THE ALGEBRA OF CELL-ZETA VALUES

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ABSTRACT. In this paper, we introduce *cell-forms* on $\mathfrak{M}_{0,n}$, which are top-dimensional differential forms diverging along the boundary of exactly one cell (connected component) of the real moduli space $\mathfrak{M}_{0,n}(\mathbb{R})$. We show that the cell-forms generate the top-dimensional cohomology group of $\mathfrak{M}_{0,n}$, so that there is a natural duality between cells and cell-forms. In the heart of the paper, we determine an explicit basis for the subspace of differential forms which converge along a given cell X. The elements of this basis are called *insertion forms*, their integrals over X are real numbers, called *cell-zeta values*, which generate a $\mathbb{Q}$-algebra called the cell-zeta algebra. By a result of F. Brown, the cell-zeta algebra is equal to the algebra of multizeta values. The cell-zeta values satisfy a family of simple quadratic relations coming from the geometry of moduli spaces, which leads to a natural definition of a formal version of the cell-zeta algebra, conjecturally isomorphic to the formal multizeta algebra defined by the much-studied double shuffle relations.

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1. Introduction

Let $n_1, \ldots, n_r \in \mathbb{N}$ and suppose that $n_r \geq 2$. The multiple zeta values (MZV's)

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R} \ , \tag{1.1}$$

were first defined by Euler, and have recently acquired much importance in their relation to mixed Tate motives. It is conjectured that the periods of all mixed Tate motives over $\mathbb{Z}$ are expressible in terms of such numbers. By an observation due to Kontsevich, every multiple zeta value can be written as an iterated integral:

$$\int_{0\leq t_1\leq \ldots \leq t_\ell \leq 1} \frac{dt_1 \ldots dt_\ell}{(\varepsilon_1 - t_1) \ldots (\varepsilon_\ell - t_\ell)} \ , \tag{1.2}$$

where $\varepsilon_i \in \{0,1\}$, and $\varepsilon_1 = 1$ and $\varepsilon_\ell = 0$ to ensure convergence, and $\ell = n_1 + \cdots + n_r$. The iterated integral (1.2) can be considered as a period on $\mathfrak{M}_{0,n}$ (with $n = \ell + 3$), or a period of the motivic fundamental group of $\mathfrak{M}_{0,4} = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, whose de Rham cohomology $H^1(\mathfrak{M}_{0,4})$ is spanned by the forms $\frac{dt}{t}$ and $\frac{dt}{1-t}$ [6], [8]. One proves that the multiple zeta values satisfy two sets of quadratic relations [5], [14], known as the regularised double shuffle relations, and it has been conjectured that these generate all algebraic relations between MZV's [4], [23]. This is the traditional point of view on multiple zeta values.

On the other hand, by a general construction due to Beilinson, one can view the iterated integral (1.2) as a period integral in the ordinary sense, but this time of the ℓ-dimensional affine scheme

$$\mathfrak{M}_{0,n} \simeq (\mathfrak{M}_{0,4})^{\ell} \backslash \{\text{diagonals}\} = \{(t_1, \ldots, t_\ell) : t_i \neq 0, 1 \ , t_i \neq t_j\} \ ,$$

where $n = \ell+3$. This is the moduli space of curves of genus 0 with n ordered marked points. Indeed, the open domain of integration $X = \{0 < t_1 < \ldots < t_\ell < 1\}$ is one of the connected components of the set of real points $\mathfrak{M}_{0,n}(\mathbb{R})$, and the integrand of (1.2) is a regular algebraic form in $H^{\ell}(\mathfrak{M}_{0,n})$ which converges on X. Thus, the study of multiple zeta values leads naturally to the study of all periods on $\mathfrak{M}_{0,n}$, which was initiated by Goncharov and Manin [3], [13]. These periods can be written

$$\int_X \omega \ , \qquad \text{where } \omega \in H^{\ell}(\mathfrak{M}_{0,n}) \text{ has no poles along } \overline{X} \ . \tag{1.3}$$

The general philosophy of motives and their periods [16] indicates that one should study relations between all such integrals. This leads to the following problems:

(1) Construct a good basis of all logarithmic ℓ-forms ω in $H^{\ell}(\mathfrak{M}_{0,n})$ whose integral over the cell X converges.
(2) Find all relations between the integrals $\int_X \omega$ which arise from natural geometric considerations on the moduli spaces $\mathfrak{M}_{0,n}$.

In this paper, we give an explicit solution to (1), and a family of relations which conjecturally answers (2). Firstly, we give an explicit description of a basis of the subspace of $H^{\ell}(\mathfrak{M}_{0,n})$ of forms convergent on the standard cell, in terms of the combinatorics of polygons. (Note that the idea of connecting differential forms with combinatorial structures has previously been explored from different aspects, in [10] and [22] for example.) The corresponding integrals are more general than (1.2), although Brown's theorem [3] proves that they do occur as $\mathbb{Q}$-linear combinations of multiple zeta values of the form (1.2).

For (2), we explore a new family of quadratic relations, which we call *product map relations*, because they arise from products of forgetful maps between moduli spaces. To this family we add two other simpler families; one arising from the dihedral subgroup of automorphisms of $\mathfrak{M}_{0,n}$ which stabilise X, and the other from a basic identity in the combinatorics of polygons. These families are sufficiently intrinsic and general to motivate the following conjecture, which we have verified computationally up through $n = 9$.

Conjecture. *The three families of relations between integrals (given explicitly in definition 2.28) generate the complete set of relations between periods of the moduli spaces* $\mathfrak{M}_{0,n}$.

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1.1. **Main results.** We give a brief presentation of the main objects introduced in this paper, and the results obtained using them.

Recall that Deligne-Mumford constructed a stable compactification $\overline{\mathfrak{M}}_{0,n}$ of $\mathfrak{M}_{0,n}$, such that $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ is a smooth normal crossing divisor whose irreducible components correspond bijectively to partitions of the set of n marked points into two subsets of cardinal ≥ 2 [7], [15]. The real part $\mathfrak{M}_{0,n}(\mathbb{R})$ of $\mathfrak{M}_{0,n}$ is not connected, but has $n!/2n$ connected components (open cells) corresponding to the different cyclic orders of the real points $0, t_1, \ldots, t_\ell, 1, \infty \in \mathbb{P}^1(\mathbb{R})$, up to dihedral permutation [9]. Thus, we can identify cells with n-sided polygons with edges labeled by $\{0, t_1, \ldots, t_\ell, 1, \infty\}$. In the compactification $\overline{\mathfrak{M}}_{0,n}(\mathbb{R})$, the closed cells have the structure of associahedra or Stasheff polytopes; the boundary of a given cell is a union of irreducible divisors corresponding to partitions given by the chords (cf. definition 3.3) in the associated polygon. The standard cell is the cell corresponding to the standard order we denote δ, given by $0 < t_1 < \ldots < t_\ell < 1$. We write $\mathfrak{M}_{0,n}^{\delta}$ for the space

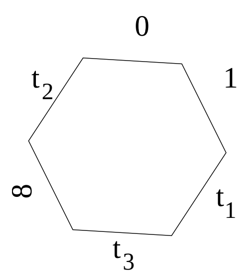
$$\overline{\mathfrak{M}}_{0,n} \setminus \{\text{all boundary divisors of } \mathfrak{M}_{0,n} \text{ except those bounding the standard cell}\}.$$

This is a smooth affine scheme introduced in [3].

1.1.1. *Polygons.* Since a cell of $\mathfrak{M}_{0,n}(\mathbb{R})$ is given by an ordering of $\{0, t_1, \ldots, t_\ell, 1, \infty\}$ up to dihedral permutation, we can identify it as above with an unoriented n-sided polygon with edges indexed by the set $\{0, t_1, \ldots, t_\ell, 1, \infty\}$.

1.1.2. *Cell-forms.* A cell-form is a holomorphic differential ℓ-form on $\mathfrak{M}_{0,n}$ with logarithmic singularities along the boundary components of the stable compactification, having the property that its singular locus forms the boundary of a single cell in the real moduli space $\mathfrak{M}_{0,n}(\mathbb{R})$.

Up to sign, the cell-form diverging on a given cell is obtained by taking the successive differences of the edges of the polygon representing that cell (ignoring ∞) as factors in the denominator. For example the cell corresponding to the cyclic order $(0, 1, t_1, t_3, \infty, t_2)$ is represented by the polygon on the left of the following figure, and the cell-form diverging along it is given on the right:



$$\longleftrightarrow \qquad \pm \frac{dt_1 dt_2 dt_3}{(t_1 - 1)(t_3 - t_1)(-t_2)}$$

Let $\mathcal{P}$ denote the $\mathbb{Q}$-vector space generated by *oriented* n-gons indexed by $\{0, 1, t_1, \ldots, t_\ell, 1, \infty\}$. The orientation fixes the sign of the corresponding cell form, and this gives a map

$$\rho : \mathcal{P} \to H^\ell(\mathfrak{M}_{0,n}). \tag{1.4}$$

In proposition 4.1 of section 4.1 we prove that this map is surjective and identify its kernel. Chapter 3 is entirely devoted to a purely combinatorial reformulation, in terms of polygons which simultaneously represent both cells and cell-forms on moduli space, of the familiar notions of convergence, divergence and residues of differential forms along divisors.

1.1.3. *Cell-form cohomology basis.* We show that cell-forms provide a good framework for studying the logarithmic differential forms on $\mathfrak{M}_{0,n}$, starting with the following result (theorem 2.12), whose proof is based on Arnol'd's well-known construction of a different basis for the cohomology group $H^\ell(\mathfrak{M}_{0,n})$.

Theorem. *The set of* 01 *cell-forms (those corresponding to polygons in which* 0 *appears next to* 1 *in the indexing of the edges) forms a basis for the cohomology group* $H^\ell(\mathfrak{M}_{0,n})$ *of top-dimensional differential forms on the moduli space.*

In particular, this shows that the cohomology group $H^\ell(\mathfrak{M}_{0,n})$ is canonically isomorphic to the subspace of $\mathcal{P}$ of polygons having 0 adjacent to 1, providing a new approach.

1.1.4. *Insertion forms.* Insertion forms (definition 4.8) are particular linear combinations of 01 cell-forms having the property given in the following theorem (theorem 4.9), one of the main results of this paper.

Theorem. *The insertion forms form a basis for the space of top-dimensional logarithmic differential forms which converge on the closure of the standard cell of* $\mathfrak{M}_{0,n}(\mathbb{R})$.

In other words, insertion forms give a basis for the cohomology group $H^\ell(\mathfrak{M}^\delta_{0,n})$ of (classes of) forms having no poles along the boundary of the standard cell of $\mathfrak{M}_{0,n}(\mathbb{R})$, so that the integral (1.3) converges, yielding a period.

The insertion forms are defined in definition 4.8, but the definition is based on the essential construction of Lyndon insertion words given in definition 3.16 and studied throughout section 3.3. The proof of this theorem uses all the polygon machinery developed in chapter 3.

1.1.5. *Cell-zeta values.* These are real numbers obtained by integrating insertion forms over the standard cell as in (1.3). They are a generalization of multiple zeta values to a larger set of periods on $\mathfrak{M}_{0,n}$, such as

$$\int_{0<t_1<t_2<t_3<1} \frac{dt_1 dt_2 dt_3}{(1-t_1)(t_3-t_1)t_2}.$$

Note that unlike the multiple zeta values, this is not an iterated integral as in (1.2).

1.1.6. *Product map relations between cell-zeta values.* Via the pullback, the maps $f : \mathfrak{M}_{0,n} \to \mathfrak{M}_{0,r} \times \mathfrak{M}_{0,s}$ obtained by forgetting disjoint complementary subsets of the marked points $t_1, \ldots, t_\ell$ yield expressions for products of cell-zeta values on $\mathfrak{M}_{0,r}$ and $\mathfrak{M}_{0,s}$ as linear combinations of cell-zeta values on $\mathfrak{M}_{0,n}$:

$$\int_{X_1} \omega_1 \int_{X_2} \omega_2 = \int_{f^{-1}(X_1 \times X_2)} f^*(\omega_1 \wedge \omega_2). \tag{1.5}$$

There is a simple combinatorial algorithm to compute the multiplication law in terms of cell-forms. This is a geometric analog of the familiar quadratic relations for multiple zeta values, and is explained in section 2.3.4.

1.1.7. *Dihedral relations between cell-zeta values.* These relations between cell-zeta values are given by

$$\int_X \omega = \int_X \sigma^*(\omega) \ , \tag{1.6}$$

where σ is an automorphism of $\mathfrak{M}_{0,n}$ which maps the standard cell to itself: $\sigma(X) = X$, and thus σ is a dihedral permutation of the marked points $\{0, 1, t_1, \ldots, t_\ell, \infty\}$.

1.1.8. *The cell-zeta value algebra $\mathcal{C}$.* The multiplication laws associated to product maps (1.5) make the space of all cell-zeta values on $\mathfrak{M}_{0,n}$, $n \geq 5$, into a $\mathbb{Q}$-algebra which we denote by $\mathcal{C}$. By Brown's theorem [3], which states essentially that all periods on $\mathfrak{M}_{0,n}$ are linear combinations of multiple zeta values, together with Kontsevitch's expression (1.2) of multiple zeta values, we obtain the following result (theorem 2.25).

Theorem. *The cell-zeta value algebra $\mathcal{C}$ is equal to the algebra of multiple zeta values $\mathcal{Z}$.*

1.1.9. *The formal cell-zeta value algebra $\mathcal{FC}$.* By lifting the previous constructions to the level of polygons along the map (1.4), we define in section 2.4 an algebra of formal cell-zeta values which we denote by $\mathcal{FC}$. It is generated by the *Lyndon insertion words* (see definition 3.16), which are formal sums of polygons corresponding to the insertion forms introduced above, subject to combinatorial versions of the product map relations (1.5) and the dihedral relations (1.6). We consider this analogous to the formal multizeta algebra $\mathcal{FZ}$, generated by formal symbols representing convergent multiple zeta values, subject only to the convergent double shuffle and Hoffmann relations ([14]). The computer calculations in low weight described in chapter 4 motivated us to make the following conjecture, which essentially says that the product map and dihedral relations (plus another simple family coming from combinatorial identities on polygons, see definition 2.28 for the complete definition of the three families of relations) generate all relations between periods of the moduli space.

Conjecture. *The formal cell-zeta algebra $\mathcal{FC}$ is isomorphic to the formal multizeta algebra $\mathcal{FZ}$.*

The paper is organized as follows. In §2, we introduce cell forms and polygons and define the three familes of relations. In §3, we define Lyndon insertion words of polygons, which may be of independent combinatorial interest. These are used to construct the insertion basis of convergent forms in §4. In §4.4, we give complete computations of this basis and the corresponding product map relations for $\mathfrak{M}_{0,n}$, where $n = 5, 6, 7$.

In the remainder of this introduction we sketch the connections between the formal cell-zeta value algebra and standard results and conjectures in the theory of multiple zeta values and mixed Tate motives.

1.2. **Relation to mixed Tate motives and conjectures.** Let $\mathcal{MT}(\mathbb{Z})$ denote the category of mixed Tate motives which are unramified over $\mathbb{Z}$ [8]. Let δ denote the standard cyclic structure on $S = \{1, \ldots, n\}$, and let B_δ denote the divisor which bounds the standard cell X. Let A_δ denote the set of all remaining divisors on $\overline{\mathfrak{M}}_{0,S} \backslash \mathfrak{M}_{0,S}$, so that $\mathfrak{M}_{0,S}^{\delta} = \overline{\mathfrak{M}}_{0,S} \setminus A_\delta$ ([3]). We write:

$$M_\delta = H^\ell(\overline{\mathfrak{M}}_{0,n} \backslash A_\delta, B_\delta \backslash (B_\delta \cap A_\delta)) \ . \tag{1.7}$$

By a result due to Goncharov and Manin [13], M_δ defines an element in $\mathcal{MT}(\mathbb{Z})$, and therefore is equipped with an increasing weight filtration W. They show that $\mathrm{gr}_\ell^W M_\delta$ is isomorphic to the de Rham cohomology $H^\ell(\mathfrak{M}_{0,n}^{\delta})$, and that $\mathrm{gr}_0^W M_\delta$ is isomorphic to the dual of the relative Betti homology $H_\ell(\overline{\mathfrak{M}}_{0,n}, B_\delta)$.

Let M be any element in $\mathcal{MT}(\mathbb{Z})$. A framing for M consists of an integer n and non-zero maps

$$v \in \mathrm{Hom}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} M) \quad \text{and} \quad f \in \mathrm{Hom}(\mathrm{gr}^W_0 M, \mathbb{Q}(0)) \ . \tag{1.8}$$

Two framed motives (M, v, f) and (M', v', f') are said to be equivalent if there is a morphism $\phi : M \to M'$ such that $\phi \circ v = v'$ and $f' \circ \phi = f$. This generates an equivalence relation whose equivalence classes are denoted $[M, v, f]$. Let $\mathcal{M}(\mathbb{Z})$ denote the set of equivalence classes of framed mixed Tate motives which are unramified over $\mathbb{Z}$, as defined in [11]. It is a commutative, graded Hopf algebra over $\mathbb{Q}$.

To every convergent cohomology class $\omega \in H^\ell(\mathfrak{M}^\delta_{0,n})$, we associate the following ℓ-framed mixed Tate motive:

$$m(\omega) = \big[M_\delta, \omega, [X]\big] \ , \tag{1.9}$$

where $[X]$ denotes the relative homology class of the standard cell. This defines a map $\mathcal{F}C \to \mathcal{M}(\mathbb{Z})$. The maximal period of $m(\omega)$ is exactly the cell-zeta value

$$\int_X \omega \ .$$

Proposition 1.1. *The dihedral symmetry relation and product map relations are motivic. In other words,*

$$\begin{aligned} m(\sigma^*(\omega)) &= m(\omega) \ , \\ m(\omega_1 \cdot \omega_2) &= m(\omega_1) \otimes m(\omega_2) \ , \end{aligned}$$

for every dihedral symmetry σ of X, and for every modular shuffle product $\omega_1 \cdot \omega_2$ of convergent forms ω_1, ω_2 on $\mathfrak{M}_{0,r}$, $\mathfrak{M}_{0,s}$ respectively.

The motivic nature of our constructions will be clear from the definitions. We therefore obtain a well-defined map m from the algebra of formal cell-zeta numbers $\mathcal{F}C$ to $\mathcal{M}(\mathbb{Z})$. On $\mathfrak{M}_{0,5}$, there is a unique element $\zeta_2 \in \mathcal{F}C$ whose period is $\zeta(2)$, which maps to 0 in $\mathcal{M}(\mathbb{Z})$.

Conjecture 1.2. $\mathcal{F}C$ is a free $\mathbb{Q}[\zeta_2]$-module, and the induced map

$$m : \mathcal{FC}/\zeta_2\mathcal{F}C \longrightarrow \widehat{\mathcal{M}}(\mathbb{Z})$$

is an isomorphism.

Since the structure of $\mathcal{M}(\mathbb{Z})$ is known, we are led to more precise conjectures on the structure of the formal cell-zeta algebra. To motivate this, let $\mathfrak{L} = \mathbb{Q}[e_3, e_5, \ldots,]$ denote the free Lie algebra generated by one element e_{2n+1} in each odd degree. Set

$$\mathfrak{F} = \mathbb{Q}[e_2] \oplus \mathfrak{L} \ .$$

The underlying graded vector space is generated by, in increasing weight:

$$e_2 \ ; \ e_3 \ ; \ e_5 \ ; \ e_7 \ ; \ [e_3, e_5] \ ; \ e_9 \ ; \ [e_3, e_7] \ ; \ [e_3, [e_5, e_3]] \, , \, e_{11} \ ; \ [e_3, e_9] \, , \, [e_5, e_7] \ ; \ \ldots \ .$$

Let $\mathcal{U}\mathfrak{F}$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{F}$. Then, setting $\widehat{\mathcal{M}}(\mathbb{Z}) = \mathcal{M}(\mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_2]$, it is known that $\widehat{\mathcal{M}}(\mathbb{Z})$ is dual to $\mathcal{U}\mathfrak{F}$. From the explicit description of $\mathfrak{F}$ given above, one can deduce that the graded dimensions $d_k = \dim_{\mathbb{Q}} \mathrm{gr}^W_k \widehat{\mathcal{M}}(\mathbb{Z})$ satisfy Zagier's recurrence relation

$$d_k = d_{k-2} + d_{k-3} \ , \tag{1.10}$$

with the initial conditions $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.

Conjecture 1.3. The dimension of the $\mathbb{Q}$-vector space of formal cell-zeta values on $\mathfrak{M}_{0,n}$, modulo all linear relations obtained from the dihedral and modular shuffle relations, is equal to d_ℓ, where $n = \ell + 3$.

