ON LINEARISED AND ELLIPTIC VERSIONS OF THE KASHIWARA-VERGNE LIE ALGEBRA

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Abstract. The goal of this article is to define a linearized or depth-graded version $\mathfrak{lkrv}$, and a closely related elliptic version $\mathfrak{krv}_{ell}$, of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ originally constructed by Alekseev and Torossian as the space of solutions to the linearized Kashiwara-Vergne problem. We show how the elliptic Lie algebra $\mathfrak{krv}_{ell}$ is related to earlier constructions of elliptic versions $\mathfrak{grt}_{ell}$ and $\mathfrak{ds}_{ell}$ of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$ respectively. Based on the known relationships between the three Lie algebra $\mathfrak{grt}$, $\mathfrak{ds}$ and $\mathfrak{krv}$, we discuss the corresponding relationships between the linearized versions, and also between the elliptic versions.

Contents

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0. Introduction

This article studies two Lie algebras closely related to the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ defined in [AT]: firstly, a depth-graded (or "linearized") version $\mathfrak{lkrv}$, and secondly, an elliptic version $\mathfrak{krv}_{ell}$. The results are motivated by the comparison of $\mathfrak{krv}$ with two other Lie algebras familiar from the theory of multiple zeta values: the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$. Our definition of $\mathfrak{lkrv}$ is an analog of the definition of the depth-graded (or linearized) double shuffle Lie algebra $\mathfrak{ls}$, whose structure has given rise to many results and conjectures, in particular the famous Broadhurst-Kreimer conjecture. Our definition of $\mathfrak{krv}_{ell}$ is an analog of the definition of the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ (cf. [S3]), which itself is related on the one hand to $\mathfrak{ls}$ and on the other to the elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{ell}$. We explore all the relations between these different objects. The main observation is that the spaces $\mathfrak{lkrv}$ and $\mathfrak{krv}_{ell}$ are defined by identical sets of properties, only applied to a different class of objects. This situation, which exactly parallels the case of $\mathfrak{ds}$ and $\mathfrak{ds}_{ell}$, reveals a close and surprising relationship between the depth-graded and the elliptic versions of the Lie algebras $\mathfrak{ds}$ and $\mathfrak{krv}$, which remains invisible without the use of mould theory as a basic tool.

Like $\mathfrak{grt}$ and $\mathfrak{ds}$, the Lie algebra $\mathfrak{krv}$ is equipped with a depth filtration; we write gr for the associated graded. We show that in analogy with the known injective map $gr\,\mathfrak{ds} \to \mathfrak{ls}$, there is an injective map $gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkrv}$ (Proposition 1.9). We also show that there is an injective Lie algebra homomorphism $\mathfrak{ls} \hookrightarrow \mathfrak{lkrv}$ (Theorem 1.11), and that the parts of these spaces of depths $d = 1, 2, 3$ are isomorphic for all weights n (Corollary 1.14), which yields the dimensions of the bigraded parts of $\mathfrak{lkrv}$ (and also $gr\,\mathfrak{krv}$) of depths $1, 2, 3$ in all weights, since these dimensions are well-known for $\mathfrak{ls}$.

Passing to the elliptic situation, we define the elliptic version $\mathfrak{krv}_{ell}$ as a subspace of derivations of the free Lie algebra on two generators, and prove that it is closed under the Lie bracket of derivations (Theorem 1.16). We also define an injective Lie algebra homomorphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ (in Theorem 1.18) under the hypothetical isomorphism (113) in analogy with the section map $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ ([E1]) and the mould-theoretic double shuffle map $\mathfrak{ds} \to \mathfrak{ds}_{ell}$ ([S3]). Finally, although we were not able to prove the existence of an injection $\mathfrak{grt}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$, we define a Lie subalgebra

$\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$ such that the following diagram commutes (Theorem 1.20)

$$\begin{array}{ccccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} & \hookrightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell} \end{array} \tag{1}$$

Here the inclusion $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$ is given in [F2] and $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ is given in [S1] [1] and see §1.6 below for other details. The main technique used for the constructions in this article is the mould theory developed by J. Écalle, to which we review in §3.

Structure of the article. §1 presents the principal results of this work. In §2, we reformulate the defining conditions of $\mathfrak{krv}$, which lead to the first definition of $\mathfrak{lkrv}$ and analyze the inclusion $gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkrv}$ (Proposition 1.9). The next section, §3, gives a brief introduction to mould theory and a translation of the defining conditions of $\mathfrak{lkrv}$ into that language, and uses mould theory to prove $\mathfrak{ls} \hookrightarrow \mathfrak{lkrv}$ (Theorem 1.11). Finally, the Lie algebra structure on $\mathfrak{grt}_{ell}$ (Theorems 1.16), the inclusion $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ (Theorem 1.18) and the commutativity of the above diagram (1) (Theorem 1.20) are given in the three subsections of §4. Appendix A furnishes a long proof of Lemma 4.24, a requisite step in establishing Theorem 1.18.

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1. Statements of main results

This section provides explicit statements of the main results concerning both the linearized and elliptic versions of the Kashiwara-Vergne Lie algebra.

1.1. Special types of derivations of $\mathfrak{lie}_2$.

Let $\mathfrak{lie}_2$ denote the degree completion of the free Lie algebra over $\mathbb{Q}$ on non-commutative variables x and y. The Lie algebra $\mathfrak{lie}_2$ has a weight grading by the degree (=*weight*) of the polynomials, and a depth grading by the y-degree (=*depth*) of the polynomials. We write $(\mathfrak{lie}_2)_n$ for the graded part of weight n, $(\mathfrak{lie}_2)^r$ for the graded part of depth r, and $(\mathfrak{lie}_2)^r_n$ for the intersection, which is finite-dimensional.

All the Lie algebras we will study in this article (the well-known ones $\mathfrak{krv}$, $\mathfrak{grt}$ and $\mathfrak{ds}$ as well as the linearized $\mathfrak{ls}$, and the spaces $\mathfrak{lkrv}$ and $\mathfrak{krv}_{ell}$ that we introduce) can be viewed either as Lie subalgebras of particular subalgebras of the derivations of $\mathfrak{lie}_2$, equipped with the bracket of derivations, or as subspaces of $\mathfrak{lie}_2$ equipped with particular Lie brackets coming from the Lie bracket of derivations. Both ways of considering our spaces are natural and useful, and we go back and forth between them as convenient for our proofs.

Let $\mathfrak{der}_2$ denote the algebra of derivations on $\mathfrak{lie}_2$. It is a Lie algebra under the Lie bracket given by the commutator of derivations. For $a, b \in \mathfrak{lie}_2$, we write $D_{b,a}$

[1] The latter inclusion was initially contingent on an assumption, the validity for the senary relation (110) for $ARI(\mathcal{F}_{\rm ser})_{\underline{al}/\underline{il}}$, originally asserted in [Ec2] without a detailed proof (cf. [FK2]). However, recent work in [EF2, Ka, S4] has unconditionally validated its validity.

for the derivation defined by $x \mapsto b$ and $y \mapsto a$. The bracket is explicitly given by

$$[D_{b,a}, D_{b',a'}] = D_{\tilde{b},\tilde{a}} \tag{2}$$

with

$$\tilde{b} = D_{b,a}(b') - D_{b',a'}(b), \quad \tilde{a} = D_{b,a}(a') - D_{b',a'}(a). \tag{3}$$

• Let $\mathfrak{oder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of derivations $D = D_{b,a}$ that annihilate the bracket $[x, y]$ and such that neither $D(x)$ nor $D(y)$ have a linear term in x. The map $\mathfrak{oder}_2 \to \mathfrak{lie}_2$ given by $D \mapsto D(x)$ is injective (see Corollary 4.9).

• Let $\mathfrak{tder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of *tangential derivations*, which are the derivations $E_{a,b}$ for elements $a, b \in \mathfrak{lie}_2$ such that a has no linear term in x and b has no linear term in y, such that

$$E_{a,b}(x) = [x, a], \quad E_{a,b}(y) = [y, b].$$

The Lie bracket is explicitly given by

$$[E_{a,b}, E_{a',b'}] = E_{\tilde{a},\tilde{b}} \tag{4}$$

where

$$\tilde{a} = [a, a'] + E_{a,b}(a') - E_{a',b'}(a), \quad \tilde{b} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b). \tag{5}$$

Let $\mathfrak{tder}_2^{(x)}$ be the Lie subalgebra of $\mathfrak{tder}_2$ consisting of tangential derivations annihilating x, i.e. those of the form

$$d_b = E_{0,b}.$$

The derivation d_b is defined by its values on x and y

$$d_b(x) = 0, \quad d_b(y) = [y, b]. \tag{6}$$

The Lie bracket on $\mathfrak{tder}_2^{(x)}$ is given by $[d_b, d_{b'}] = d_{\{b,b'\}}$, where $\{b, b'\}$ is the *Poisson* (or Ihara) bracket given by

$$\{b, b'\} = [b, b'] + d_b(b') - d_{b'}(b), \tag{7}$$

i.e. the second term of (5).

• Let $\mathfrak{sder}_2^{(x)}$ be the Lie subalgebra of $\mathfrak{tder}_2^{(x)}$ consisting of derivations $D \in \mathfrak{tder}_2$ such that

$$D(x) = 0, \quad D(y) = [y, b], \quad D(z) = [z, c]$$

with $z = -x - y$ for some $b, c \in \mathfrak{lie}_2$.

• Similarly we define $\mathfrak{tder}_2^{(y)}$ to be the Lie subalgebra of $\mathfrak{tder}_2$ consisting of tangential derivations annihilating y, i.e. those of the form $E_{b,0}$. The Lie bracket is given by

$$[E_{b,0}, E_{b',0}] = E_{\{b,b'\}^o,0} \quad \text{with} \quad \{b, b'\}^o := \{b', b\}. \tag{8}$$

This relation follows from the identities $E_{b,0} + d_b = ad(b)$ and (5), since we have $[b, b'] + E_{b,0}(b') - E_{b',0}(b) = [b', b] - d_b(b') + d_{b'}(b) = \{b', b\}$.

• Let $\mathfrak{sder}_2^{(y)}$ be the Lie subalgebra of $\mathfrak{tder}_2^{(y)}$ consisting of derivations $D \in \mathfrak{tder}_2$ such that $D(x) = [x, a], D(y) = 0, D(z) = [z, c]$ for some $a, c \in \mathfrak{lie}_2$.

• Let $\mathfrak{sder}_2^{(z)}$ denote the Lie subalgebra of $\mathfrak{tder}_2$ consisting of derivations such that

$$E_{a,b}(z) = -[x, a] - [y, b] = 0.$$

We compute

$$\begin{aligned}[E_{a,b}, E_{a',b'}](x) &= E_{a,b}([x,a'] - E_{a',b'}([x,a]) \\ &= [[x,a],a'] + [x, E_{a,b}(a')] - [[x,a'],a] - [x, E_{a',b'}(a)] \\ &= [x, E_{a,b}(a') - E_{a',b'}(a) + [a,a']]\end{aligned}$$

and similarly

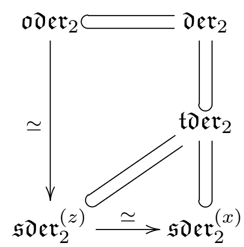
$$[E_{a,b}, E_{a',b'}](y) = [y, E_{a,b}(b') - E_{a',b'}(b) + [b,b']],$$

so the bracket of two derivations $E_{a,b}$ and $E_{a',b'}$ is given by $E_{a'',b''}$ with

(9) $$a'' = E_{a,b}(a') - E_{a',b'}(a) + [a,a'], \quad b'' = E_{a,b}(b') - E_{a',b'}(b) + [b,b'].$$

We have the following diagram showing the connections between these subspaces:

(10)



Remark 1.1. The correspondence $D_{b,a} \mapsto E_{-a,b}$ induces a linear space isomorphism

(11) $$i_{o,z} : \mathfrak{oder}_2 \xrightarrow{\simeq} \mathfrak{sder}_2^{(z)},$$

but this isomorphism is not compatible with the Lie algebra structures.

Let ν be the automorphism defined by

(12) $$\nu(x) = z = -x - y, \quad \nu(y) = y.$$

Lemma 1.2. *Conjugation by ν induces an isomorphism of Lie algebras*

(13) $$\begin{aligned}\mathfrak{sder}_2^{(z)} &\xrightarrow{\sim} \mathfrak{sder}_2^{(x)} \\ E_{a,b} &\mapsto d_{\nu(b)}.\end{aligned}$$

Proof. Recall that $E_{a,b} \in \mathfrak{sder}_2^{(z)}$ maps $x \mapsto [x,a]$ and $y \mapsto [y,b]$, and $d_{\nu(b)} \in \mathfrak{sder}_2^{(x)}$ is the Ihara derivation defined by $x \mapsto 0$, $y \mapsto [y, \nu(b)]$.

Let us first show that $d_{\nu(b)}$ is the conjugate of $E_{a,b}$ by ν, i.e. $d_{\nu(b)} = \nu \circ E_{a,b} \circ \nu$ (since ν is an involution). It is enough to show they agree on x and y, so we compute

$$\nu \circ E_{a,b} \circ \nu(x) = \nu \circ E_{a,b}(z) = 0 = d_{\nu(b)}(x)$$

and

$$\nu \circ E_{a,b} \circ \nu(y) = \nu \circ E_{a,b}(y) = \nu\big([y,b]\big) = [y, \nu(b)] = d_{\nu(b)}(y).$$

This shows that $\nu \circ E_{a,b} \circ \nu$ is indeed equal to $d_{\nu(b)}$. To show that $d_{\nu(b)}$ lies in $\mathfrak{sder}_2^{(x)}$, we check that $d_{\nu(b)}(z)$ is a bracket of z with another element of $\mathfrak{lie}_2$:

$$d_{\nu(b)}(z) = \nu \circ E_{a,b} \circ \nu(z) = \nu \circ E_{a,b}(x) = \nu([x,a]) = [z, \nu(a)].$$

The same argument goes the other way to show that conjugation by ν maps an element of $\mathfrak{sder}_2^{(x)}$ to an element of $\mathfrak{sder}_2^{(z)}$, which yields the isomorphism (13) as linear spaces. To see that it is also an isomorphism of Lie algebras, it suffices to note that conjugation by ν preserves the Lie bracket of derivations in $\mathfrak{Der}_2$, i.e.

$$\nu \circ [D_1, D_2] \circ \nu = [\nu \circ D_1 \circ \nu, \nu \circ D_2 \circ \nu],$$

since ν is an involution. Since the Lie brackets on $\mathfrak{sDer}_2^{(z)}$ and $\mathfrak{sDer}_2^{(x)}$ are just restrictions to those subspaces of the Lie bracket on the space of all derivations, conjugation by ν carries one to the other. □

1.2. **Definition of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$.** The free associative algebra $\mathrm{Ass}_2 = \mathbb{Q}\langle\langle x, y\rangle\rangle$ on non-commutative generators x, y (i.e. the ring of formal power series in x and y) is the completion with respect to the degree of the universal enveloping algebra of the free Lie algebra $\mathfrak{lie}_2$ on x and y.

Definition 1.3. (1). The *trace linear space* $\mathfrak{tr}_2$ (cf. [AT]) is defined to be the quotient of Ass_2 by the equivalence relation given between words in x and y by $w \sim w'$ if w' can be obtained from w by a cyclic permutation of the letters of the word w, and extended linearly to polynomials. The natural projection is denoted

$$tr : \mathrm{Ass}_2 \to \mathfrak{tr}_2.$$

For any polynomial $f \in \mathrm{Ass}_2$ with constant term c, we can decompose f in two ways as

$$f = c + f_x x + f_y y = c + x f^x + y f^y \tag{14}$$

for uniquely determined polynomials f_x, f_y, f^x, f^y in Ass_2.

(2). The *divergence* map is given by

$$\mathrm{div} : \begin{array}{ccc} \mathfrak{tDer}_2 & \longrightarrow & \mathfrak{tr}_2 \\ u = E_{a,b} & \longmapsto & \mathrm{tr}(a_x x + b_y y). \end{array}$$

Definition 1.4. The *Kashiwara-Vergne Lie algebra* $\mathfrak{krv}_2$ is defined to be the subspace of $\mathfrak{sDer}_2^{(z)}$ of derivations $E_{a,b}$ such that there exists a one-variable power series $h(x) \in \mathbb{Q}[x]$ of degree ≥ 2 such that

$$\mathrm{div}(E_{a,b}) = tr\big(h(x+y) - h(x) - h(y)\big). \tag{15}$$

This definition comes from [AT], where it was shown that $\mathfrak{krv}_2$ is actually a Lie subalgebra of $\mathfrak{sDer}_2^{(z)}$. This Lie algebra inherits a weight-grading from that of $\mathfrak{lie}_2$, for which $E_{a,b}$ is of weight n if b (and thus also a) is a Lie polynomial of homogeneous degree n. In particular, the weight 1 part of $\mathfrak{krv}_2$ is spanned by the single element $u = E_{y,x}$, and the weight 2 part is zero. In this article, we do not consider the weight 1 part of $\mathfrak{krv}_2$. For convenience, we set $\mathfrak{krv} = \oplus_{n\geq 3}(\mathfrak{krv}_2)_n$, where $(\mathfrak{krv}_2)_n$ denotes the weight graded part of $\mathfrak{krv}_2$ of weight n. We have

$$\mathfrak{krv}_2 = (\mathfrak{krv}_2)_1 \oplus \mathfrak{krv} = \mathbb{Q}[E_{y,x}] \oplus \mathfrak{krv}.$$

Because the other Lie algebras in the literature that are most often compared with the Kashiwara-Vergne Lie algebra have no weight 1 or weight 2 parts, it makes most sense to compare them with $\mathfrak{krv}$. Thus it is $\mathfrak{krv}$ that we study for the remainder of this article.

The Lie algebra $\mathfrak{krv}$ also inherits a depth filtration from the depth grading on $\mathfrak{lie}_2$, for which $E_{a,b}$ is of depth r if r is the smallest number of y's occurring in any monomial of b. We write $gr\,\mathfrak{krv}$ for the *completed* associated graded for this depth filtration, so that $gr\,\mathfrak{krv}$ is a Lie algebra that is bigraded for the weight and the depth; we write $gr_n^r\,\mathfrak{krv}$ for the part of weight n and depth r. Essentially, an element of $gr\,\mathfrak{krv}$ is a derivation $E_{\bar{a},\bar{b}} \in \mathfrak{sder}_2^{(z)}$ where $\bar{a}, \bar{b}$ are the lowest-depth parts (i.e. the parts of lowest y-degree) of elements $a, b \in \mathfrak{lie}_2$ such that $E_{a,b} \in \mathfrak{krv}$. If $\bar{b}$ is of homogeneous y-degree r, then $\bar{a}$ is of homogeneous y-degree $r+1$.

Example 1.5. The smallest element of $\mathfrak{krv}$ is in weight 3 and is given by $E_{a,b}$ with

$$a = [[x,y],y], \quad b = [x,[x,y]].$$

Since $\bar{a} = a$ and $\bar{b} = b$, this is also equal to $E_{\bar{a},\bar{b}} \in gr\,\mathfrak{krv}$. The next smallest element of $\mathfrak{krv}$ is in weight 5, and the depth-graded part $E_{\bar{a},\bar{b}}$ is given by

$$\bar{a} = [x,[x,[[x,y],y]]] - 2[[x,[x,y]],[x,y]], \qquad \bar{b} = [x,[x,[x,[x,y]]]].$$

By application of mould theory, $\mathfrak{krv}$ is identified with the subspace $ARI(\mathcal{F}_{\rm ser})_{al+tsen/circconst}$ of moulds given in (112) via the isomorphism (38), (115) and (121).

1.3. **The Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$.** Recall that the *Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying the famous *pentagon relation*, equipped with the Poisson bracket (7). This algebra was first introduced by Y. Ihara in [I], with three defining relations, as a particular derivation algebra of $\mathfrak{lie}_2$ (via the association $b \mapsto d_b$ as in (6)); it was subsequently shown that the pentagonal relation implies the other two in [F1].

Recall also that the *double shuffle Lie algebra* $\mathfrak{ds}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying a particular set of conditions on the coefficients called the *stuffle relations*, studied in the first place by Racinet (cf. [R]), who gave a quite difficult proof that $\mathfrak{ds}$ is also a Lie algebra under the Poisson bracket (7). This proof was later somewhat streamlined in the appendix of [F2]. Another proof with a different approach, identifying the space as a stabilizer was given in [EF1]. Putting together basic elements from Écalle's mould theory, particularly with the identification of $\mathfrak{ds}$ with $ARI(\mathcal{F}_{\rm ser})_{\underline{al}/\underline{il}}$ (cf. [S2]), also yields a completely different and very simple proof of this result ([SS]).

There is a commutative triangle of injective Lie algebra homomorphisms:

(16)
$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ & \searrow \quad \swarrow & \\ & \mathfrak{krv} & \end{array} .$$

The existence of the injection $\mathfrak{grt} \to \mathfrak{ds}$ was proven in [F1]; it is given by $b(x,y) \mapsto b(x,-y)$. The existence of the injection $\mathfrak{grt} \to \mathfrak{krv}$ was proven in [AT]; it is given by $b(x,y) \mapsto b(z,y)$ where $z = -x-y$. Finally, the existence of the injection from $\mathfrak{ds}$ to $\mathfrak{krv}$ which is given by $b(x,y) \mapsto b(z,-y)$ was proven in [S1] under the assumption of the senary relation and which was subsequently validated through independent proofs in [S4] and [EF2] (see footnote 1). We add that these morphisms respect the weight gradings and depth filtrations on all three spaces.

