda$$

where H_λ is a one dimensional E_λ-vector space. Let's fix such a $\lambda|\ell$. The action of $\mathrm{Gal}(\overline{F}|F)$ on H_λ is given by a character $\psi_\lambda : \mathrm{Gal}(F^{ab}|F) \longrightarrow E_\lambda^\times$ and thus via Artin reciprocity map a continuous character $\psi_\lambda : \mathbf{A}_F^\times/F^\times \longrightarrow E_\lambda^\times$ verifying

$$\forall v \notin S \;\; \psi_\lambda(\pi_v) = \mathrm{Frob}_v$$

and trivial on $F_\infty^\times$. Since an element in $\mu_\infty(E)$ is trivial if sufficiently near from 1 for the ℓ-adic topology it suffices to prove the composite $F^\times \xrightarrow{\Delta} E^\times \hookrightarrow E_\lambda^\times$ is continuous when $F^\times$ is equipped with the topology induced by $F^\times \hookrightarrow \prod_{v\in S_f} F_v^\times$ and $E_\lambda^\times$ with the λ-adic one. The morphism $N_{\Phi'}$ being algebraic is clearly continuous. Now

$$\forall x \in F^\times \;\; \prod_{v\notin S} \mathrm{Frob}_v^{-v(x)} = \psi_\lambda(\underbrace{x,\dots,x}_{\text{places in } S_f}, 1\dots,1,\dots)$$

and ψ_λ is continuous, thus Δ is too. $\square$

2.4.4. The Hecke character and its avatars. — Let's define now

$$\begin{array}{rcl} \chi : \mathbf{A}_F^\times = F_\infty^\times \times \mathbf{A}_{F,f}^\times & \longrightarrow & (E\otimes\mathbb{R})^\times \\ (x_\infty, x_f) & \longmapsto & N_{\Phi'}(x_\infty)^{-1}\alpha(x_f) \end{array}$$

It is clearly a Hecke character $\chi : \mathbf{A}_F^\times/F^\times \longrightarrow (E\otimes\mathbb{R})^\times$.

It gives rise for each embedding $\sigma : E \hookrightarrow \mathbb{C}$ to an algebraic Hecke character

$$\chi_\sigma : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbb{C}^\times$$

of weight $\sigma\circ N_{\Phi'}$ where $\sigma\circ N_{\Phi'} \in X^*(\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)$ since $N_{\Phi'} \in \mathrm{Hom}(\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m, \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)$ and $\sigma \in X^*(\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)$. It verifies

$$\chi_{c\sigma} = \overline{\chi_\sigma}$$

and

$$\chi_{\sigma,f} : \mathbf{A}_F^\times/F^\times \longrightarrow \sigma(E)^\times$$

(compare with lemma 1.3). For ℓ a prime number the étale cohomology of A splits

$$H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = \bigoplus_{\lambda|\ell} H_\lambda$$

accordingly to the decomposition $E\otimes_{\mathbb{Q}}\mathbb{Q}_\ell = \prod_{\lambda|\ell} E_\lambda$ and where

$$\dim_{E_\lambda} H_\lambda(A) = 1$$

This respects the Galois action thus as $\mathrm{Gal}(\overline{F}|F)$-modules

$$H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = \bigoplus_{\lambda|\ell} \psi_\lambda$$

where ψ_λ is a character, $\psi_\lambda : \mathrm{Gal}(F^{ab}|F) \longrightarrow E_\lambda^\times$ continuous for the λ-adic topology.

Proposition 2.7. — *One has* $\forall\lambda|\ell, \forall\sigma : E \hookrightarrow \overline{\mathbb{Q}}$

$$\psi_\lambda = \sigma^{-1} \circ \chi_{\sigma,\sigma(\lambda)}$$

where $\sigma(\lambda)$ *is the prime of* $\sigma(E)$ *associated to* λ, $(\chi_\sigma)_{\lambda'}$ *the* λ'*-adic compatible system attached to* χ_σ *as in section 1.2 for* λ' *going through finite places of* $\sigma(E)$*, and the right hand side means the composite* $Gal(F^{ab}|F) \xrightarrow{\chi_{\sigma,\sigma(\lambda)}} \sigma(E)_{\sigma(\lambda)} \xrightarrow[\sim]{\sigma^{-1}} E_\lambda$. *And thus*

$$H^1_{ét}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = \bigoplus_{\lambda|\ell} \sigma^{-1} \circ \chi_{\sigma,\sigma(\lambda)}$$

Proof. This is a consequence of Tchebotarev theorem since both agree on almost all Frobenius. □

Fix an algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. It induces for each $\sigma : E \hookrightarrow \overline{\mathbb{Q}}$ a prime $\lambda(\sigma)$ of $\sigma(E)$. As a corollary one obtains that

$$H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \simeq \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} \chi_{\sigma,\lambda(\sigma)}$$

This corresponds to a decomposition of the motive $h^1(A)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}} \in \text{Mot}_{/\mathbb{Q}}(\overline{\mathbb{Q}}-\text{coefficients})$ (extension of coefficients to $\overline{\mathbb{Q}}$) in

$$h^1(A) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} M(\chi_\sigma)$$

where $M(\chi_\sigma)$ is a rank one motive (this decomposition can be made precise in terms of Hodge cycles, see section 7).

Corollary 2.8. — *The motive* $h^1(A) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ *is "automorphic"*

$$L(s, A) = \prod_{\sigma:E\hookrightarrow\mathbb{C}} L(s, \chi_\sigma)$$

(with or without the archimedian factors included). In particular $L(s, A)$ *has an holomorphic extension to the complex plane and satisfies a functional equation.*

Remark 2.9. — Let's see the motive $h^1(A)$ as a rank one motive with coefficients in E. Then for each embedding $\sigma : E \hookrightarrow \overline{\mathbb{Q}}$ the motive $h^1(A) \otimes_{E,\sigma} \overline{\mathbb{Q}}$ is automorphic, attached to χ_σ

$$L(s, h^1(A) \otimes_{E,\sigma} \overline{\mathbb{Q}}) = L(s, \chi_\sigma)$$

Thus the preceding decomposition is nothing else than $h^1(A) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\sigma:E\hookrightarrow\overline{\mathbb{Q}}} h^1(A) \otimes_{E,\sigma} \overline{\mathbb{Q}}$.

3. Shimura Taniyama reciprocity law

We take the notations from section 2. Let's fix (A, ι) a simple CM abelian variety over $\overline{\mathbb{Q}}$ as in section 2 where $\iota : E \xrightarrow{\sim} \text{End}(A)_{\mathbb{Q}}$. Let $\Phi \subset \text{Hom}(E, \mathbb{C})$ be the associated CM type and $K \subset \overline{\mathbb{Q}}$ its reflex field. Recall K is the field of definition of the isogeny class of (A, ι), $\text{Gal}(\overline{\mathbb{Q}}|K) = \{\sigma \in \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \mid (A^\sigma, \iota^\sigma) \text{ is isogenous to } (A, \iota)\}$.

In section 2 we were interested in the action of the Galois group $\text{Gal}(\overline{F}|F)$ on the torsion points of A where (A, ι) is defined over F. Now we're going to look at the action of $\text{Gal}(\overline{\mathbb{Q}}|K)$.

We note $H^1_{\text{ét}}(A, \mathbf{A}_{\mathbb{Q},f})$ the restricted product of the $H^1_{\text{ét}}(A, \mathbb{Q}_\ell)$ with respect to the lattices $H^1_{\text{ét}}(A, \mathbb{Z}_\ell)$. It is a free $\mathbf{A}_{\mathbb{Q},f}$-module of rank $2\dim A$ and a rank one module over $\mathbf{A}_{\mathbb{Q},f}\otimes_{\mathbb{Q}}E = \mathbf{A}_{E,f}$.

Thus let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}|K)$. Let

$$f : (A, \iota) \longrightarrow (A^\sigma, \iota^\sigma)$$

be an isogeny. It induces an isomorphism

$$H^1_{\text{ét}}(A, \mathbf{A}_{\mathbb{Q},f}) \xrightarrow{\sigma^*} H^1_{\text{ét}}(A^{(\sigma)}, \mathbf{A}_{\mathbb{Q},f}) \xrightarrow{f^*} H^1_{\text{ét}}(A, \mathbf{A}_{\mathbb{Q},f})$$

where the first isomorphism is induced by the projection $A^\sigma = A \otimes_{\overline{\mathbb{Q}},\sigma} \overline{\mathbb{Q}} \longrightarrow A$ (invariance of étale cohomology by change of algebraic closed field). Moreover this isomorphism commutes with the action of E and thus induces an automorphism of the rank one $\mathbf{A}_{E,f}$-module $H^1_{\text{ét}}(A, \mathbf{A}_{\mathbb{Q},f})$ that is to say it is given by multiplication by an element of $\mathbf{A}_{E,f}^\times$, $f^*\sigma^* \in \mathbf{A}_{E,f}^\times$.

Now if we make another choice of isogeny f between (A, ι) and (A^σ, ι^σ) is differs by an element in $\text{End}(A, \iota)_{\mathbb{Q}}^\times = E^\times$. We thus have a well defined map

$$\text{Gal}(\overline{\mathbb{Q}}|K) \longrightarrow \mathbf{A}_{E,f}^\times / E^\times$$

and one checks immediately this is a group morphism whence in fact a morphism

$$\text{Gal}(K^{ab}|K) \longrightarrow \mathbf{A}_{E,f}^\times / E^\times$$

and via Artin reciprocity map a morphism

$$\pi_0(C_K) \longrightarrow \mathbf{A}_{E,f}^\times / E^\times$$

where $C_K = \mathbf{A}_K^\times / K^\times$, the idele class group. The purpose of the Shimura-Taniyama reciprocity law is to give the following formula.

Proposition 3.1. — *The reciprocity map $\pi_0(C_K) \longrightarrow \mathbf{A}_{E,f}^\times / E^\times$ is the inverse of the reflex norm that is to say $N_{\Phi'}^{-1}$.*

Remark 3.2. — We have $\pi_0(\mathbf{A}_K^\times / K^\times) = \mathbf{A}_K^\times / \overline{K_\infty^\times K^\times} = \mathbf{A}_{K,f}^\times / \overline{K^\times}$. The torus morphism $N_{\Phi'}$: $\text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \longrightarrow \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ thus induces a morphism $\pi_0(\mathbf{A}_K^\times / K^\times) \longrightarrow \mathbf{A}_{E,f}^\times / \overline{E^\times}$. But in fact the reflex norm factorises through Serre's torus (see section 5) $N_{\Phi'} : \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \twoheadrightarrow S^K \longrightarrow \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m$. And $S^K / w(\mathbb{G}_m)$ is anisotropic over $\mathbb{R}$ where w is the weight morphism, $(S^K / w(\mathbb{G}_m))(\mathbb{R}) \simeq U(1)^{[K:\mathbb{Q}]/2}$. From this one deduces easily $S^K(\mathbb{Q})$ is in fact closed inside $S^K(\mathbf{A}_{\mathbb{Q},f})$, and it follows that in fact the norm morphism extends to a morphism $\pi_0(\mathbf{A}_K^\times / K^\times) \longrightarrow \mathbf{A}_{E,f}^\times / E^\times$, and this is the morphism we called the reflex norm in the preceding proposition.

We now give the proof of the preceding proposition. Let's begin with a lemma.

Lemma 3.3. — *Let $F \subset \overline{\mathbb{Q}}$, $K \subset F$, a field over which (A, ι) is defined. Then the composite $\pi_0(C_F) = Gal(F^{ab}|F) \longrightarrow Gal(K^{ab}|K) \xrightarrow{\text{reciprocity map}} \mathbf{A}_{E,f}^\times / E^\times$ is given by $N_{\Phi'}^{-1} \circ N_{F/K}$.*

Proof. Let $\psi : \text{Gal}(F^{ab}|F) \longrightarrow \mathbf{A}_{E,f}^\times$ be the compatible system of λ-adic representations as in section 2.4.4. It is clear the following diagram commutes

$$\begin{array}{ccccc} \pi_0(\mathbf{A}_F^\times / F^\times) & \xrightarrow{\sim} & \text{Gal}(F^{ab}|F) & \xrightarrow{\psi} & \mathbf{A}_{E,f}^\times \\ \downarrow{\scriptstyle N_{F/K}} & & \downarrow & & \downarrow \\ \pi_0(\mathbf{A}_K^\times / K^\times) & \xrightarrow{\sim} & \text{Gal}(K^{ab}|K) & \xrightarrow{\text{rec. map}} & \mathbf{A}_{E,f}^\times / E^\times \end{array}$$

Now recall (see section 2) we constructed ψ as $\psi = \alpha . N_{\Phi'}^{-1}$ where $\alpha : \mathbf{A}_{F,f}^\times \longrightarrow E^\times$ and here $N_{\Phi'} : \mathbf{A}_{F,f}^\times \longrightarrow \mathbf{A}_{E,f}^\times$ (sorry, we note the same for $N_{\Phi'}$ and $N_{\Phi'} \circ N_{F/K}$). The result follows immediately. ☐

Lemma 3.4. — *Let $v|p$ be a place of K unramified in a field $F|K$ over which (A, ι) is defined and such that A has good reduction at all places of F dividing v. Then the restriction of the reciprocity* *map at the local Galois group Gal($K_v^{ab}|K_v$) factorizes as*

$$\begin{tikzcd} \pi_0(C_K) \arrow[r, "rec.\ map"] & \mathbf{A}_{E,f}^\times/E^\times \\ K_v^\times \arrow[u] \arrow[r, "factorization"] & (E\otimes\mathbb{Q}_p)^\times \arrow[u] \end{tikzcd}$$

Proof. Let's note for this proof (A,ι) for a model over F. Let $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the place v on K and let $w|v$ the induced place on F. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p|K_v) \hookrightarrow \mathrm{Gal}(\overline{K}|K)$ (via ν as a decomposition subgroup). Let $f : (A,\iota) \longrightarrow (A^\sigma, \iota^\sigma)$ be an isogeny whence an isogeny $f : A\otimes_F F_w \longrightarrow A^\sigma \otimes_F F_w = (A\otimes_F F_w)^\sigma$. This isogeny extends to the Neron model of of $A\otimes_F F_w$ and reduces to an isogeny

$$\overline{f} : A_{k(w)} \longrightarrow (A_{k(w)})^\sigma = (A_{k(w)})^{(p^r)}$$

where σ reduces to the Frobenius $x \mapsto x^{p^r}$. There's another isogeny, the geometric Frobenius

$$F^r : A_{k(w)} \longrightarrow (A_{k(w)})^{(p^r)}$$

that commutes with the action of $\iota(E)$ and $\iota^\sigma(E)$ and thus $\varphi = \overline{f}^{-1} \circ F^r \in \mathrm{End}(A,\iota)_{\mathbb{Q}}^\times = E^\times$. Now there's a commutative diagram

$$\begin{tikzcd} H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \widehat{\mathbb{Z}}^{(p)}) \arrow[r, "\sigma^*"] \arrow[d, equal] & H^1_{\text{ét}}(A^\sigma_{\overline{\mathbb{Q}}}, \widehat{\mathbb{Z}}^{(p)}) \arrow[r, "f^*"] \arrow[d, equal] & H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \widehat{\mathbb{Z}}^{(p)})[d] \arrow[d, "\varphi\in E^\times"] \\ H^1_{\text{ét}}(A_{\overline{k(w)}}, \widehat{\mathbb{Z}}^{(p)}) \arrow[r, "(\sigma \bmod p)^*"] \arrow[rr, bend right, "Id"'] & H^1_{\text{ét}}((A_{\overline{k(w)}})^{(p^r)}, \widehat{\mathbb{Z}}^{(p)}) \arrow[r, "(F^r)^*"] & H^1_{\text{ét}}(A_{\overline{k(w)}}, \widehat{\mathbb{Z}}^{(p)}) \end{tikzcd}$$

where $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell\neq p} \mathbb{Z}_\ell$, the vertical identifications correspond to the invariance of ℓ-adic cohomology for $\ell \neq p$ under good reduction and the equality $(F^r)^* \circ (\sigma \bmod p)^* = Id$ is nothing else than geometric Frobenius composed with the arithmetic one equals identity on étale cohomology. From this diagram one deduces the image by the reciprocity map of σ in $\mathbf{A}_{E,f}^\times/E^\times$ can be made to have trivial component outside p in $\mathbf{A}_{E,f}^\times$. $\square$