We verified this conjecture for $\mathfrak{M}_{0,n}$ for $n \leq 9$ by direct calculation (see §4.4). When $n = 9$, the dimension of the convergent cohomology $H^6(\mathfrak{M}^\delta_{0,9})$ is 1089, and after taking into account all linear relations coming from dihedral and modular shuffle products, this reduces to a vector space of dimension $d_6 = 2$.

To compare this picture with the classical picture of multiple zeta values, let $\mathcal{F}Z$ denote the formal multizeta algebra. This is the quotient of the free $\mathbb{Q}$-algebra generated by formal symbols (1.2) modulo the regularised double shuffle relations. It has been conjectured that $\mathcal{F}Z$ is isomorphic to $\widehat{\mathcal{M}}(\mathbb{Z})$, and proved (cf. [21]) that the dimensions d_ℓ are actually upper bounds for the dimensions of the weight ℓ parts of $\mathcal{F}Z$. This leads us to the second main conjecture.

Conjecture 1.4. The formal algebras $\mathcal{F}C$ and $\mathcal{F}Z$ are isomorphic.

Put more prosaically, this states that the formal ring of periods of $\mathfrak{M}_{0,n}$ modulo dihedral and modular shuffle relations, is isomorphic to the formal ring of periods of the motivic fundamental group of $\mathfrak{M}_{0,4}$ modulo the regularised double shuffle relations.

By (1.2), we have a natural linear map $\mathcal{F}Z \to \mathcal{F}C$. However, at present we cannot show that it is an algebra homomorphism. Indeed, although it is easy to deduce the regularised shuffle relation for the image of $\mathcal{F}Z$ in $\mathcal{F}C$ from the dihedral and modular shuffle relations, we are unable to deduce the regularised stuffle relations. For further detail on this question, see remark 2.29 below.

Remark 1.5. The motivic nature of the regularised double shuffle relations proved to be somewhat difficult to establish [11], [12], [21]. It is interesting that the motivic nature of the dihedral and modular shuffle relations we define here is immediate.

2. The cell-zeta value algebra associated to moduli spaces of curves

Let $\mathfrak{M}_{0,n}$, $n \geq 4$ denote the moduli space of genus zero curves (Riemann spheres) with n ordered marked points $(z_1, \ldots, z_n)$. This space is described by the set of n-tuples of distinct points $(z_1, \ldots, z_n)$ modulo the equivalence relation given by the action of PSL_2. Because this action is triply transitive, there is a unique representative of each equivalence class such that $z_1 = 0$, $z_{n-1} = 1$, $z_n = \infty$. We define simplicial coordinates $t_1, \ldots, t_\ell$ on $\mathfrak{M}_{0,n}$ by setting

$$t_1 = z_2\ , \quad t_2 = z_3\ , \quad \ldots\ , \quad t_\ell = z_{n-2}, \tag{2.1}$$

where $\ell = n - 3$ is the dimension of $\mathfrak{M}_{0,n}(\mathbb{C})$. This gives the familiar identification

$$\mathfrak{M}_{0,n} \cong \{(t_1, \ldots, t_\ell) \in (\mathbb{P}^1 - \{0, 1, \infty\})^\ell \mid t_i \neq t_j \text{ for all } i \neq j\}\ . \tag{2.2}$$

2.1. Cell forms.

Definition 2.1. Let $S = \{1, \ldots, n\}$. A *cyclic structure* γ on S is a cyclic ordering of the elements of S or equivalently, an identification of the elements of S with the edges of an oriented n-gon modulo rotations. A *dihedral structure* δ on S is an identification with the edges of an unoriented n-gon modulo dihedral symmetries.

We can write a cyclic structure as an ordered n-tuple $\gamma = (\gamma(1), \gamma(2), ..., \gamma(n))$ considered up to cyclic rotations.

Definition 2.2. Let $(z_1, \dots, z_n) = (0, t_1, \dots, t_\ell, 1, \infty)$ be a representative of a point on $\mathfrak{M}_{0,n}$ as above. Let γ be a cyclic structure on S, and let σ be the unique ordering of $z_1, \dots, z_n$ compatible with γ such that $\sigma(n) = n$. The *cell-form* corresponding to γ is defined to be the differential ℓ-form

$$\omega_\gamma = [z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}] = \frac{dt_1 \cdots dt_\ell}{(z_{\sigma(2)} - z_{\sigma(1)})(z_{\sigma(3)} - z_{\sigma(2)}) \cdots (z_{\sigma(n-1)} - z_{\sigma(n-2)})}. \tag{2.3}$$

In other words, by writing the terms of $\omega_\gamma = [z_{\sigma(1)}, ..., z_{\sigma(n)}]$ clockwise around a polygon, the denominator of a cell form is just the product of successive differences $(z_{\sigma(i)} - z_{\sigma(i-1)})$ with the two factors containing ∞ simply left out.

Remark 2.3. To every dihedral structure there correspond two opposite cyclic structures. If these are given by γ and τ, then we have

$$\omega_\gamma = (-1)^n \omega_\tau. \tag{2.4}$$

Example 2.4. Let $n = 7$, and $S = \{1, \dots, 7\}$. Consider the cyclic structure γ on S given by the order 1635724. The unique ordering σ of S compatible with γ and having $\sigma(n) = n$, is the ordering 2416357, which can be depicted by writing the elements $z_{\sigma(1)}, \dots, z_{\sigma(7)}$, or 0, 1, t_2, t_4, ∞, t_1, t_3 clockwise around a circle:

$$\gamma = (z_{\sigma(1)}, \dots, z_{\sigma(7)}) = (t_1, t_3, 0, 1, t_2, t_4, \infty).$$

The corresponding cell-form on $\mathfrak{M}_{0,7}$ is

$$\omega_\gamma = [t_1, t_3, 0, 1, t_2, t_4, \infty] = \frac{dt_1 dt_2 dt_3 dt_4}{(t_3 - t_1)(-t_3)(t_2 - 1)(t_4 - t_2)} \ .$$

The symmetric group $\mathfrak{S}(S)$ acts on $\mathfrak{M}_{0,n}$ by permutation of the marked points. It therefore acts both on the set of cyclic structures γ, and also on the ring of differential forms on $\mathfrak{M}_{0,n}$. These actions coincide for cell forms.

For any cyclic structure γ on S, let $D_\gamma \subset \mathfrak{S}(S)$ denote the group of automorphisms of the dihedral structure which underlies γ, which is a dihedral group of order $2n$.

Lemma 2.5. *For every cyclic structure γ on S, we have the formula:*

$$\sigma^*(\omega_\gamma) = \omega_{\sigma(\gamma)} \qquad \textit{for all } \sigma \in \mathfrak{S}(S) \ . \tag{2.5}$$

Proof. Consider the logarithmic n-form on $(\mathbb{P}^1)^S_*$ defined by the formula:

$$\widetilde{\omega}_\gamma = \frac{dz_1 \wedge \ldots \wedge dz_n}{(z_{\gamma(1)} - z_{\gamma(2)}) \ldots (z_{\gamma(n)} - z_{\gamma(1)})} \ . \tag{2.6}$$

It clearly satisfies $\sigma^*(\widetilde{\omega}_\gamma) = \widetilde{\omega}_{\sigma(\gamma)}$ for all $\sigma \in D_\gamma$. A simple calculation shows that $\widetilde{\omega}_\gamma$ is invariant under the action of PSL_2 by Möbius transformations. Let $\pi : (\mathbb{P}^1)^S_* \to \mathfrak{M}_{0,S}$ denote the projection map with fibres isomorphic to PSL_2. There is a unique (up to scalar multiple in $\mathbb{Q}^\times$) non-zero invariant logarithmic 3-form v on $\mathrm{PSL}_2(\mathbb{C})$ which is defined over $\mathbb{Q}$. Then, by renormalising v if necessary, we have $\omega_\gamma \wedge v = \widetilde{\omega}_\gamma$. In fact, ω_γ is the unique ℓ-form on $\mathfrak{M}_{0,S}$ satisfying this equation. We deduce that $\sigma^*(\omega_\gamma) = \omega_{\sigma(\gamma)}$ for all $\sigma \in D_\gamma$. $\square$

Each dihedral structure η on S corresponds to a unique connected component of the real locus $\mathfrak{M}_{0,n}(\mathbb{R})$, namely the component associated to the set of Riemann spheres with real marked points $(z_1, \ldots, z_n)$ whose real ordering is given by η. We denote this component by $X_{S,\eta}$ or $X_{n,\eta}$. It is an algebraic manifold with corners with the combinatorial structure of a Stasheff polytope, so we often refer to it as a *cell*. A cyclic structure compatible with η corresponds to a choice of orientation of this cell.

Definition 2.6. Let δ once and for all denote the cyclic order corresponding to the ordering $(1, 2, \ldots, n)$. We call $X_{S,\delta} = X_{n,\delta}$ the *standard cell*. It is the set of points on $\mathfrak{M}_{0,n}$ given by real marked points $(0, t_1, \ldots, t_\ell, 1, \infty)$ in that cyclic order; in simplicial coordinates it is given by the standard real simplex $0 < t_1 < \ldots < t_\ell < 1$.

The distinguishing feature of cell-forms, from which they derive their name, is given in the following proposition.

Proposition 2.7. *Let η be a dihedral structure on S, and let γ be either of the two cyclic substructures of η. Then the cell form ω_γ has simple poles along the boundary of the cell $X_{S,\eta}$ and no poles anywhere else.*

Proof. Let $D \subset \overline{\mathfrak{M}}_{0,S} \backslash \mathfrak{M}_{0,S}$ be a divisor given by a partition $S = S_1 \coprod S_2$ such that $|S_i| > 1$ for $i = 1, 2$. In [3], the following notation was introduced:

$$\mathbb{I}_D(i,j) = \mathbb{I}(\{i,j\} \subset S_1) + \mathbb{I}(\{i,j\} \subset S_2) \ ,$$

where $\mathbb{I}(A \subset B)$ is the indicator function which takes the value 1 if A is contained in B and 0 otherwise. Therefore $\mathbb{I}_D(i,j) \in \{0,1\}$. Then we have

$$2\,\mathrm{ord}_D(\omega_\gamma) = (\ell - 1) - \mathbb{I}_D(\gamma(1), \gamma(2)) - \mathbb{I}_D(\gamma(2), \gamma(3)) - \ldots - \mathbb{I}_D(\gamma(n), \gamma(1)) \ . \tag{2.7}$$

To prove this, observe that $\omega_\gamma = f_\gamma \omega_0$, where

$$f_\gamma = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{(z_i - z_{i+2})}{(z_{\gamma(i)} - z_{\gamma(i+1)})} \ ,$$

and

$$\omega_0 = \frac{dt_1 \ldots dt_\ell}{t_2(t_3 - t_1)(t_4 - t_2) \ldots (t_\ell - t_{\ell-2})(1 - t_\ell)}$$

is the canonical volume form with no zeros or poles along the standard cell defined in [3]. The proof of (2.7) follows on applying proposition 7.5 from [3].

Now, (2.7) shows that ω_γ has the worst singularities when the most possible $\mathbb{I}_D(\gamma(i), \gamma(i+1))$ are equal to 1. This happens when only two of them are equal to zero, namely

$$S_1 = \{\gamma(1), \gamma(2), \ldots, \gamma(k)\} \quad \text{and} \quad S_2 = \{\gamma(k{+}1), \gamma(k{+}2), \ldots, \gamma(n)\}, \ \ 2 \le k \le n{-}2.$$

In this case, (2.7) yields $2\mathrm{ord}_D \omega_\gamma = (\ell - 1) - (n - 2) = -2$, so $\mathrm{ord}_D \omega_\gamma = -1$. In all other cases we must therefore have $\mathrm{ord}_D \omega_\gamma \ge 0$. Thus the singular locus of ω_γ is precisely given by the set of divisors bounding the cell $X_{S,\eta}$. □

2.2. **01 cell-forms and a basis of the cohomology of $\mathfrak{M}_{0,n}$.** We first derive some useful identities between certain rational functions. Let $S = \{1, \ldots, n\}$ and let $v_1, \ldots, v_n$ denote coordinates on $\mathbb{A}^n$. For every cyclic structure γ on S, let $\langle\gamma\rangle = \langle v_{\gamma(1)}, \ldots, v_{\gamma(n)}\rangle$ denote the rational function

$$\frac{1}{(v_{\gamma(2)} - v_{\gamma(1)}) \cdots (v_{\gamma(n)} - v_{\gamma(n-1)})(v_{\gamma(1)} - v_{\gamma(n)})} \in \mathbb{Z}\Big[v_i, \frac{1}{v_i - v_j}\Big] \ . \tag{2.8}$$

We refer to such a function as a cell-function. We can extend its definition linearly to $\mathbb{Q}$-linear combinations of cyclic structures. Let $X = \{x_1, \ldots, x_n\}$ denote any alphabet on n symbols. Recall that the shuffle product [18] is defined on linear combinations of words on X by the inductive formula

$$w ш e = e ш w \quad \text{and} \quad aw ш a'w' = a(w ш a'w') + a'(aw ш w') \ , \tag{2.9}$$

where w, w' are any words in X and e denotes the empty or trivial word.

Definition 2.8. Let $A, B \subset S$ such that $A \cap B = C = \{c_1, \ldots, c_r\}$ with $r \geq 1$. Let γ_A be a cyclic order on A such that the elements $c_1, \ldots, c_r$ appear in their standard cyclic order, and let γ_B be a cyclic order on B with the same property. We write $\gamma_A = (c_1, A_{1,2}, c_2, A_{2,3}, \ldots, c_r, A_{r,1})$ and $\gamma_B = (c_1, B_{1,2}, c_2, B_{2,3}, \ldots, c_r, B_{r,1})$, where the $A_{i,i+1}$, (resp. the $B_{i,i+1}$) together with C, form a partition of A (resp. B). We denote the *shuffle product* of the two cell-functions $\langle\gamma_A\rangle$ and $\langle\gamma_B\rangle$ with respect to $c_1, \ldots, c_r$ by

$$\langle\gamma_A\rangle ш_{c_1,\ldots,c_r} \langle\gamma_B\rangle$$

which is defined to be the sum of cell functions

$$\langle c_1, A_{1,2} ш B_{1,2}, c_2, A_{2,3} ш B_{2,3}, \ldots, c_r, A_{r,1} ш B_{r,1}\rangle \ . \tag{2.10}$$

The shuffle product of two cell-functions is related to their actual product by the following lemma.

Proposition 2.9. *Let $A, B \subset S$, such that $|A \cap B| \geq 2$. Let γ_A, γ_B be cyclic structures on A, B such that the cyclic structures on $A \cap B$ induced by γ_A and γ_B coincide. If $\gamma_{A\cap B}$ denotes the induced cyclic structure on $A \cap B$, we have:*

$$\frac{\langle\gamma_A\rangle \cdot \langle\gamma_B\rangle}{\langle\gamma_{A\cap B}\rangle} = \langle\gamma_A\rangle ш_{\gamma_{A\cap B}} \langle\gamma_B\rangle \ . \tag{2.11}$$

Proof. Write the cell functions $\langle\gamma_A\rangle$ and $\langle\gamma_B\rangle$ as $\langle a_{i_1}, P_1, a_{i_2}, P_2, \ldots, a_{i_r}, P_r\rangle$ and $\langle a_{i_1}, R_1, a_{i_2}, R_2, \ldots, a_{i_r}, R_r\rangle$, where P_i, R_i for $1 \leq i \leq r$ are tuples of elements in S. Let $\Delta_{ab} = (b - a)$. We will first prove the result for $r = 2$ and $P_2, R_2 = \emptyset$:

$$\Delta_{ab}\Delta_{ba}\langle a, p_1, \ldots, p_{k_1}, b\rangle\langle a, r_1, \ldots, r_{k_2}, b\rangle = \langle a, (p_1, \ldots, p_{k_1}) ш (r_1, \ldots, r_{k_2}), b\rangle. \tag{2.12}$$

We prove this case by induction on $k_1 + k_2$. Trivially, for $k_1 + k_2 = 0$ we have

$$\Delta_{ab}\Delta_{ba}\langle a, b\rangle\langle a, b\rangle = \langle a, b\rangle.$$

Now assume the induction hypothesis that

$\Delta_{ab}\Delta_{ba}\langle a, p_2, \ldots, p_{k_1}, b\rangle\langle a, r_1, \ldots, r_{k_2}, b\rangle = \langle a, \big((p_2, \ldots, p_{k_1}) ш (r_1, \ldots, r_{k_2})\big), b\rangle$ and

$\Delta_{ab}\Delta_{ba}\langle a, p_1, \ldots, p_{k_1}, b\rangle\langle a, r_2, \ldots, r_{k_2}, b\rangle = \langle a, \big((p_1, \ldots, p_{k_1}) ш (r_2, \ldots, r_{k_2})\big), b\rangle.$

To lighten the notation, let $p_2,\ldots,p_{k_1}=\underline{p}$ and $r_2,\ldots,r_{k_2}=\underline{r}$. By the shuffle recurrence formula (2.9) and the induction hypothesis:

$$\begin{aligned}\langle a,\big((p_1,\underline{p})\,ш\,(r_1,\underline{r})\big),b\rangle &= \langle a,p_1,\big((\underline{p})\,ш\,(r_1,\underline{r})\big),b\rangle+\langle a,r_1,\big((p_1,\underline{p})\,ш\,(\underline{r})\big),b\rangle\\ &= \frac{\Delta_{p_1b}\langle p_1,\big((\underline{p})\,ш\,(r_1,\underline{r})\big),b\rangle}{\Delta_{ab}\Delta_{ap_1}}+\frac{\Delta_{r_1b}\langle r_1,\big((p_1,\underline{p})\,ш\,\underline{r}\big),b\rangle}{\Delta_{ab}\Delta_{ar_1}}\\ &= \frac{\Delta_{p_1b}\Delta_{bp_1}\Delta_{p_1b}\langle p_1,\underline{p},b\rangle\langle p_1,r_1,\underline{r},b\rangle}{\Delta_{ab}\Delta_{ap_1}}+\frac{\Delta_{r_1b}\Delta_{br_1}\Delta_{r_1b}\langle r_1,p_1,\underline{p},b\rangle\langle r_1,\underline{r},b\rangle}{\Delta_{ab}\Delta_{ar_1}}\end{aligned}$$

Using identities such as $\langle p_1,\underline{p},b\rangle=\frac{\Delta_{ap_1}\Delta_{ba}}{\Delta_{bp_1}}\langle a,p_1,\underline{p},b\rangle$, this is

$$\Big[\frac{\Delta^2_{p_1b}\Delta_{bp_1}}{\Delta_{ab}\Delta_{ap_1}}\,\frac{\Delta_{ap_1}\Delta_{ba}}{\Delta_{bp_1}}\,\frac{\Delta_{ba}\Delta_{ar_1}}{\Delta_{bp_1}\Delta_{p_1r_1}}+\frac{\Delta^2_{r_1b}\Delta_{br_1}}{\Delta_{ab}\Delta_{ar_1}}\,\frac{\Delta_{ap_1}\Delta_{ba}}{\Delta_{r_1p_1}\Delta_{br_1}}\,\frac{\Delta_{ba}\Delta_{ar_1}}{\Delta_{br_1}}\Big]\langle a,p_1,\underline{p},b\rangle\langle a,r_1,\underline{r},b\rangle$$

$$=\Delta_{ab}\Big[\frac{\Delta_{ar_1}\Delta_{bp_1}}{\Delta_{p_1r_1}}+\frac{\Delta_{br_1}\Delta_{ap_1}}{\Delta_{r_1p_1}}\Big]\langle a,p_1,\underline{p},b\rangle\langle a,r_1,\underline{r},b\rangle=\Delta_{ab}\Delta_{ba}\langle a,p_1,\underline{p},b\rangle\langle a,r_1,\underline{r},b\rangle.$$

The last equality is the Plücker relation $\Delta_{ar_1}\Delta_{bp_1}-\Delta_{br_1}\Delta_{ap_1}=\Delta_{p_1r_1}\Delta_{ba}$. This proves the identity (2.12). Now, using the identity

$$\langle a_{i_1}P_1a_{i_2}P_2\ldots a_{i_r}P_r\rangle=\Delta_{a_{i_2}a_{i_1}}\langle a_{i_1}P_1a_{i_2}\rangle\times\Delta_{a_{i_3}a_{i_2}}\langle a_{i_2}P_2a_{i_3}\rangle\times\cdots\times\Delta_{a_{i_r}a_{i_1}}\langle a_{i_r}P_ra_{i_1}\rangle,$$

the general case follows from (2.12). □

Corollary 2.10. *Let X and Y be disjoint sequences of indeterminates and let e be an indeterminate not appearing in either X or Y. We have the following identity on cell functions:*

$$\langle (X,e)\,ш_e(Y,e)\rangle=\langle X\,ш\,Y,e\rangle=0. \tag{2.13}$$

Proof. Write $X=x_1,x_2,...,x_n$ and $Y=y_1,y_2,...,y_m$. By the recurrence formula for the shuffle product and proposition 2.9, we have

$$\begin{aligned}\langle X\,ш\,Y,e\rangle &= \langle x_1,(x_2,...,x_n\,ш\,y_1,...,y_m),e\rangle+\langle y_1,(x_1,...,x_n\,ш\,y_2,...,y_m),e\rangle\\ &= \langle X,e\rangle\langle x_1,Y,e\rangle(e-x_1)(x_1-e)+\langle y_1,X,e\rangle\langle Y,e\rangle(y_1-e)(e-y_1)\\ &= \frac{(e-x_1)(x_1-e)}{(x_2-x_1)\cdots(e-x_n)(x_1-e)\;(y_1-x_1)(y_2-y_1)\cdots(e-y_m)(x_1-e)}\\ &\quad+\frac{(y_1-e)(e-y_1)}{(x_1-y_1)(x_2-x_1)\cdots(e-x_n)(y_1-e)\;(y_2-y_1)\cdots(e-y_m)(y_1-e)}\\ &= \frac{(-1)+(-1)^2}{(x_2-x_1)\cdots(e-x_n)\;(y_1-x_1)(y_2-y_1)\cdots(e-y_m)}=0\;.\end{aligned}$$

□

By specialization, we can formally extend the definition of a cell function to the case where some of the terms v_i are constant, or one of the v_i is infinite, by setting

$$\langle v_1,\ldots,v_{i-1},\infty,v_{i+1},\ldots,v_n\rangle=\lim_{x\to\infty}x^2\langle v_1,\ldots,v_{i-1},x,v_{i+1},\ldots,v_n\rangle$$

$$=\frac{1}{(v_2-v_1)\ldots(v_{i-1}-v_{i-2})(v_{i+2}-v_{i+1})\ldots(v_n-v_{n-1})(v_1-v_n)}\;.$$

This is the rational function obtained by omitting all terms containing ∞. By taking the appropriate limit, it is clear that (2.11) and (2.13) are valid in this case too. In the case where $\{v_1,\ldots,v_n\}=\{0,1,t_1,\ldots,t_\ell,\infty\}$ we have the formula

$$[v_1,\ldots,v_n]=\langle v_1,\ldots,v_n\rangle\, dt_1dt_2\ldots dt_\ell\ . \tag{2.14}$$

Definition 2.11. A 01 *cyclic (resp. dihedral) structure* is a cyclic (resp. dihedral) structure on S in which the numbers 1 and $n-1$ are consecutive. Since $z_1=0$ and $z_{n-1}=1$, a 01 cyclic (or dihedral) structure is a set of orderings of the set $\{z_1,\ldots,z_n\}=\{0,t_1,\ldots,t_\ell,1,\infty\}$, in which the elements 0 and 1 are consecutive. In these terms, each dihedral structure can be written as an ordering $(0,1,\pi)$ where π is some ordering of $\{t_1,\ldots,t_\ell,\infty\}$. To each such ordering we associate a cell-function $\langle 0,1,\pi\rangle$, which is called a 01 cell-function.