1.4. **Results on the linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkrv}$.** For $i \geq 1$, set $C_i = ad(x)^{i-1}(y)$ for $i \geq 1$, and let $\mathfrak{lie}_C$ denote the degree completion of the Lie algebra freely generated over $\mathbb{Q}$ by $C_1, C_2, \ldots$. By Lazard elimination, $\mathfrak{lie}_C$ is free on the C_i and

(17)
$$\mathfrak{lie}_2 \simeq \mathbb{Q} \cdot x \oplus \mathfrak{lie}_C.$$

Thus, Lazard elimination shows that every polynomial $b \in \mathfrak{lie}_2$ having no linear term in x can be written uniquely as a Lie polynomial in the C_i.

Definition 1.6. Let the *push-operator* be defined on monomials in x, y by

$$push(x^{a_0}yx^{a_1}y\cdots yx^{a_r}) = x^{a_r}yx^{a_0}y\cdots yx^{a_{r-1}}. \tag{18}$$

The push is considered to act trivially on constants and powers of x^n, so we can extend it to all of Ass_2 by linearity. Any element $b \in Ass_2$ is said to be

- *push-invariant* if $push(b) = b$, and
- *push-neutral* if $b^r + push(b^r) + \cdots + push^r(b^r) = 0$ for all $r \geq 1$, where b^r denotes the depth r part of b. Finally, we say that b is
- *circ-neutral* if b^y is push-neutral in depths $r > 1$ when writing $b = c + xb^x + yb^y$ $(c \in \mathbb{Q})$.

Our $\mathfrak{lkrv}$ is constructed by the push operator.

Definition 1.7. The *linearized Kashiwara-Vergne Lie algebra* $\mathfrak{lkrv}$ is the space of elements $b \in \mathfrak{lie}_C$ of degree ≥ 3 such that

(i) b is push-invariant, and

(ii) b is circ-neutral.

Our first result is that $\mathfrak{lkrv}$ forms a bigraded Lie algebra.

Proposition 1.8. *The space $\mathfrak{lkrv}$ is bigraded by weight and depth, and forms a Lie algebra under the Poisson bracket defined in (7).*

It is proved in §3.6. We note that it is also proved in [FK1] Remark 2.18 and Theorem 2.23 that the depth > 1-part of $\mathfrak{lkrv}$ forms a Lie algebra.

In §2, we show how we derive the definition of $\mathfrak{lkrv}$ via a reformulation of the defining properties of $\mathfrak{krv}$, in the sense that the defining properties of $\mathfrak{lkrv}$ are merely truncations of the two reformulated defining properties of $\mathfrak{krv}$ to their lowest-depth parts. The reformulation suffices to prove the following result on $\mathfrak{lkrv}$, whose proof is given in the end of §2.3.

Proposition 1.9. *There is an injective Lie algebra morphism*

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkrv}.$$

We speculate that these two spaces are in fact isomorphic.

In using this type of definition for $\mathfrak{lkrv}$, we are following the analogous situation of the well-known double shuffle Lie algebra $\mathfrak{ds}$ and the associated linearized double shuffle space $\mathfrak{ls}$. [2]

Definition 1.10 ([Br])**.** The *linearized double shuffle Lie algebra* $\mathfrak{ls}$ is the bigraded linear space, actually shown to be bigraded Lie algebra under the bracket (7), defined as the set of Lie polynomials $f \in \mathfrak{lie}_2$ of weight $n \geq 3$ such that the polynomial $f_y y$, rewritten in the variables $y_n = x^{n-1}y$ for $n \geq 1$, is an element of the free Lie algebra on the y_i. One also adds the extra assumption that if f is of depth 1, then it is of odd weight.

[2] It is shown in [M] that it is also isomorphic to the linear dual of Goncharov's dihedral Lie coalgebra [G1].

The above additional assumption is not needed for $\mathfrak{lkrv}$ as it follows from the push-invariance condition in the definition. By its very construction, there is an injective Lie algebra homomorphism

$$gr\,\mathfrak{ds} \hookrightarrow \mathfrak{ls}, \tag{19}$$

and it is speculated that these two spaces are isomorphic, but like for $\mathfrak{lkrv}$, this is still an open question.

The injective Lie algebra morphism (16) from $\mathfrak{ds}$ to $\mathfrak{krv}$ yields a corresponding bigraded injective map:

$$gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv} \tag{20}$$

Our next result shows that there is a Lie algebra map on the generalized spaces spaces $\mathfrak{ls}$ and $\mathfrak{lkrv}$ (without any recourse to Écalle's theorem).

Theorem 1.11. *There is a bigraded Lie algebra injection on linearized spaces*

$$\mathfrak{ls} \hookrightarrow \mathfrak{lkrv}. \tag{21}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}_n^r \simeq \mathfrak{lkrv}_n^r.$$

Via the isomorphisms

$$ma : \mathfrak{ls} \xrightarrow{\sim} ARI(\mathcal{F}_{\text{ser}})_{\underline{al}/\underline{al}} \tag{22}$$

given in Lemma 3.44 and

$$ma : \mathfrak{lkrv} \xrightarrow{\sim} ARI(\mathcal{F}_{\text{ser}})_{al+push/circneut}$$

constructed in (78), the proof of the theorem reduces to establishing an inclusion

$$ARI(\mathcal{F}_{\text{ser}})_{\underline{al}/\underline{al}} \hookrightarrow ARI(\mathcal{F}_{\text{ser}})_{al+push/circneut}$$

as formalized in Theorem 3.45.

Remark 1.12. If the isomorphisms $\mathfrak{ls} \simeq gr\,\mathfrak{ds}$ and $\mathfrak{lkrv} \simeq gr\,\mathfrak{krv}$ hold, then the map in Theorem 1.11 is the same as the map (20). Without those speculations, we can only say that the Lie algebra injection (21) should extend the map (20), fitting into a commutative diagram

$$\begin{array}{ccc} gr\,\mathfrak{ds} & \hookrightarrow & gr\,\mathfrak{krv} \\ \downarrow & & \downarrow \\ \mathfrak{ls} & \hookrightarrow & \mathfrak{lkrv}. \end{array} \tag{23}$$

Remark 1.13. We note that in [FK1], the Lie algebra $\mathfrak{lkrv}(\Gamma)$ is constructed for any finite abelian group Γ, thereby extending the definition of $\mathfrak{lkrv}$ In particular, when Γ is trivial this construction coincides with $\mathfrak{lkrv}$ for depths greater than 1 (cf. [FK1, Remark 2.18]). Moreover, Propositions 1.8 and 1.9 and Theorem 1.11 for $\mathfrak{lkrv}(\Gamma)$ were established in [FK1].

Adding a variety of known results in the depth 2 and depth 3 situations to this result, we obtain the following corollary.

Corollary 1.14. *The following spaces are isomorphic for $n \geq 3$ and $r = 1, 2, 3$:*

$$gr_n^r\mathfrak{grt} \simeq gr_n^r\mathfrak{ds} \simeq gr_n^r\mathfrak{krv} \simeq \mathfrak{ls}_n^r \simeq \mathfrak{lkrv}_n^r.$$

In particular, all of these spaces are zero when $r = 1$ or 3 and n is even, or when $r = 2$ and n is odd.

Proof. The dimensions of the spaces $gr_n^r\mathfrak{grt}$, $gr_n^r\mathfrak{ds}$ and $\mathfrak{ls}_n^r$ in depths are known to be equal to each other in depths $r \leq 3$ ([R], [G2]). Indeed more is known than merely the dimensions:

• the spaces $gr_n^1\mathfrak{grt}$, $gr_n^1\mathfrak{ds}$ and $\mathfrak{ls}_n^1$ are all 0 when n is even and 1-dimensional generated by $ad(x)^{n-1}(y)$ when n is odd;

the spaces $gr_n^2\mathfrak{grt}$, $gr_n^2\mathfrak{ds}$ and $\mathfrak{ls}_n^2$ are all 0 when n is odd and spanned by the double Poisson brackets $\{ad(x)^{p-1}(y), ad(x)^{q-1}(y)\}$ for odd $p, q \leq 3$ with $p + q = n$ when n is even;

• the spaces $gr_n^3\mathfrak{grt}$, $gr_n^3\mathfrak{ds}$ and $\mathfrak{ls}_n^3$ are all 0 when n even and spanned by the triple brackets $\{ad(x)^{p-1}(y), \{ad(x)^{q-1}(y), ad(x)^{s-1}(y)\}\}$ with odd $p, q, s \geq 3$ and $p + q + s = n$ when n is odd.

(Note that the proof for $r = 3$ and odd n is much more difficult than the proof for $r = 2$, and was discovered by Goncharov [G2]; as for the case $r \geq 4$, the analogous result is known to be false.) By Theorem 1.11, we see that $\mathfrak{lkrv}_n^r \simeq \mathfrak{ls}_n^r$ for $r = 1, 2, 3$ Finally, since it is known that $\mathfrak{grt}$ injects into $\mathfrak{krv}$ (cf. [AT]), we have a corresponding injection $gr_n^r\mathfrak{grt} \hookrightarrow gr_n^r\mathfrak{krv}$, so by Proposition 1.9, shows that $gr_n^r\mathfrak{krv}$ is sandwiched between $gr_n^r\mathfrak{grt}$ and $\mathfrak{lkrv}_n^r$, which are equal for $r = 1, 2, 3$. This concludes the proof. □

We speculate that $\mathfrak{lkrv}_n^r \simeq \mathfrak{ls}_n^r$ for all n, r, and calculations up to about $n = 15$ bear this speculation out, but we were not able to prove the isomorphism for any other cases, not even the special case $n \not\equiv r \mod 2$, where it is well-known that $gr_n^r\mathfrak{grt} = gr_n^r\mathfrak{ds} = \mathfrak{ls}_n^r = 0$ (cf. [IKZ], [Br] for classical proofs, or [S2] for the exposition of Écalle's mould-theoretic proof).

1.5. **The elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{ell}$ and the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$.** In two independent articles, H. Tsunogai [Ts] and B. Enriquez [E1] defined a Lie algebra that Enriquez calls the *elliptic Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}_{ell}$, based on the idea that just as Ihara had defined $\mathfrak{grt}$ as the algebra of derivations on $\mathfrak{lie}_2$ (identified with the braid Lie algebra on four strands) that extend to a particular type of derivation on the braid Lie algebra on five strands, $\mathfrak{grt}_{ell}$ is the Lie algebra of derivations on $\mathfrak{lie}_2$ (now identified with the genus one braid Lie algebra on two strands) that extend to a very particular type of derivation of the genus one braid Lie algebra on three strands. The construction of $\mathfrak{grt}_{ell}$ shows that it is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is a canonical surjection

$$s : \mathfrak{grt}_{ell} \to \mathfrak{grt}. \tag{24}$$

Let $\mathfrak{r}_{ell}$ denote the kernel. Enriquez [E1] showed that there also exists a Lie algebra morphism

$$\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell} \tag{25}$$

that is a section of (24), i.e. such that $\gamma \circ s = id$ on $\mathfrak{grt}$. Thus, there is a semi-direct product isomorphism

$$\mathfrak{grt}_{ell} \simeq \mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt}). \tag{26}$$

While an elliptic version $\mathfrak{ds}_{ell}$ of the double shuffle Lie algebra $\mathfrak{ds}$ was constructed in [S3] using mould theory as $\Delta\big(ARI^{\Delta}_{\underline{al}*\underline{al}}\big)$ (cf. Definition 4.12), which induces an inclusion

$$\mathfrak{ls} \hookrightarrow \mathfrak{ds}_{ell} \tag{27}$$

by definition and (22) and it is shown there that like $\mathfrak{grt}_{ell}$, $\mathfrak{ds}_{ell}$ is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is an injective Lie algebra homomorphism

$$\tilde{\gamma} : \mathfrak{ds} \to \mathfrak{ds}_{ell} \tag{28}$$

that makes the diagram

$$\begin{array}{ccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \tilde{\gamma}} \\ \mathfrak{grt}_{ell} & & \mathfrak{ds}_{ell} \\ & \searrow \quad \swarrow & \\ & \mathfrak{oder}_2 & \end{array} \tag{29}$$

commutative.

1.6. **Results on the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$.** Our definition of *elliptic Kashiwara-Vergne Lie algebra* is based on that of the linearized Lie algebra $\mathfrak{lkrv}$, differing only from Definition 1.7 by the denominator appearing in (30), which makes it impossible to express it directly in terms of Lie elements like Definition 1.7.

Definition 1.15. The *elliptic Kashiwara-Vergne linear space* $\mathfrak{krv}_{ell}$ is spanned by the elements $b \in \mathfrak{lie}_C$ such that when writing the depth r part $B^r(u_1, \ldots, u_r)$ of $ma(b) \in ARI(\mathcal{F}_{\rm ser})$ (see (45)) and setting

$$B^r_*(u_1, \ldots, u_r) := \frac{1}{u_1 \cdots u_r(u_1 + \cdots + u_r)} B^r(u_1, \ldots, u_r), \tag{30}$$

we have

(i) B^r_* is *push-invariant* for $r \geq 1$; i.e.

$$M^r(u_0, u_1, \ldots, u_{r-1}) = M^r(u_1, \ldots, u_r) \tag{31}$$

holds for $M^r = B^r$ where $u_0 = -u_1 - \cdots - u_r$, and

(ii) $swap(B_*)^r$ (cf. Definition 3.3) is *circ-neutral* up to addition of a constant $C^r \in \mathbb{Q}$ for $r > 1$; that is,

$$M^r(v_1, \ldots, v_r) + M^r(v_2, \ldots, v_r, v_1) + \cdots + M^r(v_r, v_1, \ldots, v_{r-1}) = 0 \tag{32}$$

holds for $M^r = swap(B_*)^r + C^r$.

Its mould theoretical reformulation is given as $\Delta\big(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\big)$ in Definition 4.6. The first main result on $\mathfrak{krv}_{ell}$ is of course that it is a bigraded Lie algebra, but this comes from an injective map from $\mathfrak{krv}_{ell}$ into $\mathfrak{oder}_2$ rather than into $\mathfrak{sder}_2^{(z)}$ as for $\mathfrak{lkrv}$.

Theorem 1.16. *(i) The space $\mathfrak{krv}_{ell}$ is bigraded for the weight and the depth.*

(ii) For each $b \in \mathfrak{krv}_{ell}$, there exists a unique polynomial $a \in \mathfrak{lie}_C$, called the partner *of b, such that $D_{b,a} \in \mathfrak{oder}_2$.*

(iii) The image of the injective linear map $b \mapsto D_{b,a}$ is a Lie subalgebra of $\mathfrak{oder}_2$; in other words $\mathfrak{krv}_{ell}$ is a Lie algebra under the Lie bracket

$$\langle b, b' \rangle = D_{b,a}(b') - D_{b',a'}(b) \tag{33}$$

coming from the bracket of derivations as in (2) and (3).

This theorem is proven in §4.1.

The following is an analogue of the map (27) which will be key to the comparison between $\mathfrak{lkrv}$ and $\mathfrak{krv}_{ell}$, and to the proof that $\mathfrak{lkrv}$ is a Lie algebra.

Proposition 1.17. *There is an injective linear map*

$$\mathfrak{lkrv} \hookrightarrow \mathfrak{krv}_{ell}$$
$$b(x,y) \mapsto [x, b(x,[x,y])].$$

In fact, this linear map is actually a Lie algebra homomorphism.

This claim is proven in §4.1 as Corollary 4.14.

Our second main result on $\mathfrak{krv}_{ell}$ is an analog of the existence of γ and $\tilde{\gamma}$ in the diagram (29).

Theorem 1.18. *Assume the isomorphism in* (113). *Then we obtain an injective Lie algebra morphism*

$$\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$$

This theorem will be proven in §4.2.

Based on the known injective Lie algebra homomorphisms $\mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{krv}$ evoked in §1.3 above, we believe that there are corresponding injective Lie algebra homomorphisms between the elliptic versions of these Lie algebras. However, we were not able to prove that $\mathfrak{grt}_{ell}$ as defined in [E1] injects into $\mathfrak{ds}_{ell}$ or $\mathfrak{krv}_{ell}$. To circumvent this difficulty, we define a Lie subalgebra $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$, conjecturally isomorphic to $\mathfrak{grt}_{ell}$, as follows.

Definition 1.19. For $n \geq 0$, let $\delta_{2n} \in \mathfrak{oder}_2$ denote the derivation of $\mathfrak{lie}_2$ defined by

$$\delta_{2n}(x) = ad(x)^{2n}(y), \quad \delta_{2n}([x,y]) = 0.$$

Let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{oder}_2$ generated by the δ_{2n}.

Enriquez showed in [E1] that $\delta_{2n} \in \mathfrak{r}_{ell}$ for $n \geq 0$, so $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{r}_{ell}$. Let $\mathfrak{B}$ denote the Lie ideal generated by $\mathfrak{b} \subset \mathfrak{r}_{ell}$ under the semi-direct action of $\gamma(\mathfrak{grt})$ on $\mathfrak{r}_{ell}$ of (26). We set

$$\widetilde{\mathfrak{grt}}_{ell} = \mathfrak{B} \rtimes \gamma(\mathfrak{grt}). \tag{34}$$

Our third main result on $\mathfrak{krv}_{ell}$ relates all these maps via a commutative diagram.

Theorem 1.20. *We have the following commutative diagram of injective Lie algebra homomorphisms:*

$$\begin{array}{ccccc} \mathfrak{grt} & \hookrightarrow & \mathfrak{ds} & \hookrightarrow & \mathfrak{krv} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{grt}}_{ell} & \hookrightarrow & \mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathfrak{oder}_2. & & \end{array} \tag{35}$$

The theorem will be proven in §4.3.

2. Reformulation of the definition of $\mathfrak{krv}$ and definition of the linearized Lie algebra $\mathfrak{lkrv}$

In this section, we give a convenient reformulation of the defining conditions of $\mathfrak{krv}$ given in Definition 1.4, which leads to a simple definition of the linearized version $\mathfrak{lkrv}$ that passes easily into the language of moulds which will be essential for our subsequent proofs in §3 and §4.

2.1. **The first defining condition of $\mathfrak{krv}$: specialness.** The first of the two defining conditions of $\mathfrak{krv}$ is that $\mathfrak{krv}$ lies in $\mathfrak{sder}_2^{(z)}$, i.e. elements of $\mathfrak{krv}$ are special tangential derivations having the form $E_{a,b}$ with $E_{a,b}(x) = [x,a]$, $E_{a,b}(y) = [y,b]$ and $[x,a] + [y,b] = 0$.

The following equivalent formulations of the property of specialness as properties of the polynomial b were given in [S1].

Proposition 2.1 ([S1, Theorem 2.1])**.** *Let $b \in \mathfrak{lie}_C$ be homogeneous with degree $n \geq 1$; write*

$$b = b_x x + b_y y = x b^x + y b^y.$$

Then the following are equivalent:

(i) There exists a unique element $a \in \mathfrak{lie}_C$ such that $[x,a] + [y,b] = 0$;

(ii) b is push-invariant;

(iii) $b_y = b^y$.

Proof. The claim when $n \geq 3$ can be proven as follows: The equivalent between (i) and (ii) is given in [S1, Theorem 2.1], where it is also shown that (ii) is equivalent to the antipalindrome for b_y. By [S1, (2.1)], we see that it is equivalent to (iii). The case for $n = 1, 2$ can be proved by a direct computation. □

Thanks to this proposition, we can now reformulate the first defining condition of $\mathfrak{krv}$ as follows: the pair of polynomials $a, b \in \mathfrak{lie}_C$ satisfies $[x,a] + [y,b] = 0$ if and only if b is push-invariant and a is its partner.

2.2. **The second defining condition of $\mathfrak{krv}$: divergence.** We now consider the second defining condition of $\mathfrak{krv}$, the divergence condition. Because $\mathfrak{krv}$ is weight-graded, we may restrict attention to derivations $E_{a,b}$ of homogeneous weight n, i.e. such that a and b are Lie polynomials of homogeneous degree $n \geq 3$. The second defining condition (15) then simplifies to the existence of a constant $c \in \mathbb{Q}$ such that

$$\mathrm{tr}(xa_x + yb_y) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big) \text{ in } \mathfrak{tr}_2.$$

Let us reformulate this as a condition only on b, just as we did for the first defining condition. Since $a \in \mathfrak{lie}_2$, its trace is zero and thus $\mathrm{tr}(xa_x) = \mathrm{tr}(a_x x) = -\mathrm{tr}(a_y y) = -\mathrm{tr}(ya_y)$, so

$$\mathrm{tr}(xa_x + yb_y) = \mathrm{tr}(yb_y - ya_y).$$

Since $E_{a,b} \in \mathfrak{sder}^{(z)}$, we have $[x,a] = [b,y]$. Expanding this in terms of the decompositions of a and b, we obtain

$$xa_x x + xa_y y - xa^x x - ya^y x = xb^x y + yb^y y - yb_x x - yb_y y,$$

from which we deduce that $a_y = b^x$ and $a^y = b_x$. Thus

$$\mathrm{tr}(yb_y - ya_y) = \mathrm{tr}(yb_y - yb^x) = \mathrm{tr}\big(y(b_y - b^x)\big).$$

From Proposition 2.1, we have $b_y = b^y$, so now, using the circularity of the trace, the divergence condition can be reformulated as

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n + y^n\big).$$

We use this to express it as a condition directly on $b^y - b^x$ as follows, using the push-operator defined in (18).