Let's come to the final part of the proof. Fix a field of definition $F|K$ of (A,ι). Note $b : \pi_0(\mathbf{A}_K^\times/K^\times) \longrightarrow \mathbf{A}_{E,f}^\times/E^\times$ the reciprocity map and consider $\beta = b.N_{\Phi'}$. We want to prove $\beta = 1$. We know thanks to lemma 3.3 β is trivial on $N_{F/K}(\mathbf{A}_F^\times/F^\times)$. Thus

$$\beta : \underbrace{C_K/N_{F/K}(C_F)}_{\text{finite group}} \longrightarrow \mathbf{A}_{E,f}^\times/E^\times$$

For S a sufficiently big finite set of places of K, thanks to lemma 3.4, there is a commutative diagram $\forall v \notin S$, $v|p$,

$$\begin{tikzcd} C_K/N_{F/K}(C_F) \arrow[r, "\beta"] & \mathbf{A}_{E,f}^\times/E^\times \\ K_v^\times/\mathcal{O}_{K_v}^\times \arrow[u] \arrow[r, "\beta_{|K_v^\times}"'] & (E\otimes\mathbb{Q}_p)^\times \arrow[u] \end{tikzcd}$$

By Tchebotarev it suffices to prove $\forall v \notin S$ $\beta(\pi_v) = 1$. But, still thanks to Tchebotarev, if $v \notin S, v|p$ there exists $v' \notin S, v'|p', p' \neq p$ such that $\underbrace{\beta(\pi_{v'})}_{\in(E\otimes\mathbb{Q}_p)^\times} = \underbrace{\beta(\pi_v)}_{\in(E\otimes\mathbb{Q}_{p'})^\times} \in \mathbf{A}_{E,f}^\times/E^\times$. But $(E\otimes\mathbb{Q}_p)^\times \cap (E\otimes\mathbb{Q}_{p'})^\times = \{1\} \subset \mathbf{A}_{E,f}^\times/E^\times$. $\square$

4. The general picture

4.1. The motivic side. —

4.1.1. The big motivic Galois group. — Let G be the motivic Galois group associated to the Grothendieck category of motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$ for numerical equivalence (let's suppose for a moment we know what this means, that is to say all standard conjecture are true). One has $G = \underline{\mathrm{Aut}}^{\otimes}(\omega_{Betti})$ where ω_{Betti} means the fiber functor Betti-Cohomology. One has a Tannakian equivalence

$$\mathrm{Rep}_{\mathbb{Q}}(G) \simeq \text{Motives}/\mathbb{Q} \text{ with coefficients in } \mathbb{Q}$$

and G is a pro-reductive algebraic group (reductiveness follows from the semi-simplicity of the category of motives).

There is an exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

where G^0, the neutral component classifies motives over $\overline{\mathbb{Q}}$. Thanks to Hodge conjecture this category of motives over $\overline{\mathbb{Q}}$ has a faithful functor towards polarizable $\mathbb{Q}$-Hodge structures and G^0 can be seen as a universal Mumford-Tate group for all motives.

In the preceding exact sequence $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ has to be seen has a pro-finite constant group-scheme over $\mathbb{Q}$ and $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}))$ is equivalent to the category of Artin motives over $\mathbb{Q}$ (with $\mathbb{Q}$ coefficients). The morphism $G \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ is Tannaka-dual to the inclusion of Artin motives inside the big category of motives. The morphism $G^0 \longrightarrow G$ is dual to the base change functor from motives over $\mathbb{Q}$ toward motives over $\overline{\mathbb{Q}}$.

For $L|\mathbb{Q}$ finite one can take the pull-back of the preceding extension via $\mathrm{Gal}(\overline{\mathbb{Q}}|L) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$

$$\begin{tikzcd}
1 \arrow[r] & G^0 \arrow[r] \arrow[d, equal] & {}_LG \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{\mathbb{Q}}|L) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G^0 \arrow[r] & G \arrow[r] & \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[r] & 1
\end{tikzcd}$$

Then

$$\mathrm{Rep}_{\mathbb{Q}}({}_LG) \simeq \text{Motives}/L \text{ with coefficients in } \mathbb{Q}$$

the inclusion ${}_LG \hookrightarrow G$ being dual to the base change functor scalar extensions for motives $- \otimes_{\mathbb{Q}} L$ from $\mathbb{Q}$ to L.

4.1.2. The abelianized version. — Consider the abelianization of the preceding sequence

$$1 \longrightarrow (G^0)_{ab} \longrightarrow G/(G^0)' \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

One knows how to define this exact sequence rigorously without using any of the standard conjectures, using the theory of absolute Hodge cycles of Deligne ([**4**]). Moreover one has an explicit construction of this sequence due to Serre for the connected component and Langlands for the whole extension.

This sequence is usually written as

$$1 \longrightarrow \underbrace{S}_{\text{Serre group}} \longrightarrow \underbrace{\mathcal{T}}_{\text{Taniyama group}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

where $\mathrm{Rep}_{\mathbb{Q}}(\mathcal{T})$ is equivalent to the category of "potentially abelian" motives that is to say motives such that there Mumford-Tate group is a torus. In fact if $\rho : G \longrightarrow \mathrm{GL}_n$, ρ factorises through $\mathcal{T}$ iff after a finite base change $L|\mathbb{Q}$ the restriction $\rho_{|_LG}$ has an abelian image.

4.2. The automorphic side. —

4.2.1. The big Langlands group. — Let $\mathcal{L}_{\mathbb{Q}}$ be the conjectural Langlands group. It should be a topological locally compact group and classify automorphic representations : the set of isomorphism classes of irreducible continuous representations $\mathcal{L}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ should be in bijection with cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_{\mathbb{Q}})$ (in fact there is a few conditions on such representations called Langlands parameters).

There should be an extension

$$1 \longrightarrow \mathcal{L}^0_{\mathbb{Q}} \longrightarrow \mathcal{L}_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

Moreover if for $L|\mathbb{Q}$ finite $\mathcal{L}_L$ is the pullback of the preceding by $\mathrm{Gal}(\overline{\mathbb{Q}}|L) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ then irreducible representations $\mathcal{L}_L \longrightarrow \mathrm{GL}_n(\mathbb{C})$ up to isomorphism should classify automorphic cuspidal representations of $\mathrm{GL}_n(\mathbf{A}_L)$.

More generally semi-simple non-irreducible representations of $\mathcal{L}_{\mathbb{Q}}$ or more generally $\mathcal{L}_L$ should correspond to other automorphic representations "build from Eisenstein series" that are induced from cuspidal one. Then the inclusion $\mathcal{L}_L \hookrightarrow \mathcal{L}_{\mathbb{Q}}$ should correspond dually at the level of representations to the so-called Langlands Base change functoriality from automorphic representations of $\mathrm{GL}_n(\mathbf{A}_{\mathbb{Q}})$ to the one of $\mathrm{GL}_n(\mathbf{A}_L)$ (this Langlands functoriality is only known for the moment for $L|\mathbb{Q}$ a Galois solvable extension (Arthur-Clozel)).

4.2.2. The abelianized version. — Let's consider the abelianized version of the preceding sequence

$$1 \longrightarrow (\mathcal{L}^0_{\mathbb{Q}})_{ab} \longrightarrow \mathcal{L}_{\mathbb{Q}}/(\mathcal{L}^0_{\mathbb{Q}})' \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

Then a continuous representations $\rho : \mathcal{L}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ factorizes through $\mathcal{L}_{\mathbb{Q}}/(\mathcal{L}^0_{\mathbb{Q}})'$ iff after a finite base change $\rho_{|\mathcal{L}_L}$ becomes abelian which from the automorphic point of view means after a Base change the automorphic representation comes from Eisenstein series build from Hecke characters.

In fact one knows how to define the preceding sequence using global class field theory. It is called the global Weil group

$$1 \longrightarrow W^0_{\mathbb{Q}} \longrightarrow W_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

and the same for $L|\mathbb{Q}$. Moreover there is an isomorphism $W_L^{ab} \simeq \mathbf{A}_L^\times/L^\times$ and thus abelian semi-simple representations of W_L correspond to finite collections of Hecke characters $\mathbf{A}_L^\times/L^\times \longrightarrow \mathbb{C}^\times$.

4.3. The link between both sides. — Conjecturaly some cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_{\mathbb{Q}})$ (the "algebraic one") should correspond to simple motives over $\mathbb{Q}$ with coefficients in $\overline{\mathbb{Q}}$. The automorphic cuspidal representation Π corresponds to the motive M iff $L(s,M) = L(s,\Pi)$.

In the case $n = 1$ this corresponds to what we called algebraic Hecke characters in section 1.

More generally the algebraicness condition on $\Pi = \bigotimes_v \Pi_v$ is on Π_∞ (the Langlands parameter of Π_∞ should correspond to a Hodge-structure of the motive, this Langlands parameter $W_{\mathbb{R}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ restricted to $\mathbb{C}^\times \subset W_{\mathbb{R}}$ should be algebraic and correspond to the associated Hodge structure, viewing $\mathbb{C}^\times$ as Deligne torus).

In some sens the general conjecture could be something like : the Tannakian category of semi-simple representations of the topological group $\mathcal{L}_{\mathbb{Q}}$ that are algebraic, via $\mathbb{C}^\times \hookrightarrow W_{\mathbb{R}} \longrightarrow \mathcal{L}_{\mathbb{Q}}$, is equivalent to the category of motives over $\mathbb{Q}$ with $\overline{\mathbb{Q}}$ coefficients.

In this article we are going to interest ourselves to the abelian case (or rather potentially abelian) of this conjecture.

One can prove that in fact there is a link between some representations of $W_{\mathbb{Q}}$ in $\mathrm{GL}_n(\mathbb{C})$ and potentially CM (CM stands for abelian in the motivic world) motives over $\mathbb{Q}$ with $\overline{\mathbb{Q}}$-coefficients that is to say algebraic representations of the Taniyama group $\mathcal{T} \longrightarrow \mathrm{GL}_{n/\overline{\mathbb{Q}}}$.

Now one conjectures that irreducible representations of $W_{\mathbb{Q}}$ parametrizes some cuspidal automorphic representations. This conjecture contains as a particular case Artin's conjecture. The problem being that one does not know automorphic induction for non-solvable extensions.

Nevertheless this will prove a less strong fact that is if M is potentially CM motive over $\mathbb{Q}$ in the sens of Hodge cycles then it is modular in a weak sens : $L(s,M)$ can be written as a product/quotient of L-functions of Hecke characters of finite extensions of $\mathbb{Q}$ (it is an easy adaptation of Brauer's theorem to the Weil group that any representations of $W_{\mathbb{Q}}$ is virtually a sum with $\mathbb{Z}$-coefficients of inductions of a Hecke character). In particular $L(s,M)$ has meromorphic continuation.

5. The connected Serre group

5.1. Hodge-theoretic description. — Let $\mathcal{C}$ be the category of CM $\mathbb{Q}$-Hodge structure. Here CM means polarizable and whose Mumford-Tate group is a torus (or equivalently semi-simple and M.T. group is a torus). This is a $\mathbb{Q}$-linear neutral Tannakian category. It is moreover semi-simple (thanks to the existence of polarizations). The canonical fiber functor $\omega : (V,h) \longmapsto V$, where $h : \mathbb{S} \longrightarrow \mathrm{GL}(V_{\mathbb{R}})$, should be understood as the Betti fiber functor. With the notations of the preceding sections one should have thanks to Hodge conjecture $(G^0)_{ab} \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega)$ (one will in fact see in a few seconds that any CM Hodge structure comes from a motive).

Let's describe $\mathcal{C}$. Let $(V,h) \in \mathcal{C}$ be simple. Then $E = \mathrm{End}(V,h)$ is a CM field or $\mathbb{Q}$ (this follows from the fact that $\mathrm{End}(V,h)$ is the centralizer of the M.T. group and the existence of a polarization). Consider the torus $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$. Then

$$h : \mathbb{S} \longrightarrow T_{\mathbb{R}}, \;\; w_h : \mathbb{G}_m \longrightarrow T$$

(the weight morphism that is defined over $\mathbb{Q}$) and

$$\mu_h \in X_*(T)$$

defines the Hodge filtration. One has $w_h = \mu_h^{1+c}$. Let's note

$$\mu_h = \sum_{\tau : E \hookrightarrow \mathbb{C}} a_\tau[\tau]$$

By the preceding

$$\exists w \in \mathbb{Z} \;\; \forall \tau \; a_\tau + a_{c\tau} = w$$

where $w_h = w.\sum_{\tau:E\hookrightarrow\mathbb{C}}[\tau] \in X_*(T)$.

Example 5.1. — The case when $\forall \tau \; a_\tau \in \{0,1\}$ corresponds to the case of abelian varieties with CM. Then to give oneself μ_h is equivalent to give oneself a CM type Φ, $\mu_h = \sum_{\tau\in\Phi}[\tau]$.

Let L be the field of definition of μ_h (the reflex field).

$$\mu_h : \mathbb{G}_{m\,L} \longrightarrow T_L$$

It gives rise to a diagram

$$\begin{tikzcd} \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \arrow[rr, "\mathrm{Res}_{L/\mathbb{Q}}\mu_h"] \arrow[dr, two heads] & & \mathrm{Res}_{L/\mathbb{Q}}T_L \arrow[r, "N_{L/\mathbb{Q}}", two heads] & T \\ & S^L \arrow[urr, "\varphi"'] & & \end{tikzcd}$$

where S^L is the quotient of $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m$ s.t.

$$X^*(S^L) = \{ \sum_{\tau : L \hookrightarrow \overline{\mathbb{Q}}} b_\tau[\tau] \mid \exists w \in \mathbb{Z} \, \forall \tau \; b_\tau + b_{c\tau} = w\} \subset X^*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m)$$

and the morphism $N_{L/\mathbb{Q}} \circ \mathrm{Res}_{L/\mathbb{Q}}(\mu_h)$ factorises in φ through the quotient S^L because of μ_h^{1+c} is defined over $\mathbb{Q}$. The morphism φ is a generalisation of the reflex norm defined in section 2.

Definition 5.2. — The group S^L is the Serre group attached to the CM field L.

It is canonically equipped with a cocharacter

$$\mu^L \in X_*(S^L)$$

defined by : μ_L is the composite of $[\tau_0] \in X_*(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m)$ where $\tau_0 : L \hookrightarrow \overline{\mathbb{Q}}$ is the canonical embedding (recall L is a reflex field and thus an embedded field) with the projection toward S^L. In another way

$$\forall \sum_\tau a_\tau[\tau] \in X^*(S^L) \quad <\mu^L, \sum_\tau a_\tau[\tau]>= a_{\tau_0}$$

from which it follows immediately that $(\mu^L)^{1+c}$ is defined over $\mathbb{Q}$. It thus defines a $h^L : \mathbb{S} \longrightarrow (S^L)_\mathbb{R}$, $\forall z \in \mathbb{C}^\times$ $h^L(z) = \mu^L(z)\overline{h^L(z)}$. Moreover

$$\varphi \circ \mu^L = \mu_h \quad \text{and} \quad \varphi \circ h^L = h$$

Reciprocally h^L plus the fact that the associated weight $(\mu^L)^{1+c}$ is defined over $\mathbb{Q}$ implies any representation of S^L gives rise to a CM Hodge structure whose Hodge filtration is given by the universal μ^L. This defines a tensor functor from $\mathrm{Rep}_\mathbb{Q} S^L$ to $\mathcal{C}$.

From the preceding study of simple CM Hodge structure one deduces the following lemma.

Lemma 5.3. — *There is an equivalence of Tannakian categories*

$$Rep_\mathbb{Q} S^L \xrightarrow{\sim} \mathcal{C}^L$$

the full $\otimes$-subcategory of $\mathcal{C}$ formed by CM Hodge structure whose reflex field in contained in L. The cocharacter $\mu^L \in X_*(S^L)$ is the universal Hodge cocharacter and via this equivalence the canonical fiber functor on $Rep_\mathbb{Q} S^L$ is the Betti fiber functor on $\mathcal{C}^L$.