Since 01 cell-functions corresponding to different π are clearly different, it follows that there exist exactly $(n-2)!$ distinct 01 cell-functions $\langle 0,1,\pi\rangle$. To these correspond $(n-2)!$ distinct 01 cell-forms $\omega_{(0,1,\pi)}=\langle 0,1,\pi\rangle\, dt_1\ldots dt_\ell$.

Theorem 2.12. *The set of* 01 *cell-forms* $\omega_{(0,1,\pi)}$*, where* π *denotes any ordering of* $\{t_1,\ldots,t_\ell,\infty\}$*, has cardinal* $(n-2)!$ *and forms a basis of* $H^\ell(\mathfrak{M}_{0,n},\mathbb{Q})$.

Proof. The proof is based on the following well-known result due to Arnol'd [1].

Theorem 2.13. *A basis of* $H^\ell(\mathfrak{M}_{0,n},\mathbb{Q})$ *is given by the classes of the forms*

$$\Omega(\underline{\varepsilon}):=\frac{dt_1\ldots dt_\ell}{(t_1-\varepsilon_1)\ldots(t_\ell-\varepsilon_\ell)}\ ,\quad \varepsilon_i\in E_i\ , \tag{2.15}$$

where $E_1=\{0,1\}$ *and* $E_i=\{0,1,t_1,\ldots,t_{i-1}\}$ *for* $2\leq i\leq \ell$.

It suffices to prove that each element $\Omega(\underline{\varepsilon})$ in (2.15) can be written as a linear combination of 01 cell-forms. We begin by expressing a given rational function $\frac{1}{(t_1-\varepsilon_1)\cdots(t_\ell-\epsilon_\ell)}$ as a product of cell-functions and then apply proposition 2.9. To every t_i, we associate its *type* $\tau(t_i)\in\{0,1\}$ (which depends on $\varepsilon_1,\ldots,\varepsilon_\ell$) as follows. If $\varepsilon_i=0$ then $\tau(t_i)=0$; if $\varepsilon_i=1$, then $\tau(t_i)=1$, but if $\varepsilon_i\neq 0,1$ then $\varepsilon_i=t_j$ for some $j<i$, and the type of t_i is defined to be equal to the type of t_j. Since the indices decrease, the type is well-defined.

We associate a cell-function F_i to each factor $(t_i-\varepsilon_i)$ in the denominator of $\Omega(\underline{\varepsilon})$ as follows:

$$F_i=\begin{cases}\ \ \langle 0,1,t_i,\infty\rangle & \text{if } \varepsilon_i=1\\ -\langle 0,1,\infty,t_i\rangle & \text{if } \varepsilon_i=0\\ \ \ \langle 0,1,\varepsilon_i,t_i,\infty\rangle & \text{if } \varepsilon_i\neq 1 \text{ and the type } \tau(t_i)=1\\ -\langle 0,1,\infty,t_i,\varepsilon_i\rangle & \text{if } \varepsilon_i\neq 0 \text{ and the type } \tau(t_i)=0\ .\end{cases} \tag{2.16}$$

We have

$$\Omega(\underline{\varepsilon})=\Delta\prod_{i=1}^{\ell}F_i\ ,$$

where

$$\Delta=\prod_{j|\varepsilon_j\neq 0,1}(-1)^{\tau(\varepsilon_j)-1}(\varepsilon_j-\tau(\varepsilon_j))$$

is exactly the factor occurring when multiplying cell-functions as in proposition 2.9. This product can be expressed as a shuffle product, which is a sum of cell-functions.

Furthermore each one corresponds to a cell beginning $0, 1, \ldots$ since this is the case for all of the F_i. The 01-cell forms thus span $H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q})$. Since there are exactly $(n-2)!$ of them, and since $\dim H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q}) = (n-2)!$, they must form a basis. □

2.3. **Pairs of polygons and multiplication.**

Definition 2.14. Let $S = \{1, \ldots, n\}$, and let $\mathcal{P}_S$ denote the $\mathbb{Q}$-vector space generated by the set of cyclic structures γ on S, i.e. by planar polygons with n sides indexed by S. Let $\tilde{\mathcal{P}}_S$ denote the $\mathbb{Q}$-vector space generated by the set of cyclic structures γ on S, modulo the relation $\gamma = (-1)^n \overleftarrow{\gamma}$, where $\overleftarrow{\gamma}$ denotes the cyclic structure with the opposite orientation to γ. Throughout this chapter we will study $\tilde{\mathcal{P}}_S$, but the full vector space $\mathcal{P}_S$ will be studied in chapter 3.

2.3.1. *Shuffles of polygons.* Let T_1, T_2 denote two subsets of $Z = \{z_1, \ldots, z_n\}$ satisfying:

$$\begin{aligned} T_1 \cup T_2 &= Z \\ |T_1 \cap T_2| &= 3 \, . \end{aligned} \tag{2.17}$$

Let $E = \{z_{i_1}, z_{i_2}, z_{i_3}\}$ denote the set of three points common to T_1 and T_2.

Definition 2.15. Consider elements γ_1 and γ_2 in $\tilde{\mathcal{P}}_S$ coming from a choice of cyclic structure on T_1 and T_2 respectively. For every such pair, define the *shuffle relative to the set E of three points of intersection*, $\gamma_1 ш_E \gamma_2$ by taking the unique liftings of γ_1 and γ_2 to elements $\bar{\gamma}_1$ and $\bar{\gamma}_2$ of $\mathcal{P}_S$ such that the cyclic order on E obtained by restricting the cyclic order $\bar{\gamma}_1$ on T_1 (resp. $\bar{\gamma}_2$ on T_2) is equal to the standard cyclic order on E, and setting

$$\gamma_1 ш_E \gamma_2 = \sum_{\substack{\bar{\gamma} \in \mathcal{P}_S \\ \bar{\gamma}|_{T_1} = \bar{\gamma}_1, \bar{\gamma}|_{T_2} = \bar{\gamma}_2}} \gamma, \tag{2.18}$$

where γ denotes the image in $\tilde{\mathcal{P}}_S$ of $\bar{\gamma} \in \mathcal{P}_S$.

We can write the shuffle with respect to three points using the following simple formula (compare with (2.10)). If $\{z_1, \ldots, z_n\} = \{0, 1, \infty, t_1, \ldots, t_\ell\}$ with $E = \{0, 1, \infty\}$, we write $\gamma_1 = (0, A_{1,2}, 1, A_{2,3}, \infty, A_{3,1})$ where T_1 is the disjoint union of $A_{1,2}, A_{2,3}, A_{3,1}$ and $0, 1, \infty$, and $\gamma_2 = (0, B_{1,2}, 1, B_{2,3}, \infty, B_{3,1})$, where T_2 is the disjoint union of $B_{1,2}, B_{2,3}, B_{3,1}$ and $0, 1, \infty$. Then $\gamma_1 ш_E \gamma_2$ is the sum of polygons in $\tilde{\mathcal{P}}_S$ given by

$$\gamma = (0, A_{1,2} ш B_{1,2}, 1, A_{2,3} ш B_{2,3}, \infty, A_{3,1} ш B_{3,1}) \, .$$

Example 2.16. Let $T_1 = \{0, 1, \infty, t_1, t_3\}$ and $T_2 = \{0, 1, \infty, t_2\}$. Let γ_1 and γ_2 denote the elements of $\tilde{\mathcal{P}}_S$ given by cyclic orders $(0, t_1, 1, t_3, \infty)$ and $(0, \infty, t_2, 1)$. Then we take the liftings $\bar{\gamma}_1 = (0, t_1, 1, t_3, \infty)$, $\bar{\gamma}_2 = (-1)^4(0, 1, t_2, \infty)$, and we find that

$$\gamma_1 ш \gamma_2 = (0, t_1, 1, t_2, t_3, \infty) + (0, t_1, 1, t_3, t_2, \infty) \in \tilde{\mathcal{P}}_S.$$

We will often write, for example, $(0, t_1, 1, t_2 ш t_3, \infty, t_4)$ for the right-hand side.

2.3.2. *Multiplying pairs of polygons: the modular shuffle relation.* In this section, we consider elements of $\tilde{\mathcal{P}}_S \otimes \tilde{\mathcal{P}}_S$. We use the notation (γ, η) for $\gamma \otimes \eta$ where $\gamma, \eta \in \tilde{\mathcal{P}}_S$. When γ and η are polygons (as opposed to linear combinations), we can associate a geometric meaning to a pair of polygons as follows. The left-hand polygon γ, which we will write using round parentheses, for example $(0, t_1, \ldots, t_\ell, 1, \infty)$, is associated to the real cell X_γ of the moduli space $\mathfrak{M}_{0,n}$ associated to the cyclic structure. The right-hand polygon η, which we will write using square parentheses, for example $[0, t_1, \ldots, t_\ell, 1, \infty]$, is associated to the cell-form ω_η associated to the cyclic structure. The pair of polygons will be associated to the (possibly divergent) integral $\int_{X_\gamma} \omega_\eta$. This geometric interpretation extends in the obvious way to all pairs of elements (γ, η). In the following section we will investigate in detail the map from pairs of polygons to integrals.

Definition 2.17. Given sets T_1, T_2 as in (2.17), the *modular shuffle product* on the vector space $\tilde{\mathcal{P}}_S \otimes \tilde{\mathcal{P}}_S$ is defined by

$$(\gamma_1, \eta_1) \,ш\, (\gamma_2, \eta_2) = (\gamma_1 \,ш\, \gamma_2, \eta_1 \,ш\, \eta_2), \tag{2.19}$$

for pairs of polygons $(\gamma_1, \eta_1) \,ш\, (\gamma_2, \eta_2)$, where γ_i and η_i are cyclic structures on T_i for $i = 1, 2$.

Example 2.18. The following product of two polygon pairs is given by

$$\big((0, t_1, 1, \infty, t_4), [0, \infty, t_1, t_4, 1]\big)\big((0, t_2, 1, t_3, \infty), [0, t_3, t_2, \infty, 1]\big)$$
$$= -\big((0, t_1 \,ш\, t_2, 1, t_3, \infty, t_4), [0, t_3, t_2, \infty, t_1, t_4, 1]\big).$$

Let us now explain the geometric meaning of the modular shuffle product (2.19), in terms of integrals of forms on moduli space. Recall that a *product map* between moduli spaces was defined in[3] as follows. Let T_1, T_2 denote two subsets of $Z = \{z_1, \ldots, z_n\}$ as in (2.17), Then we can consider the product of forgetful maps:

$$f = f_{T_1} \times f_{T_2} : \mathfrak{M}_{0,n} \longrightarrow \mathfrak{M}_{0,T_1} \times \mathfrak{M}_{0,T_2}\ . \tag{2.20}$$

The map f is a birational embedding because

$$\dim \mathfrak{M}_{0,S} = |S| - 3 = |T_1| - 3 + |T_2| - 3 = \dim \mathfrak{M}_{0,T_1} \times \mathfrak{M}_{0,T_2}\ .$$

If f is a product map as above and z_i, z_j, z_k are the three common points of T_1 and T_2, use an element $\alpha \in \mathrm{PSL}_2$ to map z_i to 0, z_j to 1 and z_k to ∞. Let $t_1, \ldots, t_\ell$ denote the images of $z_1, \ldots, z_n$ (excluding z_i, z_j, z_k) under α. Given the indices i, j and k, the product map is then determined by specifying a partition of $\{t_1, \ldots, t_\ell\}$ into S_1 and S_2. We use the notation $T_i = \{0, 1, \infty\} \cup S_i$ for $i = 1, 2$.

The shuffle product formula (2.19) on pairs of polygons is motivated by the formula for multiplying integrals given in the following proposition.

Proposition 2.19. *Let $S = \{1, \ldots, n\}$, and let T_1 and T_2 be subsets of S as in (2.17), of orders $r + 3$ and $s + 3$ respectively. Let ω_1 (resp. ω_2) be a cell-form on $\mathfrak{M}_{0,r}$ (resp. on $\mathfrak{M}_{0,s}$), and let γ_1 and γ_2 denote cyclic orderings on T_1 and T_2. Then the product rule for integrals is given by the following formula, called the modular shuffle relation:*

$$\int_{X_{\gamma_1}} \omega_1 \int_{X_{\gamma_2}} \omega_2 = \int_{X_{\gamma_1 \,ш\, \gamma_2}} \omega_1 \,ш\, \omega_2, \tag{2.21}$$

where $\omega_1 \,ш\, \omega_2$ converges on the cell X_γ for each term γ in $\gamma_1 \,ш\, \gamma_2$.

Proof. The subsets T_1 and T_2 correspond to a product map

$$f : \mathfrak{M}_{0,n} \to \mathfrak{M}_{0,r} \times \mathfrak{M}_{0,s}.$$

The pullback formula gives a multiplication law on the pair of integrals:

$$\int_{X_{\gamma_1}} \omega_1 \int_{X_{\gamma_2}} \omega_2 = \int_{X_{\gamma_1} \times X_{\gamma_2}} \omega_1 \wedge \omega_2 = \int_{f^{-1}(X_{\gamma_1} \times X_{\gamma_2})} f^*(\omega_1 \wedge \omega_2). \tag{2.22}$$

The preimage $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$ decomposes into a disjoint union of cells of $\mathfrak{M}_{0,n}$, which are precisely the cells given by cyclic orders of $\gamma_1 ш \gamma_2$. In other words,

$$f^{-1}(X_{\gamma_1} \times X_{\gamma_2}) = \sum_{\gamma \in \gamma_1 ш \gamma_2} X_\gamma \ ,$$

where the sum denotes a disjoint union. Now we can assume without loss of generality that $T_1 = \{0, 1, \infty, t_1, \ldots, t_k\}$, $T_2 = \{0, 1, \infty, t_{k+1}, \ldots, t_\ell\}$ and that δ_1, δ_2 are the cyclic structures on T_1, T_2 corresponding to ω_1, ω_2, respectively, where δ_1, δ_2 restrict to the standard cyclic order on $0, 1, \infty$. Then, in cell function notation,

$$f^*(\omega_1 \wedge \omega_2) = \langle \delta_1 \rangle \langle \delta_2 \rangle \, dt_1 \ldots dt_\ell = \frac{\langle \delta_1 ш_{\{0,1,\infty\}} \delta_2 \rangle}{\langle 0, 1, \infty \rangle} \, dt_1 \ldots dt_\ell = \omega_1 ш \omega_2 \ ,$$

by proposition 2.9. Since ω_1 and ω_2 converge on the closed cells $\overline{X}_{\gamma_1}$ and $\overline{X}_{\gamma_2}$ respectively, $\omega_1 \wedge \omega_2$ has no poles on the contractible set $\overline{X}_{\gamma_1} \times \overline{X}_{\gamma_2}$, and therefore $\omega_1 ш \omega_2 = f^*(\omega_1 \wedge \omega_2)$ has no poles on the closure of $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$. But $\sum_{\gamma \in \gamma_1 ш \gamma_2} X_\gamma$ is a cellular decomposition of $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$, so, in particular, $\omega_1 ш \omega_2$ can have no poles along the closure of each cell X_γ, where $\gamma \in \gamma_1 ш \gamma_2$. $\square$

2.3.3. $\mathfrak{S}(n)$ *action on pairs of polygons.* The symmetric group $\mathfrak{S}(n)$ acts on a pair of polygons by permuting their labels in the obvious way, and this extends to the vector space $\tilde{\mathcal{P}}_S \otimes \tilde{\mathcal{P}}_S$ by linearity. If $\tau : \mathfrak{M}_{0,n} \to \mathfrak{M}_{0,n}$ is an element of $\mathfrak{S}(n)$, then the corresponding action on integrals is given by the pullback formula:

$$\int_{X_\gamma} \omega_\eta = \int_{\tau(X_\gamma)} \tau^*(\omega_\eta) = \int_{X_{\tau(\gamma)}} \omega_{\tau(\eta)} \ . \tag{2.23}$$

Suppose that τ belongs to the dihedral group which preserves the dihedral structure underlying a cyclic structure γ. Let $\epsilon = 1$ if τ preserves γ, and $\epsilon = -1$ if τ reverses its orientation. We have the following *dihedral relation* between convergent integrals:

$$\int_{X_\gamma} \omega_\eta = (-1)^\epsilon \int_{X_\gamma} \tau^*(\omega_\eta) = (-1)^\epsilon \int_{X_\gamma} \omega_{\tau(\eta)}. \tag{2.24}$$

Both the formulas (2.23) and (2.24) extend to linear combinations of integrals of cell-forms as long as the linear combination converges over the integration domain. This convergence is not a consideration when working with pairs of polygons rather than integrals.

Example 2.20. The form corresponding to $\zeta(2,1)$ on $\mathfrak{M}_{0,6}$ is

$$\frac{dt_1 dt_2 dt_3}{(1-t_1)(1-t_2)t_3} = [0, 1, t_1, t_2, \infty, t_3] + [0, 1, t_2, t_1, \infty, t_3],$$

which gives $\zeta(2,1)$ after integrating over the standard cell. By applying the rotation $(1,2,3,4,5,6)$, a dihedral rotation of the standard cell, to this form, one obtains

$$\begin{aligned}[t_1,\infty,t_2,t_3,0,1]+[t_1,\infty,t_3,t_2,0,1] &= [0,1,t_1,\infty,t_2,t_3]+[0,1,t_1,\infty,t_3,t_2]\\ &= \frac{dt_1dt_2dt_3}{(1-t_1)t_2t_3},\end{aligned}$$

which gives $\zeta(3)$ after integrating over the standard cell. Therefore, we have the following relation on linear combinations of pairs of polygons:

$$\begin{aligned}&\big((0,t_1,t_2,t_3,1,\infty),[0,1,t_1,t_2,\infty,t_3]+[0,1,t_2,t_1,\infty,t_3]\big)\\ &\quad=\big((0,t_1,t_2,t_3,1,\infty),[0,1,t_1,\infty,t_2,t_3]+[0,1,t_1,\infty,t_3,t_2]\big)\end{aligned}\tag{2.25}$$

which on the level of integrals corresponds to

$$\zeta(2,1)=\int_{X_{3,\delta}}\frac{dt_1dt_2dt_3}{t_3(1-t_2)(1-t_1)}=\int_{X_{3,\delta}}\frac{dt_1dt_2dt_3}{t_3t_2(1-t_1)}=\zeta(3).$$

Remark 2.21. This identity is an example of the well-known duality relation between multiple zeta values given as follows. Every tuple $(n_1,\ldots,n_r)$ of positive integers with $n_1>1$ is uniquely associated to a word $x^{n_1-1}y\cdots x^{n_r-1}y$ in non-commutative variables x, y. Let $(m_1,\ldots,m_s)$ be the tuple thus associated to the word $xy^{n_r-1}\cdots xy^{n_1-1}$. The duality relation is

$$\zeta(n_1,\ldots,n_r)=\zeta(m_1,\ldots,m_s).$$

This relation follows from the dihedral relation above, using the reflection permutation corresponding to the reflection of the polygon $(0,1,t_1,\ldots,t_{n-3},\infty)$ over the symmetry axis through the side labeled ∞.