Definition 2.2. A polynomial $b \in Ass_2$ of homogeneous weight $n > 1$ is said to be *push-constant for the value* $c \in \mathbb{Q}$ if b does not contain the monomial y^n and for each $0 < r < n$, writing b^r for the depth r part of b, we have

$$\sum_{i=0}^{r} push^i(b^r) = c \sum_w w$$

where the sum in the right-hand factor is over all monomials of weight n and depth r (for *push*, see (18)). Equivalently, b is push-constant for the value c if it does not contain y^n and for all monomials $w \neq x^n$ with depth r and weight n, we have

$$\sum_{v \in Push(w)} (b|v) = c$$

where $(b|v)$ denotes the coefficient of the monomial v in b, and $Push(w)$ is the list (with possible repetitions) $[w, push(w), \ldots, push^r(w)]$. The monomial x^n is said to be push-constant for its own coefficient. If b is push-constant for the value $c = (b|x^n)$, then we say that b is *strictly push-constant. If $c = 0$, then b is said to be push-neutral (cf. Definition 1.6).*

Example 2.3. The simplest example of a push-constant polynomial is obtained by separating the full set of words of given degree n and depth d into push-orbits, taking one representative w from each push-orbit, and summing over the lists $Push(w)$. For example when $n = 5$ and $d = 2$, the full list of words is given by the 10 words

$$\{x^3y^2, x^2yxy, x^2y^2x, xyx^2y, xyxyx, xy^2x^2, yx^3y, yx^2yx, yxyx^2, y^2x^3\},$$

the push-orbits are given by

$$\{x^3y^2, yx^3y, y^2x^3\},\ \{x^2yxy, yx^2yx, xy^2x^2\},\ \{x^2y^2x, yxyx^2, xyx^2y\},\ \{xyxyx\},$$

the $Push$-lists are the same as the orbits except in the case of $xyxyx$ where the $Push(xyxyx) = [xyxyx, xyxyx, xyxyx]$ three times, and the polynomial b is thus given by

$$b = x^3y^2 + x^2yxy + x^2y^2x + xyx^2y + 3xyxyx + xy^2x^2 + yx^3y + yx^2yx + yxyx^2 + y^2x^3,$$

which is push-constant for the value $c = 3$. Note that since $(b|x^5) = 0$, this polynomial is not strictly push-constant.

Interesting strictly push-constant polynomials can be obtained from elements $\psi \in \mathfrak{grt}$ by writing $\psi = x\psi^x + y\psi^y$ and taking $b = \psi^y$. In this way we obtain for example the degree 4 polynomial b which is strictly push-constant for the value $c = (b|x^4) = 1$:

$$\begin{aligned} b = x^4 + 3x^3y - \frac{9}{2}x^2yx - \frac{1}{2}x^2y^2 + \frac{11}{2}xyx^2 + 2xyxy + \frac{9}{2}xy^2x + 4xy^3 - 2yx^3 \\ - \frac{1}{2}yx^2y - \frac{11}{2}yxyx - 6yxy^2 + 2y^2x^2 + 4y^2xy - y^3x. \end{aligned}$$

The following proposition shows that the divergence condition comes down to requiring that $b^y - b^x$ be push-constant.

Proposition 2.4 ([S1]). *Let $b \in Ass_2$ be a push-invariant Lie polynomial of homogeneous degree n. Then b satisfies the divergence condition*

$$\mathrm{tr}((b^y - b^x)y) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

if and only if $b^y - b^x$ is push-constant for the value nc. Furthermore, if this is the case then

$$c = \frac{1}{n}(b \,|\, x^{n-1}y). \tag{36}$$

The following proof originates from the arguments given in [S1, Section 3].

Proof. Let w be a monomial of degree n and depth $r \geq 1$, and let C_w denote the list of words obtained from w by cyclically permuting the letters, so that C_w contains exactly n words (with possible repetitions). Let C_w^y denote the list obtained from C_w by removing all words ending in x, so that C_w^y contains exactly r words. Write $C_w^y = [u_1y, \ldots, u_ry]$. Then we have the equality of lists

$$[u_1, ..., u_r] = Push(u_1).$$

Let $c_w = \mathrm{tr}(w)$, i.e. c_w is the equivalence class of w, which is the set of the words in the list C_w, without repetitions: thus C_w is nothing other than $n/|c_w|$ copies of c_w. The divergence condition

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

translates as the following family of conditions for one word in each equivalence class c_w:

$$\sum_{v \in c_w} \big((b^y - b^x)y \,|\, v\big) = c|c_w|, \tag{37}$$

where each side is the coefficient of the class c_w in the trace, i.e. the sum of the coefficients of the words in c_w in the original polynomial.

If $r>1$, we can choose a word $uy \in C_w$ that starts in y. Then from (37), the divergence condition on b implies that

$$\begin{aligned} c &= \frac{1}{|c_w|}\sum_{v\in c_w}\big((b^y-b^x)y\,|\,v\big)\\ &= \frac{1}{n}\sum_{v\in C_w}\big((b^y-b^x)y\,|\,v\big)\\ &= \frac{1}{n}\sum_{v\in C_w^y}\big((b^y-b^x)y\,|\,v\big)\\ &= \frac{1}{n}\sum_{u'\in Push(u)}((b^y-b^x)\,|\,u'). \end{aligned}$$

This is exactly the definition of b^y-b^x being push-constant for the value nc.

If $r=1$, then w is of depth 1, $|c_w|=n$ and $x^{n-1}y$ is the only word in c_w ending in y. Thus (37) comes down to

$$\big((b^y-b^x)y\,|\,x^{n-1}y\big)=nc.$$

But since b is a Lie polynomial, we have $(b|x^n)=(b^x|x^{n-1})=0$, so using $b^y=b_y$ (by Proposition 2.1), we also have

$$\big((b^y-b^x)y\,|\,x^{n-1}y\big)=(b^y-b^x\,|\,x^{n-1})=(b^y|x^{n-1})$$

$$=(b_y|x^{n-1})=(b_yy|x^{n-1}y)=(b|x^{n-1}y),$$

which proves that $nc=(b|x^{n-1}y)$ as desired. Note that this condition means that if b has no depth 1 part, then b^y-b^x is push-neutral. □

We now have a new way of expressing $\mathfrak{krv}$, which is much easier to translate into the mould language.

Definition 2.5 (cf. [S1, Theorem 1.2])**.** Let $V_{\mathfrak{krv}}$ be the completion of the linear space spanned by polynomials $b\in\mathfrak{lie}_C$ of homogeneous degree $n\geq 3$ such that

(i) b is push-invariant, and

(ii) b^y-b^x is push-constant for the value $(b\,|\,x^{n-1}y)$,

equipped with the Lie bracket

$$\{b,b'\}=[b,b']+E_{a,b}(b')-E_{a',b'}(b)$$

where a and a' are the (unique) partners of b and b' respectively.

Indeed, since Propositions 2.1 and 2.4 show that

$$\begin{aligned} \mathfrak{krv} &\ \overset{\sim}{\to}\ V_{\mathfrak{krv}}\\ E_{a,b} &\ \mapsto\ b \end{aligned} \tag{38}$$

is an isomorphism of linear spaces (shown in [S1, Theorem 1.2]) and $\mathfrak{krv}$ is known to be a Lie subalgebra of $\mathfrak{sder}_2^{(z)}$, the bracket on $V_{\mathfrak{krv}}$ is inherited directly from this and makes $V_{\mathfrak{krv}}$ into a Lie algebra (cf. [S1, Theorem 1.2]).

2.3. The linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkrv}$. Using the above isomorphism of $\mathfrak{krv}$ with the linear space $V_{\mathfrak{krv}}$ given by $E_{a,b} \mapsto b$, let us now consider the depth-graded versions of the defining conditions of $V_{\mathfrak{krv}}$, i.e. determine what these conditions say about the lowest-depth parts of elements $b \in V_{\mathfrak{krv}}$. The push-invariance is a depth-graded condition, so it restricts to the statement that the lowest depth part of b is still push-invariant; in particular, by Proposition 2.1 it admits of a unique partner $a \in \mathfrak{lie}_C$ such that $[x,a]+[y,b]=0$, i.e. such that the associated derivation $E_{a,b}$ lies in $\mathfrak{sder}_2^{(z)}$.

In the second condition, if b is of degree n and depth $r=1$ and b^1 denotes the lowest-depth part of b, then $(b^1)^y = x^{n-1}$, so the push-constance condition on b^1 is empty since $(b^1)^y = (b|x^{n-1}y)x^{n-1}$. If $r>1$, however, then $(b|x^{n-1}y)=0$ and so the push-constance condition on $b^y - b^x$ is actually push-neutrality, which implies the push-neutrality of $(b^r)^y$ alone, since $(b^r)^y$ is the only part of the expression $b^y - b^x$ of minimal depth $r-1$.

These observations lead directly to the definition of the *linearized version* $\mathfrak{lkrv}$ *of the Kashiwara-Vergne Lie algebra* given in Definition 1.7 above, and that by definition it is bigraded by weight and depth. The statement of Proposition 1.8, that $\mathfrak{lkrv}$ is a Lie algebra under the bracket coming from the bracket of derivations in $\mathfrak{sder}_2^{(z)}$, namely

$$\{b,b'\} = [b,b'] + E_{a,b}(b') - E_{a',b'}(b), \tag{39}$$

will be proved at the end of §4.1. Up to this fact, the proof of Proposition 1.9 now follows trivially from the equivalences above.

Proof of Proposition 1.9 assuming Proposition 1.8.[3] The defining properties of the associated graded $gr\,\mathfrak{krv}$ are properties satisfied by the the lowest-depth parts of elements of $\mathfrak{krv}$. We identify $\mathfrak{krv}$ with $V_{\mathfrak{krv}}$ and use the version of its defining properties expressed in Definition 2.5. Since both the properties of being a Lie element and being push-invariant respect the depth, the same properties are satisfied by elements of $gr\,\mathfrak{krv}$. For the divergence, the argument in the paragraph preceding this proof shows that it implies no condition on the lowest-depth part if the depth is 1, and it implies the push-neutrality of the lowest-depth part if the depth is >1. We do not know if this property along with being Lie and push-invariant, which together define $\mathfrak{lkrv}$, are all that is implied on the lowest-depth part of an element of $\mathfrak{krv}$ by its defining properties, but we certainly know that they all hold for the lowest-depth part, and therefore we obtain the desired inclusion of linear spaces

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkrv}.$$

We now show that this inclusion is a Lie algebra homomorphism. The space $gr(\mathfrak{krv})$ is equipped with the Lie bracket inherited from the Lie bracket on $\mathfrak{krv}$, which is itself inherited from $\mathfrak{sder}^{(z)}$ since $\mathfrak{krv}$ is a Lie subalgebra of $\mathfrak{sder}^{(z)}$; this is the bracket given explicitly in (39). Now suppose that $B, B' \in \mathfrak{krv}$ (more precisely, the derivations $E_{A,B}$ and $E_{A',B'}$ are in $\mathfrak{krv}$, where A and A' denote the uniquely determined partners of B and B' such that $E_{A,B}(z) = E_{A',B'}(z) = 0$). Let b, b' be the lowest-depth part of B and B', of depths r and s respectively, and let a and a' denote the lowest-depth parts of A and A', which are also the partners of b and b', in the sense that $E_{a,b}(z) = E_{a',b'}(z) = 0$. Since $E_{a,b}(x) = [x,a]$ and $E_{a,b}(y) = [y,b]$,

[3] Proposition 1.8 will be proven in Proposition 3.42.

this means that $[x,a]+[y,b]=0$, so a must be of depth $r+1$, and a' must be of depth $s+1$.

Now, the lowest depth part of $[E_{A,B},E_{A',B'}]$ necessarily arises from restricting the bracket to the lowest-depth parts of A,B,A',B', i.e. from the expression

$$[b,b']+E_{a,b}(b')-E_{a',b'}(b). \tag{40}$$

The term $[b,b']$ is of homogeneous depth $r+s$. However, the term $E_{a,b}(b')$, is obtained by replacing each of the s y's of each monomial b' by $[y,b]$, giving terms of depth $s+r$, and each x of b' by $[x,a]$, giving terms of depth $s+r+1$ since a is of depth $r+1$. Similarly, $E_{a',b'}(b)$ contains both terms of depth $s+r$, from replacing each y in b by $[y,b']$, and terms of depth $s+r+1$, from replacing each x in b by $[x,a']$. Thus, to get the lowest depth part of (40), we must ignore the terms coming from replacing x in b' by $[x,a]$ and replacing x in b by $[x,a']$. This comes down to applying the derivation d_b to b' (the Ihara derivation mapping $x\mapsto 0$ and $y\mapsto [y,b]$), and similarly applying $d_{b'}$ to b. So the lowest-depth part of (40) is actually given by

$$[b,b']+d_b(b')-d_{b'}(b), \tag{41}$$

which is precisely the bracket on $\mathfrak{sder}^{(x)}$, and corresponds in mould terms to the ari bracket. Since $\mathfrak{lkv}$ is a Lie algebra equipped with this bracket by Proposition 1.8, this shows that the linear map $gr(\mathfrak{krv})\to\mathfrak{lkrv}$ is actually an injective Lie algebra morphism. □

Remark 2.6. No examples of elements of $\mathfrak{lkrv}$ that are not truncations to lowest depth of elements of $\mathfrak{krv}$ are known. It would be interesting to try to prove the equality of $\mathfrak{lkrv}$ with $gr\,\mathfrak{krv}$ by starting with a polynomial $\mathfrak{lkrv}$ of depth $r>1$ and finding a way to construct a depth by depth lifting to an element of $\mathfrak{krv}$.

3. Mould theory

In this section, we recall the language of Écalle's moulds theory and reformulate the defining conditions of $\mathfrak{lkrv}$ in this language. We end the section with the proof of Theorem 1.11 and its corollary in terms of moulds. We hope that this section and the next one, which explores the elliptic version of $\mathfrak{krv}$, will illustrate the way in which moulds are powerful tools in this context.

3.1. **Moulds and various operators.**

We recall the definitions of moulds and various operators which are employed in this article.

Let $\mathcal{F}=\cup_{m\geqslant 0}\mathcal{F}_m$ denote a family of functions (see [FHK] for a precise definition). In this article, unless otherwise specified, all statements are formulated for a general family of functions $\mathcal{F}$. However, for practical purposes, it suffices to keep the following specific cases in mind:

- $\mathcal{F}_{\rm pol}=\cup_{m\geqslant 0}\mathcal{F}_{{\rm pol},m}$ with $\mathcal{F}_{{\rm pol},m}=\mathbb{Q}[u_1,\dots,u_m]$,
- $\mathcal{F}_{\rm rat}=\cup_{m\geqslant 0}\mathcal{F}_{{\rm rat},m}$ with $\mathcal{F}_{{\rm rat},m}=\mathbb{Q}(u_1,\dots,u_m)$,
- $\mathcal{F}_{\rm ser}=\cup_{m\geqslant 0}\mathcal{F}_{{\rm ser},m}$ with $\mathcal{F}_{{\rm ser},m}=\mathbb{Q}[[u_1,\dots,u_m]]$,
- $\mathcal{F}_{\rm Lau}=\cup_{m\geqslant 0}\mathcal{F}_{{\rm Lau},m}$ with $\mathcal{F}_{{\rm Lau},m}=\mathbb{Q}((u_1,\dots,u_m))$ which is the quotient field of $\mathcal{F}_{{\rm ser},m}$.

The notion of moulds was introduced by J. Écalle in [Ec1]. However this article adopts the formulation in [S1].

Definition 3.1. A *mould* valued in a family of function $\mathcal{F} = \cup_{r\geqslant 0}\mathcal{F}_r$ is a collection $A = \big(A^r(u_1,\dots,u_r)\big)_{r\geq 0}$ where each component $A^r(u_1,u_2,...,u_r)$ belongs to $\mathcal{F}_r$. The set $\mathcal{M}(\mathcal{F})$ of all moulds valued in $\mathcal{F}$ forms a $\mathbb{Q}$-linear space under componentwise addition and scalar multiplication. We denote by $ARI(\mathcal{F})$ the linear subspace consisting of all moulds with $A^0 = 0 \in \mathcal{F}_0$. We say that a mould A is *concentrated in depth* r if $A^s = 0$ for all $s \neq r$, and we let $ARI(\mathcal{F})^r \subset ARI(\mathcal{F})$ be the subspace of moulds concentrated in depth r. Thus $ARI(\mathcal{F}) = \prod_{r\geq 1} ARI(\mathcal{F})^r$.

For later use, we list several operations on $\mathcal{M}(\mathcal{F})$ below:

Definition 3.2 ([Ec2, (2.4)–(2.11)])**.** For any mould $M = (M^m(u_1,\dots,u_m))_m$ in $\mathcal{M}(\mathcal{F})$, the following $\mathbb{Q}$-linear operators are defined componentwise [4]:

$$
\begin{aligned}
&pari(M)(u_1,\dots,u_m) = (-1)^m M(u_1,\dots,u_m),\\
&mantar(M)^m(u_1,\dots,u_m) = (-1)^{m-1} M^m(u_m,\dots,u_1),\\
&push(M)^m(u_1,\dots,u_m) = M^m(-u_1-\cdots-u_m,u_1,\dots,u_{m-1}),\\
&circ(M)^m(u_1,\dots,u_m) = M^m(u_m,u_1,\dots,u_{m-1}),\\
&neg(M)^m(u_1,\dots,u_m) = M^m(-u_1,\dots,-u_m),\\
&teru(M)^m(u_1,\dots,u_m) = M^m(u_1,\dots,u_m)\\
&\qquad + \frac{1}{u_m}\left\{M^{m-1}(u_1,\dots,u_{m-2},u_{m-1}+u_m) - M^{m-1}(u_1,\dots,u_{m-2},u_{m-1})\right\}.
\end{aligned}
$$

For convenience, we also introduce $\overline{\mathcal{M}}(\mathcal{F})$ (resp. $\overline{ARI}(\mathcal{F})$), which is a copy of $\mathcal{M}(\mathcal{F})$ (resp. $ARI(\mathcal{F})$). However, to distinguish between these two, we use the parameters $v_1,v_2,\dots$ instead of $u_1,u_2,\dots$ in $\overline{\mathcal{M}}(\mathcal{F})$ and $\overline{ARI}(\mathcal{F})$. We now introduce Écalle's important *swap* operator on moulds.

Definition 3.3. The operator $swap : \mathcal{M}(\mathcal{F}) \to \overline{\mathcal{M}}(\mathcal{F})$ is defined by

$$swap(B)(v_1,\dots,v_r) = B(v_r,v_{r-1}-v_r,\dots,v_1-v_2)$$

for $B \in \mathcal{M}(\mathcal{F})$. The inverse operator mapping $\overline{\mathcal{M}}(\mathcal{F})$ to $\mathcal{M}(\mathcal{F})$ (which we also denote by *swap*, as the context is clear according to whether *swap* is acting on a mould in $\mathcal{M}(\mathcal{F})$ or one in $\overline{\mathcal{M}}(\mathcal{F})$) is given by

$$swap(C)(u_1,\dots,u_r) = C(u_1+\cdots+u_r,u_1+\cdots+u_{r-1},\dots,u_1)$$

for $C \in \overline{\mathcal{M}}(\mathcal{F})$. Thus it makes sense to write $swap \circ swap = id$.

We also need to consider an important symmetry on moulds, based on the *shuffle* operator on tuples of commutative variables, which is defined by

$$Sh\big((u_1,\dots,u_i)(u_{i+1},\dots,u_r)\big) = \Big\{(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(r)}) \,|\, \sigma \in S_r^i\Big\},$$

where S_r^i is the subset of permutations $\sigma \in S_r$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(r)$.

Definition 3.4. A mould $A \in ARI(\mathcal{F})$ is *alternal* if in each depth $r \geq 2$ we have

$$\sum_{w\in Sh((u_1,\dots,u_i)(u_{i+1},\dots,u_r))} A^r(w) = 0 \quad \text{for} \quad 1 \leq i \leq \left[\frac{r}{2}\right].$$

[4] In [Ec2], the operation *circ* is denoted by *pus*. However, to avoid confusion with *push*, we adopt the notation *circ* throughout this paper.

By convention, the alternality condition is void in depth 1, i.e. all depth 1 moulds are considered to be alternal.

Example 3.5. In depth 4, there are two alternality conditions, given by

$$A(u_1,u_2,u_3,u_4)+A(u_2,u_1,u_3,u_4)+A(u_2,u_3,u_1,u_4)+A(u_2,u_3,u_4,u_1)=0$$
$$A(u_1,u_2,u_3,u_4)+A(u_3,u_1,u_2,u_4)+A(u_3,u_4,u_1,u_2)+A(u_1,u_3,u_2,u_4)$$
$$+A(u_1,u_3,u_4,u_2)+A(u_3,u_1,u_4,u_2)=0$$

Definition 3.6. (1). We write $ARI(\mathcal{F})_{al}$ for the linear subspace of $ARI(\mathcal{F})$ consisting of alternal moulds.

(2). Let $ARI(\mathcal{F})_{al/al}$ denote the linear space of moulds that are alternal and have alternal swap and following Écalle's notation [Ec2], let $ARI(\mathcal{F})_{\underline{al}/\underline{al}}$ denote the linear subspace of $ARI(\mathcal{F})_{al/al}$ of moulds that are even in depth 1.

(3). Let $ARI(\mathcal{F})_{al*al}$ denote the linear space of moulds that are alternal and whose swap are alternal up to addition of constant-valued moulds. Following [S2], we denote by $ARI(\mathcal{F})_{\underline{al}*\underline{al}}$ the linear subspace of $ARI(\mathcal{F})_{al*al}$ moulds that are even functions in depth 1. [5]

Example 3.7. An example of a mould in ARI_{al*al} is the mould where A is the concentrated $\{u_1u_2u_3(u_1+u_2+u_3)\}^{-1}A(u_1,u_2,u_3)$ in depth 3 with

$$A(u_1,u_2,u_3)=-\frac{1}{4}u_1^3u_2+\frac{1}{4}u_1^3u_3-\frac{1}{4}u_1^2u_2^2+\frac{1}{2}u_1^2u_3^2+\frac{1}{4}u_1u_3^3-\frac{1}{4}u_2^2u_3^2-\frac{1}{4}u_2u_3^3$$
$$-\frac{1}{12}u_1^2u_2u_3+\frac{1}{6}u_1u_2^2u_3-\frac{1}{12}u_1u_2u_3^2.$$

It is easy to check that $\Delta^{-1}(A)$ is alternal, but its swap is not alternal unless one adds on the constant $\frac{1}{3}$.