Now one has

$$\mathcal{C} = \bigcup_{\substack{L|\mathbb{Q} \\ CM}} \mathcal{C}^L$$

Tannaka duality this corresponds to a projective system formed by norm morphisms

$$\forall L'|L \;\; S^{L'} \xrightarrow{N_{L'|L}} S^L$$

where norm morphism means they are the unique morphisms s.t. the following diagram commutes

$$\begin{tikzcd} \mathrm{Res}_{L'/\mathbb{Q}}\mathbb{G}_m \arrow[r, "N_{L'/L}"] \arrow[d] & \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \arrow[d] \\ S^{L'} \arrow[r] & S^L \end{tikzcd}$$

Now one defines

$$S = \varprojlim_{\substack{L|\mathbb{Q} \\ CM}} S^L$$

This is the general Serre-group, a pro-torus over $\mathbb{Q}$. And

$$\mathrm{Rep}_\mathbb{Q}(S) \xrightarrow[\otimes]{\sim} \mathcal{C}$$

And moreover one checks easily from this

$$\mathcal{C} =< H^1(A(\mathbb{C}), \mathbb{Q}) \mid A \text{ abelian variety with CM } /\mathbb{C} >^\otimes$$

(the Tannakian subcategory generated by). This follows from the fact that $X^*(S^L)$ is generated by the $\sum_\tau a_\tau[\tau]$ with $\forall \tau\ a_\tau \in \{0, 1\}$.

5.2. Shimura type description. — Consider the category whose objects are pairs (T,μ) where T is a torus over $\mathbb{Q}$ split by a CM field and $\mu \in X_*(T)$ is a cocharacter verifying that μ^{1+c} is defined over $\mathbb{Q}$ (T being split by a CM field μ^c is defined without ambiguity since complex conjugation acts the same way on $X_*(T)$ independently of the choice of any embedding into $\mathbb{C}$). The morphism of this category are the evident one. Pairs (T,μ) can be considered as some zero-dimensional Shimura data. Then

$$S = \lim_{\substack{\longleftarrow \\ (T,\mu)}} T$$

5.3. Serre's original automorphic description. — Serre's original definition ([**9**]) didn't make any reference to Hodge structures or motives. It is based on the following observation : if $\chi : \mathbf{A}_L^\times / L^\times \longrightarrow \mathbb{C}^\times$ is an algebraic character of weight ρ then there exists a congruence subgroup $\Gamma \subset \mathcal{O}_L^\times$ s.t.

$$\rho_{|\Gamma} = 1$$

where $\Gamma \subset \mathcal{O}_L^\times \subset L^\times = (\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m)(\mathbb{Q})$.

Remark 5.4. — Here a congruence subgroup means $\exists$ a modulus $\mathfrak{m}$ s.t. $\{x \in \mathcal{O}_L^\times \mid x \equiv 1 [\mathfrak{m}]\} \subset \Gamma$, or equivalently a compact open subgroup $K_f \subset \mathbf{A}_{L,f}^\times$ s.t. $L^\times \cap K_f \subset \Gamma$, or equivalently thanks to a theorem of Chevalley $[\mathcal{O}_L^\times : \Gamma] < +\infty$.

Thus ρ being algebraic it is trivial on the Zariski closure

$$\overline{\Gamma} \subset \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m$$

And thus ρ factorizes through

$$\rho \in X^*(\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m / \overline{\Gamma})$$

The algebraic group $\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m / \overline{\Gamma}$ is a torus s.t.

$$X^*(\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m / \overline{\Gamma}) = \{ \sum_{\tau : L \hookrightarrow \overline{\mathbb{Q}}} a_\tau [\tau] \mid \forall x \in \Gamma \ \prod_\tau \tau(x)^{a_\tau} = 1 \}$$

Now using Dirichlet's unit theorem one can check that for Γ a sufficiently small congruence subgroup this quotient does not depend on Γ and

$$(\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m) / \overline{\Gamma} = S^L$$

(this is equivalent to the proof of proposition 1.12). Those both definitions of S^L : Hodge theoretic/the "automorphic" one we just gave are the key point to make the link between algebraic Hecke characters and motives.

6. Serre's extension

Let $\mathfrak{m}$ be a modulus of the CM field L. Let $L^{\mathfrak{m}} | L$ the ray class field extension of modulus $\mathfrak{m}$. Let $\mathcal{O}_L^\times(\mathfrak{m})$ be the associated congruence subgroup equal to $L^\times \cap K(\mathfrak{m})$ where $K(\mathfrak{m}) \subset \mathbf{A}_{L,f}^\times$ is the compact open subgroup associated to $\mathfrak{m}$. By algebraizing the following extension

$$1 \longrightarrow L^\times / \mathcal{O}_L^\times(\mathfrak{m}) \longrightarrow \mathbf{A}_L^\times / (L_\infty^\times \times K(\mathfrak{m})) \longrightarrow \mathrm{Gal}(L^{\mathfrak{m}} | L) \longrightarrow 1$$

taking $\mathfrak{m}$ sufficiently big and the the definition of S^L as $S^L = \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m / \overline{\mathcal{O}^\times(\mathfrak{m})}$ Serre constructed in [**9**] an extension

$$1 \longrightarrow S^L \longrightarrow \mathcal{E}_{\mathfrak{m}}^L \longrightarrow \mathrm{Gal}(L^{\mathfrak{m}} | L) \longrightarrow 1$$

where $\mathcal{E}^L_{\mathfrak{m}}$ is abelian, that is to say a diagonalizable group whose neutral component is S^L, together with a morphism of extensions

$$\begin{tikzcd} 1 \arrow[r] & L^\times/\mathcal{O}_L^\times(\mathfrak{m}) \arrow[r] \arrow[d] & \mathbf{A}_L^\times/L_\infty^\times \times K(\mathfrak{m}) \arrow[r] \arrow[d] & \mathrm{Gal}(L^{\mathfrak{m}}|L) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & S^L(\mathbb{Q}) \arrow[r] & \mathcal{E}^L_{\mathfrak{m}}(\mathbb{Q}) \arrow[r] & \mathrm{Gal}(L^{\mathfrak{m}}|L) \arrow[r] & 1 \end{tikzcd}$$

This extension is constructed in the following way. Take $\sigma \mapsto a_\sigma$ a set theoretical section of $\mathbf{A}_L^\times/L_\infty^\times \times K(\mathfrak{m}) \twoheadrightarrow \mathrm{Gal}(L^{\mathfrak{m}}|L)$ and let $c_{\sigma,\tau} = a_\sigma a_\tau a_{\sigma\tau}^{-1}$ be the associated 2-cocyle with values in $L^\times/\mathcal{O}_L^\times(\mathfrak{m})$. Then as an algebraic variety over $\mathbb{Q}$

$$\mathcal{E}^L_{\mathfrak{m}} = \coprod_{\sigma\in\mathrm{Gal}(L^{\mathfrak{m}}|L)} S^L$$

and the group structure is defined via the cocyle $c_{\sigma,\tau}$. The addition morphism

$$\coprod_{\sigma\in\mathrm{Gal}(L^{\mathfrak{m}}|L)} S^L \times \coprod_{\sigma\in\mathrm{Gal}(L^{\mathfrak{m}}|L)} S^L = \coprod_{\sigma,\tau}(S^L\times S^L) \longrightarrow \coprod_{\sigma} S^L$$

being defined on the component index by (σ,τ) on the source by

$$\begin{array}{ccc} S^L\times S^L & \longrightarrow & S^L \xrightarrow{\text{component } \sigma\tau} \coprod_\nu S^L \\ (x,y) & \longmapsto & xyf(c_{\sigma,\tau}) \end{array}$$

where $f : L^\times/\mathcal{O}_L^\times(\mathfrak{m}) \longrightarrow S^L(\mathbb{Q})$. This Serre extension equipped with the preceding morphism of extensions at the level of $\mathbb{Q}$-points satisfies a universal property and is thus essentially unique. When $\mathfrak{m}$ varies this form a compatible system.

Taking the projective limit over all $\mathfrak{m}$ one obtains an extension

$$1 \longrightarrow S^L \longrightarrow \mathcal{E}^L \longrightarrow \mathrm{Gal}(L^{ab}|L) \longrightarrow 1$$

where $\mathcal{E}^L$ is pro-diagonalizable.

Moreover this extension is equipped with a continuous adelic section

$$\begin{tikzcd} \mathcal{E}^L(\mathbf{A}_{\mathbb{Q},f}) \arrow[r, two heads] & \mathrm{Gal}(L^{ab}|L) \arrow[l, bend left, "(s_\ell)_\ell"] \end{tikzcd}$$

and a “Weil-section” : a continuous section

$$\begin{tikzcd} \mathcal{E}^L(\mathbb{R}) \arrow[r, two heads] & \mathrm{Gal}(L^{ab}|L) \\ & \mathbf{A}_L^\times/L^\times = W_L^{ab} \arrow[ul, "s_\infty"] \arrow[u, two heads] \end{tikzcd}$$

More precisely for the section $s_\ell : \mathrm{Gal}(L^{ab}|L) \longrightarrow \mathcal{E}^L(\mathbb{Q}_\ell)$ one uses the same trick as we did for the construction of the ℓ-adic system associated to an algebraic Hecke character. Let $\epsilon_\ell : \mathbf{A}_L^\times \xrightarrow{\text{proj.}} (L\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times = \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m(\mathbb{Q}_\ell) \longrightarrow S^L(\mathbb{Q}_\ell)$. Note $\alpha : \mathbf{A}_L^\times \longrightarrow \mathcal{E}^L(\mathbb{Q})$ (the morphism factorizing through $L_\infty \times K(\mathfrak{m})$ for a modulus $\mathfrak{m}$ when we were working at a fixed finite modulus). Then

$$s_\ell = \alpha\epsilon_\ell^{-1} : \mathbf{A}_L^\times/L^\times \longrightarrow \mathcal{E}^L(\mathbb{Q}_\ell)$$

and being continuous it factorizes through $\pi_0(\mathbf{A}_L^\times/L^\times) = \mathrm{Gal}(L^{ab}|L)$. This is our section.

Remark 6.1. — There is a quick way to construct the extension $\mathcal{E}^L$ together with its ℓ-adic sections by applying proposition 9.3 and the last point in proposition 9.5. See the end of section 9.9.

For the section s_∞ let $\epsilon_\infty : \mathbf{A}_L^\times \twoheadrightarrow L_\infty^\times = \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m)(\mathbb{R}) \longrightarrow S^L(\mathbb{R})$ and we set

$$s_\infty = \alpha\epsilon_\infty^{-1} : \mathbf{A}_L^\times/L^\times \longrightarrow \mathcal{E}^L(\mathbb{R})$$

Now one has the following theorem.

Theorem 6.2. — *If $\rho : \mathcal{E}^L_{/E} \longrightarrow \mathbb{G}_{mE}$ is a one dimensional representation (any irreducible representation $\mathcal{E}^L_{/\overline{\mathbb{Q}}} \longrightarrow \mathbb{G}_{m\overline{\mathbb{Q}}}$ is defined over a number field E) and if we still note $\rho : \mathcal{E}^L \longrightarrow Res_{E/\mathbb{Q}}\mathbb{G}_m$ the associated morphism then*

$$\chi = \rho \circ s_\infty$$

is an algebraic Hecke character of $\mathbf{A}_L^\times/L^\times$ of weight $\rho_{|S^L}$ s.t. its associated ℓ-adic compatible system is

$$(\rho \circ s_\ell)_\ell$$

This correspondence defines a bijection between algebraic Hecke characters χ of $\mathbf{A}_L^\times/L^\times$ s.t. $Im(\chi_f) \subset E^\times$ and representations $\mathcal{E}^L_{/E} \longrightarrow \mathbb{G}_{m/E}$.

Proof. In one direction, if $\rho : \mathcal{E}^L \longrightarrow \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ it factorizes through a finite type quotient and is thus given by a $\mathcal{E}^L_{\mathfrak{m}} \longrightarrow \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ for some modulus $\mathfrak{m}$. Then it is immediately checked with the preceding formulas that $\rho \circ s_\infty$ is algebraic and the associated compatible system is $(\rho \circ s_\ell)_\ell$.

In the other direction, if $\chi : \mathbf{A}_L^\times/L^\times \longrightarrow \mathbb{C}^\times$ is algebraic with weight ρ then $\rho \in X^*(L^\times)$ has to factorize through Serre's torus by section 5.3, $\rho \in X^*(S^L)$. Let E be a number field and $\mathfrak{m}$ a modulus such that $\chi_f : \mathbf{A}_{L,f}^\times/K(\mathfrak{m}) \longrightarrow E^\times$. We take the cocyle definition of $\mathcal{E}^L_{\mathfrak{m}} = \coprod_{\sigma \in \mathrm{Gal}(L^{\mathfrak{m}}|L)} S^L$. Define the representation $\mathcal{E}^L_{\mathfrak{m}/E} \longrightarrow \mathbb{G}_{m/E}$ on the factor indexed by σ as

$$\begin{array}{ccc} S^L_{/E} & \longrightarrow & \mathbb{G}_{m/E} \\ x & \longmapsto & \chi_f(a_\sigma)\rho(x) \end{array}$$

Then one checks immediately this gives us inverse correspondences. □

Example 6.3. — In the case of CM abelian varieties as in section 2 the morphism $\rho_{|S^L}$ is the reflex norm $N_{\Phi'}$.

7. The Taniyama group

7.1. The group and its properties. — Here we only describe vaguely the results.

***Theorem 7.1* (Deligne-Langlands).** — *For $L|\mathbb{Q}$ CM Galois one can construct explicitly an extension*

$$1 \longrightarrow S^L \longrightarrow \mathcal{T}^L \longrightarrow Gal(L^{ab}|\mathbb{Q}) \longrightarrow 1$$

equipped with continuous sections

$$\begin{tikzcd} \mathcal{T}^L(\mathbf{A}_{\mathbb{Q},f}) \arrow[r] & Gal(L^{ab}|\mathbb{Q}) \arrow[l, bend left, "(s_\ell)_\ell"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T}^L(\mathbb{C}) \arrow[r] & Gal(L^{ab}|\mathbb{Q}) \\ & W_{\mathbb{Q}} \arrow[u, two heads] \arrow[ul, "s_\infty"] \end{tikzcd}$$

with compatible transition morphisms for L varying, for $L'|L$ $\mathcal{T}^{L'} \longrightarrow \mathcal{T}^L$ inducing the norm maps on the Serre groups, s.t. if

$$\mathcal{T} = \varprojlim_{L} \mathcal{T}^L$$

then $Rep_{\mathbb{Q}}(\mathcal{T})$ is equivalent to the Tannakian category of CM motives over $\mathbb{Q}$ defined in terms of Hodge cycles (here CM means whose Mumford-Tate group is a torus, or equivalently "potentially abelian", this is defined concretely as the Tannakian category generated by the $h^1(A)$ with A a potentially CM abelian variety and Artin motives). In this equivalence the canonical fiber functor on $Rep_{\mathbb{Q}}\mathcal{T}$ corresponds to the Betti cohomology fiber functor.

If the representation ρ corresponds to the motive M then

$$\rho \circ (s_\ell)_\ell \longleftrightarrow (H^\bullet_{\acute{e}t}(M, \mathbb{Q}_\ell))_\ell$$

the ℓ-adic compatible system associated through étale ℓ-adic cohomology. If $\rho \circ s_\infty$ is the associated continuous finite dimensional representation of the global Weil group $W_{\mathbb{Q}}$ then

$$L(s, M) = L(s, \rho \circ s_\infty)$$

(archimedean factors included). Moreover Serre's extension is the pullback

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{S}^L \arrow[r] \arrow[d, equal] & \mathcal{E}^L \arrow[r] \arrow[d] & Gal(L^{ab}|L) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & S^L \arrow[r] & \mathcal{T}^L \arrow[r] & Gal(L^{ab}|\mathbb{Q}) \arrow[r] & 1 \end{tikzcd}$$

and $Rep_{\mathbb{Q}}\mathcal{E}^L$ is the category of CM motives over L such that the action of $Gal(\overline{\mathbb{Q}}|L)$ on their étale ℓ-adic cohomology is abelian.