2.3.4. *Standard pairs and the product map relations.* A standard pair of polygons is a pair (δ,η) where the left-hand polygon is the standard cyclic structure. Let $S=\{1,\ldots,n\}$, and $T_1\cup T_2=S$ with $T_1\cap T_2=\{0,1,\infty\}$ be as above, and let γ_1 and γ_2 be cyclic orders on T_1 and T_2. In the present section we show how for each such γ_1,γ_2, we can modify the modular shuffle relation to construct a multiplication law on standard pairs.

Definition 2.22. Let δ_1 and δ_2 denote the standard orders on T_1 and T_2. Then there is a unique permutation τ_i mapping δ_i to γ_i such that $\tau_i(0)=0$, for $i=1,2$. The multiplication law, denoted by the symbol $\times$, and called the *product map relation*, is defined by

$$\begin{aligned}(\delta_1,\omega_1)\times(\delta_2,\omega_2) &= (\gamma_1,\tau_1(\omega_1))\,ш\,(\gamma_2,\tau_2(\omega_2))\\ &= (\gamma_1\,ш\,\gamma_2,\tau_1(\omega_1)\,ш\,\tau_2(\omega_2))\\ &= \sum_{\gamma\in\gamma_1\,ш\,\gamma_2}(\delta,\tau_\gamma^{-1}(\tau_1(\omega_1)\,ш\,\tau_2(\omega_2))),\end{aligned}\tag{2.26}$$

where for each $\gamma\in\gamma_1\,ш\,\gamma_2$, τ_γ is the unique permutation such that $\tau_\gamma(\delta)=\gamma$ and $\tau_\gamma(0)=0$.

Example 2.23. Let $S = \{0, 1, \infty, t_1, t_2, t_3, t_4\}$, $T_1 = \{0, 1, \infty, t_1, t_4\}$ and $T_2 = \{0, 1, \infty, t_2, t_3\}$. Let the cyclic orders on T_1 and T_2 be given by $\gamma_1 = (0, t_1, 1, \infty, t_4)$ and $\gamma_2 = (0, t_2, 1, t_3, \infty)$. Applying the product map relation to the pairs of polygons below yields

(2.27)
$$
\begin{aligned}
&\big((0,t_1,t_4,1,\infty),[0,1,t_1,\infty,t_4]\big) \times \big((0,t_2,t_3,1,\infty),[0,1,t_2,\infty,t_3]\big)\\
&\qquad = \big((0,t_1,1,\infty,t_4),[0,\infty,t_1,t_4,1]\big) ш \big((0,t_2,1,t_3,\infty),[0,t_3,t_2,\infty,1]\big)\\
&\qquad = -\big((0,t_1,t_2,1,t_3,\infty,t_4),[0,t_3,t_2,\infty,t_1,t_4,1]\big)\\
&\qquad\qquad - \big((0,t_2,t_1,1,t_3,\infty,t_4),[0,t_3,t_2,\infty,t_1,t_4,1]\big)\\
&\qquad = \big((0,t_1,t_2,t_3,t_4,1,\infty),[0,t_3,\infty,t_1,1,t_2,t_4] + [0,t_3,\infty,t_2,1,t_1,t_4].
\end{aligned}
$$

In terms of integrals, this corresponds to the relation

$$
\begin{aligned}
\zeta(2)^2 &= \int_{X_{5,\delta}} \frac{dt_1dt_4}{(1-t_1)t_4} \int_{X_{5,\delta}} \frac{dt_2dt_3}{(1-t_2)t_3}\\
&= \int_{X_{7,\delta}} \frac{dt_1dt_2dt_3dt_4}{t_4(t_4-t_2)(1-t_2)(1-t_1)t_3} + \frac{dt_1dt_2dt_3dt_4}{t_4(t_4-t_1)(1-t_1)(1-t_2)t_3}
\end{aligned}
\tag{2.28}
$$

We will show in §4.4 that the last two integrals evaluate to $\frac{7}{10}\zeta(2)^2$ and $\frac{3}{10}\zeta(2)^2$ respectively.

2.4. The algebra of cell-zeta values.

Definition 2.24. Let $\mathcal{C}$ denote the $\mathbb{Q}$-subvector space of $\mathbb{R}$ generated by the integrals $\int_{X_{n,\delta}} \omega$, where $X_{n,\delta}$ denotes the standard cell of $\mathfrak{M}_{0,n}$ for $n \geq 5$ and ω is a holomorphic ℓ-form on $\mathfrak{M}_{0,n}$ with logarithmic singularities at infinity (thus a linear combination of 01 cell-forms) which converges on $X_{n,\delta}$. We call these numbers *cell-zeta values*. The existence of product map multiplication laws in proposition 2.19 imply that $\mathcal{C}$ is in fact a $\mathbb{Q}$-algebra.

Theorem 2.25. *The $\mathbb{Q}$-algebra $\mathcal{C}$ of cell-zeta values is isomorphic to the $\mathbb{Q}$-algebra $\mathcal{Z}$ of multizeta values.*

Proof. Multizeta values are real numbers which can all be expressed as integrals $\int_{X_{n,\delta}} \omega$ where ω is an ℓ-form of the form

$$
\omega = (-1)^d \prod_{i=1}^{\ell} \frac{d\underline{t}}{t_i - \epsilon_i},
\tag{2.29}
$$

where $\epsilon_1 = 0$, $\epsilon_i \in \{0, 1\}$ for $2 \leq i \leq \ell - 1$, $\epsilon_\ell = 1$, and d denotes the number of i such that $\epsilon_i = 1$. Since each such form converges on $X_{n,\delta}$, the multizeta algebra $\mathcal{Z}$ is a subalgebra of $\mathcal{C}$. The converse is a consequence of the following theorem due to F. Brown [3].

Theorem 2.26. *If ω is a holomorphic ℓ-form on $\mathfrak{M}_{0,n}$ with logarithmic singularities at infinity and convergent on $X_{n,\delta}$, then $\int_{X_{n,\delta}} \omega$ is $\mathbb{Q}$-linear combination of multizeta values.*

Thus, $\mathcal{C}$ is also a subalgebra of $\mathcal{Z}$, proving the equality. □

The structure of the formal multizeta algebra, generated by symbols (formally representing integrals of the form (2.29)) subject to relations such as shuffle and stuffle relations, has been much studied. The present article provides a different approach to the study of this algebra, by turning instead to the study of a formal version of $\mathcal{C}$.

Definition 2.27. Let $|S| \geq 5$. The *formal algebra of cell-zeta values* $\mathcal{FC}$ is defined as follows. Let $\mathcal{A}$ be the vector space of formal linear combinations of standard pairs of polygons in $\tilde{\mathcal{P}}_S \otimes \tilde{\mathcal{P}}_S$

$$\sum_i a_i(\delta, \omega_i)$$

such that the associated ℓ-form $\sum_i a_i \omega_i$ converges on the standard cell $X_{n,\delta}$. Let $\mathcal{FC}$ denote the quotient of $\mathcal{A}$ by the following families of relations.

Definition 2.28. The three families of relations defining $\mathcal{FC}$ are as follows:

- *Product map relations.* These relations were defined in section 2.3. For every choice of subsets T_1, T_2 of $S = \{1, \ldots, n\}$ such that $T_1 \cup T_2 = S$ and $|T_1 \cap T_2| = 3$, and every choice of cyclic orders γ_1, γ_2 on T_1, T_2, formula (2.26) gives a multiplication law expressing the product of any two standard pairs of polygons of sizes $|T_1|$ and $|T_2|$ as a linear combination of standard pairs of polygons of size n.
- *Dihedral relations.* For σ in the dihedral group associated to δ, i.e. $\sigma(\delta) = \pm\delta$, there is a dihedral relation $(\delta, \omega) = (\sigma(\delta), \sigma(\omega))$.
- *Shuffles with respect to one element.* The linear combinations of pairs of polygons $(\delta, (A, e) ш_e (B, e))$ where A and B are disjoint of length $n-1$ are zero, as in (2.13).

With the goal of approaching the combinatorial conjectures given in the introduction, the purpose of the next chapters is to give an explicit combinatorial description of a set of generators for $\mathcal{FC}$. We do this in two steps. First we define the notion of a linear combination of polygons convergent with respect to a chord of the standard polygon δ, and thence, the notion of a linear combination of polygon convergent with respect to the standard polygon. We exhibit an explicit basis, the basis of *Lyndon insertion words and shuffles* for the subspace of such linear combinations. In the subsequent chapter, we deduce from this a set of generators for the formal cell-zeta value algebra $\mathcal{FC}$ and also, as a corollary, a basis for the subspace of the cohomology space $H^\ell(\mathfrak{M}_{0,n})$ consisting of classes of forms converging on the standard cell.

Remark 2.29. One of the most intriguing and important questions concerning $\mathcal{FC}$ is the conjectural isomorphism with the algebra of formal multizeta values $\mathcal{FZ}$ mentioned earlier in conjecture 1.4. In fact, there is a very natural "candidate map" from the generators of $\mathcal{FZ}$ to elements of $\mathcal{FC}$, coming from simply mapping the differential forms in (1.2) to the corresponding form in the convergent cohomology group $H^\ell(\mathfrak{M}^\delta_{0,n})$ (an explicit expression in terms of the basis is given in formula (4.7) below). However, in order to yield an algebra morphism, this map would have to respect the regularized double shuffle relations on the multizeta values. The shuffle relation is easy to obtain on the images, using the shuffle product maps corresponding to the partition of $(0, t_1, \ldots, t_\ell, 1, \infty)$ into $(0, t_1, \ldots, t_m, 1, \infty)$ and

$(0, t_{m+1}, \ldots, t_\ell, 1, \infty)$ for $2 \leq m \leq \ell - 2$ (cf. [3]). Likewise, one could hope that the stuffle relations would follow from the so-called stuffle product maps defined in [3]. These maps can be expressed very simply in terms of the cubical coordinates $x_1, \ldots, x_\ell$ defined by $t_1 = x_1 \cdots x_\ell$, $t_2 = x_2 \cdots x_\ell, \ldots, t_\ell = x_\ell$, as

$$(0, x_1, \ldots, x_\ell, 1, \infty) \mapsto (0, x_1, \ldots, x_m, 1, \infty) \times (0, x_{m+1}, \ldots, x_\ell, 1, \infty)$$

(it is easy to see that this is indeed a product map [3]). However, computing the product of two multizeta values as a sum using this product map yields a sum of cell-zeta values which is not obviously equal to a sum of multiple zeta values (let alone the desired stuffle sum).

By a method due to P. Cartier, the stuffle relations on multizeta values written as integrals of the differential forms ω in (1.2) written in cubical coordinates can be proved using variable changes of the form

$$\int_{[0,1]^\ell} \omega = \int_{[0,1]^\ell} \sigma^*(\omega) \tag{2.30}$$

for σ any permutation of the ℓ coordinates $x_1, \ldots, x_\ell$. We could choose to forcibly add the relations (2.30), for all forms ω such that both ω and $\sigma^*(\omega)$ are defined on $\mathfrak{M}_{0,n}$ and convergent on the standard cell. This would ensure the validity of the stuffle relations on multiple zeta values inside $\mathcal{FC}$. However, we have abstained from doing so in the hopes that some possibly weaker conditions may be deduced from our relations and imply the stuffle, hence giving a morphism $\mathcal{F}Z \to \mathcal{FC}$ with the definition of $\mathcal{FC}$ above. This certainly occurs experimentally up to $n = 9$. The paper [19] by I. Soudères takes up this question in the context of motivic multiple zeta values.

Remark 2.30. By analogy with the situation for mixed Tate motives and formal multizeta values, we expect that the formal cell-zeta value algebra will be a Hopf algebra. However, we have not yet determined an explicit coproduct.

3. Polygons and convergence

The present chapter is devoted to redefining certain familiar geometric notions from the moduli space situation: differential forms, divisors, convergence of forms on cells, divergence of forms along divisors, residues, etc., in the completely combinatorial setting of polygons.

In this setting, the twin notions of cells and cell-forms are simultaneously replaced by the single notion of a polygon, as explained in the previous chapters. Boundary divisors then correspond to chords of polygons, and the issues of divergence become entirely symmetric, with a chord of one polygon being "a bad chord" for another if the latter corresponds to a form which diverges along the divisor represented by the bad chord. This language makes it much easier to discuss residue calculations, convergence of linear combinations of polygons along bad chords, and most importantly, convergence of linear combinations of polygons with respect to the standard polygon δ. In the main result of this chapter, we exhibit an explicit basis for the space of linear combinations of polygons convergent with respect to the standard polygon, consisting of linear combinations called *Lyndon insertion words and Lyndon insertion shuffles*. This result will be key in the following chapter to determining an explicit basis for the space of holomorphic differential ℓ-forms on $\mathfrak{M}_{0,n}$ with logarithmic singularities at the boundary, that converge on the standard cell δ. The integrals of these basis elements, baptized *cell-zeta values*, form the basic generating set of our algebra of cell-zeta values, and it is the polygon construction given here that allows us to define a set of formal cell-zeta values generating the corresponding, combinatorially defined, formal cell-zeta algebra.

3.1. **Bad chords and polygon convergence.** For any finite set R of cardinality n, let $\mathcal{P}_R$ denote the $\mathbb{Q}$ vector space of linear combinations of *polygons on* R, i.e. cyclic structures on R, identified with planar polygons with edges indexed by R, as in definition 2.14 from section 2.3.

Let $\mathcal{V}$ denote the free polynomial shuffle algebra on the alphabet of positive integers, and let V be the quotient of $\mathcal{V}$ by the relations $w = 0$ if w is a word in which any letter appears more than once (these relations imply that $w \sqcup\!\sqcup\, w' = 0$ if w and w' are not disjoint). A basis for $\mathcal{V}$ is usually taken to be the set of all words w, but a theorem of Radford ([17] or [18], Theorem 6.1 (i)), gives an alternative basis for $\mathcal{V}$ which we use here.

Definition 3.1. Put the lexicographic ordering on the set of all words in a given ordered alphabet $\mathcal{A}$. A *Lyndon word* w in the alphabet is a word having the following property: for every way of cutting the word w into two non-trivial pieces w_1 and w_2 (so w is the concatenation $w_1 w_2$), the word w_2 is greater than w itself for the lexicographical order. The *Lyndon basis* for the vector space generated by words in $\mathcal{A}$ is given by Lyndon words and shuffles of Lyndon words.

Consider the image of the Lyndon basis of $\mathcal{V}$ under the quotient map $\mathcal{V} \to V$. The elements of this basis which do not map to zero remain linearly independent in V, whose basis thus consists of Lyndon words with distinct letters – such a word is Lyndon if and only if the smallest character appears on the left – and shuffles of disjoint Lyndon words with distinct letters. Throughout this chapter, we work in V, so that when we refer to a 'word', we automatically mean a word with distinct letters, and shuffles of such words are zero unless the words are disjoint. Let V_S be the subspace of V spanned by the $n!$ words of length n with distinct letters in the characters of $S = \{1, \ldots, n\}$. Then the Lyndon basis for V_S is given by the $(n-1)!$ Lyndon words of degree n and the $(n-1)\cdot(n-1)!$ shuffles of disjoint Lyndon words the union of whose letters is equal to S.

Recall from definition 2.14 that the vector space $\mathcal{P}_S$ is generated by cyclic structures on $\{1, \ldots, n\}$, identified with planar n-polygons with edges indexed by S. If we consider $(n+1)$-polygons with edges indexed by $S \cup \{d\}$ for some new letter $d \notin S$, we have a natural isomorphism

$$V_S \xrightarrow{\sim} \mathcal{P}_{S \cup \{d\}} \tag{3.1}$$

given by writing each cyclic structure on $S \cup \{d\}$ as a word on the letters of S followed by the letter d.

Definition 3.2. Let $I_S \subset \mathcal{P}_{S\cup\{d\}}$ be the subspace linearly generated by shuffles of polygons $(A \sqcup\!\sqcup\, B, d)$, where $A \cup B = S$, $A \cap B = \emptyset$ and $A, B \neq \emptyset$. Here, a shuffle of polygons simply refers to the linear combination of polygons indexed by the words in the shuffle sum $(A \sqcup\!\sqcup\, B, d)$.

Then under the isomorphism (3.1), I_S is identified with the subspace of V_S generated by the part of the Lyndon basis consisting of shuffles. By a slight abuse of notation, we use the same notation I_S for the corresponding subspaces of $\mathcal{P}_{S\cup\{d\}}$ and of V_S.

Definition 3.3. Let $D = S_1 \cup S_2$ denote a stable partition of S (partition into two disjoint subsets of order ≥ 2). Let γ be a polygon on S. We say that the partition D corresponds to a *chord of* γ if the polygon γ admits a chord which cuts γ into two pieces indexed by S_1 and S_2. The sets S_1, S_2 are called *blocks* associated to the chord D. Thus, a chord divides γ into two blocks, and the set of chords $\chi(\gamma)$ indexes the set of stable partitions which are *compatible* with γ in the sense that the subsets S_1 and S_2 of the partition are blocks of γ.

Definition 3.4. Let γ, η denote two polygons on S. We say that η is *convergent relative to* γ if there are no stable partitions of S compatible with both γ and η:

$$\chi(\gamma) \cap \chi(\eta) = \emptyset \ . \tag{3.2}$$

In other words, there exists no block of γ having the same underlying set as a block of η. If η is a polygon on S, then a block of η is said to be a *consecutive block* if its underlying set corresponds to a block of the polygon with the standard cyclic order δ. The polygon η is said to be *convergent* if it has no consecutive blocks at all, i.e., if it is convergent relative to δ. A polygon $\eta \in \mathcal{P}_{S\cup\{d\}}$ is said to be convergent if it has no chords partitioning $S \cup \{d\}$ into disjoint subsets $S_1 \cup S_2$ such that S_1 is a consecutive subset of $S = \{1, \ldots, n\}$.

Definition 3.5. We now adapt the definition of convergence for polygons in $\mathcal{P}_{S\cup\{d\}}$ to the corresponding words in V_S. A *convergent word* in the alphabet S is a word having no subword which forms a consecutive block. In other words, if $w = a_{i_1}a_{i_2}\cdots a_{i_r}$, then w is convergent if it has no subword $a_{i_j}a_{i_{j+1}}\cdots a_{i_k}$ such that the underlying set $\{a_{i_j}, a_{i_{j+1}}, \ldots, a_{i_k}\} = \{i, i+1, \ldots, i+r\} \subset \{1, \ldots, n\}$. A convergent word is in fact the image in V_S of a convergent polygon in $\mathcal{P}_{S\cup\{d\}}$ under the isomorphism (3.1).

Example 3.6. When $1 \leq n \leq 4$ there are no convergent polygons in $\mathcal{P}_S$. For $n = 5$, there is only one convergent polygon up to sign, given by $\gamma = (13524)$. The other convergent cyclic structure (14253) is just the cyclic structure (13524) written backwards. When $n = 6$, there are three convergent polygons up to sign:

$$(135264)\ , \quad (152463)\ , \quad (142635)\ .$$

There are 23 convergent polygons for $n = 7$. Note that when $n = 8$, the dihedral structure $\eta = (24136857)$ is not convergent even though no neighbouring numbers are adjacent, because $\{1, 2, 3, 4\}$ forms a consecutive block for both η and δ.

Remark 3.7. The enumeration of permutations satisfying the single condition that no two adjacent elements in γ should be consecutive (the case $k = 2$) is known as the dinner table problem and is a classic problem in enumerative combinatorics. The more general problem of convergent words (arbitrary k) seems not to have been studied previously. The problems coincide for $n \leq 7$, but the counterexample for $n = 8$ above shows that the problems are not equivalent for $n \geq 8$.

3.2. **Residues of polygons along chords.** In this section, we give a combinatorial definition on polygons generalizing the notion of the residue of a differential form at a boundary divisor along which it diverges.