3.2. **The map ma.** Let Ass_2 be the degree-completed free associative algebra on x,y. It forms a (topological) noncommutative co-commutative Hopf algebra whose coproduct is given by $\Delta(x)=x\otimes 1+1\otimes x$ and $\Delta(y)=y\otimes 1+1\otimes y$. Let $e:Ass_2\to\mathbb{Q}$ be the map taking the coefficient of x. Define

$$Ass_2^\dagger:=\{w\in Ass_2\bigm|(e\otimes id)\circ\Delta(w)=0\}. \tag{42}$$

Lemma 3.8. *(i). The subspace $Ass_2^\dagger$ constitutes a Hopf subalgebra of Ass_2.*

(ii). Let $Ass_C=\mathbb{Q}\langle\langle C\rangle\rangle=\mathbb{Q}\langle\langle C_1,C_2,\ldots\rangle\rangle$ be the degree-completion of the free non-commutative polynomial algebra on variables $C_i=\mathrm{ad}_x^{i-1}(y)$ for $i\geq 1$, for the degree given by $deg\, C_i=i$. Then we have a natural identification of Hopf algebras:

$$Ass_2^\dagger\simeq Ass_C. \tag{43}$$

(iii). The algebra Ass_2 admits the following decomposition as completed tensor products:

$$Ass_2=\mathbb{Q}[[x]]\hat{\otimes}Ass_2^\dagger$$

Proof. The above claims follow from [FHK, Lemma 46] and [FHK, Proposition 47]. □

[5] We use the notation ARI_{P_1/P_2} for the linear space of moulds having property P_1 and whose swaps have property P_2. We also use a slightly more general notation $ARI_{P_1*P_2}$ to denote the space of moulds having property P_1 and whose swap has property P_2 up to adding on a constant-valued mould.

Remark 3.9. We consider $C_{a_1,\dots,a_{r-1},a_r} \in Ass_2$ defined by

$$C_{a_1,\dots,a_{r-1},a_r} := \begin{cases} \mathrm{ad}_x(C_{a_1,\dots,a_{r-1},a_r-1}) & (a_r \geq 2), \\ C_{a_1,\dots,a_{r-1}} \cdot y & (a_r = 1), \end{cases}$$

for $r \geq 1$ and $a_1, \dots, a_r \geq 1$. By definition, the set $\{C_{a_1,\dots,a_r} \mid r \geq 1, a_1, \dots, a_r \geq 1\} \cup \{1\}$ forms a linear basis of Ass_C.

From now on, we regard Ass_C as a Hopf subalgebra of Ass_2. Let

$$\mathfrak{lie}_C = \mathfrak{lie}_2 \cap Ass_C \tag{44}$$

where we regard Ass_C as a subspace of Ass_2 under the identification (43). By [FHK, Proposition 47], it is identified with the depth ≥ 1-part of $\mathfrak{lie}_2$. This identification induces the decomposition (17) and through which we establish that this is free generated by C_i with $i \geq 1$ by Lazard elimination (cf. [Bb]).

Lemma 3.10. *(i). For $r, n \geq 1$, let $Ass_C^{(r,n)}$ denote the (finite-dimensional) subspace of Ass_C spanned by elements corresponding to monomials $C_{a_1} \cdots C_{a_r}$ with $a_1 + \cdots + a_r = n$ under the identification (43), and let $\mathcal{M}(\mathcal{F}_{\mathrm{ser}})^{(r,n)}$ denote the subspace of $\mathcal{M}(\mathcal{F}_{\mathrm{ser}})$ [6] consisting of polynomial moulds of degree $n-r$ concentrated in depth r. The map*

$$ma : \begin{array}{ccc} Ass_C^{(r,n)} & \longrightarrow & \mathcal{M}(\mathcal{F}_{\mathrm{ser}})^{(r,n)} \\ C_{a_1} \cdots C_{a_r} & \longmapsto & (-1)^{n-r} u_1^{a_1-1} \cdots u_r^{a_r-1} \end{array} \tag{45}$$

is a linear space isomorphism.

(ii). Let $\mathfrak{lie}_C^{(r,n)} = \mathfrak{lie}_2 \cap Ass_C^{(r,n)}$. For each $r \geq 1$, the map ma restricts to a (finite-dimensional) linear space isomorphism

$$ma : \mathfrak{lie}_C^{(r,n)} \to ARI(\mathcal{F}_{\mathrm{ser}})_{al}^{(r,n)}.$$

with $ARI(\mathcal{F}_{\mathrm{ser}})_{al}^{(r,n)} := ARI(\mathcal{F}_{\mathrm{ser}})_{al} \cap \mathcal{M}(\mathcal{F}_{\mathrm{ser}})^{(r,n)}$.

The above maps yield the following isomorphisms of linear spaces

$$ma : Ass_C \xrightarrow{\sim} \mathcal{M}(\mathcal{F}_{\mathrm{ser}}),$$
$$ma : \mathfrak{lie}_C \xrightarrow{\sim} ARI(\mathcal{F}_{\mathrm{ser}})_{al}.$$

Example 3.11. The mould $ma(C_3) = ma([x,[x,y]])$ is the mould concentrated in depth 1 given by u_1^2. Similarly, $ma(C_2C_1 - C_1C_2) = ma([[x,y],y])$ is the mould concentrated in depth 2 given by $(-1)^{3-2}(u_1^1u_2^0 - u_1^0u_2^1) = u_2 - u_1$.

Lemma 3.12. *Denote $mi := swap \circ ma$. Then the map $mi : Ass_C \to \overline{\mathcal{M}}(\mathcal{F}_{\mathrm{ser}})$ is bijective, and its restriction ($a_1, \dots, a_r \geq 1$ with $a_1 + \cdots + a_r = n$)*

$$mi : \begin{array}{ccc} Ass_C^{(r,n)} & \longrightarrow & \overline{\mathcal{M}}(\mathcal{F}_{\mathrm{ser}})^{(r,n)} \\ C_{a_1,a_2,\dots,a_r} & \longmapsto & (-1)^{n-r} v_r^{a_1-1} v_{r-1}^{a_2-1} \cdots v_1^{a_r-1} \end{array} \tag{46}$$

is a linear isomorphism.

Proof. We prove (46) by induction on $r \geq 1$. When $r = 1$, the claim is obvious. Assume that we have (46) for $r \leq m$, that is, we have

$$ma(C_{a_1,a_2,\dots,a_r})(u_1, \dots, u_s) = \delta_{r,s}(-1)^{n-r} u_1^{a_1-1}(u_1+u_2)^{a_2-1} \cdots (u_1 + \cdots + u_r)^{a_r-1}$$

[6] We may choose instead $\mathcal{M}(\mathcal{F}_{\mathrm{pol}})$.

for $r \leq m$ and $s \geq 0$. When $r = m+1$, by Leibnitz rule, we have

$$\begin{aligned} C_{a_1,\dots,a_m,a_{m+1}} &= \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} \mathrm{ad}_x^i(C_{a_1,\dots,a_m})\mathrm{ad}_x^{a_{m+1}-1-i}(y) \\ &= \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} C_{a_1,\dots,a_{m-1},a_m+i} C_{a_{m+1}-i}. \end{aligned}$$

Applying the map ma to both sides, we get

$$\begin{aligned} &ma(C_{a_1,\dots,a_m,a_{m+1}})(u_1,\dots,u_s) \\ &= \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} ma(C_{a_1,\dots,a_{m-1},a_m+i}) \times ma(C_{a_{m+1}-i})(u_1,\dots,u_s) \\ &= \delta_{m+1,s} \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} ma(C_{a_1,\dots,a_{m-1},a_m+i})(u_1,\dots,u_m) \cdot ma(C_{a_{m+1}-i})(u_{m+1}) \\ &= \delta_{m+1,s} \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} (-1)^{a_1+\cdots+a_m+i-m} u_1^{a_1-1} \cdots (u_1+\cdots+u_{m-1})^{a_{m-1}-1} \\ &\qquad \cdot (u_1+\cdots+u_m)^{a_m+i-1} \cdot (-1)^{a_{m+1}-i-1} u_{m+1}^{a_{m+1}-i-1} \\ &= \delta_{m+1,s}(-1)^{a_1+\cdots+a_{m+1}-(m+1)} u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\cdots+u_m)^{a_m-1} \\ &\qquad \cdot \sum_{i=0}^{a_{m+1}-1} \binom{a_{m+1}-1}{i} (u_1+\cdots+u_m)^i u_{m+1}^{a_{m+1}-i-1} \\ &= \delta_{m+1,s}(-1)^{a_1+\cdots+a_{m+1}-(m+1)} u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\cdots+u_{m+1})^{a_{m+1}-1}. \end{aligned}$$

Applying the map *swap* to both sides, we have

$$\begin{aligned} &mi(C_{a_1,\dots,a_m,a_{m+1}})(v_1,\dots,v_s) \\ &= swap \circ ma(C_{a_1,\dots,a_m,a_{m+1}})(v_1,\dots,v_s) \\ &= \delta_{m+1,s}(-1)^{a_1+\cdots+a_{m+1}-(m+1)} v_{m+1}^{a_1-1} v_m^{a_2-1} \cdots v_1^{a_{m+1}-1}. \end{aligned}$$

So we obtain (46) for $r = m+1$. Hence, we finish the proof. □

Definition 3.13. Let β denote the *backwards writing operator* on words in x, y, meaning that $\beta(m)$ is obtained from a word m by writing it from right to left. The operator β extends to polynomials by linearity.

Remark 3.14. By (43), the map ma in (45) can be regarded as

$$ma(b)(u_1,\dots,u_r) = \sum_{\underline{a}=(0,a_1,\dots,a_r)} k_{\underline{a}} u_1^{a_1}(u_1+u_2)^{a_2}\cdots(u_1+\cdots+u_r)^{a_r}$$

for $r \geq 0$ and $b \in Ass_2^\dagger$ with

$$b = \sum_{r\geq 0} \sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}} x^{a_0} y x^{a_1} \cdots y x^{a_r}.$$

So by Definition 3.3, we have

$$swap(ma(b))(v_1,\dots,v_r) = \sum_{\underline{a}=(0,a_1,\dots,a_r)} k_{\underline{a}} v_r^{a_1} v_{r-1}^{a_2} \cdots v_1^{a_r}.$$

3.3. **Ari bracket.** We explain the Lie bracket ari on $ARI(\mathcal{F})$ and explain how it is related with the bracket $\{.,.\}$ of (7).

We begin by introducing the standard mould multiplication that Écalle denotes mu.

Definition 3.15. Let $\mathcal{F}$ be a family of functions. For moulds A, B in $\mathcal{M}(\mathcal{F})$, we define the multiplication

$$mu(A,B)(u_1,\ldots,u_r) = \sum_{i=0}^{r} A(u_1,\ldots,u_i)B(u_{i+1},\ldots,u_r).$$

By mu, $\mathcal{M}(\mathcal{F})$ forms a $\mathbb{Q}$-algebra. The associated Lie bracket lu in $ARI(\mathcal{F})$ is defined by

$$lu(A,B) = mu(A,B) - mu(B,A).$$

We write $ARI(\mathcal{F})_{lu}$ for $ARI(\mathcal{F})$ viewed as a Lie algebra for the lu-bracket.

The identical formulas yield a multiplication and Lie algebra (also called mu and lu) on $\overline{ARI}(\mathcal{F})$.

Remark 3.16. If f and g are power series in Ass_C and $A = ma(f)$, $B = ma(g)$ in $\mathcal{M}(\mathcal{F}_{\mathrm{ser}})$, then mu is a mould translation of the usual non-commutative multiplication, and lu the usual Lie bracket:

$$mu(A,B) = ma(fg), \qquad lu(A,B) = ma\big([f,g]\big).$$

In order to define Écalle's ari-bracket, we first introduce three derivations of $ARI(\mathcal{F})_{lu}$ associated to a given mould $A \in ARI(\mathcal{F})$. It is non-trivial to prove that these operators are actually derivations (cf. [S2, Prop. 2.2.1]).

Definition 3.17 ([Ec2, §2.2])**.** Let $\mathcal{F}$ be a family of functions. Let $B \in ARI(\mathcal{F})$. Then the derivation $amit(B)$ of $ARI(\mathcal{F})_{lu}$ is given by

$$\big(amit(B)\cdot A\big)(u_1,\ldots,u_r) = \sum_{0\le i<j<r} A(u_1,\ldots,u_i,u_{i+1}+\cdots+u_{j+1},u_{j+2},\ldots,u_r)B(u_{i+1},\ldots,u_j),$$

and the derivation $anit(B)$ is given by

$$\big(anit(B)\cdot A\big)(u_1,\ldots,u_r) = \sum_{0<i<j\le r} A(u_1,\ldots,u_{i-1},u_i+\cdots+u_j,u_{j+1},\ldots,u_r)B(u_{i+1},\ldots,u_j).$$

We also have corresponding derivations $\overline{amit}(B)$ and $\overline{anit}(B)$ of $\overline{ARI}(\mathcal{F})_{lu}$ for $B \in \overline{ARI}(\mathcal{F})$, given by the formulas

$$\big(\overline{amit}(B)\cdot A\big)(v_1,\ldots,v_r) = \sum_{0\le i<j<r} A(v_1,\ldots,v_i,v_{j+1},\ldots,v_r)B(v_{i+1}-v_{j+1},\ldots,v_j-v_{j+1}),$$

$$\big(\overline{anit}(B)\cdot A\big)(v_1,\ldots,v_r) = \sum_{0<i<j\le r} A(v_1,\ldots,v_i,v_{j+1},\ldots,v_r)B(v_{i+1}-v_i,\ldots,v_j-v_i).$$

Finally, Écalle defines the derivation $arit(B)$ on $ARI(\mathcal{F})_{lu}$ by

$$arit(B) = amit(B) - anit(B),$$

and the ari-bracket on $ARI(\mathcal{F})$ by

$$ari(A,B) = arit(B)\cdot A - arit(A)\cdot B + lu(A,B), \tag{47}$$

as well as the derivation $\overline{arit}$ on $\overline{ARI}(\mathcal{F})_{lu}$ and the bracket $\overline{ari}$ on $\overline{ARI}(\mathcal{F})$ by the same formulas with overlines.

Proposition 3.18. *Let $\mathcal{F}$ be a family of functions. Then we have the following:*

(1). The linear space $ARI(\mathcal{F})$ forms a Lie algebra under the ari-bracket (47).

(2). The linear subspace $ARI(\mathcal{F})_{al}$ forms a Lie subalgebra of $ARI(\mathcal{F})$.

(3). The linear subspace $ARI(\mathcal{F})_{al/al}$ forms a Lie subalgebra of $ARI(\mathcal{F})$.

(4). The linear subspace $ARI(\mathcal{F})_{\underline{al}*\underline{al}}$ forms a Lie subalgebra of $ARI(\mathcal{F})$.

Proof. The proof of claim (1) is given in [S2], although the proof of the key formula (2.2.10) looks unprovided therein. A complete proof, together with an extended version of the claim, is given in [FK1, Proposition 1.14]. The claim (2) (resp. (3)) is first proven in [SS, Appendix A] (resp. [S2, Theorem 2.5.6]), while its generalization is systematically developed in [FK1, Proposition 1.15 (resp. Proposition 1.24)]. The claim (4) can be deduced from the claim (3). □

Remark 3.19. The definitions of *amit*, *anit*, *arit* and *ari* are generalizations to all moulds of familiar derivations of Ass_C. Indeed, if $f, g \in Ass_C$ and $A = ma(f)$, $B = ma(g)$ in $\mathcal{M}(\mathcal{F}_{\rm ser})$, then

$$amit(B) \cdot A = ma\big(D_g^l(f)\big)$$

where D_g^l is defined by $x \mapsto 0$, $y \mapsto gy$,

$$anit(B) \cdot A = ma\big(D_g^r(f)\big)$$

where D_g^r is defined by $x \mapsto 0$, $y \mapsto yg$, and thus

$$arit(B) \cdot A = ma\big(-d_g(f)\big)$$

where d_g is the Ihara derivation $x \mapsto 0$, $y \mapsto [y, g]$ (see (6)), and

$$ari(A, B) = ma\big([f, g] + d_f(g) - d_g(f)\big) = ma\big(\{f, g\}\big). \tag{48}$$

corresponds to the Ihara or Poisson Lie bracket (7) on $\mathfrak{lie}_C$. (See [S2], Corollary 3.3.4).

Lemma 3.20. *The map $b \mapsto -E_{b,0}$ induces a Lie algebra isomorphism*

$$i_y : (\mathfrak{lie}_C, \{,\}) \xrightarrow{\sim} \mathfrak{tder}_2^{(y)}. \tag{49}$$

Proof. Bijectivity is clear. By (8), we have

$$[-E_{b,0}, -E_{b',0}] = -E_{\{b,b'\},0},$$

which shows that i_y is a Lie algebra homomorphism. □

Denote $Ass_{C,\geq 1}$ to be the subspace of Ass_C which is the depth $\geq$ 1-part of Ass_C, that is,

$$Ass_C = \mathbb{Q}1 \oplus Ass_{C,\geq 1}.$$

Note that the space $Ass_{C,\geq 1}$ is generated by $\{C_{a_1} \cdots C_{a_r} \mid r \geq 1, a_1, \ldots, a_r \geq 1\}$.

Lemma 3.21. *The map ma induces the Lie algebra isomorphism:*

$$ma : (Ass_{C,\geq 1}, \{,\}) \to (ARI(\mathcal{F}_{\rm ser}), ari),$$
$$ma : (\mathfrak{lie}_C, \{,\}) \to (ARI(\mathcal{F}_{\rm ser})_{al}, ari).$$

Proof. This follows from the above arguments and Lemma 3.10. □

Definition 3.22. Let $\mathcal{F}$ be a family of functions. We denote by $GARI(\mathcal{F})$ the set of all moulds A in $\mathcal{M}(\mathcal{F})$ with constant term 1, that is, $A^0 = 1 \in \mathcal{F}_0$. Similarly, we introduce $\overline{GARI}(\mathcal{F})$ which is a copy of $GARI(\mathcal{F})$.

To introduce an operator $\overline{ganit}$, we prepare the following notations: Set $\mathbf{v} = (v_1, \ldots, v_r)$, and let $W_{\mathbf{v}}$ denote the set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks

$$d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s \tag{50}$$

for $s \geq 1$, where with the possible exception of $\mathbf{b}_s$, the $\mathbf{a}_i$ and $\mathbf{b}_i$ are non-empty. Thus for instance, when $r = 2$ there are two decompositions in $W_{\mathbf{v}}$, namely $\mathbf{a}_1 = (v_1, v_2)$ and $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2)$, and when $r = 3$ there are four decompositions, three for $s = 1$: $\mathbf{a}_1 = (v_1, v_2, v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1, v_2)(v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2, v_3)$, and one for $s = 2$: $\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 = (v_1)(v_2)(v_3)$.

Definition 3.23 ([Ec2, §2.2])**.** For any mould $Q \in \overline{GARI}(\mathcal{F})$, we define an operation $\overline{ganit}(Q)$ given by

$$\big(\overline{ganit}(Q) \cdot P\big)(\mathbf{v}) = \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s \in W_{\mathbf{v}}} Q(\lfloor\mathbf{b}_1) \cdots Q(\lfloor\mathbf{b}_s)\ P(\mathbf{a}_1 \cdots \mathbf{a}_s), \tag{51}$$

where if $\mathbf{b}_i$ is the chunk $(v_k, v_{k+1}, \ldots, v_{k+l})$, then we use the notation

$$\lfloor\mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1}). \tag{52}$$

We note that in [Ko, Theorem 3.7] it is shown that $\overline{ganit}(Q)$ is an automorphism of the Lie algebra $\overline{ARI}(\mathcal{F})_{lu}$.

3.4. **The special mould *pal* and Écalle's fundamental identity.**

We are now ready to introduce the fundamental identity of Écalle in Remark 3.27, which is the key to the proof of Theorem 1.18 given in §4.2.

Definition 3.24. Let constants $c_r \in \mathbb{Q}$, $r \geq 1$, be defined by setting $f(x) = 1 - e^{-x}$ and expanding $f_*(x) = \sum_{r\geq 1} c_r x^{r+1}$, where $f_*(x)$ is the *infinitesimal generator* of $f(x)$, defined by

$$f(x) = \Big(exp\big(f_*(x)\frac{d}{dx}\big)\Big) \cdot x.$$

Let

$$lopil \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})_{\overline{ari}}$$

be the mould defined by $lopil(v_1) = -\frac{1}{2v_1}$ and for $r \geq 2$ by the simple expression

$$lopil(v_1, \ldots, v_r) = c_r\, \frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r} \tag{53}$$

Set

$$pil = exp_{\overline{ari}}(lopil) \in GARI(\mathcal{F}_{\mathrm{Lau}})$$

where $exp_{\overline{ari}}$ denotes the exponential map associated to the pre-Lie law $\overline{preari}$ given by

$$\overline{preari}(A, B) = \overline{arit}(B) \cdot A + mu(A, B) \qquad \text{on} \quad \overline{ARI}(\mathcal{F}),$$

and set

$$pal = swap(pil) \in GARI(\mathcal{F}_{\mathrm{Lau}}).$$

The mould *lopil* is easily seen to be both alternal and circ-neutral (see Definition 3.33 and Example 3.34). It is also known (although surprisingly difficult to show) that the mould

$$lopal = log_{ari}(pal) \in ARI(\mathcal{F}_{\mathrm{Lau}})$$

is alternal (cf. [Ec3, §4], or [S2, Chap. 4]). Thus the moulds *pil* and *pal* are both exponentials of alternal moulds; this is called being *symmetral*. The inverses of *pal* in $GARI(\mathcal{F}_{\rm Lau})$ and *pil* in $\overline{GARI}(\mathcal{F}_{\rm Lau})$ are given by

$$invpal = exp_{ari}(-lopal), \qquad invpil = exp_{\overline{ari}}(-lopil).$$

Remark 3.25. The key maps we will be using in our proof are the adjoint operators associated to *pal* and *pil*, given by

$$Ad_{ari}(pal) = exp\big(ad_{ari}(lopal)\big), \qquad Ad_{\overline{ari}}(pil) = exp\big(ad_{\overline{ari}}(lopil)\big), \tag{54}$$

where $ad_{ari}(P)\cdot Q = ari(P,Q)$. The inverses of these adjoint actions are given by

$$Ad_{ari}(invpal) = exp\big(ad_{ari}(-lopal)\big), \quad Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big). \tag{55}$$

These adjoint actions produce remarkable transformations of certain mould properties into others, and form the heart of much of Écalle's theory of multizeta values.