Remark 7.2. — The Taniyama group is not abelian as was the non-connected Serre group $\mathcal{E}^L$. In particular $\mathrm{Gal}(L^{ab}|\mathbb{Q})$ acts non-trivially on S^L in the Taniyama extension (but this action factorizes through $\mathrm{Gal}(L|\mathbb{Q})$ since Serre's extension is abelian). This action of $\mathrm{Gal}(L|\mathbb{Q})$ has to be understood in the way $\mathrm{Gal}(L|\mathbb{Q})$ permutes CM-Hodge structures whose relfex field is contained in L. For example if (A, ι) is a simple CM abelian variety with CM type (E, Φ) over $\overline{\mathbb{Q}}$ whose reflex field is contained in L then $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ (A^σ, ι^σ) is associated to $(E, \sigma\Phi)$. Thus $\mathrm{Gal}(L|\mathbb{Q})$ acts on isogeny classes of simple CM abelian varieties whose relfex field is contained in L via $\Phi \longmapsto \sigma\Phi$. This reflects this algebraic action of $\mathrm{Gal}(L|\mathbb{Q})$ on S^L. More precisely if $\mu^L \in X_*(S^L)$ is the universal Hodge cocharacter, $\forall \sigma \in \mathrm{Gal}(L|\mathbb{Q})$ $(\mu^L)^\sigma$ has its weight defined over $\mathbb{Q}$ and thus by the universal property of (S^L, μ^L) there is a unique automorphism of S^L such that trough it $\mu^L \longmapsto (\mu^L)^\sigma$. This reflects the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on the category of CM $\mathbb{Q}$-Hodge structures by permuting the Hodge filtration.

The proof of the preceding theorem is divided in two parts :

- Langlands constructed in [**7**] a candidate by constructing explicitly a cocycle defining the preceding extension together with its adelic and Weil sections. The idea is to "algebrize" the extension of global Weil group

$$1 \longrightarrow \mathbf{A}_L^\times / L^\times \longrightarrow W_{L/\mathbb{Q}} \longrightarrow Gal(L|\mathbb{Q}) \longrightarrow 1$$

  given by the fundamental class of class-field theory $u_{L/\mathbb{Q}} \in H^2(L|\mathbb{Q}, \mathbf{A}_L^\times/L^\times)$ ore rather to use the more subtle extension

$$1 \longrightarrow (\mathbf{A}_L^\times / L^\times)^0 \longrightarrow W_{L/\mathbb{Q}} \longrightarrow \mathrm{Gal}(L^{ab}|\mathbb{Q}) \longrightarrow 1$$

- Deligne checked in article IV of [**4**] Langlands extension has to be the one given by the theory of Tannakian categories and his theory of Hodge cycles. This is not an explicit computation, the proof uses all the "rigidity" properties of the extension equipped with its sections and the properties of the Serre extension we defined in the preceding section.

7.2. Automorphic consequences. —

Corollary 7.3. — *For each number field L and an algebraic Hecke character χ of $\mathbf{A}_L^\times/L^\times$ there exists a rank one CM motive $M(\chi)$ with $\overline{\mathbb{Q}}$-coefficients such that $L(s, M(\chi)) = L(s,\chi)$. The λ-adic compatible system associated to χ like in section 1 is the one given by the system of realizations of $M(\chi)$ on its étale cohomology. The weight of χ corresponds to the CM-Hodge structure associated to $M(\chi)$.*

In fact one can prove more

Proposition 7.4. — *The correspondence $\chi \mapsto M(\chi)$ defines a bijection between algebraic Hecke characters of $\mathbf{A}_L^\times/L^\times$ and isomorphism classes of rank one CM motives with $\overline{\mathbb{Q}}$-coefficients over L.*

Remark 7.5. — Any abelian variety A over L having complex multiplication over L by E defines an algebraic Hecke character χ (section 2) and $M(\chi) \simeq h^1(A)$ (after a choice of embedding of E in $\mathbb{C}$). There is an evident restriction on the algebraic Hecke characters induced by such A, the one on the weight. But having done this restriction the author does not know how to distinguish the algebraic χ s.t. $M(\chi)$ is of the form $h^1(A)$ and the others.

Corollary 7.6. — *Let M be a CM motive over $\mathbb{Q}$ with $\overline{\mathbb{Q}}$-coefficients, in the sens of Hodge cycles. Then $\exists \rho$ a finite dimensional representation of the global Weil group $W_{\mathbb{Q}}$ s.t.*

$$L(s, M) = L(s,\rho)$$

Now one conjectures

Conjecture 7.7. — *For each irreducible representation $\rho : W_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ there exists a cuspidal automorphic representation Π of $GL_n(\mathbf{A}_{\mathbb{Q}})$ s.t.*

$$L(s,\rho) = L(s,\Pi)$$

This conjecture would imply that any CM motive is "automorphic". The problem is that we don't know the Base Change and Automorphic Induction functoriality of Langlands (it is known only for solvable extensions [**1**]). It contains as a particular case Artin's conjecture (Artin motives are CM). But we know the preceding conjecture at least "virtually" as it is the case for Artin conjecture.

Proposition 7.8. — *There is a version of Brauer's theorem for representations of the Weil group*

$$Groth(Rep_{\mathbb{C}} W_{\mathbb{Q}}) = < Ind_{W_{\mathbb{Q}}}^{W_E} \chi >$$

where E goes through finite extensions of $\mathbb{Q}$ and χ through Hecke characters of $W_{E/E} = \mathbf{A}_E^\times/E^\times$.

Corollary 7.9. — *Let M be a CM motive over $\mathbb{Q}$ with $\overline{\mathbb{Q}}$-coefficients. Then M is "virtually automorphic" : there exists a finite set I and collections for $i \in I$ $E_i|\mathbb{Q}, a_i \in \mathbb{Z}, \chi_i : \mathbf{A}_{E_i}^\times/E_i^\times \longrightarrow \mathbb{C}^\times$ s.t.*

$$L(s, M) = \prod_{i\in I} L(s,\chi_i)^{a_i}$$

in particular $L(s, M)$ has meromorphic continuation to the complex plane and satisfies a functional equation.

Remark 7.10. — The ϵ-factors appearing in the functional equation admit a decomposition as a product of local ones at all places that do not depend on a decomposition like in proposition 7.8 but only on the local Galois ℓ-representations associated to M at all finite places, and the associated $\mathbb{R}$-Hodge structure equipped with its action of Gal($\mathbb{C}|\mathbb{R}$) for the infinite place ([**3**], [**10**]).

Now let's dig a little bit more into the link between the Weil and Taniyama groups.

Definition 7.11. — Let $\rho : W_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a continuous representation. We say it is algebraic if via $\mathbb{C}^\times \hookrightarrow W_{\mathbb{R}} \longrightarrow W_{\mathbb{Q}}$ the associated representation of $\mathbb{C}^\times$ seen as Deligne's torus $\mathbb{S}$ is an algebraic representation that is to say a direct sum of copies of characters of the form $z^p\overline{z}^q$ with $p, q \in \mathbb{Z}$.

Any algebraic representation of $W_{\mathbb{Q}}$ is semi-simple. In fact if $W^1_{\mathbb{Q}} = \ker(W_{\mathbb{Q}} \longrightarrow W^{ab}_{\mathbb{Q}} = \mathbf{A}^\times_{\mathbb{Q}}/\mathbb{Q}^\times \xrightarrow{\|\cdot\|} \mathbb{R}^\times_+)$ then $W^1_{\mathbb{Q}}$ is compact, $W_{\mathbb{Q}}/W^1_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{R}^\times_+$ and the composite $\mathbb{C}^\times \hookrightarrow W_{\mathbb{R}} \longrightarrow W_{\mathbb{Q}} \longrightarrow \mathbb{R}^\times_+$ is $z \mapsto z\overline{z}$.

Now we have the following whose proof is left as an exercise to the reader.

Proposition 7.12. — *The correspondence $Rep_{\mathbb{C}}(\mathcal{T}) \ni \rho \longmapsto \rho \circ s_\infty$ defines a Tannakian category equivalence between CM motives over $\mathbb{Q}$ with coefficients in $\mathbb{C}$ and algebraic representations of $W_{\mathbb{Q}}$. For $L|\mathbb{Q}$ CM and Galois it restricts to an equivalence between CM motives over $\mathbb{Q}$ such that the Galois action on their étale cohomology becomes abelian over L and algebraic representations of $W_{L/\mathbb{Q}}$. If ρ is an algebraic representation of $W_{\mathbb{Q}}$ associated to the motive M over $\mathbb{Q}$ then the Hodge-structure equipped with its action of Gal$(\mathbb{C}|\mathbb{R})$ associated to M is the one attached to the composite representation $W_{\mathbb{R}} \longrightarrow W_{\mathbb{Q}}$ with ρ, the local factor at a prime p of the action of Galois on the étale cohomology of M is deduced from the composite $W_{\mathbb{Q}_p} \longrightarrow W_{\mathbb{Q}}$ with ρ*

In the preceding proposition the category of CM motives with coefficients in $\mathbb{C}$ is equivalent to the category whose objects are motives with coefficients in $\overline{\mathbb{Q}}$ and we tensor homomorphisms by $\mathbb{C}$. In fact if G is a pro-reductive group-scheme over $\overline{\mathbb{Q}}$ then the category $\mathrm{Rep}_{\mathbb{C}}(G_{\mathbb{C}})$ is equivalent to the one whose objects are those of $\mathrm{Rep}_{\overline{\mathbb{Q}}}(G)$ and we tensor homomorphisms by $- \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

8. Fontaine-Mazur conjecture in the abelian case

This is the following theorem that is a reciprocal to the statements of section 1.2

Theorem 8.1. — *Let E be a number field and $\lambda|\ell$ a place of E. Let $(\rho_\lambda)_\lambda$ be a λ-adic semi-simple representation of Gal$(L^{ab}|L)$ such that*

- *It is unramified at almost all places and at this places the characteristic polynomial of Frobenius is E-rational*
- *It is potentially cristalline at all places of L dividing ℓ*

Then $(\rho_\lambda)_\lambda$ is the λ-adic system attached to an algebraic Hecke character, thus to a representation of the non-connected Serre group $\mathcal{E}^L$ and thus this is the system of realizations of a CM motives over L.

This theorem is proved in [**9**] when L is the composite of quadratic extensions of $\mathbb{Q}$ (a transcendence result was laking for the general case). For the general case one needs a transcendence result proved by Waldschmidt, it is given by Henniart in [**6**].

In the book [**9**] what we call potentially cristalline is called locally algebraic. From the modern point of view of Fontaine's theory potentially cristalline is the good terminology.

9. Conjugation of CM motives and thorough study of the Taniyama group

9.1. Notations. — We use the theory of Hodge cycles of Deligne (see article I by Deligne in [**4**]). With it one can construct a Tannakian category of Motives generated by motives of abelian varieties and Artin motives over a subfield of $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}} \subset \mathbb{C}$. On this category of motives Hodge conjecture is forced to be true : the Betti fiber functor toward polarizable $\mathbb{Q}$-Hodge structure is fully faithful. Moreover we will use Deligne's theorem that any Hodge cycle is absolutely Hodge to study those categories.

Definition 9.1. — We note $\mathrm{CM}_{\mathbb{Q}}$ resp. $\mathrm{CM}_{\overline{\mathbb{Q}}}$, the Tannakian category of CM motives over $\mathbb{Q}$, resp. $\overline{\mathbb{Q}}$, with $\mathbb{Q}$-coefficients. By definition $\mathrm{CM}_{\mathbb{Q}}$ is the category generated by (potentially) CM abelian varieties over $\mathbb{Q}$ and Artin motives over $\mathbb{Q}$. For $L \subset \overline{\mathbb{Q}}$, $L|\mathbb{Q}$ a Galois CM field we note $\mathrm{CM}^L_{\overline{\mathbb{Q}}}$, resp. $\mathrm{CM}^L_{\mathbb{Q}}$ for the sub-category of $\mathrm{CM}_{\overline{\mathbb{Q}}}$, resp. $\mathrm{CM}_{\mathbb{Q}}$, formed by motives whose reflex field is contained in L that is to say its Mumford-Tate group is split by L. The category $\mathrm{CM}^L_{\mathbb{Q}}$ is generated by CM abelian varieties over $\mathbb{Q}$ whose relfex field is contained in L and Artin motives.

There are equivalences between Tannakian categories

$$\mathrm{CM}^L_{\overline{\mathbb{Q}}} \xrightarrow[\sim]{\text{Betti}} \text{CM } \mathbb{Q}\text{-Hodge structures reflex field} \subset L \xleftarrow{\sim} \mathrm{Rep}_{\mathbb{Q}} S^L$$

where the right equivalence is induced by $h^L \in \mathrm{Hom}(\mathbb{S}, S^L_{/\mathbb{R}})$ (see section 5.1) and through which the Betti fiber functor $\omega_B : \mathrm{CM}^L_{\overline{\mathbb{Q}}} \longrightarrow \mathbb{Q}$-v.s. corresponds to the canonical one on $\mathrm{Rep}_{\mathbb{Q}} S^L$. Thus $S^L \simeq \underline{\mathrm{Aut}}^{\otimes}\omega_B$. The Hodge filtration defines a $\otimes$-functor

$$\mathrm{CM}^L_{\overline{\mathbb{Q}}} \simeq \mathrm{Rep}_{\mathbb{Q}} S^L \longrightarrow \text{Filtered } \mathbb{C}\text{-vector spaces}$$

This is given by $\mu^L \in X_*(S^L)$ (see section 5.1).

We have the same for $S = \varprojlim_L S^L$ and $\mathrm{CM}_{\overline{\mathbb{Q}}} = \bigcup_L \mathrm{CM}^L_{\overline{\mathbb{Q}}}$.

9.2. Algebraic action on Serre group. — As explained in remark 7.2 there is an "algebraic action" $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(L|\mathbb{Q}) \longrightarrow \mathrm{Aut}(S^L)$ denoted $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\ \ S^L \ni x \longmapsto x^\sigma$. From now we note the "arithmetic actions", for example the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on $S^L(\overline{\mathbb{Q}})$ or on $X^*(S^L)$, by $x \longmapsto \sigma.x$. Both actions, the algebraic and the arithmetic one, commute but one has to be careful with them.

Recall $X^*(S^L) = \{\sum_{\tau : L \hookrightarrow \overline{\mathbb{Q}}} a_\tau[\tau] \mid \exists w \in \mathbb{Z}\ \forall \tau\ a_\tau + a_{c\tau} = w\}$. Then $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ the algebraic action of σ is given by $X^*\left((-)^\sigma\right)\left(\sum_\tau a_\tau[\tau]\right) = \sum_\tau a_\tau[\tau\sigma]$.

One has $(-)^\sigma \circ \mu^L = \sigma^{-1}.\mu^L$ and the following diagram commutes

$$\begin{tikzcd}
(V,\rho) \arrow[d, mapsto] & \mathrm{Rep}_{\mathbb{Q}}(S^L) \arrow[r, "\sim"] \arrow[d] & \mathrm{CM}^L_{\overline{\mathbb{Q}}} \arrow[r, "\text{Hodge Fil.}"] \arrow[d] & \text{Filtered } \mathbb{C}\text{-v.s.} \arrow[d] & (V,\mathrm{Fil}) \arrow[d, mapsto] \\
(V, \rho\circ(-)^\sigma) & \mathrm{Rep}_{\mathbb{Q}}(S^L) \arrow[r, "\sim"] & \mathrm{CM}^L_{\overline{\mathbb{Q}}} \arrow[r, "\text{Hodge Fil.}"] & \text{Filtered } \mathbb{C}\text{-v.s.} & (V^\sigma, \mathrm{Fil}^\sigma)
\end{tikzcd}$$

9.3. The extension. — Let M, resp. M^L, be the group of $\otimes$-automorphisms of the Betti fiber functor on $\mathrm{CM}_{\mathbb{Q}}$, resp. $\mathrm{CM}^L_{\mathbb{Q}}$. There is a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & S^L \arrow[r] & M^L \arrow[r] & \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[r] & 1
\end{tikzcd}$$

and the lower one is the push-forward of the upper one by $S \twoheadrightarrow S^L$. Tannaka dually this sequences correspond to the following type of sequences

$$\text{Artin motives} \longrightarrow \text{Motives over } \mathbb{Q} \xrightarrow{\text{Scalar extension}} \text{Motives over } \overline{\mathbb{Q}}$$

The only difficulty to prove the preceding type of sequence is exact is exactness in the middle. For this one has to use Deligne's theorem that any Hodge cycle is absolutely Hodge (see proposition 6.23 of article II in [**4**]).

Since S^L is abelian the preceding extension gives rise to an action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on S^L. This is the preceding algebraic action (see next section for a quick justification).