Definition 3.8. (*Polygon residues*) For every stable partition D of S given by $S = S_1 \cup S_2$, we define a residue map on polygons

$$\mathrm{Res}^p_D : \mathcal{P}_S \longrightarrow \mathcal{P}_{S_1 \cup \{d\}} \otimes_{\mathbb{Q}} \mathcal{P}_{S_2 \cup \{d\}}$$

as follows. Let η be a polygon in $\mathcal{P}_S$. If the partition D corresponds to a chord of η, then it cuts η into two subpolygons η_i $(i = 1, 2)$ whose edges are indexed by the set S_i and an edge labelled d corresponding to the chord D. We set

$$\mathrm{Res}^p_D(\eta) = \begin{cases} \eta_1 \otimes \eta_2 & \text{if } D \text{ is a chord of } \eta \\ 0 & \text{if } D \text{ is not a chord of } \eta. \end{cases} \tag{3.3}$$

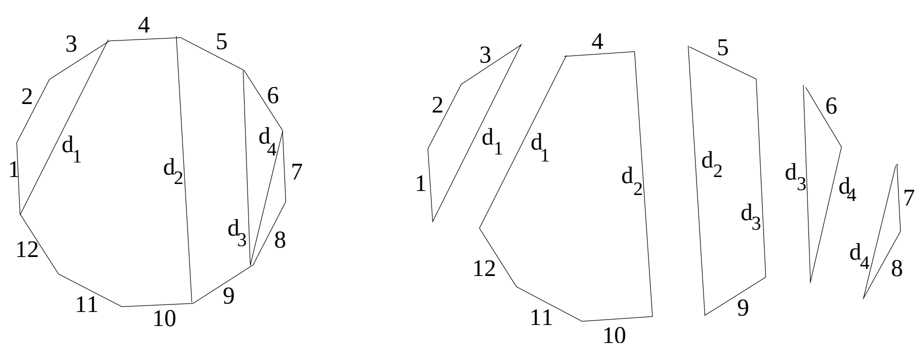
More generally, we can define the residue for several disjoint chords simultaneously. Let $S = S_1 \cup \cdots \cup S_{r+1}$ be a partition of S into $r + 1$ disjoint subsets with $r \geq 2$. For $1 \leq i \leq r$, let D_i be the partition of S into the two subsets $(S_1 \cup \cdots S_i) \cup (S_{i+1} \cup \cdots \cup S_{r+1})$. For any polygon $\eta \in \mathcal{P}_S$, we say that η admits the chords $D_1, \ldots, D_r$ if there exist r chords of η, disjoint except possibly for endpoints, partitioning the edges of η into the sets $S_1, \ldots, S_{r+1}$. If η admits the chords $D_1, \ldots, D_r$, then these chords cut η into $r+1$ subpolygons $\eta_1, \ldots, \eta_{r+1}$. Let T_i denote the set indexing the edges of η_i, so that each T_i is a union of S_i and elements of the set $\{d_1, \ldots, d_r\}$ of indices of the chords. The *composed residue map*

$$\mathrm{Res}^p_{D_1,\ldots,D_r} : \mathcal{P}_S \to \mathcal{P}_{T_1} \otimes \cdots \otimes \mathcal{P}_{T_r}$$

is defined as follows:

$$\mathrm{Res}^p_{D_1,\ldots,D_r}(\eta) = \begin{cases} \eta_1 \otimes \cdots \otimes \eta_{r+1} & \text{if } \eta \text{ admits } D_1, \ldots, D_r \text{ as disjoint chords} \\ 0 & \text{if } \eta \text{ does not admit } D_1, \ldots, D_r \end{cases} \tag{3.4}$$

Examples 3.9. In this example, $n = 12$ and the partition of S given by D_1, D_2, D_3 and D_4 is $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 10, 11, 12\}$, $S_3 = \{5, 9\}$, $S_4 = \{6\}$, $S_5 = \{7, 8\}$.



We have $T_1 = S_1 \cup \{d_1\}$, $T_2 = S_2 \cup \{d_1, d_2\}$, $T_3 = S_3 \cup \{d_2, d_3\}$, $T_4 = S_4 \cup \{d_3, d_4\}$, $T_5 = S_5 \cup \{d_4\}$. The composed residue map $\mathrm{Res}^p_{D_1,D_2,D_3,D_4}$ maps the standard polygon $\delta = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$ to the tensor product of the five subpolygons shown in the figure.

The definition of the residue allows us to extend the definition of convergence of a polygon to linear combinations of polygons.

Definition 3.10. (*Polygon divergence along the standard polygon: bad chords*) Let E be a partition of $S \cup \{d\}$ into two subsets, one of which is a consecutive subset $T = \{i, i+1, \ldots, i+j\}$ of S for the standard order, and let η be a polygon. We say that E is a *bad chord* for η, or eqiuvalently η is a *bad polygon* for E, if $E \in \chi(\eta)$ (this expresses the idea that the cell-form corresponding to η diverges along the boundary divisor, corresponding to E, of the standard cell δ). If $\eta = \sum_i a_i\eta_i$, then we say that E is a bad chord for η if any η_i is a bad polygon for E.

Definition 3.11. (*Polygon convergence along the standard polygon*) The linear combination $\eta = \sum_i a_i\eta_i$ is said to *converge* along the chord E of the standard polygon (or along the corresponding consecutive subset T) if the residue satisfies

$$\mathrm{Res}^p_E(\eta) \in I_T \otimes \mathcal{P}_{S\setminus T\cup\{d\}\cup\{e\}}, \tag{3.5}$$

where I_T is as in definition 3.2. A linear combination η is *convergent* (along the standard polygon) if it converges along all of its bad chords.

The goal of the following section is to define a set of particular linear combinations of polygons, the Lyndon insertion words and Lyndon insertion shuffles, which are convergent, and show that they are linearly independent. In the section after that, we will prove that this set forms a basis for the convergent subspace of $\mathcal{P}_{S\cup\{d\}}$.

3.3. **The Lyndon insertion subspace.**

Definition 3.12. Let a $1n$-word be a word of length n in the distinct letters of $S = \{1, \ldots, n\}$ in which the letter 1 appears just to the left of the letter n, and let $W_S \subset V_S \simeq \mathcal{P}_{S\cup\{d\}}$ denote the subspace generated by these words. The space W_S is of dimension $(n-1)!$.

The following lemma will show that $V_S = W_S \oplus I_S$, where I_S is the subspace of shuffles of definition 3.2.

Lemma 3.13. *Fix two elements a_1 and a_2 of $S = \{1, \ldots, n\}$.*

Let

$$\tau = \sum_i c_i\eta_i \in V_S,$$

where the η_i run over the words of length n in V_S such that a_1 is the leftmost character of η_i (resp. the η_i run over the words where a_1 appears just to the left of a_2 in η_i). Then $\tau \in I_S$ if and only if $c_i = 0$ for all i.

Proof. The assumption $\tau \in I_S$ means that we can write $\tau = \sum_i c_i u_i ш v_i$ for non-empty words u_i and v_i. Considering this in the space $\mathcal{P}_{S\cup\{d\}}$ isomorphic to V_S, it is a sum of cyclic structures $\sum_i c_i(u_i, d) ш (v_i, d)$ shuffled with respect to the point d. Choose any bijection $\rho : \{1, \ldots, n, d\} \to \{0, 1, \infty, t_1, \ldots, t_{n-2}\}$ which maps d to 0 and a_1 to 1 (resp. which maps a_1 to 0 and a_2 to 1). Define a linear map from $\mathcal{P}_{S\cup\{d\}}$ to $H^{n-2}(\mathfrak{M}_{0,n+1})$ by first renumbering the indices $(1, \ldots, n, d)$ of each polygon $\eta \in \mathcal{P}_{S\cup\{d\}}$ as $(0, 1, \infty, t_1, \ldots, t_{n-2})$ via ρ, then mapping the renumbered polygon to the corresponding cell-form (same cyclic order). By hypothesis, $\tau = \sum_i c_i\eta_i$

maps to a sum $\omega_\tau = \sum_i c_i\omega_{\eta_i}$ of 01 cell forms. Since τ is a shuffle with respect to one point, we know by (2.13) that $\omega_\tau = 0$. But the 01 cell-forms ω_{η_i} are linearly independent by theorem 2.12. Therefore each $c_i = 0$. $\square$

Recall that the shuffles of disjoint Lyndon words form a basis for I_S; we call them *Lyndon shuffles*. A *convergent Lyndon shuffle* is a shuffle of convergent Lyndon words.

Definition 3.14. We will recursively define the set $\mathcal{L}_S$ of *Lyndon insertion shuffles* in I_S. If $S = \{1\}$, then $\mathcal{L}_S = \emptyset$. If $S = \{1, 2\}$ then $\mathcal{L}_S = \{1 ш 2\}$. In general, if D is any (lexicographically ordered) alphabet on m letters and $S = \{1, \ldots, m\}$, we define $\mathcal{L}_D$ to be the image of $\mathcal{L}_S$ under the order-preserving bijection $S \to D$ corresponding to the ordering of D.

Assume now that $S = \{1, \ldots, n\}$ with $n > 2$, and that we have constructed all of the sets $\mathcal{L}_{\{1,\ldots,i\}}$ with $i < n$. Let us construct $\mathcal{L}_S$. The elements of these sets are constructed by taking convergent Lyndon shuffles on a smaller alphabet, and making "insertions" into every letter except for the leftmost letter of each Lyndon word in the shuffle, according to the following explicit procedure. Let $T = \{a_1, \ldots, a_k\}$ be an alphabet with $3 \le k \le n$ letters, ordered by the lexicographical ordering $a_1 < \cdots < a_k$, and choose a convergent Lyndon shuffle γ of length k in the letters of T. Write γ as a shuffle of $s > 1$ convergent Lyndon words in disjoint letters:

$$\gamma = (a_{i_1} \cdots a_{i_{k_1}})\, ш\, (a_{i_{k_1+1}} \cdots a_{i_{k_2}})\, ш \cdots ш\, (a_{i_{k_{s-1}+1}} \cdots a_{i_{k_s}})$$

where $1 \le k_1 < k_2 < \cdots < k_s = k$. Choose integers $v_1, \ldots, v_k \ge 1$ such that $\sum_i v_i = n$ and such that for each of the indices $l = i_1, i_{k_1+1}, \ldots, i_{k_{s-1}+1}$ of the leftmost characters of the s convergent Lyndon words in γ, we have $v_l = 1$. For $1 \le i \le k$, let D_i denote an alphabet $\{b^i_1, \ldots, b^i_{v_i}\}$. When $v_i = 1$, insert b^i_1 into the place of the letter a_i in γ; when $v_i > 1$, choose any element V_i from $\mathcal{L}_{D_i}$, and insert this V_i into the place of the letter a_i.

The result is a sum of words in the alphabet $\cup_{i=1}^k D_i$. Note that this alphabet is of cardinal n and equipped with a natural lexicographical ordering given by the ordering $D_1, \ldots, D_k$ and the orderings within each alphabet D_i. We can therefore renumber this alphabet as $1, \ldots, n$. Since it is a sum of shuffles, the renumbered element lies in I_S, and we call it a *Lyndon insertion shuffle* on S. The original convergent Lyndon shuffle γ on T is called the framing; together with the integers v_i, we call this the fixed structure of the insertion shuffle. We define $\mathcal{L}_S$ to be the set of all Lyndon insertion shuffles on S, constructed by varying the choice of $3 \le k \le n$, the convergent Lyndon shuffle γ on k letters, the numbers $v_1, \ldots, v_k$ and the elements V_i for each $v_i > 1$ in every possible way.

In the special case where $k = n$, we have $v_i = 1$ for $1 \le i \le k$ and there are no non-trivial insertions. The corresponding elements of $\mathcal{L}_S$ are thus just convergent Lyndon shuffles.

Example 3.15. We have

$$\mathcal{L}_{\{1,2\}} = \{1 ш 2\}$$

$$\mathcal{L}_{\{1,2,3\}} = \{1 ш 2 ш 3,\ 2 ш 13\}$$

$$\mathcal{L}_{\{1,2,3,4\}} = \{1 ш 2 ш 3 ш 4,\ 13 ш 2 ш 4,\ 14 ш 2 ш 3,\ 24 ш 1 ш 3,$$
$$3 ш 142,\ 13 ш 24,\ 1(3 ш 4) ш 2\}$$

The last element of $\mathcal{L}_{\{1,2,3,4\}}$ is obtained by taking $T = \{1,2,3\}$ and $\gamma = 13 ш 2$. We can only insert in the place of the character 3 since 1 and 2 are leftmost letters of the Lyndon words in $13 ш 2$. As for what can be inserted in the place of 3, the only possible choices are $k = 1$, $v_1 = 2$, $D_1 = \{b_1, b_2\}$, and $V_1 = b_1 ш b_2$, the unique element of $\mathcal{L}_{D_1}$. The natural ordering on the alphabet $\{T \setminus 3\} \cup D_1$ is given by $(1,2,b_1,b_2)$ since $b_1 ш b_2$ is inserted in the place of 3, so we renumber b_1 as 3 and b_2 as 4, obtaining the new element $1(3 ш 4) ш 2 = 134 ш 2 + 143 ш 2 = 2134 + 1234 + 1324 + 1342 + 2143 + 1243 + 1423 + 1432$.

For $n = 5$, $\mathcal{L}_{\{1,2,3,4,5\}}$ has 34 elements. Of these, 25 are convergent Lyndon shuffles which we do not list. The remaining nine elements are obtained by insertions into the smaller convergent Lyndon shuffles: they are given by

$$\begin{cases} 2 ш 1(4 ш 35),\ 2 ш 1(3 ш 4 ш 5) & \text{insertions into } 2 ш 13 \\ 3 ш 1(4 ш 5)2,\ 4 ш 15(2 ш 3) & \text{insertions into } 3 ш 142 \\ 13 ш 2(4 ш 5),\ 1(3 ш 4) ш 25 & \text{insertions into } 13 ш 24 \\ 1(3 ш 4) ш 2 ш 5 & \text{insertion into } 13 ш 2 ш 4 \\ 1(4 ш 5) ш 2 ш 3 & \text{insertion into } 14 ш 2 ш 3 \\ 2(4 ш 5) ш 1 ш 3 & \text{insertion into } 24 ш 1 ш 3. \end{cases}$$

Definition 3.16. We now define a complementary set, the set $\mathcal{W}_S$ of *Lyndon insertion words*. Let a *special convergent word* $w \in V_S$ denote a convergent word of length n in S such that in the lexicographical ordering $(1, \dots, n, d)$, the polygon (cyclic structure) $\eta = (w, d)$ satisfies $\chi(\delta) \cap \chi(\eta) = \emptyset$; in other words, the polygon η has no chords in common with the standard polygon. This condition is a little stronger than asking w to be a convergent word (for instance, 13524 is a convergent word but not a special convergent word, since $13524d$ has a bad chord $\{2,3,4,5\}$). The first elements of $\mathcal{W}_S$ are given by the special convergent $1n$-words. The remaining elements of $\mathcal{W}_S$ are the *Lyndon insertion words* constructed as follows. Take a special convergent word w' in a smaller alphabet $T = \{a_1, \dots, a_k\}$ with $k < n$ such that a_1 appears just to the left of a_{k-1}, and choose positive integers $v_1, \dots, v_k$ such that $v_1 = v_k = 1$ and $\sum_i v_i = n$. As above, we let $D_i = \{b_1^i, \dots, b_{v_i}^i\}$ for $1 \le i \le k$, and choose an element D_i of $\mathcal{L}_{D_i}$ for each i such that $v_i > 1$. For i such that $v_i = 1$, insert b_1^i in the place of a_i in w', and for i such that $v_i > 1$ insert D_i in the place of a_i. We obtain a sum of words w'' in the letters $\cup D_i$. This alphabet has a natural lexicographic ordering $D_1, \dots, D_k$ as above, so we can renumber its letters from 1 to n, which transforms w'' into a sum of words $w \in V_S$ called a *Lyndon insertion word*. Note that by construction, the result is still a sum of $1n$-words. The set $\mathcal{W}_S$ consists of the special convergent words and the Lyndon insertion words.

Remark 3.17. It follows from lemma 3.13 that the intersection of the subspace $\langle \mathcal{W}_S \rangle$ in V_S with the subspace I_S of shuffles is equal to zero.

Example 3.18. We have

$$\mathcal{W}_{\{1,2\}} = \emptyset,\ \ \mathcal{W}_{\{1,2,3\}} = \emptyset,\ \ \mathcal{W}_{\{1,2,3,4\}} = \{3142\},$$
$$\mathcal{W}_{\{1,2,3,4,5\}} = \{24153, 31524, (3 ш 4)152, 415(2 ш 3)\}$$

The last two elements of $\mathcal{W}_{\{1,2,3,4,5\}}$ are obtained by taking $v_1 = 1, v_2 = 1, v_3 = 2, v_4 = 1$ and $v_1 = 1, v_2 = 2, v_3 = 1, v_4 = 1$ and creating the corresponding Lyndon insertion word with respect to 3142.

Theorem 3.19. *The set $\mathcal{W}_S \cup \mathcal{L}_S$ of Lyndon insertion words and shuffles is linearly independent.*

Proof. We will prove the result by induction on n. Since $\mathcal{L}_S \subset I_S$ and we saw by lemma 3.13 that the space generated by $\mathcal{W}_S$ has zero intersection with I_S, we only have to show that that both $\mathcal{W}_S$ and $\mathcal{L}_S$ are linearly independent sets. We begin with $\mathcal{L}_S$. Since $\mathcal{L}_{\{1,2\}}$ contains a single element, we may assume that $n > 2$.

Let $W = A_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup A_r$ be a Lyndon shuffle, with $r > 1$. We define its *fixed structure* as follows. Replace every maximal consecutive block (not contained in any larger consecutive block) in each A_i by a single letter. Then W becomes becomes a convergent Lyndon shuffle W' in a smaller alphabet T' on k letters, which is equipped with an inherited lexicographical ordering. If $T = \{1, \ldots, k\}$, then under the order-respecting bijection $T' \to T$, W' is mapped to a convergent Lyndon shuffle V in T, called the framing of W. The fixed structure is given by the framing together with the set of integers $\{v_i \mid 1 \leq i \leq k\}$ defined by $v_i = 1$ if that letter in T does not correspond to a maximal block, and v_i is the length of the maximal block if it does. Thus we have $v_1 + \cdots + v_k = n$. We can extend this definition to the fixed structure of a Lyndon insertion shuffle, since by definition this is a linear combination of Lyndon shuffles all having the same fixed structure, and we recover the framing and fixed structure of the insertion shuffle given in the definition.

Example 3.20. If W is the Lyndon shuffle $1546 \sqcup\!\sqcup 237$, we replace the consecutive blocks 23 and 546 by letters b_1 and b_2, obtaining the convergent shuffle $W' = 1b_2 \sqcup\!\sqcup b_1 7$ in the alphabet $T' = \{1, b_1, b_2, 7\}$; renumbering this as $1, 2, 3, 4$ we obtain $V = 13 \sqcup\!\sqcup 24 \in \mathcal{L}_{\{1,2,3,4\}}$. The fixed structure is given by $13 \sqcup\!\sqcup 24$ and integers $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 1$.

The Lyndon insertion shuffles $(1, (3 \sqcup\!\sqcup 4)) \sqcup\!\sqcup (2, 5)$ and $(1, 3) \sqcup\!\sqcup (2, (4 \sqcup\!\sqcup 5))$ have the same framing $13 \sqcup\!\sqcup 24$, but since $(v_1, v_2, v_3, v_4) = (1, 1, 2, 1)$ for the first one and $(1, 1, 1, 2)$ for the second, they do not have the same fixed structure. The Lyndon insertion shuffles $(1, (5) \sqcup\!\sqcup (3, 4, 6)) \sqcup\!\sqcup (2, 7)$ and $(1, (3, 5) \sqcup\!\sqcup (4, 6)) \sqcup\!\sqcup (2, 7)$ have the same associated framing $13 \sqcup\!\sqcup 24$ and the same integers $(v_1, v_2, v_3, v_4) = (1, 1, 4, 1)$. so they have the same fixed structure.

For any fixed structure, given by a convergent Lyndon shuffle γ on an alphabet T of length k and associated integers $v_1, \ldots, v_k$ with $v_1 + \cdots + v_k = n$, let $L(\gamma, v_1, ..., v_k)$ be the subspace of V_S spanned by Lyndon shuffles with that fixed structure. Since Lyndon shuffles are linearly independent, we have

$$V_S = \bigoplus L(\gamma, v_1, ..., v_k)$$

Now, as we saw above, a Lyndon insertion shuffle is a linear combination of Lyndon shuffles all having the same fixed structure, so every element of $\mathcal{W}_S \cup \mathcal{L}_S$ lies in exactly one subspace $L(\gamma, v_1, \ldots, v_k)$. Thus, to prove that the elements of $\mathcal{L}_S$ are linearly independent, it is only necessary to prove the linear independence of Lyndon insertion shuffles with the same fixed structure. If all of the $v_i = 1$, then the fixed structure is just a single convergent Lyndon shuffle on S, and these are linearly independent. So let $(\gamma, v_1, \ldots, v_k)$ be a fixed structure with not all of the v_i equal to 1, and let $\omega = \sum_q c_q \omega_q$ be a linear combination of Lyndon insertion shuffles of fixed structure $\gamma, v_1, \ldots, v_k$.