To discuss the fundamental identity, we prepare the following two moulds.

Definition 3.26. Let $pic \in \overline{GARI}(\mathcal{F}_{\rm Lau})$ be the mould defined by

$$pic(v_1,\dots,v_r) = \frac{1}{v_1\cdots v_r}.$$

Similarly let $poc \in \overline{GARI}(\mathcal{F}_{\rm Lau})$ be the mould defined by

$$poc(v_1,\dots,v_r) = \frac{-1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)}. \tag{56}$$

Remark 3.27. Écalle's *fundamental identity* [7] relates the two adjoint actions of (54). Valid for all push-invariant moulds R, it is given by

$$swap\cdot Ad_{ari}(pal)\cdot R = \overline{ganit}(pic)\cdot Ad_{\overline{ari}}(pil)\cdot swap(R). \tag{57}$$

Since it is shown

$$\overline{ganit}(pic)^{-1} = \overline{ganit}(poc) \tag{58}$$

as automorphisms of $\overline{ARI}(\mathcal{F}_{\rm Lau})_{lu}$ in the proof of [B, Lemma 4.37], we can rewrite the above identity (57) as

$$\overline{ganit}(poc)\cdot swap\cdot Ad_{ari}(pal)\cdot R = Ad_{\overline{ari}}(pil)\cdot swap(R).$$

Letting $N = Ad_{ari}(pal)\cdot R$, i.e. $R = Ad_{ari}(invpal)\cdot N$, we further rewrite it in terms of N as

$$Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)\cdot swap(N) = swap\cdot Ad_{ari}(invpal)\cdot N, \tag{59}$$

which is valid whenever $R = Ad_{ari}(invpal)\cdot N$ is push-invariant. This identity will be used later in the proof of Proposition 4.21.

[7]This identity, indicated in a personal communication by Écalle, follows from the first fundamental identity given in (2.62) of [Ec2] (see also (2.8.4) of [S2]). The complete statement and proof is given in [S2] Theorem 4.5.2; the proof relies among other things on a basic fact of mould theory stated by Écalle and used constantly in the mould literature, namely that the operator $\overline{ganit}(pic)$ transforms alternal moulds in $\overline{ARI}(\mathcal{F})$ to alternil moulds. A full proof of this fact was not written down until the recent article [Ko] by N. Komiyama, see Corollary 3.25.

3.5. **Push-invariance and the first defining relation of $\mathfrak{lkrv}$.** We show how to reformulate the first defining property of elements of $\mathfrak{lkrv}$ in terms of moulds.

Definition 3.28. A mould $B \in ARI(\mathcal{F})$ is said to be *push-invariant* if

$$push\,(B) = B.$$

We denote the subset of push-invariant moulds in $ARI(\mathcal{F})$ by $ARI(\mathcal{F})_{push}$.

Proposition 3.29. *Let $\mathcal{F}$ be a family of functions. The subspace $ARI(\mathcal{F})_{push}$ forms a Lie subalgebra of $ARI(\mathcal{F})$ under the ari-bracket.*

Proof. A complete demonstration of this result, along with its generalization to broader cases, is presented in [FK1, Proposition 1.28]. □

Then the following proposition shows that this notion is precisely the translation into mould terms of the property of push-invariance for a Lie polynomial given in Definition 1.6 above.

Proposition 3.30. *Define*

$$Ass_{C,\geq 1}^{push} := \{b \in Ass_{C,\geq 1} \mid push(b) = b\}.$$

Then the map ma induces an isomorphism of linear spaces

$$ma : Ass_{C,\geq 1}^{push} \xrightarrow{\sim} ARI(\mathcal{F}_{\rm ser})_{push}.$$

Whence Ass_C^{push} forms a Lie algebra under the bracket $\{,\}$ and we have a Lie algebra isomorphism

$$ma : (Ass_{C,\geq 1}^{push}, \{,\}) \xrightarrow{\sim} (ARI(\mathcal{F}_{\rm ser})_{push}, ari).$$

Proof. By Lemma 3.21, it is enough to show that $b \in Ass_{C,\geq 1}$ is push-invariant if and only if $ma(b)$ is a push-invariant mould, i.e. $push(ma(b)) = ma(b)$.

Let $b \in Ass_{C,\geq 1}$ with depth 1. We only prove the case $b = C_{a+1}$ for $a \geq 0$. Then we have

$$\begin{aligned} push(b) &= \sum_{k=0}^{a}(-1)^k\binom{a}{k}push(x^{a-k}yx^k) = \sum_{k=0}^{a}(-1)^k\binom{a}{k}x^k y x^{a-k} \\ &= (-1)^a\sum_{k=0}^{a}(-1)^k\binom{a}{k}x^{a-k}yx^k = (-1)^a b. \end{aligned}$$

So b is push-invariant if and only if a is even. When $b = C_{a+1}$, by (45), the associated mould $ma(b)$ is given as

$$ma(b)(u_1,\dots,u_r) = \begin{cases} (-1)^a u_1^a & (r=1), \\ 0 & (r \neq 1), \end{cases}$$

so a is even if and only if $ma(b)$ is push-invariant, that is, we obtain the claim for depth 1.

Now, let $b \in Ass_{C,\geq 1}$ with depth $r \geq 2$, and put $f = yb \in Ass_{C,\geq 1}$. The associated moulds $ma(f)$ and $ma(b)$ are related by the formula

$$\begin{aligned} ma(f)(u_0,u_1,...,u_r) &= ma(C_1 b)(u_0,\dots,u_r) = ma(C_1)(u_0)ma(b)(u_1,...,u_r) \\ &= ma(b)(u_1,...,u_r). \end{aligned} \tag{60}$$

We write
$$b=\sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}}\, x^{a_0}yx^{a_1}\cdots yx^{a_r},$$
then we have
$$f=x^0yb=\sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}}\, x^0yx^{a_0}yx^{a_1}\cdots yx^{a_r}.$$
By Remark 3.14, we get
$$swap(ma(f))(v_0,\dots,v_r)=\sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}} v_r^{a_0}v_{r-1}^{a_1}\cdots v_1^{a_{r-1}}v_0^{a_r}. \tag{61}$$
Because we have
$$push(b)=\sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}}\, x^{a_r}yx^{a_0}\cdots yx^{a_{r-1}}=\sum_{\underline{a}=(a_0,\dots,a_r)} k_{a_1,\dots,a_r,a_0}\, x^{a_0}yx^{a_1}\cdots yx^{a_r},$$
$b=push(b)$ if and only if $k_{\underline{a}}=k_{(a_1,\dots,a_r,a_0)}$ for each $\underline{a}$, this is equivalent to
$$\begin{aligned} swap\big(ma(f)\big)(v_0,\dots,v_r) &= \sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}} v_r^{a_0}v_{r-1}^{a_1}\cdots v_1^{a_{r-1}}v_0^{a_r} \\ &= \sum_{\underline{a}=(a_0,\dots,a_r)} k_{(a_1,\dots,a_r,a_0)} v_r^{a_0}v_{r-1}^{a_1}\cdots v_1^{a_{r-1}}v_0^{a_r} \\ &= \sum_{\underline{a}=(a_0,\dots,a_r)} k_{\underline{a}} v_r^{a_r}v_{r-1}^{a_0}\cdots v_1^{a_{r-2}}v_0^{a_{r-1}} = swap\big(ma(f)\big)(v_r,v_0,\dots,v_{r-1}) \end{aligned} \tag{62}$$
by (61). Using the definition of the *swap*, we rewrite (62) in terms of $ma(f)$ as
$$ma(f)(v_r,v_{r-1}-v_r,\dots,v_0-v_1)=ma(f)(v_{r-1},v_{r-2}-v_{r-1},\dots,v_0-v_1,v_r-v_0). \tag{63}$$
By making the change of variables
$$v_k=\begin{cases} u_0+\cdots+u_{r-k-1} & (0\le k\le r-1),\\ u_0+\cdots+u_r & (k=r),\end{cases}$$
in this equation, we obtain
$$ma(f)(u_0+\cdots+u_r,-u_1-\cdots-u_r,u_1,\dots,u_{r-1})=ma(f)(u_0,u_1,\dots,u_r). \tag{64}$$
Finally, using relation (60), we write this in terms of $ma(b)$ as
$$ma(b)(-u_1-\cdots-u_r,u_1,\dots,u_{r-1})=ma(b)(u_1,\dots,u_r). \tag{65}$$
which is just the condition of mould push-invariance $ma(b)$ in depth r. The last claim follows from Lemma 3.21 and Proposition 3.29. □

For a family $\mathcal{F}$ of functions, we define
$$ARI(\mathcal{F})_{al+push}:=ARI(\mathcal{F})_{al}\cap ARI(\mathcal{F})_{push},$$
which forms a Lie algebra under the *ari*-bracket by Propositions 3.18 and 3.29.

Corollary 3.31. *The linear subspace*
$$\mathfrak{lie}_C^{push}:=Ass_{C,\ge 1}^{push}\cap\mathfrak{lie}_C$$
forms Lie subalgebra of $\mathfrak{lie}_C$ *(cf. (44)) under the bracket* $\{,\}$ *and we have a Lie algebra isomorphism*
$$ma:(\mathfrak{lie}_C^{push},\{\ ,\ \})\xrightarrow{\sim}(ARI(\mathcal{F}_{\rm ser})_{al+push},ari). \tag{66}$$

Proof. Since the bracket $\{,\}$ is compatible with ari under the map ma by Lemma 3.21, the result follows from Proposition 3.30. $\square$

We also obtain the following corollary.

Corollary 3.32. *The map i_y in Lemma 3.20 induces Lie algebra inclusion*

$$i_y : (\mathfrak{lie}_C^{push}, \{\ ,\ \}) \hookrightarrow \mathfrak{tder}_2^{(y)}. \tag{67}$$

Proof. This follows from Lemma 3.21 and Corollary 3.31. $\square$

3.6. **Circ-neutrality and the second defining relation of $\mathfrak{krv}$.** We show how to reformulate the second defining property of elements of $\mathfrak{krv}$ in terms of moulds and prove Proposition 1.8.

For $B \in \overline{ARI}(\mathcal{F})$ with $\mathcal{F} = \mathcal{F}_{\mathrm{pol}}$ or $\mathcal{F}_{\mathrm{ser}}$, there is a unique decomposition $\{B_n\}_{n\geq 1} \subset \overline{ARI}(\mathcal{F}_{\mathrm{pol}})$ such that $\deg B_n(v_1, \dots, v_r) = n - r$ for $1 \leq r \leq n$ and

$$B = \sum_{n=1}^{\infty} B_n.$$

Actually, if B is represented by

$$B(v_1, \dots, v_r) = \sum_{k_1,\dots,k_r=0}^{\infty} \langle B \mid k_1, \dots, k_r \rangle v_1^{k_1} \cdots v_r^{k_r}$$

for $r \geq 1$, each mould B_n ($n \geq 1$) can be represented by

$$B_n(v_1, \dots, v_r) = \sum_{\substack{k_1+\cdots+k_r=n-r \\ k_i \geq 0}} \langle B \mid k_1, \dots, k_r \rangle v_1^{k_1} \cdots v_r^{k_r}$$

for $1 \leq r \leq n$, and $B_n(v_1, \dots, v_r) = 0$ for $r > n$.

Definition 3.33. (1). A mould $B \in \overline{ARI}(\mathcal{F})$ is said to be *circ-neutral* [8] (cf. [Ec2, (2.73)]) if for $r > 1$ we have

$$\sum_{i=0}^{r-1} circ^i(B)(v_1, \dots, v_r) = 0.$$

For $circ$, see Definition 3.2.

(2). Let $B \in \overline{ARI}(\mathcal{F})$ with $\mathcal{F} = \mathcal{F}_{\mathrm{pol}}$ or $\mathcal{F}_{\mathrm{ser}}$, and consider the decomposition $\{B_n\}_{n\geq 1}$ of B as above. The mould B is called *circ-constant for the sequence* $\{c_n\}_{n>1} \subset \mathbb{Q}$ if we have[9]

$$\sum_{i=0}^{r-1} circ^i(B_n)(v_1, \dots, v_r) = c_n \left(\sum_{\substack{a_1+\cdots+a_r=n-r \\ a_i \geq 0}} v_1^{a_1} \cdots v_r^{a_r} \right) \tag{68}$$

for $n \geq 2$ and for $1 < r \leq n$. If $c_n = 0$ for any $n \geq 2$, we say that B is *circ-neutral*.

[8] This property is referred to as "pus-neutral" in [FK1, Definition 1.26] (respectively, [Ec2, (2.73)]) when it holds for all $r \geq 1$ (respectively, for all $r > 1$). The distinction between these two definitions disappears when the depth-one component vanishes; for example, this is the case under the push-invariant condition.

[9] When $r = n$, we note that

Example 3.34. We put $v_k := v_{k-r}$ for $k > r$. By (53), for $r > 1$, we have

$$\begin{aligned}\sum_{i=0}^{r-1} circ^i(lopil)(v_1,\dots,v_r) &= \sum_{i=0}^{r-1} lopil(v_{1+i},\dots,v_{r+i})\\ &= c_r \sum_{i=0}^{r-1} \frac{v_{1+i}+\cdots+v_{r+i}}{v_{1+i}(v_{1+i}-v_{2+i})\cdots(v_{r-1+i}-v_{r+i})v_{r+i}}\\ &= c_r \frac{v_1+\cdots+v_r}{(v_1-v_2)\cdots(v_{r-1}-v_r)(v_r-v_1)} \sum_{i=0}^{r-1} \frac{v_{r+i}-v_{r+1+i}}{v_{1+i}v_{r+i}}.\end{aligned}$$

Because $v_{r+1+i} = v_{1+i}$ for $0 \le i \le r-1$, we get

$$\sum_{i=0}^{r-1} \frac{v_{r+i}-v_{r+1+i}}{v_{1+i}v_{r+i}} = \sum_{i=0}^{r-1} \left(\frac{1}{v_{1+i}} - \frac{1}{v_{r+i}} \right) = 0,$$

so we obtain

$$\sum_{i=0}^{r-1} circ^i(lopil)(v_1,\dots,v_r) = 0$$

for $r > 1$, that is, the mould *lopil* is circ-neutral.

The following is the corresponding definition in Ass_2-side.

Definition 3.35. (1). We define the $\mathbb{Q}$-linear map $circ : Ass_2 \to Ass_2$ by

$$circ(x^{a_0}yx^{a_1}yx^{a_2}\cdots yx^{a_r}) := x^{a_0}yx^{a_r}yx^{a_1}\cdots yx^{a_{r-1}} \tag{69}$$

for $r \ge 0$ and $a_0, a_1, \dots, a_r \ge 0$. In particular, if $a_0 = 0$ so that we have a monomial of the form $yx^{a_1}\cdots yx^{a_r}$, then the *circ*-operator cyclically permutes the "chunks" yx^{a_i}. It is obvious that this map is a bijection. By direct calculation, the above map induces a linear isomorphism on $Ass_{C,\ge 1}$.

(2). For $b \in Ass_{C,\ge 1}$, $n \ge 1$ and for $1 < r \le n$, let b_n^r be the depth r and the weight n part of b, and put $b_{n,0} := \sum_{r=2}^{n-1} b_n^r$. We say that the element $b \in Ass_{C,\ge 1}$ is *circ-constant for the sequence* $\{c_n\}_{n>1} \subset \mathbb{Q}$, if the following three conditions hold for all $n > 1$:

(I). the polynomial $b_{n,0}^y$ is push-constant for c_n (cf. Definition 2.2),

(II). $\langle b \,|\, y^n \rangle = \frac{c_n}{n}$.

Here we denote $b_{n,0} = xb_{n,0}^x + yb_{n,0}^y$. If $c_n = 0$ for $n > 1$, then we say that b is *circ-neutral.*

Observe that if $b \in Ass_{C,\ge 1}$ and b^r denotes the depth r-part of b for $r \ge 1$, then b is circ-neutral if and only if

$$\sum_{i=0}^{r-1} circ^i(b^r) = 0$$

for $r \ge 1$.

We note that the above definition of circ-neutrality agrees with the one appearing in Definition 1.6.

Example 3.36. Let $\psi \in \mathfrak{grt}$ be homogeneous of degree n with $(\psi|x^{n-1}y) = 1$. We saw in Example 2.3 that writing $b = \psi = x\psi^x + y\psi^y$, the polynomial ψ^y is push-constant for 1. We have $(\psi^y|x^4) = 1$, so the polynomial $b' = y\psi^y + \frac{1}{n}y^n$ is strictly push-constant for 1. This polynomial b' is given by $b' = yb'_0 + \frac{1}{n}y^n$ where b'_0 is the degree 4 push-constant polynomial ψ^y given explicitly in Example 2.3.

For an example of a circ-constant mould for the sequence $\{\delta_{n,5}\}_{n>1}$ in homogeneous degree 5, we take $B = swap\big(ma(\psi)\big)$, which has the same coefficients as the above polynomial $y\psi^y$: it is given by

$$\begin{aligned}
B(v_1) &= v_1^4 \\
B(v_1, v_2) &= -2v_1^3 + \frac{11}{2}v_1^2 v_2 - \frac{9}{2}v_1 v_2^2 + 3v_2^3 \\
B(v_1, v_2, v_3) &= 2v_1^2 - \frac{11}{2}v_1 v_2 - \frac{1}{2}v_2^2 + \frac{9}{2}v_1 v_3 + 2v_2 v_3 - \frac{1}{2}v_3^2 \\
B(v_1, v_2, v_3, v_4) &= -v_1 + 4v_2 - 6v_3 + 4v_4 \\
B(v_1, v_2, v_3, v_4, v_5) &= \frac{1}{5}.
\end{aligned}$$

The following result proves that the circ-constance of a homogeneous Lie polynomial b and that of the associated mould $ma(b)$ are related as above.

Lemma 3.37. *The following diagram is commutative:* [10]

$$\begin{array}{ccc}
Ass_{C,\geq 1} & \xrightarrow{circ} & Ass_{C,\geq 1} \\
{\scriptstyle mi}\downarrow & & \downarrow{\scriptstyle mi} \\
\overline{ARI}(\mathcal{F}_{\mathrm{ser}}) & \xrightarrow{circ} & \overline{ARI}(\mathcal{F}_{\mathrm{ser}}).
\end{array} \tag{70}$$

Proof. Below we will show prove $circ \circ mi(f) = mi \circ circ(f)$ for $f \in Ass_{C,\geq 1}$ with

$$f = \sum_{r\geq 1} \sum_{\underline{a}=(a_0,a_1,\dots,a_r)} \langle\, f \,|\, \underline{a} \,\rangle\; x^{a_0} y x^{a_1} y x^{a_2} \cdots y x^{a_r}.$$

In other word, we will prove

$$circ \circ mi(f)(v_1, \dots, v_r) = mi \circ circ(f)(v_1, \dots, v_r) \tag{71}$$

for $r \geq 1$.

When $r = 1$, three maps $circ$, $swap$ and ma form identity maps, so we obtain (71). Let $r \geq 2$. By Remark 3.14, we have

$$\begin{aligned}
circ \circ mi(f)(v_1, \dots, v_r) &= mi(f)(v_r, v_1, \dots, v_{r-1}) \\
&= \sum_{\underline{a}=(0,a_1,\dots,a_r)} \langle\, f \,|\, \underline{a} \,\rangle\; v_{r-1}^{a_1} v_{r-2}^{a_2} \cdots v_1^{a_{r-1}} v_r^{a_r}.
\end{aligned} \tag{72}$$

On the other hand, we have

$$\begin{aligned}
circ(f) &= \sum_{r\geq 1} \sum_{\underline{a}=(a_0,a_1,\dots,a_r)} \langle\, f \,|\, \underline{a} \,\rangle\; circ(x^{a_0} y x^{a_1} y x^{a_2} \cdots y x^{a_r}) \\
&= \sum_{r\geq 1} \sum_{\underline{a}=(a_0,a_1,\dots,a_r)} \langle\, f \,|\, \underline{a} \,\rangle\; x^{a_0} y x^{a_r} y x^{a_1} \cdots y x^{a_{r-1}} \\
&= \sum_{r\geq 1} \sum_{\underline{a}=(a_0,a_1,\dots,a_r)} \langle\, f \,|\, a_0, a_2, \dots, a_r, a_1 \,\rangle\; x^{a_0} y x^{a_1} y x^{a_2} \cdots y x^{a_r},
\end{aligned}$$

[10] See Lemma 3.12 for the definition of the map mi.

so we get

$$\langle\, circ(f) \,|\, \underline{a} \,\rangle = \langle\, f \,|\, a_0, a_2, \ldots, a_r, a_1 \,\rangle$$

for $a_0, a_1, \ldots, a_r \geq 0$. By Remark 3.14, we have

$$\begin{aligned} mi \circ circ(f)(v_1, \ldots, v_r) &= \sum_{\underline{a}=(0,a_1,\ldots,a_r)} \langle\, circ(f) \,|\, \underline{a} \,\rangle \; v_r^{a_1} v_{r-1}^{a_2} \cdots v_2^{a_{r-1}} v_1^{a_r} \\ &= \sum_{\underline{a}=(0,a_1,\ldots,a_r)} \langle\, f \,|\, 0, a_2, \ldots, a_r, a_1 \,\rangle \; v_r^{a_1} v_{r-1}^{a_2} \cdots v_2^{a_{r-1}} v_1^{a_r} \\ &= \sum_{\underline{a}=(0,a_1,\ldots,a_r)} \langle\, f \,|\, \underline{a} \,\rangle \; v_r^{a_r} v_{r-1}^{a_1} \cdots v_2^{a_{r-2}} v_1^{a_{r-1}}. \end{aligned}$$

By comparing this and (72), we obtain (71) for $r \geq 2$. □

Proposition 3.38. *Let $b \in Ass_{C,\geq 1}$. The following three statements are equivalent:*
(i). The mould $B = mi(b) \in \overline{ARI}(\mathcal{F}_{\mathrm{ser}})$ is circ-constant for $\{c_n\}_{n>1} \subset \mathbb{Q}$.
(ii). The element $b \in Ass_{C,\geq 1}$ satisfies

$$\sum_{i=0}^{r-1} circ^i(b_n^r) = (-1)^{n-r} c_n \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_i \geq 1}} C_{a_r, a_{r-1}, \ldots, a_1} \right) \tag{73}$$

for all $n \geq 2$ and $1 < r \leq n$. Here the element $b_n^r \in Ass_{C,\geq 1}$ means the depth r and the weight n part of b.
(iii). The element $b \in Ass_{C,\geq 1}$ is circ-constant for $\{c_n\}_{n>1} \subset \mathbb{Q}$.