9.4. The effect of conjugation on Betti cohomology. — Let $\omega_B : \mathrm{CM}^L_{\overline{\mathbb{Q}}} \longrightarrow \mathbb{Q}$-v.s. be the Betti fiber functor.

For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ there is a $\otimes$-equivalence $M \longmapsto M^\sigma$ from $\mathrm{CM}^L_{\overline{\mathbb{Q}}}$ to itself. This follows from Deligne's theorem that any Hodge cycles is absolutely Hodge.

We let $\omega_{B,\sigma}$ be the composite fiber functor $\omega_{B,\sigma}(M) = \omega_B(M^\sigma)$.

Both fiber functor ω_B and $\omega_{B,\sigma}$ become isomorphic over $\overline{\mathbb{Q}}$, either because of the general theory of Tannakian categories or because

$$\omega_B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq \omega_{dR}, \;\; \omega_{B,\sigma} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq \omega_{dR} \otimes_{\overline{\mathbb{Q}},\sigma} \overline{\mathbb{Q}} \text{ and } \omega_{dR} \simeq \omega_{dR} \otimes_{\overline{\mathbb{Q}},\sigma} \overline{\mathbb{Q}}$$

where ω_{dR} is the de Rham fiber functor (this equalities allows one to verify easily the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on S^L induced by the preceding extension induces the algebraic one given in section 9.2).

The difference between ω_B and $\omega_{B,\sigma}$ is measured by the $\underline{\mathrm{Aut}}^\otimes(\omega_B) = S^L$-torsor $\underline{\mathrm{Isom}}^\otimes(\omega, \omega_{B,\sigma})$ whose isomorphism class lies in $H^1(\overline{\mathbb{Q}}|\mathbb{Q}, S^L)$ (Galois cohomology for the arithmetic action). In the exact sequence

$$1 \longrightarrow S^L \longrightarrow M^L \xrightarrow{\pi} \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

this torsor is the S^L-torsor $\pi^{-1}(\sigma)$.

Since S^L splits over L this class lies in fact in $H^1(L|\mathbb{Q}, S^L)$ and thus there is an isomorphism

$$\omega_B \otimes_{\mathbb{Q}} L \simeq \omega_{B,\sigma} \otimes_{\mathbb{Q}} L$$

well defined up to an element of $\underline{\mathrm{Aut}}^\otimes(\omega_B \otimes L) = S^L(L)$.

Concretely since S^L splits over L by Hilbert 90 the following sequence is exact

$$1 \longrightarrow S^L(L) \longrightarrow M^L(L) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$$

Now for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ let $a_\sigma \in M^L(L)$ be a lift of σ. Then $\forall \tau \in \mathrm{Gal}(L|\mathbb{Q})$ $\tau.a_\sigma$ still lifts σ (the Galois action on the pro-constant group scheme $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ is trivial) and thus $c_\tau = (\tau.a_\sigma)a_\sigma^{-1} \in S^L(L)$ and

$$(c_\tau)_\tau \in Z^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(L))$$

Then $\omega_{B,\sigma}$ is obtained from ω_B by twisting by the cocyle $(c_\tau)_\tau$. Concretely this means

$$\omega_{B,\sigma}(X) = (\omega_B(X) \otimes_{\mathbb{Q}} L)^{\mathrm{Gal}(L|\mathbb{Q})}$$

where $\tau \in \mathrm{Gal}(L|\mathbb{Q})$ acts on $\omega_B(X) \otimes_{\mathbb{Q}} L$ via $c_\tau \otimes \tau$ ($S^L(L)$ acts naturally on $\omega_B(X) \otimes_{\mathbb{Q}} L$ and $c_\tau \in S^L(L)$).

9.5. Étale cohomology. —

9.5.1. The ℓ-adic section. — The étale ℓ-adic cohomology fiber functor for all ℓ is noted

$$\omega_{\text{ét}} : \mathrm{CM}^L_{\overline{\mathbb{Q}}} \longrightarrow \mathbf{A}_{\mathbb{Q},f}\text{-free mod. of finite rank}$$

There is a canonical isomorphism $\omega_B \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f} \xrightarrow{\sim} \omega_{\text{ét}}$. Moreover $\omega_{\text{ét}}$ factorizes through the $\otimes$-category of continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on free $\mathbf{A}_{\mathbb{Q},f}$-modules of finite rank. From this one deduces the existence of a continuous section s^L

$$\begin{tikzcd} M^L(\mathbf{A}_{\mathbb{Q},f}) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[l, bend right, "s^L"'] \end{tikzcd}$$

such that if $X \in \mathrm{CM}^L_{\overline{\mathbb{Q}}}$ via the action of $M^L(\mathbf{A}_{\mathbb{Q},f}) = \mathrm{Aut}(\omega_B \otimes \mathbf{A}_{\mathbb{Q},f}) \simeq \mathrm{Aut}(\omega_{\text{ét}})$ on $\omega_B(X) \otimes \mathbf{A}_{\mathbb{Q},f}$ the one of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ given by s^L is the Galois action on étale cohomology.

9.5.2. Conséquence of the invariance of étale cohomologie under conjugation. — Étale cohomology is invariant under change of algebraically closed field and thus if $\omega_{\text{ét},\sigma}$ is the fiber functor $M \longmapsto \omega_{\text{ét}}(M^\sigma)$ on $\mathrm{CM}^L_{\mathbb{Q}}$ there is a canonical isomorphism $\omega_{\text{ét}} \xrightarrow{\sim} \omega_{\text{ét},\sigma}$ (the isomorphism noted $\sigma^* : H^1_{\text{ét}}(A, \mathbf{A}_{\mathbb{Q},f}) \xrightarrow{\sim} H^1_{\text{ét}}(A^\sigma, \mathbf{A}_{\mathbb{Q},f})$ in section 3). This induces a canonical isomorphism

$$\omega_B \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f} \xrightarrow{\sim} \omega_{B,\sigma} \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f}$$

and thus

$$[\underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma})] \in \ker\left(H^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(L)) \longrightarrow H^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(\mathbf{A}_{L,f}))\right)$$

or

$$[\underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma})] \in \mathrm{Im}\left(\left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})} \longrightarrow H^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(L))\right)$$

In fact one can be more precise since the isomorphism $\omega_B \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f} \xrightarrow{\sim} \omega_{B,\sigma} \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f}$ is canonical, which means not defined up to an element of $S^L(\mathbf{A}_{\mathbb{Q},f})$. In other words the cocycle $(c_\tau)_\tau$ constructed in section 9.4 is canonically trivialized in $S^L(\mathbf{A}_{L,f})$: there is a uniquely defined $g \in S^L(\mathbf{A}_{L,f})$ s.t. $c_\tau = gg^{-\tau}$. This implies there is a canonical lift $\bar{b}^L(\sigma)$ of $[\underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma})]$

$$\bar{b}^L(\sigma) \in \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$$

(arithmetic action of $\mathrm{Gal}(L|\mathbb{Q})$). Now at the level of cocyles, taking the notations at the end of section 9.4, one has

$$\bar{b}^L(\sigma) = a_\sigma s^L(\sigma)^{-1} \bmod S^L(L)$$

Moreover if $L'|L$ one checks $\bar{b}^{L'}(\sigma) \mapsto \bar{b}^L(\sigma)$ via $S^{L'} \to S^L$.

9.5.3. When σ varies. — Let $\sigma_1, \sigma_2 \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. The composite of the canonical isomorphism of fiber functors on $\mathrm{CM}^L_{\mathbb{Q}}$ $\sigma_2^* : \omega_{\text{ét}} \xrightarrow{\sim} \omega_{\text{ét},\sigma_2}$ and $\sigma_1^* : \omega_{\text{ét},\sigma_2} \xrightarrow{\sim} \omega_{\text{ét},\sigma_1\sigma_2}$ induces the canonical one $(\sigma_1\sigma_2)^* : \omega_{\text{ét}} \xrightarrow{\sim} \omega_{\sigma_1\sigma_2}$. Moreover there is a torsor isomorphism

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma_2}) \overset{S^L}{\wedge} \underline{\mathrm{Isom}}^{\otimes}(\omega_{B,\sigma_2}, \omega_{B,\sigma_1\sigma_2}) \simeq \underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma_1\sigma_2})$$

This is resumed in the following formula

$$\forall \sigma_1, \sigma_2 \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \;\; \bar{b}(\sigma_1\sigma_2) = \bar{b}(\sigma_1)\bar{b}(\sigma_2)^{\sigma_1} \in \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$$

where here $(-)^{\sigma_1}$ means the algebraic action. More concretely this formula is checked directly on the formula $\bar{b}^L(\sigma) \equiv a_\sigma s^L(\sigma)^{-1}$.

9.6. The effect of conjugation on Betti cohomology with coefficients in $\mathbb{R}$. — .
Here we diverge from Deligne's approach (lemma 3 in article IV of [**4**]).

9.6.1. Preliminaries on polarized Hodge structures. — The category $\mathrm{Hodge}_{\mathbb{R}}$ of $\mathbb{R}$-Hodge structures is naturally a polarized Tate triple (see sections 4 and 5 in the article II by Deligne and Milne in [**4**]).

Let $\mathbb{V}$ be the non-neutral $\mathbb{R}$-linear Tannakian category formed by graded $\mathbb{C}$-vector spaces $V = \bigoplus_{w\in\mathbb{Z}} V^\omega$ equipped with a semi-linear automorphism a s.t. $a^2 = (-1)^w$ on V^w (see example 5.3 in article II of [**4**]). The associated band is the group $\mathbb{G}_m$ over $\mathbb{R}$ that gives the grading and the isomorphism class of $\mathbb{V}$ is the class of its gerb of fiber functors in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{G}_m) = \{\pm 1\}$ which is the non-trivial class. The category $\mathbb{V}$ is naturally a Tate triple, the polarizations of a homogeneous (V, a) of weight w being the $(-1)^w$-symmetric positive definite forms on V.

Now if $(\mathbb{T}, w, T)$ is a Tate triple such that $w(-1) \neq 1$ by theorem 5.20 of II in [**4**] there is up to isomorphism a unique exact faithful functor $\mathbb{V} \longrightarrow \mathbb{T}$ compatible with the Tate triples structures and sending the natural polarization of $\mathbb{V}$ to the one of $\mathbb{T}$. In particular there is such a functor $\mathrm{Hodge}_{\mathbb{R}} \longrightarrow \mathbb{V}$. If V is an $\mathbb{R}$-Hodge structure on associates $(V_{\mathbb{C}}, a) \in \mathbb{V}$ where the grading is given by the weight of the Hodge structure and a is $(-1)^p$-times complex conjugation on $V^{p,q}$. That means to (V, h) one associates $(V_{\mathbb{C}}, \mu_h(-1) \otimes c)$.

If T is a torus over $\mathbb{R}$ equipped with $h : \mathbb{S} \longrightarrow T$ there is associated a morphism

$$\begin{array}{rcl} \xi_{\mu_h} : \mathrm{Rep}_{\mathbb{R}} T & \longrightarrow & \mathbb{V} \\ (V, \rho) & \longmapsto & (V_{\mathbb{C}}, \rho(\mu_h(-1)) \otimes c) \end{array}$$

where the grading on $V_{\mathbb{C}}$ is given by $\rho \circ w_h$.

The category $\mathrm{CM}^L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is a polarized Tate triple, the polarizations being induced by ample line bundles (or rather their Chern classes since we work with Hodge cycles). There is thus associated a morphism $\mathrm{CM}^L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{V}$ well defined up to isomorphism, sending this polarization to the canonical one of $\mathbb{V}$. Via the equivalence given by $\omega_B \otimes_{\mathbb{Q}} \mathbb{R}$ between $\mathrm{CM}^L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathrm{Rep}_{\mathbb{R}} S^L$ this morphism is exactly ξ_{μ^L}

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{R}} S^L \arrow[r, "\sim"] \arrow[rrr, bend right, "\xi_{\mu^L}"'] & \text{CM } \mathbb{Q}\text{-Hodge struct.}_{\text{ref. field in L}} & \mathrm{CM}^L_{\mathbb{Q}} \arrow[l, "\sim"'] \arrow[r, "\text{polarization}"] & \mathbb{V} \end{tikzcd}$$

9.6.2. Computation of the isomorphism class of $\underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R})$. — We compute for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$

$$[\underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R})] \in H^1(\mathbb{R}, S^L)$$

using polarizations. In fact there is a diagram

$$\begin{tikzcd} & & \text{CM } \mathbb{Q}\text{-Hodge structures} \arrow[dr, "F"] & \\ \mathrm{Rep}_{\mathbb{R}} S^L \arrow[r, "\sim"] & \mathrm{CM}^L_{\mathbb{Q}} \otimes \mathbb{R} \arrow[ur, "\omega_B"] \arrow[rr, "polarization"] \arrow[dr, "\omega_{B,\sigma}"'] & & \mathbb{V} \\ & & \text{CM } \mathbb{Q}\text{-Hodge structures} \arrow[ur, "F"'] & \end{tikzcd}$$

where both morphisms ω_B and $\omega_{B,\sigma}$ send the natural polarizations of $\mathrm{CM}^L_{\mathbb{Q}} \otimes \mathbb{R}$ to the canonical one on CM $\mathbb{Q}$-Hodge structures and the functor F from CM$\mathbb{Q}$-Hodge structures to $\mathbb{V}$ is induced by the natural polarization that is to say $F(V, h) = (V_{\mathbb{C}}, \mu_h(-1) \otimes c)$ (see the preceding section). Thus by unicity up to isomorphism of the morphism $\mathrm{CM}^L_{\mathbb{Q}} \longrightarrow \mathbb{V}$ sending polarizations to polarizations one deduces this diagram is commutative up to isomorphism. Now the composite $\mathrm{Rep}_{\mathbb{R}} S^L \longrightarrow \mathbb{V}$ through the upper path is given by ξ_{μ^L} (see preceding section). Now the lower composite is given by $\xi_{\sigma.\mu^L} * \underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R})$ (twist by a $S^L_{\mathbb{R}}$-torsor, two morphisms $\mathrm{Rep}_{\mathbb{R}} T \longrightarrow \mathbb{V}$ inducing the same band morphism differ by a T-torsor, to F_1, F_2 one associated $\underline{\mathrm{Isom}}(F_1, F_2)$). Both being isomorphic

$$\xi_{\mu^L} \simeq \xi_{\sigma.\mu^L} * \underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R}) \Longrightarrow \underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R}) \simeq \underline{\mathrm{Isom}}(\xi_{\sigma.\mu^L}, \xi_\mu)$$

But one computes easily $[\underline{\mathrm{Isom}}(\xi_{\sigma.\mu^L}, \xi_\mu)] \in H^1(\mathbb{R}, S^L)$ is given by the class of the cocycle $c \longmapsto (-1)^{(\sigma-1).\mu^L}$.

Corollary 9.2. — *The class* $[\underline{\mathrm{Isom}}(\omega_B \otimes \mathbb{R}, \omega_{B,\sigma} \otimes \mathbb{R})] \in H^1(\mathbb{R}, S^L)$ *is given by the one of the cocycle* $c \longmapsto (-1)^{(\sigma-1).\mu^L}$.

9.7. Equivalence between the construction of the Taniyama extension and the understanding of conjugation of CM motives over $\overline{\mathbb{Q}}$. — We have the following proposition

Proposition 9.3. — *Let T be a torus over $\mathbb{Q}$ split by the Galois extension $L|\mathbb{Q}$. Let G be a profinite group seen as a pro-constant group scheme over $\mathbb{Q}$. Suppose we are given an "algebraic" action $G \longrightarrow Aut(T)$ factorizing through a finite quotient of G. Then the following are equivalent:*

- *To give oneself an extension $1 \longrightarrow T \longrightarrow E \longrightarrow G \longrightarrow 1$ inducing the fixed algebraic action of G on T together with a continuous section $s : G \longrightarrow E(\mathbf{A}_{\mathbb{Q},f})$.*

– *To give oneself a map* $\bar{b} : G \longrightarrow [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})}$ *verifying*

$$\forall \sigma_1, \sigma_2 \in \mathcal{G} \;\; \bar{b}(\sigma_1\sigma_2) = \bar{b}(\sigma_1)\bar{b}(\sigma_2)^{\sigma_1}$$

where $(-)^\sigma$ *means the algebraic of* σ, *that is to say a one cocycle for the algebraic action*

$$\bar{b}(-) \in Z^1\left(G, [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})}\right)$$

and such that there exists a continuous lift of the map $\bar{b}$ *from* G *to* $T(\mathbf{A}_{L,f})$.