Break up the tuple $(1, \ldots, n)$ into k successive tuples

$$B_1 = (1, \ldots, v_1),\ B_2 = (v_1 + 1, \ldots, v_1 + v_2), \ldots, B_k = (v_1 + \cdots + v_{k-1} + 1, \ldots, n).$$

Let $i_1, \ldots, i_m$ be the indices such that $B_{i_1}, \ldots, B_{i_m}$ are the tuples of length greater than 1. These tuples correspond to the insertions in the Lyndon insertion shuffles of type $(\gamma, v_1, \ldots, v_k)$. For $1 \leq j \leq m$, let $T_j = \{B_{i_j}\} \cup \{d_j\}$. This element d_j is the index of the chord D_j corresponding to the consecutive subset B_{i_j}, which is a chord of the standard polygon and also of every term of ω. The chords $D_1, \ldots, D_r$ are disjoint and cut each term of ω into $m+1$ subpolygons, m of which are indexed by T_j, and the last one of which is indexed by $T' = S \backslash \{B_{i_1} \cup \cdots \cup B_{i_m}\} \cup \{d_1, \ldots, d_m\}$. Thus we can take the composed residue map

$$\mathrm{Res}^p_{D_1, \ldots, D_m}(\omega) \in \mathcal{P}_{T_1} \otimes \cdots \otimes \mathcal{P}_{T_m} \otimes \mathcal{P}_{T'}.$$

Let us compute this residue.

The alphabet T' is of length k and has a natural ordering corresponding to a bijection $\{1, \ldots, k\} \to T'$. Let γ' be the image of γ under this bijection, i.e. the framing. Let $P_1^q, \ldots, P_m^q$ be the insertions corresponding to the m tuples $B_{i_1}, \ldots, B_{i_m}$ in each term of $\omega = \sum_q c_q \omega_q$. Each P_j^q lies in $\mathcal{L}_{B_{i_j}}$. The image of the composed residue map is then

$$\mathrm{Res}^p_{D_1, \ldots, D_m}(\omega) = \sum_q c_q (P_1^q, d_1) \otimes \cdots \otimes (P_m^q, d_m) \otimes \gamma'. \tag{3.6}$$

Now assume that $\omega = \sum_q c_q \omega_q = 0$, and let us show that each $c_q = 0$. We have

$$\sum_q c_q (P_1^q, d_1) \otimes \cdots \otimes (P_m^q, d_m) \otimes \gamma' = 0,$$

and since γ' is fixed, we have

$$\sum_q c_q (P_1^q, d_1) \otimes \cdots \otimes (P_m^q, d_m) = 0.$$

But for $1 \leq j \leq m$, the P_j^q lie in $\mathcal{L}_{B_{i_j}}$ and thus, by the induction hypothesis, the distinct P_j^q for fixed j and varying q are linearly independent. Since d_i is the largest element in the lexicographic alphabet T_i, the sums (P_j^q, d_j) are also linearly independent for fixed j and varying q, because if $\sum_q e_q (P_j^q, d_j) = 0$ then $\sum_q e_q P_j^q = 0$ simply by erasing d_j. The tensor products are therefore also linearly independent, so we must have $c_q = 0$ for all q. This proves that $\mathcal{L}_S$ is a linearly independent set.

We now prove that $\mathcal{W}_S$ is a linearly independent set. For this, we construct the framing and fixed structure of a Lyndon insertion word of length n in $\mathcal{W}_S$ just as above, by replacing consecutive blocks with single letters, obtaining a word in a smaller alphabet T' and a set of integers corresponding to the lengths of the consecutive blocks. For instance, replacing the consecutive block $(3 ш 4)$ in the Lyndon insertion word $(3 ш 4)152$ by the letter b_1 gives a convergent word $b_1 152$ in the alphabet $(1, 2, b_1, 5)$; renumbering this as $(1, 2, 3, 4)$ gives the framing as 3124 and the associated integers as $v_1 = 2, v_2 = 1, v_3 = 2, v_4 = 1$. For every fixed structure of this type, now given as a convergent word γ of length $k < n$ together with integers $v_1, \ldots, v_k$, we let $W(\gamma, v_1, \ldots, v_k)$ denote the subspace of V_S generated by Lyndon insertion words with the fixed structure $(\gamma, v_1, \ldots, v_k)$. As above, the spaces $W(\gamma, v_1, \ldots, v_k)$ do not intersect, so $\mathcal{W}_S = \oplus W(\gamma, v_1, \ldots, v_k)$, and we have only to show that the set of Lyndon insertion words with a given fixed structure is a linearly independent set. So assume that we have some linear combination $\sum_q c_q w_q = 0$, where the w_q are all Lyndon insertion words of given fixed structure

$(\gamma, v_1, \ldots, v_k)$. If $k = n$, then these insertion words are just words, so they are linearly independent and $c_q = 0$ for all q. So assume that at least one $v_i > 1$. We proceed exactly as above. Breaking up the tuple $(1, \ldots, n)$ into tuples $B_1, \ldots, B_k$ as above, and letting $D_1, \ldots, D_m$, T_j and T' denote the same objects as before, we compute the composed residue of $\sum_q c_q w_q$ and obtain (3.6). Then because all of the insertions P_i^q lie in $\mathcal{L}_{B_{i_j}}$ and we know that these sets are linearly independent, we find as above that $c_q = 0$ for all q. $\square$

3.4. **Convergent linear combinations of polygons.**

Definition 3.21. Let $S = \{1, \ldots, n\}$. Let J_S be the subspace of $\mathcal{P}_{S\cup\{d\}}$ spanned by the set $\mathcal{L}_S$ of Lyndon insertion shuffles, and let K_S be the subspace of $\mathcal{P}_{S\cup\{d\}}$ spanned by the set $\mathcal{W}_S$ of Lyndon insertion words.

We prove the main convergence results in two separate theorems, concerning the subspaces I_S and W_S of $V_S \simeq \mathcal{P}_{S\cup\{d\}}$ respectively (cf. definitions 3.2 and 3.12).

Theorem 3.22. *An element $\omega \in I_S \subset \mathcal{P}_{S\cup\{d\}}$ is convergent if and only if $\omega \in J_S$.*

Proof. Step 1. The easy direction. One direction of this theorem is easy. Since J_S is spanned by Lyndon insertion shuffles, which lie in I_S, we only need to show that any Lyndon insertion shuffle is convergent. If it is a shuffle of convergent Lyndon words, then there are no consecutive blocks in any of the words. Therefore if the letters of any consecutive subset T of S appear as a block in any term of ω, it must be because they appeared in more than one of the convergent words which are shuffled together. So these letters appear as a shuffle, so the residue lies in $I_T \otimes \mathcal{P}_{S\setminus T\cup\{d\}}$, which by definition 3.11 means that ω is convergent. Now, if we are dealing with a Lyndon insertion shuffle with non-trivial insertions, then there are two kinds of bad chords: those corresponding to these insertions, and those corresponding to consecutive subsets of the insertion sets. For example, in the Lyndon insertion shuffle

$$\omega = (2 ш 1(4 ш 35, d) = 2 ш (1435 + 1345 + 1354) = \tag{3.7}$$

$$21435 + 12435 + 14235 + 14325 + 14352 + 21345 + 12345 + 13245+$$
$$13425 + 13452 + 21354 + 12354 + 13254 + 13524 + 13542,$$

in which $(4 ш 35)$ is inserted into the Lyndon shuffle $2 ш 13$, and we write ω in V_S rather than $\mathcal{P}_{S\cup\{d\}}$ to avoid adding the index d to the end of every word above. The bad chord 345 corresponds to the insertion, and the bad chords 34 and 45 appear in certain terms of the shuffle within the insertion. For the latter type, since they appear inside an insertion which is itself a shuffle, their letters only appear in shuffle combinations within the insertion (for instance $1435 + 1345 = 1(3 ш 4)5$ in the example above), so the residue along these chords is a shuffle. But also, for the bad chords corresponding to an insertion set, the insertion itself lies in $\mathcal{L}_T \subset I_T$, and is precisely one factor of the residue, which is thus also a shuffle. For example, the residue in the example above along the chord $E = 345$ comes from considering only the terms in (3.7) which have $\{3, 4, 5\}$ as a consecutive subset, i.e. the terms which are polygons admitting the chord 345, namely

$$\omega = 21435 + 12435 + 14352 + 21345 + 12345 + 13452 + 21354 + 12354 + 13542$$

$$= 21(435 + 345 + 354) + 12(435 + 345 + 354) + 1(435 + 345 + 354)2$$
$$= 21(4 ш 35) + 12(4 ш 35) + 1(4 ш 35)2$$

and the residue is thus simply

$$\mathrm{R}es_{345}(\omega) = (4 ш 35) \otimes (21e + 12e + 1e2),$$

where e labels the chord E, and the insertion itself is the left-hand factor. Since insertions always lie in $\mathcal{L}_T$, they are always shuffles, therefore ω converges along the corresponding chords.

Step 2. The other direction: Induction hypothesis and base case. Assume now that ω is convergent and lies in I_S, so that we can write $\omega = \sum_i a_i\omega_i$ where each $\omega_i = (A_1^i ш \cdots ш A_{r_i}^i, d)$ is a Lyndon shuffle, $r_i > 1$. We say that a consecutive block appearing in any A_j^i is maximal if the same block does not appear in that factor or in any other factor inside a bigger consecutive block. Factors may appear which contain more than one consecutive block, but the maximal blocks are disjoint.

We prove the result by induction on the length of the alphabet $S = \{1, \ldots, n\}$. The smallest case is $n = 3$, since for $n = 2$, the polygons are triangles and have no chords. For $n = 3$, let

$$\omega = c_1(12 ш 3, d) + c_2(13 ш 2, d) + c_3(1 ш 2 ш 3, d) + c_4(23 ш 1, d)$$

be a linear combination of all the Lyndon shuffles for $n = 3$. The bad chords are $E = \{1, 2\}$, $F = \{2, 3\}$. We have

$$\mathrm{Res}_E^p(\omega) = c_1(1, 2, e) \otimes (e ш 3, d) + c_2(1 ш 2, e) \otimes (e, 3, d)$$
$$+c_3(1 ш 2, e) \otimes (e ш 3, d) + c_4(1 ш 2, e) \otimes (e, 3, d).$$

For this to converge means that the left-hand parts of the two right-hand tensor factors $(e, 3, d)$ and $(e ш 3, d)$ must lie in $I_{\{1,2\}}$. Since three of the four left-hand parts already lie in $I_{\{1,2\}}$, the fourth one must as well, which must mean that $c_1 = 0$. This is the condition for ω to converge on E. Now let us consider $F = \{2, 3\}$. We have

$$\mathrm{Res}_F^p(\omega) = c_1(2 ш 3, f) \otimes (1, f, d) + c_2(2 ш 3, f) \otimes (1, f, d)$$
$$+c_3(2 ш 3, e) \otimes (1 ш f, d) + c_4(2, 3, f) \otimes (1 ш f, d).$$

This gives $c_4 = 0$ as the condition for ω to converge on F. Therefore, we find that ω is a linear combination of $13 ш 2$ and $1 ш 2 ш 3$, which are exactly the elements of the basis $\mathcal{L}_{\{1,2,3\}}$ of J_S. This settles the base case $n = 3$.

The induction hypothesis is that for every alphabet $S' = \{1, \ldots, i\}$ with $i < n$, if $\omega \in I_{S'}$ is convergent, then $\omega \in J_{S'}$.

Step 3. Construction of the insertion terms $(S_{[i]}, e) \in I_T$. Now let $S = \{1, \ldots, n\}$ and assume that $\omega \in I_S$ is convergent. Write ω as a linear combination of Lyndon shuffles

$$\omega = \sum_i c_i\omega_i = \sum_i c_i(A_1^i ш A_2^i ш \cdots A_{r_i}^i, d).$$

If no consecutive block appears in any A_j^i, then ω is a linear combination of convergent Lyndon words, so it is in J_S by definition. Assume some consecutive blocks do appear, and consider a maximal consecutive block T, corresponding to a bad chord E. Decompose $\omega = \gamma_1 + \gamma_2$ where γ_k is the sum $\sum_{i\in I_k} c_i\omega_i$, with I_1 the set of indices i for which T appears as a block in some A_j^i, which by reordering shuffled pieces we may assume to be A_1^i, and I_2 is the set of indices for which T does not appear as a block in any A_j^i. Then because letters of T appear scattered in different A_j^i in each term of γ_2, any time they appear as a block in a term of γ_2, they must appear in several terms as a shuffle combination, so $\mathrm{Res}_E^p(\gamma_2) \in I_T \otimes \mathcal{P}_{S\setminus T\cup\{e\}\cup\{d\}}$. Thus γ_2 converges along E. Since we are assuming that ω is convergent, γ_1 must then also converge, so we must have

$$\mathrm{Res}_E^p(\gamma_1) \in I_T \otimes \mathcal{P}_{S\setminus T\cup\{d\}\cup\{e\}}. \tag{3.8}$$

For each $i \in I_1$, write $A_1^i = B_1^i Y^i C_1^i$, where Y^i consists of the letters of T in some order and B_1^i is a (possibly empty) Lyndon word.

We have

$$\mathrm{Res}_E^p(\gamma_1) = \sum_{i\in I_1} c_i(Y^i, e) \otimes (B_1^i e C_1^i \text{ ш } A_2^i \text{ ш } \cdots \text{ ш } A_{r_i}^i, d) \tag{3.9}$$

Note that the alphabet $(S \setminus T) \cup \{e\}$ corresponding to all of the right-hand factors has the lexicographic ordering inherited from S by deleting the consecutive block of letters T and replacing it with the unique character e. Thus, all of the words appearing in the shuffles of the right-hand factors are Lyndon words. Indeed, the A_j^i, $j > 1$, are Lyndon by definition, the words $B_1^i e C_1^i$ with non-empty B_1^i are Lyndon because of the assumption that $A_1^i = B_1^i Y^i C_1^i$ is a Lyndon word and therefore the smallest character appears on the left of B_1^i, and the words eC_1^i which appear when B_1^i is empty are Lyndon because $A_1^i = Y^i C_1^i$ is Lyndon and the characters of Y^i (i.e. those of T) are consecutive, so they are all smaller than those appearing in C_1^i; thus e is less than any character of C_1^i in the inherited ordering. Thus, all of the right-hand factors of (3.9) are Lyndon shuffles.

Putting an equivalence relation on I_1 by letting $i \sim i'$ if the right-hand factors of (3.9) are equal, and letting $[i]$ denote the equivalence classes for this relation, we write the residue as

$$\mathrm{Res}_E^p(\gamma_1) = \sum_{[i]\subset I_1} \big(\sum_{i\in[i]} c_i(Y^i, e)\big) \otimes (B_1^{[i]} e C_1^{[i]} \text{ ш } A_2^{[i]} \text{ ш } \cdots \text{ ш } A_{r_{[i]}}^{[i]}, d). \tag{3.10}$$

Since the right-hand factors in the sum over $[i]$ are distinct Lyndon shuffles, the set of right-hand factors forms a linearly independent set. Therefore by (3.8), we must have

$$(S_{[i]}, e) = \sum_{i\in[i]} c_i(Y^i, e) \in I_T \tag{3.11}$$

for each $[i] \subset I_1$.

Let us show that $(S_{[i]}, e) = 0$ whenever $B_1^{[i]}$ is empty. For all $i \in I_1$ such that B_1^i is empty, we have $A_1^i = Y^i C_1^i$, and since these are all Lyndon words, the smallest character of T, say a, is always on the left of Y^i, so we can write $Y^i = aY_0^i$ and $A_1^i = aY_0^i C_1^i$ for all such i. Then for an equivalence class $[i]$ of such i, the $(S_{[i]}, e)$ of (3.11) can be written

$$(S_{[i]}, e) = \sum_{i\in[i]} c_i(Y^i, e) = \sum_{i\in[i]} c_i(aY_0^i, e) \in I_T.$$

But by lemma 3.13, a sum of words all having the same character (here a) on the left and the same character (here e) on the right cannot be a shuffle unless it is zero, so $(S_{[i]}, e) = 0$ if $B_1^{[i]}$ is empty.

Step 4. Proof that the insertion terms $(S_{[i]}, e)$ lie in J_T. For this, we first need to show that $(S_{[i]}, e)$ converges on every subchord of E, i.e. every consecutive subset inside the set T, before applying the induction hypothesis. Let E' be a subchord of E, corresponding to a consecutive block T' strictly contained in T.

Decompose the set of indices I_1 into two subsets I_3 and I_4, where I_3 contains the indices $i \in I_1$ such that T' appears as a consecutive block inside the block T appearing in A_1^i, and I_4 contains the indices $i \in I_1$ such that the letters of T' do not appear consecutively inside the block T. Similarly, partition I_2, the set of indices in the sum $\omega = \sum_i c_i\omega_i$ for which T does not appear as a block in A_1^i, into two sets I_5 and I_6, where I_5 contains the indices $i \in I_2$ such that T' appears as a block in some A_j^i which we may assume to be A_1^i, and I_6 contains the indices $i \in I_2$ of the terms in which T' does not appear as a block in any A_j^i. We have corresponding decompositions $\gamma_1 = \gamma_3 + \gamma_4$, $\gamma_2 = \gamma_5 + \gamma_6$.

As before, T' must appear as a shuffle in γ_6, so γ_6 converges along E'. As for γ_4, since T' does not appear as either a block or a shuffle, the residue along E' is 0. Since by assumption $\omega = \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$ converges along E', we see that $\gamma_3 + \gamma_5$ must converge along E'. Let us show that in fact both γ_3 and γ_5 converge along E'.

Write $A_1^i = R^i Z^i S^i$ for every $i \in I_3 \cup I_5$, where Z^i is a word in the letters of T'. Note that R^i is Lyndon, and non-empty by the identical reasoning to that used above to show that B_1^i is non-empty. Then for $k = 3, 5$, we have

$$\mathrm{Res}^p_{E'}(\gamma_k) = \sum_{i \in I_k} c_i(Z^i, e') \otimes (R^i e' S^i ш A_2^i ш \cdots ш A_{r_i}^i, d). \tag{3.12}$$

For $k = 3, 5$, put the equivalence relation on I_k for which $i \sim i'$ if the right-hand factors of (3.12) are equal, and let $\langle i \rangle$ denote the equivalence classes for this relation. Note that because for $i \in I_3$, T' appears as a block of T, the word B_1^i must appear as the left-hand part of R^i, and the word C_1^i must appear as the right-hand part of S^i. Therefore, in particular, the new equivalence relation is strictly finer than the old, i.e. the equivalence class $[i]$ breaks up into a finite union of equivalence classes $\langle i \rangle$. The residues for $k = 3, 5$ can now be written

$$\mathrm{Res}^p_{E'}(\gamma_k) = \sum_{\langle i \rangle \subset I_k} \Big(\sum_{i \in \langle i \rangle} c_i(Z^i, e')\Big) \otimes (R^{\langle i \rangle} e' S^{\langle i \rangle} ш A_2^{\langle i \rangle} ш \cdots ш A_{r_{\langle i \rangle}}^{\langle i \rangle}). \tag{3.13}$$

Then since the right-hand factors for each k are distinct Lyndon shuffles, they are linearly independent. Furthermore, none of the right-hand factors occurring in the sum for $k = 3$ can ever occur in the sum for $k = 5$ for the following reason: the Lyndon words $R^i e' S^i$ appearing for $k = 3$ all have the letters of $T \setminus T'$ grouped around e', whereas none of the Lyndon words $R^i e' S^i$ have this property. Therefore all the right-hand factors from the residues of γ_3 and γ_5 together form a linearly independent set, so we find that all the left-hand factors

$$\sum_{i \in \langle i \rangle \subset I_k} (Z^i, e') \in I_{T'}, \tag{3.14}$$

so that both γ_3 and γ_5 converge along E'. In particular, this means that both γ_1 and γ_2 converge along E'.