Proof. First, we prove the equivalence between (i) and (ii). Note that, by direct calculation, we have

$$B_n(v_1, \ldots, v_r) = mi\,(b_n^r)\,(v_1, \ldots, v_r)$$

for $n \geq 1$. We apply the map mi to both sides of (73). Then the left-hand side of (73) is calculated as follows:

$$\begin{aligned} \sum_{i=0}^{r-1} mi \circ circ^i(b_n^r)(v_1, \ldots, v_r) &= \sum_{i=0}^{r-1} circ^i(mi(b_n^r))(v_1, \ldots, v_r) \\ &= \sum_{i=0}^{r-1} circ^i(B_n)(v_1, \ldots, v_r). \end{aligned} \tag{74}$$

Here, we used Lemma 3.37 in the first equality. On the other hand, the right-hand side of (73) is calculated as follows:

(75)

$$\begin{aligned} &(-1)^{n-r} c_n \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_i \geq 1}} mi(C_{a_r, a_{r-1}, \ldots, a_1}) \right) \\ &= (-1)^{n-r} c_n \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_i \geq 1}} (-1)^{n-r} v_r^{a_r - 1} \cdots v_1^{a_1 - 1} \right) = c_n \left(\sum_{\substack{a_1+\cdots+a_r=n-r \\ a_i \geq 0}} v_1^{a_1} \cdots v_r^{a_r} \right) \end{aligned}$$

Here, we used (46) in the first equality. By combining (74) and (75), we know that (73) is equivalent to the following equation:

$$\sum_{i=0}^{r-1} circ^i(B_n)(v_1,\dots,v_r) = c_n \left(\sum_{\substack{a_1+\cdots+a_r=n-r \\ a_i\geq 0}} v_1^{a_1}\cdots v_r^{a_r} \right).$$

This is nothing but the equation (68), that is, the mould $B = mi(b)$ is circ-constant for $\{c_n\}_{n>1}$. Hence, condition (i) is equivalent to condition (ii).

Next, we prove the equivalence between (iii) and (i) (or (ii)). Firstly, we prove the equivalence between (73) for $r = n$ and (II) in Definition 3.35.(2). We calculate both sides of (73) for $r = n$. Because $b_n^n = \langle b \,|\, y^n \rangle y^n$, the left-hand side of (73) is calculated as follows:

$$\sum_{i=0}^{n-1} circ^i(b_n^n) = \sum_{i=0}^{n-1} \langle b \,|\, y^n \rangle y^n = n\langle b \,|\, y^n \rangle y^n.$$

On the other hand, the right-hand side of (73) is calculated as follows:

$$(-1)^{n-n} c_n \left(\sum_{\substack{a_1+\cdots+a_n=n \\ a_i\geq 1}} C_{a_n,a_{n-1},\dots,a_1} \right) = c_n C_{\underbrace{1,1,\dots,1}_{n}} = c_n y^n.$$

Comparing these coefficients, we get $\langle b \,|\, y^n \rangle = \frac{c_n}{n}$. This means condition (II) in Definition 3.35.(2).

Secondly, we prove the equivalence between (73) for $1 < r < n$ and (I) in Definition 3.35.(2). We calculate both sides of (73) for $1 < r < n$. Because we have $b_n^r = x(b_n^r)^x + y(b_n^r)^y$ and $circ(yf) = y \cdot push(f)$ for $f \in Ass_C$, the left hand side of (73) is given below:

$$\begin{aligned}
\sum_{i=0}^{r-1} circ^i(b_n^r) &= \sum_{i=0}^{r-1} circ^i\big(x(b_n^r)^x\big) + \sum_{i=0}^{r-1} circ^i\big(y(b_n^r)^y\big) \\
&= y \cdot \sum_{i=0}^{r-1} push^i\big((b_n^r)^y\big) + \sum_{i=0}^{r-1} circ^i\big(x(b_n^r)^x\big).
\end{aligned}$$

On the other hand, since the only term in $C_{a_r,a_{r-1},\dots,a_1}$ that starts with y is $(-1)^{a_1+\cdots+a_r-r} yx^{a_r-1}\cdots yx^{a_1-1}$, the right-hand side of (73) is given as follows:

$$\begin{aligned}
&(-1)^{n-r} c_n \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_i\geq 1}} C_{a_r,a_{r-1},\dots,a_1} \right) \\
&= (-1)^{n-r} c_n \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_i\geq 1}} (-1)^{a_1+\cdots+a_r-r} yx^{a_r-1}\cdots yx^{a_1-1} \right) + \varphi \\
&= y \cdot \left(c_n \sum_{\substack{a_1+\cdots+a_r=n-r \\ a_i\geq 0}} x^{a_r} yx^{a_{r-1}}\cdots yx^{a_1} \right) + \varphi.
\end{aligned}$$

Here φ is its terms which belong to $x \cdot Ass_2$. By comparing the terms which start with y, we get

$$\sum_{i=0}^{r-1} push^i\big((b_{n,0}^r)^y\big) = \sum_{i=0}^{r-1} push^i\big((b_n^r)^y\big) = c_n \sum_{\substack{a_1+\cdots+a_r=n-r\\ a_i\geq 0}} x^{a_1}yx^{a_2}\cdots yx^{a_r},$$

for $1<r<n$ where we put $b_{n,0} := \sum_{s=2}^{n-1} b_n^s$. Because $(b^r)^y = (b^y)^{r-1}$ for $b \in Ass_{C,\geq 1}$ and $r \geq 1$, we get

$$\sum_{i=0}^{r-1} push^i\big((b_{n,0}^y)^{r-1}\big) = c_n \sum_{\substack{a_1+\cdots+a_r=n-r\\ a_i\geq 0}} x^{a_1}yx^{a_2}\cdots yx^{a_r},$$

for $1<r<n$. By substituting r with $r+1$, we obtain the following:

$$\sum_{i=0}^{r} push^i\big((b_{n,0}^y)^{r}\big) = c_n \sum_{\substack{a_0+\cdots+a_r=(n-1)-r\\ a_i\geq 0}} x^{a_0}yx^{a_1}\cdots yx^{a_r},$$

for $0<r<n-1$. Since the weight of $b_{n,0}^y$ is $n-1$, this equation means that $b_{n,0}^y$ is push-constant for c_n. Hence, we obtain (I) in Definition 3.35.(2).

On the other hand, we assume (I) in Definition 3.35.(2), that is, we assume

$$\sum_{i=0}^{r-1} push^i\big((b_{n,0}^y)^{r-1}\big) = c_n \sum_{\substack{a_1+\cdots+a_r=n-r\\ a_i\geq 0}} x^{a_1}yx^{a_2}\cdots yx^{a_r}, \tag{76}$$

for $1<r<n$. By $b_{n,0}^r \in Ass_{C,\geq 1}^{(r,n)}$, there exists $\langle b_{n,0}^r \mid a_r,\dots,a_1\rangle \in \mathbb{Q}$ such that

$$b_{n,0}^r = \sum_{\substack{a_1+\cdots+a_r=n-r\\ a_k\geq 0}} \langle b_{n,0}^r \mid a_r,\dots,a_1\rangle C_{a_r+1,a_{r-1}+1,\dots,a_1+1}$$

for $1<r<n$. We have $(C_{a_r+1,\dots,a_1+1})^y = (-1)^{n-r}x^{a_r}yx^{a_{r-1}}\cdots yx^{a_1}$ with $a_1+\cdots+a_r = n-r$, so for $1<r<n$, we get

$$\begin{aligned}
&\sum_{i=0}^{r-1} push^i\big((b_{n,0}^y)^{r-1}\big) = \sum_{i=0}^{r-1} push^i\big((b_{n,0}^r)^y\big)\\
&= \sum_{i=0}^{r-1} push^i\left(\sum_{\substack{a_1+\cdots+a_r=n-r\\ a_k\geq 0}} \langle b_{n,0}^r \mid a_r,\dots,a_1\rangle (-1)^{n-r}x^{a_r}yx^{a_{r-1}}\cdots yx^{a_1}\right)\\
&= \sum_{i=0}^{r-1}\sum_{\substack{a_1+\cdots+a_r=n-r\\ a_k\geq 0}} \langle b_{n,0}^r \mid a_r,\dots,a_1\rangle (-1)^{n-r}x^{a_{r+i}}yx^{a_{r-1+i}}\cdots yx^{a_{1+i}}\\
&= \sum_{\substack{a_1+\cdots+a_r=n-r\\ a_k\geq 0}} \left((-1)^{n-r}\sum_{i=0}^{r-1}\langle b_{n,0}^r \mid a_{r+i},\dots,a_{1+i}\rangle\right) x^{a_1}yx^{a_2}\cdots yx^{a_r}.
\end{aligned}$$

By comparing this with (76), we get

$$(-1)^{n-r}\sum_{i=0}^{r-1}\langle b_{n,0}^r \mid a_{r+i},\dots,a_{1+i}\rangle = c_n \tag{77}$$

for $1 < r < n$. By using (46) and $b^r_{n,0} = b^r_n$ for $1 < r < n$, we have

$$\begin{aligned}
&\sum_{i=0}^{r-1} circ^i \circ mi(b^r_n)(v_1, \dots, v_r) \\
&= \sum_{i=0}^{r-1} circ^i \left(\sum_{\substack{a_1+\cdots+a_r=n \\ a_k \geq 1}} \langle b^r_{n,0} \mid a_r, \dots, a_1 \rangle (-1)^{n-r} v_1^{a_1-1} v_2^{a_2-1} \cdots v_r^{a_r-1} \right) \\
&= \sum_{i=0}^{r-1} \sum_{\substack{a_1+\cdots+a_r=n \\ a_k \geq 1}} (-1)^{n-r} \langle b^r_{n,0} \mid a_r, \dots, a_1 \rangle v_{1+i}^{a_1-1} v_{2+i}^{a_2-1} \cdots v_{r+i}^{a_r-1} \\
&= \sum_{\substack{a_1+\cdots+a_r=n \\ a_k \geq 1}} (-1)^{n-r} \left(\sum_{i=0}^{r-1} \langle b^r_{n,0} \mid a_{r+i}, \dots, a_{1+i} \rangle \right) v_1^{a_1-1} v_2^{a_2-1} \cdots v_r^{a_r-1} \\
&= c_n \cdot \sum_{\substack{a_1+\cdots+a_r=n \\ a_k \geq 1}} v_1^{a_1-1} v_2^{a_2-1} \cdots v_r^{a_r-1}.
\end{aligned}$$

Here, we used (77) in the last equality. This equation means (68) for $1 < r < n$. Since (68) has been shown to be equivalent to (73) in the proof of the equivalence between (i) and (ii), the proof is now complete. □

Corollary 3.39. *Let $b \in \mathfrak{lie}_C$ be of homogeneous weight $n \geq 3$. Then b is circ-neutral if and only if $mi(b)$ is a circ-neutral mould.*

Proof. By putting $c_n = 0$ for $n > 1$ in Proposition 3.38, we obtain this claim. □

The notion of circ-constance will play a role later in §4.2. The concept of circ-neutrality also plays a fundamental role in mould theory.

Definition 3.40. Let $\mathcal{F}$ be a family of functions. We define [11]

$$ARI(\mathcal{F})_{push/circneut} \subset ARI(\mathcal{F})$$

as the linear subspace of $ARI(\mathcal{F})$ consisting of moulds which are push-invariant and whose swaps are circ-neutral.

An important property of $ARI(\mathcal{F})_{circneut}$ is the compatibility with the *ari*-bracket.

Proposition 3.41. *The space $\overline{ARI}(\mathcal{F})_{circneut}$ of circ-neutral moulds in $\overline{ARI}$ forms a Lie subalgebra of $\overline{ARI}(\mathcal{F})$ under the $\overline{ari}$-bracket. Further, the space $ARI(\mathcal{F})_{push/circneut}$ forms a Lie subalgebra of $ARI(\mathcal{F})$ under the ari-bracket.*

Proof. A direct complete proof of this result, along with its generalization, is presented in [FK1, Proposition 1.30 and Theorem 1.32]. □

Proposition 3.42. *Let $\mathcal{F}$ be a family of functions. We define*

$$ARI(\mathcal{F})_{al+push/circneut} := ARI(\mathcal{F})_{al} \cap ARI(\mathcal{F})_{push/circneut}.$$

Then it forms a Lie algebra under the ari-bracket.

[11] In the case when $\mathcal{F} = \mathcal{F}_{\mathrm{rat}}$, it is $ARI(\{e\})_{push/pusnu}$ in the notation of [FK1].

Proof. It follows from Propositions 3.18 and 3.41. □

Proposition 3.43. *The map ma gives a linear space isomorphism*

$$ma : \mathfrak{lkrv} \xrightarrow{\sim} ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut},$$

Proof. The first claim is because we showed that a polynomial b lies in $\mathfrak{lkrv}$, i.e. b is a Lie polynomial that is push-invariant and circ-neutral, if and only if the associated mould $ma(b)$ is alternal (by Lemma 3.10 (ii)), push-invariant (by Proposition 3.30) and its swap is circ-neutral (by Proposition 3.38). The second claim follows from Lemma 3.21 and Proposition 3.42. □

Proof of Proposition 1.8. By the above proposition with Lemma 3.21 and Proposition 3.42, we see that $\mathfrak{lkrv}$ forms a Lie algebra under the bracket $\{,\}$. It is immediate to see that it is bigraded by weight and depth. □

We note that there is a Lie algebra isomorphism

$$ma : (\mathfrak{lkrv}, \{,\}) \xrightarrow{\sim} (ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut}, ari). \tag{78}$$

3.7. **The inclusion $\mathfrak{ls} \hookrightarrow \mathfrak{lkrv}$ (Theorem 1.11).** In order to prove the theorem, we first reformulate the statement in terms of moulds and give its proof.

Lemma 3.44. *The map ma gives a Lie algebra isomorphism*

$$ma : \mathfrak{ls} \xrightarrow{\sim} ARI(\mathcal{F}_{\mathrm{ser}})_{\underline{al}/\underline{al}}.$$

For the left hand side, see Definition 1.10 and the right hand side, see Definition 3.6.

Proof. It has been demonstrated in [S2] that the result holds, and this claim has been further extended in [FK1] □

Theorem 3.45. *Let $\mathcal{F}$ be a family of functions. There is an inclusion of mould subspaces*

$$ARI(\mathcal{F})_{\underline{al}/\underline{al}} \subset ARI(\mathcal{F})_{al+push/circneut}. \tag{79}$$

Moreover in depths $r \leq 3$, the following holds for any family $\mathcal{F}$

$$ARI(\mathcal{F})^r \cap ARI(\mathcal{F})_{\underline{al}/\underline{al}} = ARI(\mathcal{F})^r \cap ARI(\mathcal{F})_{al+push/circneut}.$$

Proof. The result that $A \in ARI_{\underline{al}/\underline{al}}(\mathcal{F})$ is push-invariant can be found in [S2, Lemma 2.5.5]. and that $swap(A)$ is circ-neutral can be found in [FK1, Proposition 3.12].

Let us now prove the isomorphism in the cases $r = 1, 2, 3.$: The case $r = 1$ is trivial. Indeed, elements in depth one components of elements in both $ARI(\mathcal{F})_{\underline{al}/\underline{al}}$ and $ARI(\mathcal{F})_{al+push/circneut}$ must satisfy only one condition: they must be even functions.

Now consider the case $r = 2$. Let $A \in ARI(\mathcal{F})_{al+push/circneut}$ be concentrated in depth 2. The circ-neutral property of the swap is explicitly given in depth 2 by $swap(A)(v_1, v_2) + swap(A)(v_2, v_1) = 0$. But this is also the alternality condition on $swap(A)$, so $A \in ARI(\mathcal{F})_{al/al}$. The isomorphism in depth 2 is thus trivial.

Finally, we consider the case $r = 3$. Let $A \in ARI(\mathcal{F})_{al+push/circneut}$ be concentrated in depth 3, and let $B = swap(A)$. Again, we only need to show that B is alternal, which in depth 3 means that B must satisfy he single equation

$$B(v_1, v_2, v_3) + B(v_2, v_1, v_3) + B(v_2, v_3, v_1) = 0. \tag{80}$$

The circ-neutrality condition on B is given by

$$B(v_1, v_2, v_3) + B(v_3, v_1, v_2) + B(v_2, v_3, v_1) = 0. \tag{81}$$

It is enough to show that B satisfies the equality

$$B(v_1, v_2, v_3) = B(v_3, v_2, v_1), \tag{82}$$

since applying this to the middle term of (81) immediately yields the alternality property (80) in depth 3. So let us show how to prove (82).

We rewrite the push-invariance condition in the v_i, which gives

$$\begin{aligned} B(v_1, v_2, v_3) &= B(v_2 - v_1, v_3 - v_1, -v_1) && (83)\\ &= B(v_3 - v_2, -v_2, v_1 - v_2) && (84)\\ &= B(-v_3, v_1 - v_3, v_2 - v_3). && (85) \end{aligned}$$

Making the variable change exchanging v_1 and v_3, this gives

$$\begin{aligned} B(v_3, v_2, v_1) &= B(v_2 - v_3, v_1 - v_3, -v_3) && (86)\\ &= B(v_1 - v_2, -v_2, v_3 - v_2) && (87)\\ &= B(-v_1, v_3 - v_1, v_2 - v_1). && (88) \end{aligned}$$

By (83), the term $B(v_2 - v_1, v_3 - v_1, -v_1)$ is circ-neutral with respect to the cyclic permutation of v_1, v_2, v_3, so we have

$$B(v_2 - v_1, v_3 - v_1, -v_1) = -B(v_3 - v_2, v_1 - v_2, -v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \tag{89}$$

But the circ-neutrality of B also lets us cyclically permute the three arguments of B, so we also have

$$-B(v_3 - v_2, v_1 - v_2, -v_2) = B(-v_2, v_3 - v_2, v_1 - v_2) + B(v_1 - v_2, -v_2, v_3 - v_2).$$

Using (83) and substituting this into the right-hand side of (89) yields

$$\begin{aligned} B(v_1, v_2, v_3) = B(-v_2, v_3 - v_2, v_1 - v_2)& \\ + B(v_1 - v_2, -v_2, v_3 - v_2) - B(v_1 - v_3, v_2 - v_3, -v_3).& && (90) \end{aligned}$$

Now, exchanging v_1 and v_2 in (88) gives

$$B(v_3, v_1, v_2) = B(-v_2, v_3 - v_2, v_1 - v_2),$$

and doing the same with (86) gives

$$B(v_3, v_1, v_2) = B(v_1 - v_3, v_2 - v_3, -v_3).$$

Substituting these two expressions as well as (87) into the right-hand side of (90), we obtain the desired equality (82). This concludes the proof of Theorem 3.45. □

The result of Theorem 3.45 is related to the injective map

$$\mathfrak{ds}_{ell} \hookrightarrow \mathfrak{krv}_{ell} \tag{91}$$

discussed in the introduction (cf. (1)).

Corollary 3.46. *Let $\mathcal{F}$ be a family of functions. The inclusion of Theorem 3.45 extends to an inclusion of spaces*

$$ARI(\mathcal{F})_{\underline{al}*\underline{al}} \subset ARI(\mathcal{F})_{al+push*circneut},$$

where $ARI(\mathcal{F})_{\underline{al}*\underline{al}}$ is given in Definition 3.6 and $ARI(\mathcal{F})_{al+push*circneut}$ is the set of moulds which are alternal and push-invariant and with swap being circ-neutral up to addition of a constant-valued mould.

Proof. Let $A \in ARI(\mathcal{F})_{\underline{al}*\underline{al}}$. We know by Lemma 2.5.5 of [S2] that A is push-invariant. Let $B := swap(A)$ and let B_0 be the constant mould such that $B + B_0$ is alternal. Then by Theorem 3.45, $B + B_0$ is also circ-neutral, and therefore by definition $B = swap(A)$ is ∗circ-neutral, i.e. circ-neutral up to addition of a constant-valued mould. □

The above corollary will be used in the next section.