To the extension $1 \to T \to E \to G \to 1$ *one associates the map* $\bar{b}$ *defined in the following way. Since* T *splits over* L *there is an exact sequence* $1 \to T(L) \to E(L) \to G \to 1$. *Let* $\sigma \mapsto a_\sigma$ *be a set theoretical section of* $E(L) \twoheadrightarrow G$. *Then*

$$\bar{b}(\sigma) = a_\sigma s(\sigma)^{-1} \;\; mod \; T(L)$$

Moreover if $\pi : E \longrightarrow G$ *then* $\forall \sigma \in G$ *the isomorphism class of the* T*-torsor* $\pi^{-1}(\sigma)$ *is given by the boundary map*

$$[T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})} \longrightarrow \ker\left(H^1(L|\mathbb{Q}, T(L)) \longrightarrow H^1(L|\mathbb{Q}, T(\mathbf{A}_{L,f}))\right)$$

The image of $\bar{b}(-)$ *in* $Z^1(G, H^1(L|\mathbb{Q}, T))$ *determines completely the isomorphism class of the extension, that is to say without its adelic section.*

In the preceding proposition when we say "To give oneself an extension..." we mean up to a unique isomorphism since the adelic section rigidifies the situation. The proof is given in the article by Milne and Shih in [**4**] in the particular case $T = S^L$, $G = \mathrm{Gal}(L^{ab}|\mathbb{Q})$ and the algebraic action given in section 9.2. But this has nothing to do with this particular case ! It relies on a lengthy, but elementary, cocycle computation.

Example 9.4. — Let $1 \to S^L \to \mathcal{E}^L \to \mathrm{Gal}(L^{ab}|L) \to 1$ be Serre's extension constructed in section 6. Then the associated map $\bar{b} : \mathrm{Gal}(L^{ab}|L) \longrightarrow \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$ is the composite

$$\mathrm{Gal}(L^{ab}|L) = \pi_0(C_L) \xrightarrow{\mu^L} S^L(\mathbf{A}_{L,f})/\overline{S^L(L)} \xrightarrow{N_{L/\mathbb{Q}}} S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}) \subset \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$$

This follows easily from the fact that the quotient $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \twoheadrightarrow S^L$ defining Serre's group, as in section 5.1, is the composite

$$\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}}\mu^L} \mathrm{Res}_{L/\mathbb{Q}}S^L_{/L} \xrightarrow{N_{L/\mathbb{Q}}} S^L$$

where here $N_{L/\mathbb{Q}}$ means the norm for the arithmetic action. The fact that its image lies in $S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q})$ is naturally interpreted in the last point of the next proposition.

One verifies moreover :

Proposition 9.5. — *Consider an exact sequence like in the preceding proposition associated to the map* $\bar{b}$. *Suppose we have a diagram*

$$\begin{tikzcd} & & & G' \arrow[d] & \\ 1 \arrow[r] & T \arrow[r] \arrow[d] & E \arrow[r] & G \arrow[r] & 1 \\ & T' & & & \end{tikzcd}$$

The the push-forward of the extension via $T \longrightarrow T'$ *with its composite adelic section corresponds to the composite cocycle*

$$G \xrightarrow{\bar{b}} [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})} \longrightarrow [T'(\mathbf{A}_{L,f})/T'(L)]^{Gal(L|\mathbb{Q})}$$

The pull-back via $G' \longrightarrow G$ *together with its deduced adelic section corresponds to the composite*

$$G' \longrightarrow G \xrightarrow{\bar{b}} [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})}$$

Moreover there is an easily defined Yoneda group structure on such type of extensions equipped with an adelic section. This structure corresponds to the group structure of $Z^1\left(G, [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})}\right)$.

The sequence $1 \longrightarrow T(\mathbb{Q}) \longrightarrow E(\mathbb{Q}) \longrightarrow G \longrightarrow 1$ *is exact, that is to say each connected component of* E *has a rational point, iff the map* $\bar{b}$ *takes values in* $T(\mathbf{A}_{\mathbb{Q},f})/T(\mathbb{Q}) \subset [T(\mathbf{A}_{L,f})/T(L)]^{Gal(L|\mathbb{Q})}$.

Let's apply this to the motivic extension $1 \to S^L \to M^L \to \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow 1$ together with its section s^L. The associated map $\bar{b}$ is the one noted $\bar{b}^L$ in the preceding sections. Thus the preceding propositions tells us it is equivalent to construct the extension M^L, or to understand $\mathrm{CM}^L_{\mathbb{Q}}$ equipped with the Betti cohomology fiber functor, and to understand the different S^L-torsors $\underline{\mathrm{Isom}}^{\otimes}(\omega_B, \omega_{B,\sigma})$ defined by CM-motives over $\overline{\mathbb{Q}}$ when σ varies together with their trivialization given by étale cohomology over $\mathbf{A}_{\mathbb{Q},f}$. Moreover by the preceding proposition the compatibility when L varies is resumed in the formula $\phi^{L'/L} \circ \bar{b}^{L'} = \bar{b}^L$ where $\phi_{L'/L} : S^{L'} \longrightarrow S^L$.

From the formula : $\bar{b}^L(\sigma_1\sigma_2) = \bar{b}^L(\sigma_1)\bar{b}^L(\sigma_2)^{\sigma_1}$ one deduces $\bar{b}^L$ factorizes through $\mathrm{Gal}(L^{ab}|\mathbb{Q})$. And thus by proposition 9.5 one finds

Corollary 9.6. — *There exists an extension* $1 \longrightarrow S^L \longrightarrow \mathcal{T}^L \longrightarrow Gal(L^{ab}|\mathbb{Q}) \longrightarrow 1$ *called the Taniyama extension together with a continuous splitting* $s^L : Gal(L^{ab}|\mathbb{Q}) \longrightarrow \mathcal{T}^L(\mathbf{A}_{\mathbb{Q},f})$ *and a morphism of extensions*

$$\begin{tikzcd}
1 \arrow[r] & S^L \arrow[r] \arrow[d, equal] & M^L \arrow[r] \arrow[d, two heads] & Gal(\overline{\mathbb{Q}}|\mathbb{Q}) \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & S^L \arrow[r] & \mathcal{T}^L \arrow[r] & Gal(L^{ab}|\mathbb{Q}) \arrow[r] & 1
\end{tikzcd}$$

inducing an isomorphism between the upper one and the pull-back of the lower one through $Gal(\overline{\mathbb{Q}}|\mathbb{Q}) \twoheadrightarrow Gal(L^{ab}|\mathbb{Q})$, *and compatible with the* ℓ*-adic sections.*

Its motivic meaning is the following. Since $M^L \longrightarrow \mathcal{T}^L$ is an epimorphism $\mathrm{Rep}_{\mathbb{Q}}(\mathcal{T}^L) \longrightarrow \mathrm{Rep}_{\mathbb{Q}}(M^L) = \mathrm{CM}^L_{\mathbb{Q}}$ is fully faithful. Then $\mathrm{Rep}_{\mathbb{Q}}(\mathcal{T}^L)$ is the subcategory formed by CM-motives over $\mathbb{Q}$ whose reflex field is contained in L and such that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on their étale cohomology becomes abelian on $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$. This follows from the easy fact that a representation ρ of $M^L = \mathcal{T}^L \times_{\mathrm{Gal}(L^{ab}|\mathbb{Q})} \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ factorizes through $\mathcal{T}^L$ iff $\rho \circ s^L$ factorizes through $\mathrm{Gal}(L^{ab}|\mathbb{Q})$.

Remark 9.7. — The author does not know any "concrete" set of motives to add to Artin motives split by L^{ab} and generating the Tannakian subcategory $\mathrm{Rep}_{\mathbb{Q}}(\mathcal{T}^L) \subset \mathrm{CM}^L_{\mathbb{Q}}$, as we did by saying $\mathrm{CM}^L_{\mathbb{Q}}$ is generated by CM abelian varieties whose reflex field is contained in L and Artin motives.

Remark 9.8. — The preceding corollary implies any CM motive over $\overline{\mathbb{Q}}$ whose reflex field is contained in L is the direct factor of a motive of the form $X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ where the Galois action on the étale cohomology of X is abelian when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$ (this follows from the fact that $S^L \longrightarrow \mathcal{T}^L$ is a monomorphism, see proposition 2.21 (b) of II in [**4**]). The author does not know any concrete proof of this for the $h^1(A)$ where (A, ι) is a CM abelian variety over $\overline{\mathbb{Q}}$. This fact seems to use Deligne's theorem that any Hodge cycle on an abelian variety is absolutely Hodge.

Thus from a logical point of view one can't construct the Taniyama extension from the beginning by saying "let's take the group classifying all CM motives over $\mathbb{Q}$...whose ℓ-adic Galois action restricted to L is abelian" since to prove the map $S^L \longrightarrow \mathcal{T}^L$ is a monomorphism one has to use the preceding fact deduced itself form the preceding corollary.

Remark 9.9. — The preceding remark is more or less in contradiction with proposition 6.28 of II in [**4**] that seems false to the author. In fact exactness in the middle of the exact sequence in this proposition 6.28 would imply if A is a CM abelian variety over $\mathbb{Q}$, if $B = \mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{Q}}$, a product of matrix algebras with coefficients in some CM fields, then the Galois action $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{Aut}(B)$ should define an Artin motive split over L^{ab} (see the proof of point (c) in proposition 6.23 of II in [**4**]). Thus the CM action of B on $A_{\overline{\mathbb{Q}}}$ should be defined on $A_{L'}$ for $L'|L$ abelian. But this is in general false, although this is true if A is absolutely simple that is to say B is a CM field.

9.8. We already know the isomorphism class of the Taniyama extension. — One checks S^L satisfies Hasse principle (proposition C.1 in IV of [**4**]). Thus the only obstruction for ω_B and $\omega_{B,\sigma}$ not to be isomorphic comes from the Betti cohomology with coefficients in $\mathbb{R}$.

Corollary 9.10. — *For $\sigma \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ the isomorphism class of the torsor between ω_B and $\omega_{B,\sigma}$ is the unique class in $H^1(\mathbb{Q}, S^L)$ that become trivial at finite places and is given by the class of the cocyle in corollary 9.2 at the archimedian one. This thus determines uniquely the composite $Gal(L^{ab}|\mathbb{Q}) \xrightarrow{\bar{b}^L} [S^L(\mathbf{A}_{L,f})/S^L(L)]^{Gal(L|\mathbb{Q})} \longrightarrow H^1(\mathbb{Q}, S^L)$, and thus by the last point in proposition 9.3 the isomorphism class of the extension $1 \to S^L \to \mathcal{T}^L \to Gal(L^{ab}|\mathbb{Q}) \to 1$ without its adelic section.*

It remains to know explicitly all the Taniyama extension with its adelic section. This is next section.

9.9. Langlands construction of the map $\bar{b}^L(-)$. — Langlands constructed in [**7**] a map $\bar{b}^L$ from $\mathrm{Gal}(L^{ab}|\mathbb{Q})$ to $\left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$ for varying L satisfying all the properties asked in the previous section. This is done in details in the article by Milne and Shih in [**4**]. Let's try to explain it in a more transparent than in [**7**] and [**4**] where it is buried in awful computations.

This construction is based on the extensions coming from the global Weil groups, themselves coming from the system of fundamental classes from class field theory ans their compatibilities under inflation/restriction. This uses the full power of class field theory, not only Artin reciprocity map.

The starting point is the morphism of extensions

$$(*) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & C_L^0 & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L^{ab}|\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \longrightarrow & C_L & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L|\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

where $C_L^0 = \bigcap_{K|L} N_{K/L} C_K = (W_{L/\mathbb{Q}})^0$. The lower one is given by the fundamental class of class field theory $u_{L/\mathbb{Q}} \in H^2(L|\mathbb{Q}, C_L)$. The upper one is more complicated to construct since it uses the whole construction of the Weil group and thus not only one fundamental class but a compatible system of them under inflation/restriction.

Now we want to construct a cocycle $\bar{b}^L$ for the algebraic action

$$\bar{b}^L \in Z^1(\mathrm{Gal}(L^{ab}|\mathbb{Q}), \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})})$$

such that when restricted to $\mathrm{Gal}(L^{ab}|L)$ is gives back Serre's extension that is to say $\bar{b}^L_{|\mathrm{Gal}(L^{ab}|L)}$ is induced by the composite

$$C_L \xrightarrow{\mu^L} S^L(\mathbf{A}_L)/S^L(L) \xrightarrow{\text{proj.}} S^L(\mathbf{A}_{L,f})/S^L(L) \xrightarrow{N_{L/\mathbb{Q}}} S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q})$$

when one takes the π_0 (see remark 3.2 and example 9.4). Let's note

$$A = \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{\mathrm{Gal}(L|\mathbb{Q})}$$

Thus one is looking for a 1-cocycle for the algebraic action $\bar{b}^L$ such that under restriction

$$\mathrm{Res}^{\mathrm{Gal}(L^{ab}|\mathbb{Q})}_{\mathrm{Gal}(L^{ab}|L)} : Z^1(\mathrm{Gal}(L^{ab}|\mathbb{Q}), A) \longrightarrow Z^1(\mathrm{Gal}(L^{ab}|L), A) = \mathrm{Hom}(\mathrm{Gal}(L^{ab}|L), A)$$

it gives us the preceding formula. The trick consists in first defining a cocycle in $Z^1(W_{L/\mathbb{Q}}, A)$ (algebraic action via $W_{L/\mathbb{Q}} \longrightarrow \mathrm{Gal}(L|\mathbb{Q})$) such that under restriction

$$\mathrm{Res}^{W_{L/\mathbb{Q}}}_{C_L} : Z^1(W_{L/\mathbb{Q}}, A) \longrightarrow Z^1(C_L, A) = \mathrm{Hom}(C_L, A)$$

it gives us $N_{L/\mathbb{Q}} \circ \text{proj.} \circ \mu^L$ and then to notice that since it is continuous and A is totally discontinuous it factorizes through $\pi_0(W_{L/\mathbb{Q}}) = \mathrm{Gal}(L^{ab}|\mathbb{Q})$ (look at the morphism of extensions denoted (*)).

Let's note

$$\mu_f^L = \text{proj.} \circ \mu^L : C_L \longrightarrow S^L(\mathbf{A}_{L,f})/S^L(L)$$

One remarks

$$N_{L/\mathbb{Q}}(\mu_f^L(x)) := \sum_{\sigma \in \mathrm{Gal}(L|\mathbb{Q})} \sigma.\mu_f^L(x) = \sum_{\sigma \in \mathrm{Gal}(L|\mathbb{Q})} \mu_f^L(x^\sigma)^{\sigma^{-1}}$$

Now one has the following lemma

Lemma 9.11. — *Let G be a group. Let H be a finite index distinguished subgroup of G and B a G/H-module, this action of g on B being noted b^g. Then the composite*

$$Hom(H,B) = H^1(H,B) \xrightarrow{Cor_H^G} H^1(G,B) \xrightarrow{Res_H^G} H^1(H,B)$$

is $f \longmapsto [h \mapsto \sum_{\sigma \in G/H} f(h^\sigma)^{\sigma^{-1}}]$

Thus setting $G = W_{L/\mathbb{Q}}, H = C_L, B = S^L(\mathbf{A}_{L,f})/S^L(L)$ with its algebraic action, the corestriction of the cocyle μ_f^L seems to be a good candidate ! For the preceding lemma we used the following (the author used "décalage cohomologique" to find those formulas).