Now, to determine that the $(S_{[i]}, e)$ of (3.11) converge along E', we will use (3.10) to compute the composed residue map $\mathrm{Res}^p_{E,E'}(\gamma_1)$. We are only concerned with the set of indices $I_1 = I_3 \cup I_4$ in (3.10). For each $i \in I_3$, write $Y^i = U^i Z^i V^i$ where Z^i is a word in the letters of T', so that $R^i = B^iU^i$, $S^i = V^iC^i$, and $A_1^i = B^iU^iZ^iV^iC^i$. Then by (3.12), we have

$$\mathrm{Res}^p_E(\gamma_1) = \sum_{[i]\in I_3}\Big(\sum_{i\in[i]} c_i(U^iZ^iV^i,e)\Big)\otimes\big(B_1^{[i]}eC_1^{[i]} \,ш\, A_2^{[i]} \,ш\, \cdots \,ш\, A_{r_{[i]}}^{[i]},d\big)+$$

$$\sum_{[i]\in I_4}\Big(\sum_{i\in[i]} c_i(Y^i,e)\Big)\otimes\big(B_1^{[i]}eC_1^{[i]} \,ш\, A_2^{[i]} \,ш\, \cdots \,ш\, A_{r_{[i]}}^{[i]},d\big).$$

The terms for $i\in I_4$ converge along T', so they vanish when taking the composed residue, and we find

$$\mathrm{Res}^p_{E,E'}(\gamma_1) = \sum_{[i]\in I_3}\Big(\sum_{i\in[i]} c_i(Z^i,e')\otimes(U^ie'V^i,e)\Big)\otimes\big(B_1^{[i]}eC_1^{[i]} \,ш\, A_2^{[i]} \,ш\, \cdots \,ш\, A_{r_{[i]}}^{[i]},d\big).$$

Since for each $[i]\subset I_3$, the right-hand factors are as usual distinct and linearly independent, this means that for each $[i]\subset I_3$,

$$\mathrm{Res}^p_{E'}(S_{[i]},e) = \sum_{i\in[i]} c_i(Z^i,e')\otimes(U^ie'V^i,e)\in\mathcal{P}_{T'\cup\{e'\}}\otimes\mathcal{P}_{T\setminus T'\cup\{e'\}\cup\{e\}}.$$

Now, the equivalence relation on $i\in[i]\subset I_3$ given by $i\sim i'$ if $U^i=U^{i'}$ and $V^i=V^{i'}$ is the same as the equivalence relation $i\sim i'$ if $R^i=R^{i'}$ and $S^i=S^{i'}$ since $R^i=B^iU^i$ and $S^i=V^iC^i$. So the classes $\langle i\rangle$ correspond to sets of i for which U^i and V^i are identical. Thus for each $[i]\subset I_3$, we can write

$$\mathrm{Res}^p_{E'}(S_{[i]},e) = \sum_{\langle i\rangle\subset[i]}\Big(\sum_{i\in\langle i\rangle} c_i(Z^i,e')\Big)\otimes(U^{\langle i\rangle}e'V^{\langle i\rangle},e),$$

where the right-hand factors are all distinct words. Then (3.14) shows that this sum lies in $I_{T'}\otimes\mathcal{P}_{T\setminus T'\cup\{e'\}\cup\{e\}}$, so in fact $(S_{[i]},e)$ converges along E'. For $[i]\subset I_4$, we have saw that $\mathrm{Res}^p_{E'}((S_{[i]},e))=0$ since T' never occurs as a block for $i\in I_4$. Thus $(S_{[i]},e)$ converges along E' for all $[i]\subset I_1$.

Since we have just shown that $(S_{[i]},e)$ converges along every subchord E' of E, i.e. along the chords corresponding to every consecutive subblock T' of T, we see that each term $(S_{[i]},e)$ is convergent along all its bad chords. Thus, by the induction hypothesis, $(S_{[i]},e)\in J_T$.

Step 5. Construction of the insertions. The above construction shows that we can write $\omega=\gamma_1+\gamma_2$ with

$$\gamma_1 = \sum_{[i]\in I_1} c_{[i]}\big(B^{[i]}S_{[i]}C^{[i]} \,ш\, A_2^{[i]} \,ш\, \cdots \,ш\, A_{r_i}^{[i]},d\big)$$

with $S_{[i]}\in J_T$. This means that the maximal block T, which appeared only in γ_1, has been replaced by an insertion in the sense of the definition of Lyndon insertion shuffles. To conclude the proof of the theorem, we successively replace each of the maximal blocks in ω by insertion terms in the same way, in any order, since maximal blocks are disjoint. The final result displays ω as a linear combination of convergent Lyndon shuffles and Lyndon insertion shuffles, so $\omega\in J_S$. $\square$

The following theorem is the exact analogy of the previous one, but with the actual shuffles in I_S replaced by the words in W_S that have 1 just to the left of n, and the set of Lyndon insertion shuffles replaced by Lyndon insertion words, which considerably simplifies the proof.

Theorem 3.23. *Let $\eta \in W_S \subset \mathcal{P}_{S\cup\{d\}}$. Then η is convergent if and only if $\eta \in K_S = \langle \mathcal{W}_S \rangle$.*

Proof. The proof that $\omega \in K_S$ is convergent is exactly as at the beginning of the proof of the previous theorem. So consider the other direction. Let $\omega \in W_S$, so that we can write

$$\omega = \sum_i a_i \eta_i$$

where each η_i is a $1n$-polygon (a $1n$-word concatenated with d), and assume ω is convergent. The only possible bad chords for ω are the consecutive blocks appearing in the η_i. Let T be a subset of S corresponding to a maximal consecutive block.

Lemma 3.24. *No maximal consecutive block having non-trivial intersection with $\{1, n\}$ can appear in any of the $1n$-words η_i of ω.*

Proof. If T is a maximal block containing both 1 and n, then $T = \{1, \dots, n\}$ which does not correspond to a chord.

Assume now that $T = \{m, \dots, n\}$ with $m > 1$. Write $\eta_i = (K^i, 1, n, Z^i, H^i, d)$ where Z^i is an ordering of $\{m, \dots, n-1\}$. Let E be the chord corresponding to T. We have

$$\mathrm{Res}^p_E(\sum_i a_i \eta_i) = \sum_i a_i (n, Z^i, e) \otimes (K^i, 1, e, H^i, d).$$

Convergence implies that for any constant words K, H, the sum

$$\sum_{i | K^i = K, H^i = H} a_i (n, Z^i, e) \in I_T. \tag{3.15}$$

But by lemma 3.13, it is impossible for a sum of words all having the same character on the left to be equal to a shuffle.

The case where $T = \{1, \dots, m\}$ with $m < n$ is identical, except for an easy adaptation of lemma 3.13 to show that a sum of words all having the same character on the right cannot be equal to a shuffle. □

Now we can complete the proof of the theorem. Let $\omega = \sum_i a_i \eta_i$ be a sum of $1n$-words which converges, and consider a maximal consecutive block $T \subset \{2, \dots, n-1\}$. Let I_1 be the set of indices i such that η_i contains the block T and I_2 the other indices. For $i \in I_1$, write $\eta_i = (K^i, Z^i, H^i, d)$ where Z^i is an ordering of T. Then

$$\mathrm{Res}^p_T(\omega) = \sum_{i \in I_1} a_i (Z^i, e) \otimes (K^i, e, H^i, d).$$

Let $i \sim i'$ be the equivalence relation on I_1 given by $K^i = K^{i'}$ and $H^i = H^{i'}$. Then

$$\mathrm{Res}^p_T(\omega) = \sum_{[i] \in I_1} \big(\sum_{i \in [i]} a_i (Z^i, e) \big) \otimes (K^{[i]}, e, H^{[i]}, d),$$

and the right-hand factors are all distinct (linearly independent) words, so by the assumption that ω convergence along E, we have

$$(S_{[i]}, e) = \sum_{i \in [i]} a_i (Z^i, e) \in I_T$$

for each $[i] \subset I_1$. Therefore we can write ω as

$$\omega = \sum_{[i]\subset I_1} a_i(K^{[i]}, S_{[i]}, H^{[i]}, d) + \sum_{i\in I_2} a_i\eta_i,$$

with the maximal block T replaced by the insertion $S_{[i]}$. We prove that $S_{[i]} \in J_T$ exactly as in the proof of the previous theorem: considering a maximal consecutive block $T' \subset T$ occurring in a factor of $S_{[i]}$, one shows that $S_{[i]}$ converges along T' if and only if ω converges along T'. Since ω does converge by assumption, $S_{[i]}$ also converges, and since this holds for all consecutive blocks $T' \subset T$, $S_{[i]}$ converges on all its subdivisors and therefore $S_{[i]} \in J_S = \langle \mathcal{L}_S \rangle$. Finally, one deals with the disjoint maximal blocks appearing in ω one at a time until no blocks at all remain, expressing ω explicitly as a linear combination of Lyndon insertion words. $\square$

A summary of the results in this chapter. We introduced the following spaces, where $S = \{1, \ldots, n\}$:

- V_S: the $\mathbb{Q}$-vector space generated by words in S having distinct letters
- I_S: the $\mathbb{Q}$-vector space generated by shuffles of disjoint words of V_S (definition 3.2)
- $\mathcal{L}_S$: the set of Lyndon insertion shuffles (definition 3.14), which are linearly independent (theorem 3.19)
- J_S: the subspace of I_S spanned by $\mathcal{L}_S$, which forms the set of convergent elements of I_S (theorem 3.22)
- W_S: the $\mathbb{Q}$-vector space generated by words in V_S, so that by Radford's theorem, we have $V_S = I_S \oplus W_S$
- $\mathcal{W}_S$: the set of Lyndon insertion words (definition 3.16), which are linearly independent (theorem 3.19)
- K_S: the subspace spanned by $\mathcal{W}_S$, which forms the set of convergent elements of W_S (theorem 3.23).

4. Explicit generators for $\mathcal{FC}$ and $H^\ell(\mathfrak{M}^\delta_{0,n})$

In this chapter, we show that the map from polygons to cell-forms is surjective, and compute its kernel. From this and the previous chapter, we will conclude that the pairs (δ, ω), where ω runs through the set $\mathcal{W}_S$ of Lyndon insertion words for $n \geq 5$, form a generating set for the formal cell-zeta algebra $\mathcal{FC}$. In the final section, we show that the images of the elements of $\mathcal{W}_S$ in the cohomology $H^\ell(\mathfrak{M}_{0,n})$ yield an explicit basis for the convergent cohomology $H^\ell(\mathfrak{M}^\delta_{0,n})$, determine its dimension, and compute the cohomology basis explicitly for small values of n. We recall that $\mathfrak{M}^\delta_{0,n}$ is defined in section 1.1, and that by the "convergent cohomology", we mean the cohomology classes of ℓ-forms with logarithmic singularities which converge on the closure of the standard cell.

4.1. From polygons to cell-forms.

Let $S = \{1, \ldots, n\}$. The bijection $\rho : S \cup \{d\} \to \{0, t_1, \ldots, t_{\ell+1}, 1, \infty\}$ given by associating the elements $1, \ldots, n, d$ to $0, t_1, \ldots, t_{\ell+1}, 1, \infty$ respectively, induces a map f from polygons to cell-forms:

$$\eta = (\sigma(1), \ldots, \sigma(n), d) \xrightarrow{f} \omega_\eta = [\rho(\sigma(1)), \ldots, \rho(\sigma(n)), \infty].$$

The map f extends by linearity to a map from $\mathcal{P}_{S\cup\{d\}}$ to the cohomology group $H^{n-2}(\mathfrak{M}_{0,n+1})$. The purpose of this section is to prove that f is a surjection, and to determine its kernel.

Recall that $I_S \subset \mathcal{P}_{S\cup\{d\}}$ denotes the subvector space of $\mathcal{P}_{S\cup\{d\}}$ spanned by the *shuffles with respect to the element* d, namely by the linear combinations of polygons

$$(S_1 ш S_2, d)$$

for all partitions $S_1 \coprod S_2$ of S.

Proposition 4.1. *Let $S = \{1, \ldots, n\}$. Then the cell-form map*

$$f : \mathcal{P}_{S\cup\{d\}} \longrightarrow H^{n-2}(\mathfrak{M}_{0,n+1})$$

is surjective with kernel equal to the subspace I_S.

Proof. The surjectivity is an immediate consequence of the fact that 01 cell-forms form a basis of $H^{n-2}(\mathfrak{M}_{0,n+1})$ (theorem 2.12), since all such cell-forms are the images under f of polygons having the edge labelled 1 next to the one labelled n.

Now, I_S lies in the kernel of f by the corollary to proposition 2.9. So it only remains to show that the kernel of f is equal to I_S. But this is a consequence of counting the dimensions of both sides. By theorem 2.12, we know that the dimension of $H^{n-2}(\mathfrak{M}_{0,n+1})$ is equal to $(n-1)!$. As for the dimension of $\mathcal{P}_{S\cup\{d\}}/I_S$, recall from the beginning of chapter 3 that $\mathcal{P}_{S\cup\{d\}} \simeq V_S$, which can be identified with the graded n part of the quotient of the polynomial algebra on S by the relation $w = 0$ for all words w containing repeated letters. Thus V_S is the vector space spanned by words on n distinct letters, so it is of dimension $n!$. But instead of taking a basis of words, we can take the Lyndon basis of Lyndon words (words with distinct characters whose smallest character is on the left) and shuffles of Lyndon words. The subspace I_S is exactly generated by the shuffles, so the dimension of the quotient is given by the number of Lyndon words on S, namely $(n-1)!$. Therefore $\mathcal{P}_{S\cup\{d\}}/I_S \simeq H^{n-2}(\mathfrak{M}_{0,n+1})$. □

Remark 4.2. The above proof has an interesting consequence. Since the map from polygons to differential forms does not depend on the role of d, the kernel cannot depend on d, and any other element of $S \cup \{d\}$ could play the same role. Therefore I_S, which is defined as the space generated by shuffles with respect to the element d, is equal to the space generated by shuffles of elements of $S \cup \{d\}$ with respect to any element of S; it is simply the subspace generated by *shuffles with respect to one element* of $S \cup \{d\}$.

Corollary 4.3. *Let $W_S \subset \mathcal{P}_{S\cup\{d\}}$ be the subset of polygons corresponding to $1n$-words (concatenated with d). Then*

$$f : W_S \simeq H^{n-2}(\mathfrak{M}_{0,n+1}).$$

Proof. The proof follows from the fact that $\mathcal{P}_{S\cup\{d\}} = W_S \oplus I_S$. □

4.2. **Generators for $\mathcal{FC}$.** By definition, $\mathcal{FC}$ is generated by all linear combinations of pairs of polygons $\sum_i a_i(\delta, \omega_i)$ whose associated differential form converges on the standard cell, but modulo the relation (among others) that shuffles are equal to zero. In other words, since $\mathcal{P}_{S\cup\{d\}} = W_S \oplus I_S$, we can redefine $\mathcal{FC}$ to be generated by linear combinations $\sum_i a_i(\delta, \omega_i)$ such that $\sum_i a_i\omega_i \in W_S$ and such that the associated differential form converges on the standard cell.

The following proposition states that the notion of the residue of a polygon and the residue of the corresponding cell-form coincide. In order to state it, we must recall that one can define the map

$$\rho : \mathcal{P}_S \longrightarrow \Omega^\ell(\mathfrak{M}_{0,S}) \ ,$$

from polygons labelled by S to cell forms in a coordinate-free way (one can do this directly from equation (2.6)). In §1, this map was defined in explicit coordinates by fixing any three marked points at $0, 1$ and ∞. This essence of lemma 2.5 is that ρ is independent of the choice of three marked points, and is thus coordinate-free.

Proposition 4.4. *Let $S = \{1, \ldots, n\}$ and let D be a stable partition $S_1 \cup S_2$ of S corresponding to a boundary divisor of $\mathfrak{M}_{0,n}$, with $|S_1| = r$ and $|S_2| = s$. Let ρ denote the usual map from polygons to cell-forms. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{P}_S \arrow[r, "\rho"] \arrow[d, "\mathrm{Res}^p_D"'] & H^\ell(\mathfrak{M}_{0,n}) \arrow[d, "\mathrm{Res}_D"] \\
\mathcal{P}_{S_1\cup\{d\}} \otimes \mathcal{P}_{S_2\cup\{d\}} \arrow[r, "\rho\otimes\rho"] & H^{r-2}(\mathfrak{M}_{0,r+1}) \otimes H^{s-2}(\mathfrak{M}_{0,s+1}).
\end{tikzcd}$$

In other words, the usual residue of differential forms corresponds to the combinatorial residue of polygons.

Proof. Let $\eta \in \mathcal{P}_S$ be a polygon, and let ω_η be the associated cell-form. If D is not compatible with ω_η, then ω_η has no pole on D by proposition 2.7, so $\mathrm{Res}_D(\omega) = 0$.

We shall work in explicit coordinates, bearing in mind that this does not affect the answer, by the remarks above. Therefore assume that η is the polygon numbered with the standard cyclic order on $\{1, \ldots, n\}$, and that D is compatible with η. The corresponding cell-form is given in simplicial coordinates by $[0, t_1, \ldots, t_\ell, 1, \infty]$. By applying a cyclic rotation, we can assume that D corresponds to the partition

$$S_1 = \{1, 2, 3, \ldots, k+1\} \quad \text{and} \quad S_2 = \{k+2, \ldots, n-1, n\}$$

for some $1 \leq k \leq \ell$. In simplicial coordinates, D corresponds to the blow-up of the cycle $0 = t_1 = \cdots = t_k$. We compute the residue of ω_η along D by applying the variable change $t_1 = x_1 \ldots x_\ell, \ldots, t_{\ell-1} = x_{\ell-1}x_\ell, t_\ell = x_\ell$ to the form $\omega_\eta = [0, t_1, \ldots, t_\ell, 1, \infty]$. The standard cell X_η is given by $\{0 < x_1, \ldots, x_\ell < 1\}$. In these coordinates, the divisor D is given by $\{x_k = 0\}$, and the form ω_η becomes

$$\omega_\eta = \frac{dx_1 \ldots dx_\ell}{x_1(1-x_1)\ldots x_\ell(1-x_\ell)}. \tag{4.1}$$

The residue of ω_η along $x_k = 0$ is given by

$$\frac{dx_1 \ldots dx_{k-1}}{x_1(1-x_1)\ldots x_{k-1}(1-x_{k-1})} \otimes \frac{dx_{k+1} \ldots dx_\ell}{x_{k+1}(1-x_{k+1})\ldots x_\ell(1-x_\ell)} \ . \tag{4.2}$$

Changing back to simplicial coordinates via $x_1 = a_1/a_2, \ldots, x_{k-2} = a_{k-2}/a_{k-1}$, $x_{k-1} = a_{k-1}$, and $x_\ell = b_\ell$, $x_{\ell-1} = b_{\ell-1}/b_\ell, \ldots, x_{k+1} = b_k/b_{k+1}$ defines simplicial coordinates on $D \cong \mathfrak{M}_{0,r+1} \times \mathfrak{M}_{0,s+1}$. The standard cells induced by η are $(0, a_1, \ldots, a_{k-1}, 1, \infty)$ on $\mathfrak{M}_{0,r+1}$ and $(0, b_k, \ldots, b_\ell, 1, \infty)$ on $\mathfrak{M}_{0,s+1}$. If we compute (4.2) in these new coordinates, it gives precisely

$$[0, a_1, \ldots, a_{k-1}, 1, \infty] \otimes [0, b_k, \ldots, b_\ell, 1, \infty] \ ,$$

which is the tensor product of the cell forms corresponding to the standard cyclic orders η_1, η_2 on $S_1 \cup \{d\}$ and $S_2 \cup \{d\}$ induced by η. Therefore $\rho(\mathrm{Res}^p_D \eta) = \mathrm{Res}_D \omega_\eta$.

To conclude the proof of the proposition, it is enough to notice that applying $\sigma \in \mathfrak{S}(n)$ to the formula $\mathrm{Res}_D \omega_\eta = \omega_{\eta_1} \otimes \omega_{\eta_2}$ yields

$$\mathrm{Res}_{\sigma(D)} \sigma^*(\omega_\eta) = \mathrm{Res}_{\sigma(D)} \omega_{\sigma(\eta)} = \sigma^*(\omega_{\eta_1}) \otimes \sigma^*(\omega_{\eta_2}) = \omega_{\sigma(\eta_1)} \otimes \omega_{\sigma(\eta_2)}.$$

Here, $\sigma(\eta_i)$ is the cyclic order induced by $\sigma(\eta)$ on the set $\sigma(S_1) \cup \{\sigma(d)\}$, where $\sigma(d)$ corresponds to the partition $S = \sigma(S_1) \cup \sigma(S_2)$. Thus $\rho(\mathrm{Res}^p_{\sigma(D)} \sigma(\eta)) = \mathrm{Res}_{\sigma(D)} \omega_{\sigma(\eta)}$ for all $\sigma \in \mathfrak{S}(n)$, which proves that $\rho(\mathrm{Res}^p_D \gamma) = \mathrm{Res}_D \omega_\gamma$ for all cyclic structures $\gamma \in \mathcal{P}_S$, and all divisors D. □

Corollary 4.5. *A linear combination $\eta = \sum_i a_i \eta_i \in W_S \subset \mathcal{P}_{S \cup \{d\}}$ converges with respect to the standard polygon if and only if its associated form ω_η converges on the standard cell.*

Proof. We first show that

$$\mathrm{Res}^p_D(\eta) \in I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}} + \mathcal{P}_{S_1 \cup \{d\}} \otimes I_{S_2} \ , \tag{4.3}$$

if and only if ω_η converges along the corresponding divisor D in the boundary of the standard cell. If (4.3) holds, then by proposition 4.1 together with the previous proposition, $\mathrm{Res}_D(\omega_\eta) = 0$. Conversely, if $\mathrm{Res}_D(\omega_\eta) = 0$ for a divisor D in the boundary of the standard cell, then by the previous proposition, $\mathrm{Res}^p_D(\eta) \in \mathrm{Ker}(\rho \otimes \rho)$, which is exactly equal to $I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}} + \mathcal{P}_{S_1 \cup \{d\}} \otimes I_{S_2}$.