Proof of Theorem 1.11. By Theorem 3.45, there is an inclusion in terms of moulds as

$$ARI(\mathcal{F}_{\mathrm{ser}})_{\underline{al}/\underline{al}} \subset ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut}.$$

By (22) and (78), the theorem is proven. □

Remark 3.47. We speculate that the inclusions of Theorem 3.45 and Corollary 3.46 may extend to isomorphisms. But even the proof of the simple equality (82) is surprisingly complicated in depth 3, let alone in higher depth. Computational experiments suggest the following general pattern: If $A \in ARI(\mathcal{F})_{al+push/circneut}$ and $B = swap(A)$, then B is mantar-invariant (cf. Definition 3.2), that is, for all $r > 0$, we have

$$B(v_1, \dots, v_r) = (-1)^{r-1} B(v_r, \dots, v_1). \tag{92}$$

The mantar invariance (92) would also yield the following useful partial result, which is the mould analog for $\mathfrak{lkrv}$ of a result that is well-known for $\mathfrak{ls}$, namely that the bigraded part $\mathfrak{ls}_n^r = 0$ when $n \not\equiv r \bmod 2$.

Lemma 3.48. *Fix $1 \le r \le n$. Let $A \in ARI(\mathcal{F}_{\mathrm{ser}})^{(r,n)} \cap ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut}$ and let $B = swap(A)$. Assume that B satisfies (92). Then if $n - r$ is odd, $A = 0$.*

For $ARI(\mathcal{F}_{\mathrm{ser}})^{(r,n)}$, see Lemma 3.10.

Proof. We denote by *mantar* the operator on $\overline{ARI}(\mathcal{F}_{\mathrm{ser}})$, which is defined in the same manner as on ARI, except that the variables v_i are used instead of u_i Then it is easy to check the following identity of operators noted by Écalle:

$$neg \circ push = mantar \circ swap \circ mantar \circ swap,$$

Let $A \in ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut}$; then A is push-invariant, so applying the left-hand operator to A gives $neg(A)$. Assuming (92) for $B = swap(A)$, i.e. assuming that $B = mantar(B)$, we see that applying the right-hand operator to A fixes A since on the one hand $swap \circ swap = id$ and on the other, $mantar(A) = A$ for all alternal moulds (cf. [S2], Lemma 2.5.3). Thus A must satisfy $neg(A) = A$, i.e. if $A \neq 0$ then the degree $d = n - r$ of A must be even. □

The following parity result, which is the analogy for $\mathfrak{lkrv}$ of the similar well-known result on $\mathfrak{ls}$.

Corollary 3.49. *If the swaps of all elements of $ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut}$ are mantar-invariant, then $ARI(\mathcal{F}_{\mathrm{ser}})^{(r,n)} \cap ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut} = 0$ whenever $r-n$ is odd, i.e.*

$$\mathfrak{lkrv}_n^r = 0 \quad \text{when } n \not\equiv r \bmod 2$$

Proof. By Proposition 3.43, there is an isomorphism

$$\mathfrak{lkrv}_n^r \simeq ARI(\mathcal{F}_{\mathrm{ser}})^{(r,n)} \cap ARI(\mathcal{F}_{\mathrm{ser}})_{al+push/circneut},$$

of each bigraded piece. Thus our claim follows from Lemma 3.48. □

4. The elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$

In this section we follow the procedure of [S3] for the double shuffle Lie algebra to define a natural candidate for the elliptic Kashiwara-Vergne Lie algebra, closely related to the linearized Kashiwara-Vergne Lie algebra, and give some of its properties (Theorem 1.16).

4.1. Definition of the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$.

We first introduce the *Dari*-bracket, a secondary Lie bracket structure on $ARI(\mathcal{F}_{\mathrm{Lau}})$. We then demonstrate that $\mathfrak{krv}_{ell}$ is closed under this bracket, thereby establishing its Lie algebraic structure.

4.1.1. *Dari bracket.*

Definition 4.1. Take $\mathcal{F} = \mathcal{F}_{\mathrm{Lau}}$. We define the linear map

$$\Delta : ARI(\mathcal{F}_{\mathrm{Lau}}) \to ARI(\mathcal{F}_{\mathrm{Lau}})$$

given by

$$\Delta(A)(u_1,\dots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)A(u_1,\dots,u_r) \tag{93}$$

for $r \geq 1$. We define

$$Dari(A,B) = \Delta\Big(ari\big(\Delta^{-1}(A),\Delta^{-1}(B)\big)\Big). \tag{94}$$

This definition implies that Δ gives an isomorphism of Lie algebras

$$\Delta : ARI(\mathcal{F}_{\mathrm{Lau}})_{ari} \xrightarrow{\sim} ARI(\mathcal{F}_{\mathrm{Lau}})_{Dari}. \tag{95}$$

Here $ARI(\mathcal{F}_{\mathrm{Lau}})_{Dari}$ is the Lie algebra $ARI(\mathcal{F}_{\mathrm{Lau}})$ equipped with *Dari*-bracket.

Remark 4.2. It is shown in [S3], Prop. 3.2.1 that we have a second definition for the *Dari*-bracket, which is more complicated but sometimes very useful in certain proofs. Let

$$dar : ARI(\mathcal{F}_{\mathrm{Lau}}) \to ARI(\mathcal{F}_{\mathrm{Lau}})$$

denote the mould operator defined by

$$dar(A)(u_1,\dots,u_r) = u_1\cdots u_r\, A(u_1,\dots,u_r).$$

We begin by introducing, for each $A \in ARI(\mathcal{F})$, an associated derivation $Darit(A)$ of $ARI(\mathcal{F})_{lu}$ by the following formula:

$$Darit(A) = dar \circ \Big(-arit\big(\Delta^{-1}(A)\big) + ad\big(\Delta^{-1}(A)\big)\Big) \circ dar^{-1}, \tag{96}$$

where $ad(A)\cdot B = lu(A,B)$. Then $Dari$ corresponds to the bracket of derivations, in the sense that

$$Dari(A,B) = Darit(A)\cdot B - Darit(B)\cdot A. \tag{97}$$

Definition 4.3. Let $ARI(\mathcal{F}_{\rm Lau})^{\Delta}$ denote the space of moulds A such that $\Delta(A)$ (for Δ, see (93)) is in $ARI(\mathcal{F}_{\rm ser})$, i.e. the denominator of A is "at worst" $u_1\cdots u_r(u_1+\cdots+u_r)$. We write $ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{\mathcal{P}}$ for the space of moulds in $ARI(\mathcal{F}_{\rm Lau})^{\Delta}\cap ARI(\mathcal{F}_{\rm Lau})_{\mathcal{P}}$, where $\mathcal{P}$ may represent any (or no) properties on moulds in $ARI(\mathcal{F}_{\rm Lau})$; we will consider properties $\mathcal{P}$ such as for example al, $push$, combinations of these etc.

Lemma 4.4. *The space $ARI(\mathcal{F}_{\rm Lau})^{\Delta}$ forms a Lie algebra under the ari-bracket* (7).

Proof. It follows from [E2, Proposition 4.2]. □

Lemma 4.5. *The space $ARI(\mathcal{F}_{\rm ser})$ forms a Lie algebra under the Dari-bracket* (94).

Proof. Let $A,B\in ARI(\mathcal{F}_{\rm ser})$. Then we have $\Delta^{-1}(A),\Delta^{-1}(B)\in ARI(\mathcal{F}_{\rm Lau})^{\Delta}$. By Lemma 4.4, $ari(\Delta^{-1}(A),\Delta^{-1}(B))\in ARI(\mathcal{F}_{\rm Lau})^{\Delta}$. Therefore $\Delta(ari(\Delta^{-1}(A),\Delta^{-1}(B)))\in ARI(\mathcal{F}_{\rm ser})$. By (94), we get $Dari(A,B)\in ARI(\mathcal{F}_{\rm ser})$. □

For convenience, when considering the Lie algebra structure on $ARI(\mathcal{F}_{\rm ser})$ with respect to the Dari-bracket, we denote it by $ARI(\mathcal{F}_{\rm ser})_{Dari}$,

Definition 4.6. *The mould-version elliptic Kashiwara-Vergne linear space* is the subspace of $ARI(\mathcal{F}_{\rm ser})$ given by

$$\Delta\big(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\big).$$

The elliptic Kashiwara-Vergne linear space $\mathfrak{krv}_{ell}(\subset\mathfrak{lie}_C)$ introduced in Definition 1.15 agrees with the above mould version under the map ma given in (45).

Lemma 4.7. *The following equality holds:*

$$ma\big(\mathfrak{krv}_{ell}\big) = \Delta\big(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\big). \tag{98}$$

Proof. By Definition 1.15, $b\in\mathfrak{lie}_C$ is in $\mathfrak{krv}_{ell}$ if and only if $B_* = \Delta^{-1}\circ ma(b)$ is in $ARI(\mathcal{F}_{\rm Lau})_{al+push*circneut}$. Our claim follows because we have $ma(\mathfrak{lie}_C) = ARI(\mathcal{F}_{\rm ser})_{al}$ and $\Delta(ARI(\mathcal{F}_{\rm ser})_{al}) = ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al}$. □

The operator Δ trivially respects push-invariance of moulds, so the space $\mathfrak{krv}_{ell}$ lies in the space $\mathfrak{lie}_C^{push}$ of push-invariant elements of $\mathfrak{lie}_C$ (cf. Definition 1.6). We will now show that the subspace $\mathfrak{krv}_{ell}$ is actually a Lie subalgebra of $\mathfrak{lie}_C^{push}$, which is itself a Lie algebra under a new Lie bracket in Corollary 4.9 below, of which a more explicit version (with a formula for the partner) is proved in [S3] (Lemma 2.1.1).

Lemma 4.8. *Let $b\in\mathfrak{lie}_C$. Then $b\in\mathfrak{lie}_C^{push}$ if and only if there exists a unique element $a\in\mathfrak{lie}_C$ (the partner of b), such that if $D_{b,a}$ is the derivation of $\mathfrak{lie}_2$ defined by $x\mapsto b$, $y\mapsto a$ (as introduced in §1.1), then $D_{b,a}$ annihilates $[x,y]$.*

By identifying $\mathfrak{lie}_C^{push}$ with the space $\mathfrak{oder}_2$ (see §1.1) of derivations that annihilate $[x,y]$, we see that $\mathfrak{lie}_C^{push}$ is a Lie algebra under the bracket of derivations. We state this as a corollary.

Corollary 4.9. *The map $b \mapsto D_{b,a}$ gives an isomorphism of linear spaces*

$$i_o : \mathfrak{lie}_C^{push} \xrightarrow{\sim} \mathfrak{oder}_2 \tag{99}$$

whose inverse is $D_{b,a} \mapsto D_{b,a}(x) = b$, and this becomes a Lie algebra isomorphism when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket

$$\langle b, b' \rangle = [D_{b,a}, D_{b',a'}](x) = D_{b,a}(b') - D_{b',a'}(b). \tag{100}$$

We note that the above derivation $\langle , \rangle$ is different from the derivation $\{ , \}$ in (7). We compare the *Dari*-bracket to the bracket $\langle \ , \ \rangle$ on $\mathfrak{lie}_C^{push}$.

Proposition 4.10. *The isomorphism* ma *in* (66) *is also compatible with the Lie brackets $\langle \cdot, \cdot \rangle$ and Dari in the sense that*

$$ma\big(\langle b, b' \rangle\big) = Dari\big(ma(b), ma(b')\big). \tag{101}$$

Consequently $ARI(\mathcal{F}_{\rm ser})_{al+push}$ forms a Lie algebra under Dari-bracket. Moreover, we have a Lie algebra isomorphism

$$ma : (\mathfrak{lie}_C^{push}, \langle \ , \ \rangle) \to (ARI(\mathcal{F}_{\rm ser})_{al+push}, Dari).$$

Proof. Since we know that the map ma in (66) is bijective, it is enough to prove that it is a Lie algebra homomorphism (101). The key point is the following non-trivial result, which is one of the main results of [BS]: if D_1 and D_2 lie in $\mathfrak{oder}_2$, then the map

$$\begin{aligned} \Psi : \mathfrak{oder}_2 &\to ARI(\mathcal{F}_{\rm Lau})_{ari} \\ D &\mapsto \Delta^{-1}\big(ma\big(D(x)\big)\big), \end{aligned} \tag{102}$$

is an injective Lie algebra homomorphism, i.e.

$$\Delta^{-1}\Big(ma\big([D_1, D_2](x)\big)\Big) = ari\Big(\Delta^{-1}\big(ma(D_1(x))\big), \Delta^{-1}\big(ma(D_2(x))\big)\Big)$$

(see Theorem 3.5 of [BS]). Applying Δ to both sides of this and using (94), this is equivalent to

$$ma\big([D_1, D_2](x)\big) = Dari\Big(ma\big(D_1(x)\big), ma\big(D_2(x)\big)\Big). \tag{103}$$

By Lemma 4.5, we see that

$$\Delta \circ \Psi : \mathfrak{oder}_2 \to ARI(\mathcal{F}_{\rm ser})_{Dari} \tag{104}$$

is a Lie algebra homomorphism. We saw in Corollary 4.9 that we have a Lie isomorphism $i_o : (\mathfrak{lie}_C^{push}, \langle , \rangle) \xrightarrow{\sim} \mathfrak{oder}_2$ when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket (100), so by composition, we have an injective Lie algebra homomorphism

$$b \overset{i_o}{\mapsto} D_{b,a} \overset{\Psi}{\mapsto} \Delta^{-1}\big(ma(D_{b,a}(x))\big) \overset{\Delta}{\mapsto} ma(b)$$

(where i_o is as in Corollary 4.9 and Ψ is as in (102)) is an injective Lie algebra homomorphism $\mathfrak{lie}_C^{push} \to ARI(\mathcal{F}_{\rm ser})_{Dari}$, which proves the result. □

It looks remarkable that $\mathfrak{lie}_C^{push}$ (and whence $ARI(\mathcal{F}_{\rm ser})_{al+push}$) is encoded with two Lie algebraic structures:

Remark 4.11. (i). From Corollary 3.31 and Proposition 4.10, we deduce that $\mathfrak{lie}_C^{push}$ forms a Lie algebra with respect to both the brackets $\{,\}$ and $\langle,\rangle$ and $ARI(\mathcal{F}_{\mathrm{ser}})_{al+push}$ inherits a Lie algebraic structure under both the *ari* and *Dari*-brackets. Furthermore, there exist sequences of Lie algebra maps:

$$(ARI(\mathcal{F}_{\mathrm{ser}})_{al+push}, ari) \overset{ma}{\simeq} (\mathfrak{lie}_C^{push}, \{\ ,\ \}) \overset{i_y}{\hookrightarrow} \mathfrak{tder}_2^{(y)},$$
$$(ARI(\mathcal{F}_{\mathrm{ser}})_{al+push}, Dari) \overset{ma}{\simeq} (\mathfrak{lie}_C^{push}, \langle\ ,\ \rangle) \overset{i_o}{\simeq} \mathfrak{oder}_2.$$

(ii). By combining the $\mathbb{Q}$-linear (but not Lie algebraic) isomorphism $i_{o,z}$ in (11), we obtain another identification of $\mathbb{Q}$-linear spaces

$$i_z: \quad \mathfrak{lie}_C^{push} \simeq \mathfrak{sder}_2^{(z)}$$

which is also obtained from Proposition 2.1. We note that the partner a in Proposition 2.1 corresponds to $-a$ in Lemma 4.8. Since one can show that

$$i_z(b) \circ \nu = i_y(b) + \mathrm{ad}(-b)$$

where ν is the involution defined in (12), it follows that the Lie bracket on $\mathfrak{sder}_2^{(z)}$ given by (4), which also induces a new Lie algebra structure $\mathfrak{lie}_C^{push}$, is nevertheless related to the bracket $\{\ ,\ \}$.

4.1.2. *Lie algebra structure on* $\mathfrak{krv}_{ell}$ *(Theorem 1.16).* The Lie algebra $(\mathfrak{lie}_C^{push}, \langle,\rangle)$ contains the elliptic Kashiwara-Vergne space $\mathfrak{krv}_{ell}$ as a linear subspace. This leads to our first main theorem concerning $\mathfrak{krv}_{ell}$ (Theorem 4.13), from which Theorem 1.16 is deduced. As a corollary (Corollary 4.14), we obtain the inclusion $\mathfrak{lkrv} \to \mathfrak{krv}_{ell}$ which establishes Proposition 1.17.

Before stating them, let us recall here the similar definition from of the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ from [S3], whose construction was the inspiration for the definition of the elliptic Kashiwara-Vergne Lie algebra.

Definition 4.12. *The mould-version elliptic double shuffle Lie algebra* was defined in [S3] as the linear subspace

$$\Delta\big(ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{\underline{al}*\underline{al}}\big) \subset ARI(\mathcal{F}_{\mathrm{ser}}).$$

The elliptic double shuffle Lie algebra is the subspace $\mathfrak{ds}_{ell} \subset \mathfrak{lie}_C$ such that

$$ma(\mathfrak{ds}_{ell}) = \Delta\big(ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{\underline{al}*\underline{al}}\big).$$

It was also shown in [S3] that

$$\mathfrak{ds}_{ell} \subset \mathfrak{lie}_C^{push},$$

and that $\mathfrak{ds}_{ell}$ is closed under the Lie bracket $\langle\ ,\ \rangle$ of (100).

By Corollary 3.46, we know that $ARI(\mathcal{F}_{\mathrm{Lau}})_{\underline{al}*\underline{al}} \subset ARI(\mathcal{F}_{\mathrm{Lau}})_{al+push*circneut}$, so we obtain the inclusions

$$ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{\underline{al}*\underline{al}} \subset ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{al+push*circneut}$$

and thus

$$\Delta\big(ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{\underline{al}*\underline{al}}\big) \subset \Delta\big(ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}_{al+push*circneut}\big).$$

In view of the above definitions of the elliptic double shuffle (in Definition 4.12) and Kashiwara-Vergne Lie algebras (in Definitions 1.15 and 4.6), this gives us the linear space inclusion

$$\mathfrak{ds}_{ell} \hookrightarrow \mathfrak{krv}_{ell}: \qquad b \mapsto b \tag{105}$$

announced in the introduction (see (1)). We will show in the following theorem that $\mathfrak{krv}_{ell}$ is also a Lie algebra.

Theorem 4.13. *The linear space* $\mathfrak{krv}_{ell}$ *forms a Lie algebra under the Lie bracket* $\langle\ ,\ \rangle$ *in* (100)*. And the following is a sequence of Lie algebra inclusions:*

$$\mathfrak{ds}_{ell} \subset \mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}.$$

Proof. In view of the fact that $\mathfrak{ds}_{ell}$ is known to be a Lie subalgebra of the Lie algebra $(\mathfrak{lie}_C^{push}, \langle\ ,\ \rangle)$, it remains only to show that the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the bracket $\langle\ ,\ \rangle$.

Step 1. By Proposition 4.10, the map ma gives an injective Lie algebra morphism

$$(\mathfrak{lie}_C^{push}, \langle\ ,\ \rangle) \to ARI(\mathcal{F}_{\text{ser}})_{Dari}.$$

Thus proving that $\mathfrak{krv}_{ell}$ is closed under $\langle , \rangle$ is equivalent to proving that its image $\Delta\big(ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push*circneut}\big)$ under the map ma is closed under the $Dari$-bracket. Since we saw above that

$$\Delta^{-1}: ARI(\mathcal{F}_{\text{Lau}})_{Dari} \to ARI(\mathcal{F}_{\text{Lau}})_{ari}$$

is a Lie algebra homomorphism it is equivalent to show that $ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push*circneut}$ is a Lie subalgebra of $ARI(\mathcal{F}_{\text{Lau}})_{ari}$.

Step 2. The space $ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push}$ is a Lie algebra under ari. Indeed, the definition of Δ shows that this operator does not change the properties of push-invariance or alternality, i.e. $\Delta^{-1}(ARI(\mathcal{F}_{\text{Lau}})_{al+push}) = ARI(\mathcal{F}_{\text{Lau}})_{al+push}$. Restricted to $\mathcal{F}_{\text{ser}}$-valued moulds, we have $\Delta^{-1}(ARI(\mathcal{F}_{\text{ser}})_{al+push}) = ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push}$. Since Δ is an isomorphism from $ARI(\mathcal{F}_{\text{Lau}})_{ari}$ to $ARI(\mathcal{F}_{\text{Lau}})_{Dari}$ by virtue of (95) and $ARI(\mathcal{F}_{\text{ser}})_{al+push}$ is a Lie subalgebra of $ARI(\mathcal{F}_{\text{ser}})_{Dari}$ by Proposition 4.10, its image $ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push}$ under Δ^{-1} is thus a Lie subalgebra of $ARI(\mathcal{F}_{\text{Lau}})_{ari}$.

Step 3. We now complete the proof of Theorem 4.13 by showing that the space $ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push*circneut}$ is a Lie algebra under ari.

Let A, B lie in $ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push*circneut}$, and let us show that $ari(A,B)$ lies in the same space. By Step 2, we know that $ari(A,B) \in ARI(\mathcal{F}_{\text{Lau}})^{\Delta}_{al+push}$, so we only need to show that $swap\big(ari(A,B)\big)$ is $*$circ-neutral[12]. But we will show that in fact this mould is actually circ-neutral. To see this, let A_0 and B_0 be the constant-valued moulds such that $swap(A)+A_0$ and $swap(B)+B_0$ are circ-neutral. Since $A+A_0$ and $B+B_0$ are push-invariant, by Proposition 3.41, we have

$$\overline{ari}\big(swap(A)+A_0, swap(B)+B_0\big) \in \overline{ARI}(\mathcal{F}_{\text{Lau}})_{circneut}.$$

Using the identity

$$swap\big(ari(M,N)\big) = \overline{ari}\big(swap(M), swap(N)\big),$$

[12] We recall that $*$circ-neutral means circ-neutral up to addition of a constant mould, (see footnote[5]).

valid whenever M and N are push-invariant moulds (cf. [S], (2.5.6)), as well as the fact that constant-valued moulds are both push and swap invariant, we have

$$\begin{aligned}\overline{ari}\big(swap(A)+A_0, swap(B)+B_0\big) &= \overline{ari}\big(swap(A+A_0), swap(B+B_0)\big)\\ &= swap\cdot ari(A+A_0, B+B_0)\\ &= swap\cdot ari(A,B)+swap\cdot ari(A,B_0)+swap\cdot ari(A_0,B)+swap\cdot ari(A_0,B_0)\\ &= swap\cdot ari(A,B)\end{aligned}$$

since the definition of the ari-bracket shows that $ari(C,M)=0$ whenever C is a constant-valued mould. So $swap\cdot ari(A,B)$ is circ-neutral, which follows $ari(A,B)\in ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}$. The proof of Theorem 4.13 is completed. □

Proof of Theorem 1.16. The claim (i) is immediate, since the conditions of push-invariance and circ-neutrality up to addition of constants are both homogeneous.