Lemma 9.12. — *With the hypothesis of the preceding lemma let $(w_\sigma)_{\sigma \in G/H}$, $w_\sigma \in G$, be a set of representative of classes in G/H. Then if $g \in G$ define $\forall \sigma \in G/H \ \ h_{g,s} \in H$ by $w_\sigma g = h_{g,\sigma} w_{\sigma g}$. Then $Cor_H^G(f)$ is the class of the 1-cocyle*

$$g \longmapsto \sum_{\sigma \in G/H} f(h_{g,\sigma})^{\sigma^{-1}}$$

A change of representative $(w_\sigma)_\sigma$ to $(u_\sigma w_\sigma)_\sigma$ with $u_\sigma \in H$ changes the preceding cocyle by the coboundary of the 0*-cycle $\sum_{\sigma \in G/H} \sigma^{-1}.f(u_\sigma)$.*

Keep the notations of the lemma. Suppose we have moreover an "arithmetic" action noted $g.b$, $g \in G$ $b \in B$ of G/H on B that commutes with the preceding one. For $B = S^L(A_{L,f})/S^L(L)$ we will take the arithmetic action. It thus induces an "arithmetic" action of G/H on $H^1(H,B)$ and $H^1(G,H)$.
Recall there is another action of G/H on $H^1(H,B)$ sending the class of the cocycle φ to the one of $\varphi(g^{-1} \bullet g)^g$ (the action appearing in the Hochschild-Serre spectral sequence related to $H \lhd G$). By generalities on group cohomology the morphism $\mathrm{Cor}_H^G : H^1(H,B) \longrightarrow H^1(G,B)$ is G/H-invariant that is to say $\mathrm{Cor}_H^G([\varphi(\bullet)]) = \mathrm{Cor}_H^G([\varphi(g^{-1} \bullet g)^g])$. Thus if $f \in \mathrm{Hom}(H,B)$ satisfies $\forall g \in G \ \forall h \in H \ \ f(h)^{g^{-1}} = g.f(ghg^{-1})$ then $\mathrm{Cor}_H^G(f) \in H^1(G,B)$ has to be fixed under the arithmetic action of G on $H^1(G,B)$. This hypothesis is satisfied by μ_f^L.

The problem is now we want to produce not a cocyle class but a real cocyle and we want it to be fixed under the arithmetic action, not up to a coboundary. This is provided by the following lemma that resumes all the computations done in the article by Milne and Shih in[**4**].

Lemma 9.13. — *Let $f \in Hom(H,B)$ satisfy $\forall \sigma \in G/H \ \forall h \in H \ \ f(h)^{\sigma^{-1}} = \sigma.f(h^{\sigma^{-1}})$. Suppose we have furthermore a central subgroup $\mathbb{Z}/2\mathbb{Z} = \{1,c\} \subset G/H$ an a morphism of extensions*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] & G \arrow[r] & G/H \arrow[r] & 1 \\
1 \arrow[r] & E \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & \{1,c\} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

such that the composite $E \longrightarrow H \xrightarrow{f} B$ is zero. Suppose $\forall \sigma \in G/H\ \forall h \in H\quad f(h)^{\sigma(c+1)} = f(h)^{c+1}$. Then there is a "canonical" way to define a "lift" of the corestriction $Cor_H^G(f)$ that is fixed under the arithmetic action

$$\widetilde{Cor_H^G}(f) \in Z^1(G, B^{G/H})$$

where $B^{G/H}$ means fixed under the arithmetic action. Take a set $\Phi \subset G/H$ s.t. $G/H = \Phi \coprod c\Phi$. Chose liftings $(w_\sigma)_{\sigma\in\Phi}$, $w_\sigma \in G$, $w_c \in G$ lifting c s.t. it come from an element of F via $F \to G$. Define then $\forall \sigma \in \Phi\ w_{\sigma c} = w_\sigma w_c$. Then with the notations of lemma 9.12

$$\widetilde{Cor_H^G}(f) = \sum_{\sigma\in G/H} f(h_{g,\sigma})^{\sigma^{-1}}$$

independently of the choices of the different liftings.

The proof is a straightforward computation. Applying it to $f = \mu_f^L$ and the morphism of extensions

$$\begin{tikzcd} 1 \arrow[r] & C_L \arrow[r] & W_{L/\mathbb{Q}} \arrow[r] & \mathrm{Gal}(L|\mathbb{Q}) \arrow[r] & 1 \\ 1 \arrow[r] & L_v^\times \arrow[r] \arrow[u] & W_{L_v/\mathbb{R}} \arrow[r] \arrow[u] & \mathrm{Gal}(L_v|\mathbb{R}) \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

for some archimedian place v of L one obtains Langlands cocyle

$$\bar{b}^L = \widetilde{\mathrm{Cor}_{C_L}^{W_{L/\mathbb{Q}}}}(\mu_f^L) \in Z^1_{\mathrm{cont}}(W_{L/\mathbb{Q}}, A) = Z^1_{cont}(\underbrace{\pi_0(W_{L/\mathbb{Q}})}_{\mathrm{Gal}(L^{ab}|\mathbb{Q})}, A)$$

where "cont" means continuous cocyle.

Concretely, let's chose a spiting of the lower extension in (*) $(w_\tau)_{\tau\in\mathrm{Gal}(L|\mathbb{Q})}$, $w_\tau \in W_{L/\mathbb{Q}}$, verifying

- $w_1 = 1$
- Via $L \subset \mathbb{C}$ we have a morphism $W_{\mathbb{C}/\mathbb{R}} \longrightarrow W_{L/\mathbb{Q}}$ and one asks w_c lies in its image
- For a CM type Φ of L one has $\forall \tau \in \Phi\ w_{\tau c} = w_\tau w_c$

Let $\sigma \in \mathrm{Gal}(L^{ab}|\mathbb{Q})$ and lift it through the upper exact sequence in (*) to a $\widetilde{\sigma} \in W_{L/\mathbb{Q}}$. Then define $c_{\tau,\widetilde{\sigma}}$ by

$$c_{\tau,\widetilde{\sigma}} w_{\tau\sigma} = w_\tau \widetilde{\sigma} \in C_L$$

Now Langlands definition is

$$\bar{b}^L(\sigma) = \prod_{\tau\in\mathrm{Gal}(L|\mathbb{Q})} c_{\tau,\widetilde{\sigma}}^{\tau.\mu^L} \bmod S^L(L_\infty)S^L(L) \in S^L(\mathbf{A}_{L,f})/S^L(L)$$

9.10. Tate's construction. — We refer to [**8**] for a detailed description of this construction. It is more concrete than Langlands one in the sens that it does not use the global Weil group, but it has a non-constructive ingredient inside (proposition 2.3 and lemma 2.4 of [**8**]). Moreover one can not use it to construct the Weil group-section noted s_∞ in theorem 7.1.

9.11. Deligne's theorem. — Deligne checked in the article IV of [**4**] that the extension equipped with its adelic section defined by Langlands map $\bar{b}^L(-)$ of section 9.9 is isomorphic to the one attached to the Motivic Galois group extension $1 \to S^L \to M^L \to \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to 1$. Let's give a sketch of Deligne's proof. Let's note $\bar{b}_1^L$ for the map attached to the motivic Galois extension defined in section 9.5 and $\bar{b}_2^L$ for Langland's one of section 9.5. One has to prove $\bar{b}_1^L = \bar{b}_2^L$. We note $\beta^L = \bar{b}_1^L(\bar{b}_2^L)^{-1}$. Thanks to proposition 9.5 this corresponds to take a Yoneda difference between the motivic Galois extension and Langlands one.

Let $1 \to S^L \to \mathcal{T}_L^L \to \mathrm{Gal}(L^{ab}|L) \to 1$ be the pullback of the motivic extension given by the Taniyama group (see corollary 9.6) via $\mathrm{Gal}(\overline{\mathbb{Q}}|L) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. It classifies CM motives over L whose Mumford-Tate group is split by L and such that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$ on their étale cohomology is abelian. Deligne begins by proving there is a an isomorphism

$$\begin{tikzcd}
1 \arrow[r] & S^L \arrow[r] \arrow[d, equal] & \mathcal{E}^L \arrow[r] \arrow[d, "\simeq"] & \mathrm{Gal}(L^{ab}|L) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & S^L \arrow[r] & \mathcal{T}_L^L \arrow[r] & \mathrm{Gal}(L^{ab}|L) \arrow[r] & 1
\end{tikzcd}$$

where the upper extension is Serre's extension. The morphism $\mathcal{E}^L \longrightarrow \mathcal{T}_L^L$ is given Tannaka dually by theorem 6.2. In fact for each simple CM motive over L with coefficients in some number field such that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$ on its étale cohomology is abelian there is attached an algebraic Hecke character χ of $\mathbf{A}_L^\times/L^\times$ such that the étale cohomology Galois action is given by the associated λ-adic compatible system $(\chi_\lambda)_\lambda$. In fact, a compatible system of λ-adic representations is associated to an algebraic Hecke character iff it is the case after a finite base change (exercise). Thus for a given motive it suffices to prove it after a finite base extension of L and then this follows from the case of CM abelian varieties (section 2) since they generate "potentially" (after killing Artin motives) our category of motives.

Applying proposition 9.5, since $\bar{b}_2^L$ gives back by construction Serre's extension, one deduces β factorizes $\beta^L : \mathrm{Gal}(L|\mathbb{Q}) \longrightarrow [S^L(\mathbf{A}_{L,f})/S^L(L)]^{\mathrm{Gal}(L|\mathbb{Q})}$.

Thanks to corollary 9.10, the fact that S^L satisfies Hasse principle and an explicit computation based on Langlands explicit cocyle showing the image of $\bar{b}_2(\sigma)$ in $H^1(\mathbb{R}, S^L)$ is the same as the one given in corollary 9.2 one deduces $\forall\sigma\ \beta^L(\sigma)$ goes to zero in $H^1(\mathbb{Q}, S^L)$ and thus the image of β^L lies in $S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q})$. This means thanks to proposition 9.5 the Yoneda difference between our two extensions is induced by an extension

$$1 \longrightarrow S^L \longrightarrow \Delta^L \longrightarrow \mathrm{Gal}(L|\mathbb{Q}) \longrightarrow 1$$

with a section $s^L : \mathrm{Gal}(L|\mathbb{Q}) \longrightarrow \Delta^L(\mathbf{A}_{\mathbb{Q},f})$ such that $1 \to S^L(\mathbb{Q}) \to \Delta^L(\mathbb{Q}) \to \mathrm{Gal}(L|\mathbb{Q}) \to 1$ is exact.

To finish we have to prove the image of $s^L : \mathrm{Gal}(L|\mathbb{Q}) \longrightarrow \Delta^L(\mathbf{A}_{\mathbb{Q},f})$ lies in $\Delta^L(\mathbb{Q})$. The trick is to use complex conjugation $c \in \mathrm{Gal}(L|\mathbb{Q})$, a central element since L is CM. One has $s^L(c) \in \Delta^L(\mathbb{Q})$. In fact this is the case for the motivic extension since complex conjugation already acts on Betti cohomology and this action is the same as the one on étale cohomology via the comparison isomorphism. For Langlands extension this follows from the explicit formula for the cocycle. Thus if L^+ is the maximal totally real subfield in L, $\mathrm{Gal}(L|L^+) = \{1, c\}$,

$$\beta^L \in \ker(Z^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q})) \xrightarrow{\mathrm{Res}} Z^1(\mathrm{Gal}(L|L^+), S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}))$$

Now there is an inflation/restriction exact sequence associated to $\mathrm{Gal}(L|L^+) \lhd \mathrm{Gal}(L|\mathbb{Q})$

$$\begin{tikzcd}
0 \arrow[r] & Z^1(\mathrm{Gal}(L^+|\mathbb{Q}), (S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}))^{c=Id}) \arrow[r, "\mathrm{Inf}"] & Z^1(\mathrm{Gal}(L|\mathbb{Q}), S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q})) \arrow[d, "\mathrm{Res}"] \\
& & Z^1(\mathrm{Gal}(L|L^+), S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}))
\end{tikzcd}$$

Thus β^L takes values in $(S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}))^{c=Id}$. But now (lemma F.5 in Deligne's article)

$$\ker(H^1(\mathbb{Z}/2\mathbb{Z}, S^L(\mathbb{Q})) \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, S^L(\mathbf{A}_{\mathbb{Q},f}))) = 0$$

where $\mathbb{Z}/2\mathbb{Z} = \{1, c\}$. Thus $(S^L(\mathbf{A}_{\mathbb{Q},f})/S^L(\mathbb{Q}))^{c=Id} = S^L(\mathbf{A}_{\mathbb{Q},f})^{c=Id}/S^L(\mathbb{Q})^{c=Id}$ and

$$\beta^L \in Z^1\left(\mathrm{Gal}(L^+|L), S^L(\mathbf{A}_{\mathbb{Q},f})^{c=Id}/S^L(\mathbb{Q})^{c=Id}\right)$$

Now Deligne uses the fact our β^L are compatible when L varies under the morphisms $S^{L_2} \longrightarrow S^{L_1}$ for $L_2|L_1$ an extension of CM fields Galois over $\mathbb{Q}$ (this is the case for $\bar{b}_1^L$ and $\bar{b}_2^L$). Thus to prove β^L is trivial we can suppose L is sufficiently big and contains an imaginary quadratic subfield L_0, for example $L_0 = \mathbb{Q}(i)$. Since $L_0^+ = \mathbb{Q}$ one has $\beta^{L_0} = 1$ and thus

$$\forall \sigma \in \mathrm{Gal}(L^+|\mathbb{Q}) \ \ \beta_\ell^L(\sigma) \in \ker\left(S^L(\mathbf{A}_{\mathbb{Q},f})^{c=Id}/S^L(\mathbb{Q})^{c=Id} \longrightarrow S^{L_0}(\mathbf{A}_{\mathbb{Q},f})^{c=Id}/S^{L_0}(\mathbb{Q})^{c=Id}\right)$$

Then (formula F.5.1 in Deligne's article) one checks the weight morphism $w^L : \mathbb{G}_m \longrightarrow S^L$ induces diagrams

$$\begin{tikzcd} \mathbb{Q}^\times \arrow[r, "w", "\sim"'] \arrow[d, hook] & S^{L_0}(\mathbb{Q})^{c=Id} \arrow[d, hook] \\ \mathbf{A}_{\mathbb{Q},f}^\times \arrow[r, "w", "\sim"'] & S^{L_0}(\mathbf{A}_{\mathbb{Q},f})^{c=Id} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & \mathbb{Q}^\times \arrow[r, "w"] \arrow[d, hook] & S^L(\mathbb{Q})^{c=Id} \arrow[r] \arrow[d] & \mu_2(L^+) \arrow[r, "N_{L^+/\mathbb{Q}}"] \arrow[d] & \mu_2(\mathbb{Q}) \arrow[d] \\ 1 \arrow[r] & \mathbf{A}_{\mathbb{Q},f}^\times \arrow[r, "w"] & S^L(\mathbf{A}_{\mathbb{Q},f})^{c=Id} \arrow[r] & \mu_2(\mathbf{A}_{L^+,f}) \arrow[r, "N_{L^+/\mathbb{Q}}"] & \mu_2(\mathbf{A}_{\mathbb{Q},f}) \end{tikzcd}$$

where the rows are exact. From this one deduces

$$\beta^L(\sigma) \in \ker(\mu_2(\mathbf{A}_{L^+,f}) \xrightarrow{N_{L^+/\mathbb{Q}}} \mu_2(\mathbf{A}_{\mathbb{Q},f}))/\ker(\mu_2(L) \xrightarrow{N_{L^+/\mathbb{Q}}} \mu_2(\mathbb{Q}))$$

Now, taking CM fields $L_1|L$, our element $\beta^L(\sigma)$ has to be in the image of

$$\ker(\mu_2(\mathbf{A}_{L_1^+,f}) \xrightarrow{N_{L_1^+/\mathbb{Q}}} \mu_2(\mathbf{A}_{\mathbb{Q},f})) \xrightarrow{N_{L_1/L}} \ker(\mu_2(\mathbf{A}_{L^+,f}) \xrightarrow{N_{L^+/\mathbb{Q}}} \mu_2(\mathbf{A}_{\mathbb{Q},f}))/\ker(\mu_2(L) \xrightarrow{N_{L^+/\mathbb{Q}}} \mu_2(\mathbb{Q}))$$

and the intersection of those images for all $L_1|L$ is easily seen to be zero.

10. Application to the conjugation of CM abelian varieties under a general automorphism of $\mathbb{C}$

Let (A, ι) be a simple CM abelian variety over $\overline{\mathbb{Q}}$, $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_{\mathbb{Q}}$, with CM type Φ. The purpose of this section is to explain the generalisation of Shimura-Taniyama reciprocity law, proposition 3.1, to automorphisms not fixing the reflex field.

Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. Let L be a Galois CM field containing the reflex field. To (A, ι) is associated $\rho : S^L \longrightarrow E^\times$ such that $\rho \circ \mu^L = \mu_\Phi$. The S^L-torsor $\underline{\mathrm{Isom}}(\omega_B, \omega_{B,\sigma})$ of section 9.4 becomes trivial after pushing it by ρ

$$[E^\times \overset{\rho, S^L}{\wedge} \underline{\mathrm{Isom}}(\omega_B, \omega_{B,\sigma})] = 0 \in H^1(\mathbb{Q}, E^\times) = 0$$

From this and the preceding sections one deduces

Theorem 10.1 **(Deligne-Langlands-Shimura-Taniyama)**. — *For $\sigma \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})$ and an isomorphism*

$$f : H_B^1(A, \mathbb{Q}) \xrightarrow{\sim} H_B^1(A^\sigma, \mathbb{Q})$$

compatible with the actions ι and ι^σ, thus well defined up to an element of $E^\times$, the diagram

$$\begin{tikzcd} H_B^1(A,\mathbb{Q}) \otimes \mathbf{A}_{\mathbb{Q},f} \arrow[rr, "f \otimes Id", "\sim"'] \arrow[d, ".\rho(\bar{b}^L(\sigma))"'] & & H_B^1(A^\sigma,\mathbb{Q}) \otimes \mathbf{A}_{\mathbb{Q},f} \arrow[d, "\simeq"] \\ H_B^1(A,\mathbb{Q}) \otimes \mathbf{A}_{\mathbb{Q},f} \arrow[r, "comparison", "\sim"'] & H_{\acute{e}t}^1(A, \mathbf{A}_{\mathbb{Q},f}) \arrow[r, "\sigma^*"] & H^1(A^\sigma, \mathbf{A}_{\mathbb{Q},f}) \end{tikzcd}$$

commutes up to $E^\times$, where $\bar{b}^L(\sigma) \in \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{Gal(L|\mathbb{Q})}$ is Langlands cocyle (section 9.9) and

$$\rho(\bar{b}^L(\sigma)) \in \left[(E \otimes \mathbf{A}_{L,f})^\times/(E \otimes L)^\times\right]^{Gal(L|\mathbb{Q})} = \mathbf{A}_{E,f}^\times/E^\times$$

Remark 10.2. — One can give an explicit formula for the image of $\rho(\bar{b}^L(\sigma))$ in $\mathbf{A}_{E,f}^\times/\overline{E^\times}$ that does not use ρ. For this take Langlands explicit cocyle construction in section 9.9 but replace (S^L, μ^L) by $(E^\times, \mu_\Phi)$. See remark 3.2 too.

11. Zero dimensional Shimura varieties

The zero dimensional Shimura varieties we consider are moduli spaces of CM motives. There are more general zero dimensional Shimura varieties that are no moduli of anything.

11.1. Definitions. — Let T be a torus over $\mathbb{Q}$ split by a CM field.

Definition 11.1. — A CM motive over $\overline{\mathbb{Q}}$ with a T-structure is an exact faithful $\otimes$-functor $M : \mathrm{Rep}_{\mathbb{Q}}T \longrightarrow \mathrm{CM}_{\overline{\mathbb{Q}}}$.

Let $\mu \in X_*(T)$ a cocharacter whose weight μ^{1+c} is defined over $\mathbb{Q}$, $\mu^{1+c} \in X_*(T)^{\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})}$. We say M has CM type μ if the composite functor

$$\mathrm{Rep}_{\mathbb{Q}}(T) \xrightarrow{M} \mathrm{CM}_{\overline{\mathbb{Q}}} \xrightarrow{\text{Hodge Fil.}} \text{Filtered } \overline{\mathbb{Q}}\text{-vector spaces}$$

is the one given by μ and both functors

$$\mathrm{Rep}_{\mathbb{R}}(T_{\mathbb{R}}) \xrightarrow{M\otimes Id} \mathrm{CM}_{\overline{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\text{induced by the polarization}} \mathbb{V}$$

and $\xi_\mu : \mathrm{Rep}_{\mathbb{R}}T \longrightarrow \mathbb{V}$ (see section 9.6.1) are isomorphic, where the polarization on $\mathrm{CM}_{\overline{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{R}$ is the usual one.

Example 11.2. —

- If (E, Φ) is a CM field with a CM type, $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ and $\mu = \mu_\Phi = \sum_{\tau\in\Phi}[\tau]$, one obtains couples (A, ι) where A is an abelian variety over $\mathbb{Q}$, $\iota : E \xrightarrow{\sim} \mathrm{End}(A)_{\mathbb{Q}}$ and the associated CM type is Φ (here the second condition on the polarization and ξ_μ is empty since $H^1(\mathbb{R}, T) = 0$).
- If $T = \{x \in E^\times \mid xx^* \in \mathbb{Q}^\times\}$ and $\mu = \mu_\Phi$ as before one obtains triples (A, ι, λ) where (A, ι) has CM type Φ and λ is a polarization. In fact without the last hypotheses on the polarization and ξ_μ one obtains $\mathbb{Q}$-Hodge structures V with an action of E such that $\dim_E V = 1$, their CM type is Φ and equipped with a symplectic product $< .\,, . >: V \otimes V^\vee \longrightarrow \mathbb{Q}(-1)$ satisfying $\forall x \in E \;\; < x.\,, . >=< .\,, x^*. >$. Last condition is equivalent to the positivity of $< ., h(i). >$ that is to say $< .\,, . >$ is a polarization.

Definition 11.3. — Let M be a CM motive over $\overline{\mathbb{Q}}$ with T-structure. We note $H^\bullet(M, \mathbf{A}_{\mathbb{Q},f})$ for the composite functor

$$\mathrm{Rep}_{\mathbb{Q}}(T) \xrightarrow{M} \mathrm{CM}_{\overline{\mathbb{Q}}} \xrightarrow{\omega_{\text{ét}}} \mathbf{A}_{\mathbb{Q},f}\text{-mod. free of finite rank}$$

Definition 11.4. — Let $\xi : \mathrm{Rep}_{\mathbb{Q}}(T) \longrightarrow \mathbf{A}_{\mathbb{Q},f}$-mod. be the canonical functor $(V, \rho) \longmapsto V \otimes_{\mathbb{Q}} \mathbf{A}_{\mathbb{Q},f}$. A level structure η on M, if it exists, is an isomorphism of tensor functors $\eta : \xi \xrightarrow{\sim} H^\bullet_{\text{ét}}(M, \mathbf{A}_{\mathbb{Q},f})$.

Definition 11.5. — For a $\mu \in X_*(T)$ whose weight is defined over $\mathbb{Q}$ we define $\mathrm{Sh}(T, \mu)$ as the set of isomorphism classes of couples (M, η) where M is a CM-motive with T-structure over $\overline{\mathbb{Q}}$ with CM type μ and η a level structure on M.

There is an action "by Hecke correspondences" of $T(\mathbf{A}_{\mathbb{Q},f}) = \mathrm{Aut}^\otimes\xi$ on $\mathrm{Sh}(T, \mu)$ given by $(M, \eta) \longmapsto (M, \eta \circ t)$.

11.2. Classification. — Let ω_B be the Betti fiber functor on $\mathrm{CM}_{\overline{\mathbb{Q}}}$ and ω_{can} the canonical one on $\mathrm{Rep}_{\mathbb{Q}}T$. A CM motive over $\overline{\mathbb{Q}}$ with T-structure defines a morphism between the bands of $\mathrm{CM}_{\overline{\mathbb{Q}}}$ and the one of $\mathrm{Rep}_{\mathbb{Q}}T$. Both being abelian this defines a morphism $S \longrightarrow T$ where S is Serre's torus. Now the isomorphism classes of CM motives with T-structure associated to a same morphism of bands $S \longrightarrow T$ are in bijection with $H^1(\mathbb{Q}, T)$. This correspondence is given by sending M to the class of the torsor $\underline{\mathrm{Isom}}^\otimes(\omega_{can}, \omega_B \circ M)$. The automorphism group of a given M is canonically isomorphic to $T(\mathbb{Q})$ (à priori this would be $T'(\mathbb{Q})$ where T' is an inner form of

T, but T being abelian...).

Now, given $\mu \in X_*(T)$ as before, the M having CM type μ are those such that via $S \longrightarrow T$ the universal Hodge character in $X_*(S)$ is sent to μ and their associated class in $H^1(\mathbb{Q},T)$ becomes trivial in $H^1(\mathbb{R},T)$ (this can be deduced from subsection 9.6.1). From this one deduces.

Proposition 11.6. — *Let (T,μ) be as before. Let L be a CM field over which μ is defined. The isomorphism classes of CM motives with T-structure having CM type μ and having a level structure are in bijection with couples formed by a morphism $S^L \longrightarrow T$ s.t. $\mu^L \mapsto \mu$ and a class in* $\ker^1(\mathbb{Q},T)$. *The associated class in* $\ker^1(\mathbb{Q},T)$ *of M is the one of the torsor* $\underline{\mathrm{Isom}}^{\otimes}(\omega_{can},\omega_B\circ M)$. *The automorphism group of a given M is canonically isomorphic to $T(\mathbb{Q})$.*

11.3. Uniformization. — From the preceding proposition one deduces

$$\mathrm{Sh}(T,\mu)/T(\mathbf{A}_{\mathbb{Q},f}) \simeq \ker^1(\mathbb{Q},T)$$

And the stabilizor of a (M,η) in $T(\mathbf{A}_{\mathbb{Q},f})$ is $T(\mathbb{Q})$. Thus there is a non-canonical $T(\mathbf{A}_{\mathbb{Q},f})$-bijection

$$\mathrm{Sh}(T,\mu) = \coprod_{\ker^1(\mathbb{Q},T)} T(\mathbb{Q})\backslash T(\mathbf{A}_{\mathbb{Q},f})$$

the non-canonicity following from the fact for each class in $\ker^1(\mathbb{Q},T)$ we have to fix a base point (M,η) associated to this class.

We can make this bijection more canonical. Let $L|\mathbb{Q}$ be a Galois extension splitting T. Let $(M_1,\eta_1),(M_2,\eta_2) \in \mathrm{Sh}(T,\mu)$. Then the $T = \underline{\mathrm{Aut}}(M_1)$-torsor $\underline{\mathrm{Isom}}(M_1,M_2)$ becomes trivial over L, is canonically trivialized over $\mathbf{A}_{\mathbb{Q},f}$ by the level structures and trivial over $\mathbb{R}$. It thus defines an element of $[T(L)\backslash T(\mathbf{A}_{L,f})]^{\mathrm{Gal}(L|\mathbb{Q})}$ whose image in $\ker(H^1(\mathbb{Q},T) \longrightarrow H^1(\mathbf{A}_{\mathbb{Q},f},T))$ lies in $\ker^1(\mathbb{Q},T)$. This makes $\mathrm{Sh}(T,\mu)$ a $[T(L)\backslash T(\mathbf{A}_L)]^{\mathrm{Gal}(L|\mathbb{Q})}/T(\mathbb{R})$-torsor. And a choice of base point in $\mathrm{Sh}(T,\mu)$ defines an $T(\mathbf{A}_{\mathbb{Q},f})$-equivariant bijection

$$\mathrm{Sh}(T,\mu) \simeq [T(L)\backslash T(\mathbf{A}_L)]^{\mathrm{Gal}(L|\mathbb{Q})}/T(\mathbb{R})$$

11.4. Galois action of an automorphism fixing the reflex field and the Shimura-Taniyama reciprocity law. — Let E be the field of definition of μ.

If M is a CM motive over $\overline{\mathbb{Q}}$ with T-structure and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ one defines M^σ as the composite of M with $X \longmapsto X^\sigma$ from $\mathrm{CM}_{\overline{\mathbb{Q}}}$ to itself. One defines in the same way the Galois action on level structures using the canonical isomorphism of fiber functors $\omega_{\text{ét}} \xrightarrow{\sim} \omega_{\text{ét},\sigma}$ on $\mathrm{CM}_{\overline{\mathbb{Q}}}$.

Then $\mathrm{Gal}(\overline{\mathbb{Q}}|E)$ acts on $\mathrm{Sh}(T,\mu)$ via $(M,\eta) \longmapsto (M^\sigma,\eta^\sigma)$ and this commutes with the action of $T(\mathbf{A}_{\mathbb{Q},f})$.

We have seen in section 9.4 that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|E)$ then the fiber functors ω_B and $\omega_{B,\sigma}$ on $\mathrm{CM}^E_{\overline{\mathbb{Q}}}$ are isomorphic. From this one deduces the action of $\mathrm{Gal}(\overline{\mathbb{Q}}|E)$ on $\mathrm{Sh}(T,\mu)$ stabilizes each component in the decomposition

$$\mathrm{Sh}(T,\mu) \simeq \coprod_{\ker^1(\mathbb{Q},T)} T(\mathbb{Q})\backslash T(\mathbf{A}_{\mathbb{Q},f})$$

Now one deduces from the results of section 9 (here we use only the motivic interpretation of Serre's extension and not the whole Taniyama group) the following generalization of proposition 3.1

Proposition 11.7. — *Let $\rho : S^E \longrightarrow T$ be the unique morphism such that $\mu^E \mapsto \mu$. The action of Gal($E^{ab}|E$) on each component of the preceding decomposition of Sh(T,μ) is given via class field theory by translation by the following Hecke correspondence*

$$\pi_0(C_E) \xrightarrow{\mu^E} S^E(\mathbf{A}_{E,f})/S^E(E) \xrightarrow{N_{E/\mathbb{Q}}} S^E(\mathbf{A}_{\mathbb{Q},f})/S^E(\mathbb{Q}) \xrightarrow{\rho^{-1}} T(\mathbf{A}_{\mathbb{Q},f})/T(\mathbb{Q})$$

Remark 11.8. — Usually what is called a Shimura variety is associated to a compact open subgroup $K \subset T(\mathbf{A}_{\mathbb{Q},f})$ and is $\mathrm{Sh}_K(T,\mu) = \mathrm{Sh}(T,\mu)/K$. Then $\mathrm{Sh}_\infty(T,\mu) := \varprojlim_K Sh_K(T,\mu)$ is uniformized by

$$\mathrm{Sh}_\infty(T,\mu) \simeq \coprod_{\ker^1(\mathbb{Q},T)} \overline{T(\mathbb{Q})}\backslash T(\mathbf{A}_{\mathbb{Q},f})$$

On can explicit the preceding reciprocity law on $\mathrm{Sh}_\infty(T,\mu)$ without using Serre's torus. It is given by translation via

$$\pi_0(C_E) \xrightarrow{\mu} T(\mathbf{A}_{E,f})/\overline{T(E)} \xrightarrow{N_{E/\mathbb{Q}}} T(\mathbf{A}_{\mathbb{Q},f})/\overline{T(\mathbb{Q})}$$

11.5. Galois action of a general automorphism and the Deligne-Langlands-Shimura-Taniyama reciprocity law. — Let (T,μ) be as before. We note $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}).\mu$ for the orbit of μ. There is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ on

$$\coprod_{\mu' \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}).\mu} \mathrm{Sh}(T,\mu')$$

The purpose of the general reciprocity law is to explicit this action.

Theorem 11.9. — *Let $L \subset \overline{\mathbb{Q}}$ be a Galois CM splitting field of T. Let $\rho : S^L \longrightarrow T$ be s.t. $\rho \circ \mu^L = \mu$. Fix a base point in $Sh(T,\mu)$. Then for each $\mu' \in Gal(\overline{\mathbb{Q}}|\mathbb{Q}).\mu$ there exists an equivariant decomposition*

$$Sh(T,\mu') \xrightarrow{\sim} [T(L)\backslash T(\mathbf{A}_L)]^{Gal(L|\mathbb{Q})}/T(\mathbb{R})$$

such that the the conjugation morphism $\forall \sigma \in Gal(\overline{\mathbb{Q}}|\mathbb{Q})$

$$\begin{array}{rcl} Sh(T,\mu) & \longrightarrow & Sh(T,\sigma.\mu) \\ (M,\eta) & \longmapsto & (M^\sigma,\eta^\sigma) \end{array}$$

is given by translation by $\rho(\bar{b}(\sigma))^{-1}$ where

$$\bar{b} : Gal(L^{ab}|\mathbb{Q}) \longrightarrow \left[S^L(\mathbf{A}_{L,f})/S^L(L)\right]^{Gal(L|\mathbb{Q})}$$

is Langlands cocyle defined in section 9.9.

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Laurent Fargues, CNRS-université Paris-Sud • *E-mail :* `laurent.fargues@math.u-psud.fr`