We now show that (4.3) is equivalent to the convergence of η. But since $\eta \in W_S$, the argument of lemma 3.24 implies that (4.3) holds automatically for any D which intersects $\{1, n\}$ non-trivially. If D intersects $\{1, n\}$ trivially, then we can assume that $\{1, n\} \subset S_2$. In that case, the fact that $W_{S_2} \cap I_{S_2} = 0$ (lemma 3.13) implies that (4.3) is equivalent to the apparently stronger condition

$$\mathrm{Res}^p_D(\eta) \in I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}} \ ,$$

and thus η converges along S_1 in the sense of definition (3.5). This holds for all divisors D and thus completes the proof of the corollary. □

Corollary 4.6. *The Lyndon insertion words of $\mathcal{W}_S$ form a generating set for $\mathcal{FC}$. Furthermore, $\mathcal{FC}$ is defined by subjecting this generating set to only two sets of relations (cf. definition 2.28)*

- *dihedral relations*
- *product map relations*

Remark 4.7. The third relation from definition 2.28 is not needed because we have restricted attention from all linear combinations of pairs of polygons to only those in the basis $\mathcal{W}_S$, where such shuffles do not occur.

4.3. The insertion basis for $H^\ell(\mathfrak{M}^\delta_{0,n})$.

Definition 4.8. Let an *insertion form* be the sum of 01-cell forms obtained by renumbering the Lyndon insertion words of $\mathcal{W}_S$ via $(1, \dots, n, d) \to (0, t_1, \dots, t_{\ell+1}, 1, \infty)$.

Theorem 4.9. *The insertion forms form a basis for $H^{n-2}(\mathfrak{M}^\delta_{0,n+1})$.*

This is an immediate corollary of all the preceding results.

It is interesting to attempt to determine the dimension of the spaces $H^\ell(\mathfrak{M}^\delta_{0,n})$. The most important numbers needed to compute these are the numbers $c_0(n)$ of special convergent words (convergent 01 cell-forms) on $\mathfrak{M}_{0,n}$. These can be computed by counting the number of polygons indexed by symbols $(0, t_1, \ldots, t_\ell, 1, \infty)$ (or $(1, \ldots, n)$) which are convergent with respect to the standard cyclic order and also have the index 0 next to 1 (or 1 next to $n-1$); in other words, the number of cyclic orders having 0 next to 1 and in which no k consecutive labels occur as a single block of k consecutive elements of the cyclic order. By direct counting, we find $c_0(4) = 0$, $c_0(5) = 1$, $c_0(6) = 2$, $c_0(7) = 11$, $c_0(8) = 64$, $c_0(9) = 461$.

Proposition 4.10. *Set $I_1 = 1$, and let I_r denote the cardinal of the set $\mathcal{L}_{\{1,\ldots,r\}}$ for $r \geq 2$ given in definition 3.14. The dimensions $d_n = \dim H^\ell(\mathfrak{M}^\delta_{0,n})$ are given by*

$$d_n = \sum_{r=5}^{n} \sum_{i_1+\cdots+i_{r-3}=n-3} I_{i_1} \ldots I_{i_r} c_0(r) \ , \tag{4.4}$$

where the inner sum is over all partitions of $(n-3)$ into $(r-3)$ strictly positive integers. This formula can be written as follows in terms of generating series. Let $I(x) = \sum_{n=1}^\infty I_n x^n = x + x^2 + 2x^3 + 7x^4 + \cdots$, and let $C(x) = \sum_{r=5}^\infty c_0(r) x^{r-3} = x^2 + 2x^3 + 11x^4 + 64x^5 + \cdots$. Then if $D(x) = \sum_{n=5}^\infty d_n x^{n-3}$, we have the identity

$$D(x) = C\big(I(x)\big).$$

Proof. This recursive counting formula is a direct consequence of the definition, counting all possible ways of making insertions into the $c_0(r)$ convergent 01-cell forms for $5 \leq r \leq n$. □

Remark 4.11. We have $I_1 = I_2 = 1$, $I_3 = 2$, $I_4 = 7$, $I_5 = 34$, $I_6 = 206$ (see example 3.15). The formula gives

$$\begin{cases} d_5 = I_1^2 c_0(5) = 1 \ , \\ d_6 = I_1 I_2 c_0(5) + I_2 I_1 c_0(5) + I_1^3 c_0(6) = 1 + 1 + 2 = 4 \ , \\ d_7 = I_1 I_3 c_0(5) + I_2^2 c_0(5) + I_3 I_1 c_0(5) + I_1^2 I_2 c_0(6) + I_1 I_2 I_1 c_0(6) + I_2 I_1^2 c_0(6) + c_0(7) \\ \quad = 5c_0(5) + 3c_0(6) + c_0(7) = 5 + 6 + 11 = 22 \ . \end{cases}$$

The authors thank Don Zagier for the restatement of formula (4.4) in terms of generating series. In the forthcoming preprint [2], the following remarkably simple identity concerning the d_n is proven. Let $E(x) = x - x^2 - \sum_{n=4}^\infty d_n x^{n-1}$, and set $F(x) = \sum_{n=1}^\infty (n-1)! x^n$. Then

$$E\big(F(x)\big) = x,$$

in other words $E(x)$ is the formal inversion of the power series $F(x)$.

While the present paper was in the final stages of correction, a preprint [20] appeared in which a sequence of numbers d_n, of which the first ones are equal to the d_n defined above, are discovered and interpreted in terms of free Lie operads. In this paper, the authors give the same expression for the generating series of their d_n as the inverse of $F(x)$, thus their result provides a new interpretation of the dimensions d_n.

Note that the formula (4.4) gives the dimensions as sums of positive terms. A very different formula for $\dim H^\ell(\mathfrak{M}^\delta_{0,n})$ is given in [2] using point-counting methods. The relations between the proof in [20], the geometry of moduli spaces, the intermediate power series $I(x)$ and $C(x)$, and the counting method in [2], will be discussed in a forthcoming paper.

4.4. **The insertion basis for $\mathfrak{M}_{0,n}$, $5 \leq n \leq 9$.** In this section we list the insertion bases in low weights. In the case $\mathfrak{M}_{0,5}$, there is a single convergent cell form:

$$\omega = [0, 1, t_1, \infty, t_2]. \tag{4.5}$$

The corresponding period integral is the cell-zeta value:

$$\zeta(\omega) = \int_{(0,t_1,t_2,1,\infty)} [0, 1, t_1, \infty, t_2] = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1 dt_2}{(1-t_1)t_2} = \zeta(2) \ .$$

Here we use the notation of round brackets for cells in the moduli space $\mathfrak{M}_{0,n}$ introduced in section 2.3.4: the cell $(0, t_1, t_2, 1, \infty)$ is the same as the cell $X_{5,\delta}$ corresponding to the standard dihedral order on the set $\{0, t_1, t_2, 1, \infty\}$. Since $C_0(5)$ is 1-dimensional, the space of periods in weight 2, namely the weight 2 graded part $\mathcal{C}_2$ of the algebra of cell-zeta values $\mathcal{C}$ of section 2.4, is just the 1-dimensional space spanned by $\int_{X_{5,\delta}} \omega = \zeta(2)$.

4.4.1. *The case* $\mathfrak{M}_{0,6}$. The space $C(6)$ is four-dimensional, generated by two 01-convergent cell-forms (the first row in the table below) and two forms (the second row in the table below) which come from inserting $\mathcal{L}_{1,2} = \{1 \text{ ш } 2\}$ and $\mathcal{L}_{2,3} = \{2 \text{ ш } 3\}$ into the unique convergent 01 cell form on $\mathfrak{M}_{0,5}$ (4.5). The position of the point ∞ plays a special role. It gives rise to another grading, corresponding to the two columns in the table below, since ∞ can only occur in two positions.

$C_0(6)$	$\omega_{1,1} = [0, 1, t_2, \infty, t_1, t_3]$	$\omega_{1,2} = [0, 1, t_1, t_3, \infty, t_2]$
$C_1(6)$	$\omega_{2,1} = [0, 1, t_1, \infty, t_2 \text{ ш } t_3]$	$\omega_{2,2} = [0, 1, t_1 \text{ ш } t_2, \infty, t_3]$

We therefore have four generators in weight 3. There are no product relations on $\mathfrak{M}_{0,6}$, so in order to compute the space of cell-zeta values, we need only compute the action of the dihedral group on the four differential forms. In particular, the order 6 cyclic generator $0 \mapsto t_1 \mapsto t_2 \mapsto t_3 \mapsto 1 \mapsto \infty \mapsto 0$ sends

$$\omega_{1,1} \mapsto -\omega_{2,1} - \omega_{2,2}, \quad \omega_{1,2} \mapsto \omega_{1,1}, \quad \omega_{2,1} \mapsto -\omega_{1,2} - \omega_{2,1}, \quad \omega_{2,2} \mapsto \omega_{2,1}.$$

Thus, letting X denote the standard cell $X_{6,\delta} = (0, t_1, t_2, t_3, 1, \infty)$, we have $\int_X \omega_{1,1} = \int_X \omega_{1,2}$, $\int_X \omega_{2,1} = \int_X \omega_{2,2}$ and $2 \int_X \omega_{2,2} = \int_X \omega_{1,2}$, so in fact the periods form a single orbit under the action of the cyclic group of order 6 on $H^\ell(\mathfrak{M}^\delta_{0,S})$. We deduce that the space of periods of weight 3 is of dimension 1, generated for instance by $\int \omega_{2,1}$. Since $\omega_{2,1}$ is the standard form for $\zeta(3)$, we have

$$\begin{aligned}
\zeta(0, 1, t_2, \infty, t_1, t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_2)(t_1-t_3)t_3} = 2\,\zeta(3) \ , \\
\zeta(0, 1, t_1, t_3, \infty, t_2) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)(t_1-t_3)t_2} = 2\,\zeta(3) \ , \\
\zeta(0, 1, t_1, \infty, t_2 \text{ ш } t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2 t_3} = \zeta(3) \ , \\
\zeta(0, 1, t_1 \text{ ш } t_2, \infty, t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)(1-t_2)t_3} = \zeta(3) \ ,
\end{aligned}$$

Note that $\omega_{2,2}$ is the standard form usually associated to $\zeta(2,1)$, so that we have recovered the well-known identity $\zeta(2,1)=\zeta(3)$, which is normally obtained using stuffle, shuffle and Hoffmann relations on multizetas.

4.4.2. *The case* $\mathfrak{M}_{0,7}$. The insertion basis is listed in the following table. It consists of 22 forms, eleven of which lie in $C_0(7)$, six of which come from making one insertion into a convergent 01 cell-form from $C_0(6)$ (using $\mathcal{L}_{1,2}=\{1 ш 2\}$ and $\mathcal{L}_{2,3}=\{2 ш 3\}$), and five of which come from making two insertions into the unique convergent 01 cell-form from $C_0(5)$ (which also uses $\mathcal{L}_{1,2,3}=\{1 ш 2 ш 3, 2 ш 13\}$ and $\mathcal{L}_{2,3,4}=\{2 ш 3 ш 4, 3 ш 24\}$).

$C_0(7)$	$[0,1,t_2,\infty,t_3,t_1,t_4]$ $[0,1,t_2,\infty,t_4,t_1,t_3]$ $[0,1,t_3,\infty,t_1,t_4,t_2]$	$[0,1,t_1,t_3,\infty,t_2,t_4]$ $[0,1,t_1,t_3,\infty,t_4,t_2]$ $[0,1,t_2,t_4,\infty,t_1,t_3]$ $[0,1,t_3,t_1,\infty,t_2,t_4]$ $[0,1,t_3,t_1,\infty,t_4,t_2]$	$[0,1,t_1,t_4,t_2,\infty,t_3]$ $[0,1,t_2,t_4,t_1,\infty,t_3]$ $[0,1,t_3,t_1,t_4,\infty,t_2]$
$C_1(7)$	$[0,1,t_2,\infty,t_1,t_3 ш t_4]$ $[0,1,t_3,\infty,t_1 ш t_2,t_4]$	$[0,1,t_1,t_4,\infty,t_2 ш t_3]$ $[0,1,t_2 ш t_3,\infty,t_1,t_4]$	$[0,1,t_1 ш t_2,t_4,\infty,t_3]$ $[0,1,t_1,t_3 ш t_4,\infty,t_2]$
$C_2(7)$	$[0,1,t_1,\infty,t_3 ш (t_2,t_4)]$ $[0,1,t_1,\infty,t_2 ш t_3 ш t_4]$	$[0,1,t_1 ш t_2,\infty,t_3 ш t_4]$	$[0,1,t_2 ш (t_1,t_3),\infty,t_4]$ $[0,1,t_1 ш t_2 ш t_3,\infty,t_4]$

The standard multizeta forms can be decomposed into sums of insertion forms as follows:

$$
\begin{aligned}
&\frac{dt_1dt_2dt_3dt_4}{(1-t_1)t_2t_3t_4} &&=[0,1,t_1,\infty,t_2 ш t_3 ш t_4]\\
&\frac{dt_1dt_2dt_3dt_4}{(1-t_1)(1-t_2)t_3t_4} &&=[0,1,t_1 ш t_2,\infty,t_3 ш t_4]\\
&\frac{dt_1dt_2dt_3dt_4}{(1-t_1)t_2(1-t_3)t_4} &&=[0,1,t_1,t_3,\infty,t_2,t_4]+[0,1,t_1,t_3,\infty,t_4,t_2]+\\
& &&\quad [0,1,t_3,t_1,\infty,t_2,t_4]+[0,1,t_3,t_1,\infty,t_4,t_2]\\
&\frac{dt_1dt_2dt_3dt_4}{(1-t_1)(1-t_2)(1-t_3)t_4} &&=[0,1,t_1 ш t_2 ш t_3,\infty,t_4]
\end{aligned}
\tag{4.6}
$$

In general, the standard multizeta form having factors $(1-t_{i_1}),\ldots,(1-t_{i_r})$ (with $i_1=1$) and $t_{j_1},\ldots,t_{j_s}$ (with $j_s=n$) in the denominator is equal to the shuffle form:

$$
[0,1,t_{i_1} ш \cdots ш t_{i_r},\infty,t_{j_1} ш \cdots ш t_{j_s}],
\tag{4.7}
$$

so to decompose it into insertion forms it is simply necessary to decompose the shuffles $t_{i_1} ш \cdots ш t_{i_r}$ and $t_{j_1} ш \cdots ш t_{j_s}$ into linear combinations of Lyndon insertion shuffles.

Computer computation confirms that the space of periods on $\mathfrak{M}_{0,7}$ is of dimension 1 and is generated by $\zeta(2)^2$. Indeed, up to dihedral equivalence, there are six product maps on $\mathfrak{M}_{0,7}$, given by

$$\begin{cases} (0,t_1,t_2,t_3,t_4,1,\infty) \mapsto (0,t_1,t_2,1,\infty)\times(0,t_3,t_4,1,\infty) \\ (0,t_1,t_2,1,t_3,t_4,\infty) \mapsto (0,t_1,t_2,1,\infty)\times(0,1,t_3,t_4,\infty) \\ (0,t_1,t_2,1,t_3,\infty,t_4) \mapsto (0,t_1,t_2,1,\infty)\times(0,1,t_3,\infty,t_4) \\ (0,t_1,t_2,1,t_3,\infty,t_4) \mapsto (0,t_1,1,t_3,\infty)\times(0,t_2,1,\infty,t_4) \\ (0,t_1,t_2,t_3,1,t_4,\infty) \mapsto (0,t_1,t_2,1,\infty)\times(0,t_3,1,t_4,\infty) \\ (0,t_1,t_2,1,t_3,t_4,\infty) \mapsto (0,t_1,1,t_3,\infty)\times(0,t_2,1,t_4,\infty) \end{cases} \tag{4.8}$$

Following the algorithm from section 2.3.4, we have six associated relations between the integrals of the 22 cell-forms. Then, explicitly computing the dihedral action on the forms yields a further set of linear equations, and it is a simple matter to solve the entire system of equations to recover the 1-dimensional solution. It also provides the value of each integral of an insertion form as a rational multiple of any given one; for instance all the values can be computed as rational multiples of $\zeta(2)^2$. In particular, we easily recover the usual identities

$$\zeta(4)=\frac{2}{5}\zeta(2)^2,\quad \zeta(3,1)=\frac{1}{10}\zeta(2)^2,\quad \zeta(2,2)=\frac{3}{10}\zeta(2)^2,\quad \zeta(2,1,1)=\frac{2}{5}\zeta(2)^2.$$

4.4.3. *The cases* $\mathfrak{M}_{0,8}$ *and* $\mathfrak{M}_{0,9}$. There are 64 convergent 01 cell-forms in on $\mathfrak{M}_{0,8}$, and the dimension of $H^5(\mathfrak{M}^\delta_{0,8})$ is 144. The remaining 80 forms are obtained by Lyndon insertion shuffles as follows:

- 44 forms obtained by making the four insertions $(t_1 ш t_2, t_3, t_4, t_5)$, $(t_1, t_2 ш t_3, t_4, t_5)$, $(t_1, t_2, t_3 ш t_4, t_5)$, $(t_1, t_2, t_3, t_4 ш t_5)$ into the eleven 01 cell-forms of $\mathfrak{M}_{0,7}$
- 12 forms obtained by the six insertion possibilities $(t_1 ш t_2 ш t_3, t_4, t_5)$, $(t_2 ш t_1 t_3, t_4, t_5)$, $(t_1, t_2 ш t_3 ш t_4, t_5)$, $(t_1, t_3 ш t_2 t_4, t_5)$, $(t_1, t_2, t_3 ш t_4 ш t_5)$, $(t_1, t_2, t_4 ш t_3 t_5)$ into the two 01 cell-forms of $\mathfrak{M}_{0,6}$
- 6 forms obtained by the three insertion possibilities $(t_1 ш t_2, t_3 ш t_4, t_5)$, $(t_1 ш t_2, t_3, t_4 ш t_5)$, $(t_1, t_2 ш t_3, t_4 ш t_5)$ into the two 01 cell-forms of $\mathfrak{M}_{0,6}$
- 4 forms obtained by the four insertions $(t_1 ш t_2 ш t_3, t_4 ш t_5)$, $(t_2 ш t_1 t_3, t_4 ш t_5)$, $(t_1 ш t_2, t_3 ш t_4 ш t_5)$, $(t_1 ш t_2, t_4 ш t_3 t_5)$ into the single 01 cell-form of $\mathfrak{M}_{0,5}$
- 14 forms obtained by the fourteen insertions $(t_1 t_3 ш t_2 t_4, t_5)$ $(t_3 ш t_1 t_4 t_2, t_5)$ $(t_1 t_3 ш t_2 ш t_4, t_5)$ $(t_1 t_4 ш t_2 ш t_3, t_5)$ $(t_2 t_4 ш t_1 ш t_3, t_5)$ $(t_2 ш t_1 (t_3 ш t_4), t_5)$ $(t_1 ш t_2 ш t_3 ш t_4, t_5)$ $(t_1, t_2 t_4 ш t_3 t_5)$ $(t_1, t_4 ш t_2 t_5 t_3)$ $(t_1, t_2 t_4 ш t_3 ш t_5)$ $(t_1, t_2 t_5 ш t_3 ш t_4)$ $(t_1, t_3 t_5 ш t_2 ш t_4)$ $(t_1, t_3 ш t_2 (t_4 ш t_5))$ $(t_1, t_2 ш t_3 ш t_4 ш t_5)$ into the single 01 cell-form of $\mathfrak{M}_{0,5}$.

The case of $\mathfrak{M}_{0,9}$ is too large to give explicitly. There are 461 convergent 01 cell-forms, and $\dim H^6(\mathfrak{M}^\delta_{0,9}) = 1089$. An interesting phenomenon occurs first in the case $\mathfrak{M}_{0,9}$; namely, this is the first value of n for which convergent (but not 01) cell-forms do not generate the cohomology. The 1463 convergent cell-forms for $\mathfrak{M}_{0,9}$ generate a subspace of dimension 1088.

For $5 \le n \le 9$, computer computations have confirmed the main conjecture, namely: *for* $n \le 9$, *the weight* $n-3$ *part* $\mathcal{FC}_{n-3}$ *of the formal cell-zeta algebra* $\mathcal{FC}$ *is of dimension* d_{n-3}, *where* d_n *is given by the Zagier formula* $d_n = d_{n-2} + d_{n-3}$ *with* $d_0 = 1$, $d_1 = 0$, $d_2 = 1$.

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