The claims (ii) and (iii) follow directly from Corollary 4.9 and Theorem 4.13. □

Corollary 4.14. *The linear map* Δ *defined in* (93) *gives a Lie algebra morphism*

(106)
$$\Delta:(ARI(\mathcal{F}_{\rm ser})_{al+push/circneut}, ari)\to(\Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}), Dari),$$

which induces a Lie algebra morphism

$$(\mathfrak{lkrv},\{\ ,\ \})\hookrightarrow(\mathfrak{krv}_{ell},\langle\ ,\ \rangle)$$
$$b(x,y)\mapsto[x,b(x,[x,y])]. \tag{107}$$

Proof. For the first statement, composing the inclusion map

$$\begin{aligned}ARI(\mathcal{F}_{\rm ser})_{al+push/circneut} &= ARI(\mathcal{F}_{\rm ser})\cap ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push/circneut}\\ &\subset ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\end{aligned} \tag{108}$$

with the operator Δ, considered as an injective linear map on moulds gives an injective linear map

$$\Delta: ARI(\mathcal{F}_{\rm ser})_{al+push/circneut}\hookrightarrow\Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}).$$

It is shown in Step 3 of the proof of Theorem 4.13 that $ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}$ is a Lie algebra under the ari-bracket, and in Proposition 3.42 that $ARI(\mathcal{F}_{\rm ser})_{al+push/circneut}$ is a Lie subalgebra of it. A basic property of the linear map Δ is that it transforms the ari-bracket into the $Dari$-bracket (cf. (94)), so the space $\Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut})$ is a Lie algebra under the $Dari$-bracket. Thus the map in (106) is a Lie algebra morphism from a Lie subalgebra of $ARI(\mathcal{F}_{\rm ser})_{ari}$ to a Lie subalgebra of $ARI(\mathcal{F}_{\rm Lau})_{Dari}$.

Finally, by (78) we have

$$ma(\mathfrak{lkrv})=ARI(\mathcal{F}_{\rm ser})_{al+push/circneut}\subset ARI(\mathcal{F}_{\rm ser})_{ari}$$

and by (98) we have

$$ma(\mathfrak{krv}_{ell})=\Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut})\subset ARI(\mathcal{F}_{\rm ser})_{Dari},$$

so (106) translates directly under ma^{-1} to a Lie algebra homomorphism $\mathfrak{lkrv}\to\mathfrak{krv}_{ell}$. The presentation (107) follows from (93) and (45). □

Whence Proposition 1.17 is proved. □

4.2. **The map $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ (Theorem 1.18).** In this subsection we prove our next main result on the elliptic Kashiwara-Vergne Lie algebra, which is analogous to known results on the elliptic Grothendieck-Teichmüller Lie algebra of [E1] and the elliptic double shuffle Lie algebra of [S3]. §4.3 below is devoted to connections between these three situations.

To state the assumption of our result, we prepare some definitions on moulds.

Definition 4.15. (i) Let

$$teru : ARI(\mathcal{F}) \to ARI(\mathcal{F})$$

be the operator defined for $A \in ARI(\mathcal{F})$ as follows: $teru(A)$ is equal to A in depths 0 and 1, and for depths $r > 1$, we have

$$(109)\quad teru(A)(u_1, \dots, u_r) = A(u_1, \dots, u_r) + \frac{1}{u_r}\Big(A(u_1, \dots, u_{r-2}, u_{r-1} + u_r) - A(u_1, \dots, u_{r-2}, u_{r-1})\Big).$$

(ii) A mould $A \in ARI(\mathcal{F})$ is said to satisfy *the senary relation* (cf. (3.64) in §3.5 of [Ec2]) if

$$(110)\qquad teru(A) = push \circ mantar \circ teru \circ mantar(A),$$

and the *twisted senary relation* if

$$(111)\qquad teru \circ pari(B) = push \circ mantar \circ teru \circ pari(B).$$

(for $pari$ see Definition 3.2), that means that $pari(A)$ satisfies the senary relation.

(iii) We define the mould subspace

$$(112)\qquad ARI(\mathcal{F})_{al+tsen/circconst} \qquad (\text{resp. } ARI(\mathcal{F})_{al+tsen*circconst})$$

to be the subspace of moulds $A \in ARI(\mathcal{F})$ such that $swap(A)$ is circ-constant (see Definition 3.33) (resp. up to adding a constant mould) and A satisfies the twisted senary relation (111).

Remark 4.16. Observe that if $swap(A) \in \overline{ARI}(\mathcal{F})$ for $\mathcal{F} = \mathcal{F}_{\rm ser}$ or $\mathcal{F}_{\rm pol}$ is a polynomial of homogeneous degree n which is circ-constant up to addition of a constant-valued mould, then the constant-valued mould is uniquely determined as being the mould whose only non-zero value is the constant value $\frac{c}{n}$ in depth n, where c is given by

$$swap(A)(v_1) = cv_1^{n-1}.$$

Theorem 1.18. *Assume* [13] *that the adjoint action $Ad_{ari}(pal) : ARI(\mathcal{F}_{\rm Lau})_{al} \to ARI(\mathcal{F}_{\rm Lau})_{al}$ of the mould $pal \in GARI(\mathcal{F}_{\rm Lau})$ (cf. Definition 3.24) restricts to a bijection between*

$$(113)\qquad Ad_{ari}(pal) : ARI(\mathcal{F}_{\rm Lau})_{push} \xrightarrow{\sim} ARI(\mathcal{F}_{\rm Lau})_{sen},$$

where $ARI(\mathcal{F}_{\rm Lau})_{push}$ denotes the set of push-invariant moulds (31)*, and $ARI(\mathcal{F}_{\rm Lau})_{sen}$ denotes the set of moulds satisfying the senary relation* (110)*. Then there exists an injective Lie algebra morphism*

$$(114)\qquad \mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}.$$

The proof constructs the morphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ in four main steps as follows.

[13] This assertion is announced to hold in [Ka].

Step 1. We first consider a twisted version of the Kashiwara-Vergne Lie algebra, or rather of the associated polynomial space $V_{\mathfrak{krv}}$ of Definition 2.5, via the map

$$\begin{aligned} \nu : V_{\mathfrak{krv}} &\overset{\sim}{\to} W_{\mathfrak{krv}} \\ f &\mapsto \nu(f), \end{aligned} \tag{115}$$

where ν is the automorphism of Ass_2 defined by (12). In paragraph 4.2.1, we prove that $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson or Ihara bracket, and give a description of $W_{\mathfrak{krv}}$ via two properties, the "twisted" versions of the two defining properties of $V_{\mathfrak{krv}}$ given in Definition 2.5.

Step 2. In paragraph 4.2.2, we study the mould space $ma\big(W_{\mathfrak{krv}}\big)$. Thanks to the compatibility of the *ari*-bracket with the Poisson bracket (see (48)), this space is a Lie subalgebra of $ARI(\mathcal{F}_{\rm ser})_{ari}$. Just as we reformulated the defining properties of $\mathfrak{lkrv}$ in mould terms in §3, proving that $ma(\mathfrak{lkrv}) = ARI(\mathcal{F}_{\rm ser})_{al+push/circneut}$, in (78), we will reformulate the defining properties of $W_{\mathfrak{krv}}$ in mould terms and show that

$$ma\big(W_{\mathfrak{krv}}\big) = ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}. \tag{116}$$

Step 3. In paragraph 4.2.3, we consider the map

$$\Xi := Ad_{ari}(invpal) \circ pari : ARI(\mathcal{F}_{\rm Lau})_{ari} \to ARI(\mathcal{F}_{\rm Lau})_{ari},$$

and we show that it gives an injective Lie algebra homomorphism

$$ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \xrightarrow{pari} ARI(\mathcal{F}_{\rm ser})_{al+sen*circconst} \xhookrightarrow{Ad_{ari}(invpal)} ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut} \tag{117}$$

of subalgebras of $ARI(\mathcal{F}_{\rm ser})_{ari}$.

Step 4. The final step in the paragraph 4.2.4 is to compose (117) with the Lie algebra homomorphism $\Delta : ARI(\mathcal{F}_{\rm Lau})_{ari} \to ARI(\mathcal{F}_{\rm Lau})_{Dari}$, obtaining an injective Lie algebra homomorphism

$$ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \to \Delta\big(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\big),$$

where the left-hand space is a subalgebra of $ARI(\mathcal{F}_{\rm ser})_{ari}$ and the right-hand one of $ARI(\mathcal{F}_{\rm ser})_{Dari}$. Since the right-hand space is equal to $ma(\mathfrak{krv}_{ell})$, the desired injective Lie algebra homomorphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ is obtained by composing all the maps described above, as shown in the following diagram:

$$\begin{array}{ccc}
\mathfrak{krv} & & \\
\text{by (38)} \big\downarrow & & \\
V_{\mathfrak{krv}} & & \mathfrak{krv}_{ell} \\
\text{by (115)} \big\downarrow \nu & & ma^{-1} \big\uparrow \text{by (98)} \\
W_{\mathfrak{krv}} & & \Delta\big(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}\big) \\
\text{by (116)} \big\downarrow ma & & \Delta \big\uparrow \\
ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} & \xrightarrow[\text{by (117)}]{\Xi} & ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}
\end{array} \tag{118}$$

4.2.1. *Step 1: The twisted space* $W_{\mathfrak{krv}}$.

Proposition 4.17. *Let*

$$W_{\mathfrak{krv}} = \nu(V_{\mathfrak{krv}})$$

(for $V_{\mathfrak{krv}}$*, see Definition 2.5). Then* $W_{\mathfrak{krv}}$ *is a Lie algebra under the Poisson bracket* $\{\ ,\ \}$ *in* (7).

Proof. We use Lemma 1.2 to complete the proof of Proposition 4.17. Write

$$\mathfrak{krv}^{\nu} = \{\nu \circ E \circ \nu \,|\, E \in \mathfrak{krv}\} \subset \mathfrak{sder}_2^{(x)}.$$

By restricting the isomorphism (13) to the subspace $\mathfrak{krv} \subset \mathfrak{sder}_2^{(z)}$, we obtain a commutative diagram of isomorphisms of linear spaces

$$\begin{array}{ccc} \mathfrak{krv} & \longrightarrow & \mathfrak{krv}^{\nu} \\ \big\downarrow & & \big\downarrow \\ V_{\mathfrak{krv}} & \xrightarrow{\ \nu\ } & W_{\mathfrak{krv}} \end{array}$$

where the left-hand vertical arrow is the isomorphism (38) mapping $E_{a,b} \mapsto b$, and the right-hand vertical map sends an Ihara derivation d_f to f. Equipping $W_{\mathfrak{krv}}$ with the Lie bracket inherited from $\mathfrak{krv}^{\nu}$ makes this into a commutative diagram of Lie isomorphisms. But this bracket is nothing other than the Poisson bracket $\{\ ,\ \}$ in (7) since $\mathfrak{krv}^{\nu} \subset \mathfrak{sder}_2^{(x)}$. □

We now give a characterization of $W_{\mathfrak{krv}}$ by two defining properties which are the twists by ν defined in (12) of those defining $V_{\mathfrak{krv}}$. Recall that β is the the backwards operator given in Definition 3.13.

Proposition 4.18. *The space* $W_{\mathfrak{krv}}$ *is the space spanned by polynomials* $b \in \mathfrak{lie}_C$*, of homogeneous degree* $n \geq 3$*, such that*

(i) $b_y - b_x$ *is* anti-palindromic*, i.e.* $\beta(b_y - b_x) = (-1)^{n-1}(b_y - b_x)$*, and*

(ii) $b + \frac{c}{n} y^n$ *with* $c = (b|x^{n-1}y)$ *is* circ-constant *for* $\{c_k\}_{k>1}$ *with* $c_k = \delta_{k,n} \cdot (b|x^{n-1}y)$.

Proof. Let $b \in W_{\mathfrak{krv}}$. Put $f = \nu(b)$, so we have $f \in V_{\mathfrak{krv}}$. Then the property that $b_y - b_x$ is anti-palindromic is precisely equivalent to the push-invariance of f (this is proved as the equivalence of properties (iv) and (v) of Theorem 2.1 of [S1]). This proves (i).

For (ii), we note that since $f \in V_{\mathfrak{krv}}$, $f^y - f^x$ is push-constant for the value $c = (f|x^{n-1}y) = (-1)^{n-1}(b|x^{n-1}y)$. We have

$$b(x,y) = xb^x(x,y) + yb^y(x,y),$$

so

$$f(x,y) = b(z,y) = zb^x(z,y) + yb^y(z,y) = -xb^x(z,y) - yb^x(z,y) + yb^y(z,y).$$

Thus since $f(x,y) = xf^x(x,y) + yf^y(x,y)$, this gives

$$f^x = -b^x(z,y) \ \text{ and } \ f^y = -b^x(z,y) + b^y(z,y),$$

so

$$f^y - f^x = b^y(z,y) = \nu(b^y).$$

Thus to prove the result, it suffices to prove that the following statement: if $g \in Ass_C$ is a polynomial of homogeneous degree n that is push-constant for $(-1)^{n-1}c$, then $\nu(g)$ is circ-constant for c, since taking $g = f^y - f^x$ then shows that $\nu(g) = b^y$ is circ-constant for c. The proof of this statement is straightforward using the substitution $z = -x - y$ (but see the proof of Lemma 3.3 in [S1] for details). To complete the proof of (ii), we note that when $f \in V_{\mathfrak{krv}}$ is of even degree n we have $c = 0$. In fact this follows from Corollary 1.14, which states that $\mathfrak{lkrv}_n^1 = 0$ when n is even; this means that there are no elements in $\mathfrak{krv}$ of even weight n and depth 1, so there are no such elements in $V_{\mathfrak{krv}}$. Since c is the coefficient of the depth 1 term $x^{n-1}y$, we have $c = 0$ when n is even. This completes the proof of (ii). □

4.2.2. *Step 2: The mould version $ma(W_{\mathfrak{krv}})$.* The space $ma(W_{\mathfrak{krv}})$ is closed under the *ari*-bracket by (48), since $W_{\mathfrak{krv}}$ is closed under the Poisson bracket by Proposition 4.17.

Let $b \in W_{\mathfrak{krv}}$ and let $B = ma(b)$. Then since b is a Lie polynomial, B is an alternal polynomial mould. Let us give the mould reformulations of properties (i) and (ii) of Proposition 4.18. The second property is easy since we already showed, in Proposition 3.38, that a polynomial b is circ-constant if and only if $swap(B)$ is circ-constant.

Expressing the first property in terms of moulds is more complicated and calls for an identity discovered by Écalle. We need to use the mould operators *mantar* and *pari* in Definition 3.2.

The operator *pari* extends the operator $y \mapsto -y$ on polynomials to all moulds, and *mantar* extends the operator $f \mapsto (-1)^{n-1}\beta(f)$.

Lemma 4.19. *Let $b \in \mathfrak{lie}_C$. Assume $\deg b > 2$. Then the following are equivalent:*

(i) $b_y - b_x$ is anti-palindromic;

(ii) if $B = ma(b)$, then B satisfies the twisted senary relation (111).

(Note that since b is a Lie element, B and $pari(B)$ are alternal and thus *mantar*-invariant, so we can drop the right-hand *mantar* from the senary relation (110).)

Proof. It suffices to prove the statement for an element b of homogeneous degree n. The statement is a consequence of the following result, proved in Proposition A.3 of the Appendix of [S1] (see also [FK2, Lemma 4.2]). Let $\tilde{b} \in \mathfrak{lie}_C$ and let $\tilde{B} = ma(\tilde{b})$ with $n > 2$. Write $\tilde{b} = \tilde{b}_x x + \tilde{b}_y y$ as usual. Then for each depth r part $(\tilde{b}_x + \tilde{b}_y)^r$ of the polynomial $\tilde{b}_x + \tilde{b}_y$ $(1 \le r \le n-1)$, the anti-palindromic property

$$(\tilde{b}_x + \tilde{b}_y)^r = (-1)^{n-1}\beta(\tilde{b}_x + \tilde{b}_y)^r \tag{119}$$

translates directly to the following relation on $\tilde{B}$:

$$teru(\tilde{B})(u_1, \ldots, u_r) = push \circ mantar \circ teru(\tilde{B})(u_1, \ldots, u_r). \tag{120}$$

Let us deduce the equivalence of the claims (i) and (ii) from that of (119) and (120). Let $\tilde{b}$ be defined by $\tilde{b}(x, y) = b(x, -y)$. This implies that $(b_x)^r = (-1)^r(\tilde{b}_x)^r$, $(b_y)^r = (-1)^{r-1}(\tilde{b}_y)^r$, and $\tilde{B} = pari(B)$. Thus $b_y - b_x$ is anti-palindromic if and only if $\tilde{b}_y + \tilde{b}_x$ is, i.e. if and only if (119) holds for $\tilde{b}$, which is the case if and only if (120) holds for $\tilde{B}$, which is equivalent to (111) for B. This proves the lemma. □

Corollary 4.20. *We have the isomorphism of Lie algebras*

$$ma : W_{\mathfrak{krv}} \xrightarrow{\sim} ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \subset ARI(\mathcal{F}_{\rm ser})_{ari} . \tag{121}$$

Proof. By Proposition 4.18, the space $W_{\mathfrak{krv}}$ is the space of Lie polynomials b satisfying (i) $b_y - b_x$ is antipalindromic and (ii) $b + \frac{c}{n} y^n$ for $c = (b|x^{n-1}y)$ is circ-constant for $\{c_k\}_{k>1}$ with $c_k = \delta_{k,n} \cdot (b|x^{n-1}y)$. By Lemma 4.19, property (i) is equivalent to the fact that $pari(B)$ satisfies the senary relation (110). By Proposition 3.38 the fact that b is circ-constant is equivalent to $swap(B)$ being circ-constant (and Remark 4.16 shows that the constant is necessarily unique and the same). But by Definition 4.15, $ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}$ is precisely the space of alternal polynomial moulds satisfying precisely these two mould properties. □

4.2.3. *Step 3: Construction of the map* Ξ. In this section we finally arrive at the main step of the construction of our map $\mathfrak{krv} \to \mathfrak{krv}_{ell}$, namely the construction of the map Ξ given in the following proposition.

Proposition 4.21. *The operator* $\Xi = Ad_{ari}(invpal) \circ pari$ *gives an injective Lie algebra homomorphism of Lie subalgebras of* $ARI(\mathcal{F}_{\rm Lau})_{ari}$*:*

$$\Xi : ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \hookrightarrow ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}. \tag{122}$$

Proof. We have already shown that both spaces are Lie subalgebras of $ARI(\mathcal{F}_{\rm Lau})_{ari}$, the first in Corollary 4.20 and the second in §4.1.2. Furthermore, since $pari$ and $Ad_{ari}(invpal)$ are both invertible and respect the ari-bracket, the proposed map is a Lie algebra map invertible on its image, and therefore injective. Thus it remains only to show that the image of $ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}$ under Ξ really lies in the target space $ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}$. We will show separately that if $B \in ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}$ and $A = \Xi(B)$, then

(i) A is push-invariant,

(ii) A is alternal,

(iii) $swap(A)$ is $*$circ-neutral (i.e. circ-neutral up to addition of a constant-valued mould),

(iv) $A \in ARI(\mathcal{F}_{\rm Lau})^{\Delta}$.

Proof of (i): A is push-invariant. This statement follows directly on Écalle's senary property (110). Indeed, since B satisfies (111), $\tilde{B} := pari(B)$ satisfies (120), so (113) implies that $Ad_{ari}(invpal)(\tilde{B}) = \Xi(B) = A$ is push-invariant. □

Proof of (ii): A is alternal. The subspace of alternal moulds $ARI(\mathcal{F}_{\rm Lau})_{al}$ is closed under ari (cf. [SS]), so $exp_{ari}(ARI(\mathcal{F}_{\rm Lau})_{al})$ forms a subgroup of $GARI(\mathcal{F}_{\rm Lau})_{gari}$, which we denote by $GARI(\mathcal{F}_{\rm Lau})^{as}_{gari}$ (the superscript *as* stands for *symmetral*). The mould *pal* is known to be symmetral (cf. [Ec3], or in more detail [S2], Theorem 4.3.4). Thus, since $GARI(\mathcal{F}_{\rm Lau})^{as}_{gari}$ is a group, the *gari*-inverse mould *invpal* is also symmetral. Therefore the adjoint action $Ad_{ari}(invpal)$ on $ARI(\mathcal{F}_{\rm Lau})$ restricts to an adjoint action on the Lie subalgebra $ARI(\mathcal{F}_{\rm Lau})_{al}$ of alternal moulds. If B is alternal, then $pari(B)$ is alternal, and so $A = \Xi(B)$ is alternal. This completes the proof of (ii). □

For the assertions (iii) and (iv), we will make use of Écalle's fundamental identity in the version (59) given in §3.4, with $N = pari(B)$ (recall that (59) is valid whenever $Ad_{ari}(invpal) \cdot N$ is push-invariant, which is the case for $pari(B)$ thanks to (i) above). The key point is that the operators $\overline{ganit}(poc)$ and $Ad_{\overline{ari}}(invpil)$ on the left-hand side of (59) are better adapted to tracking the circ-neutrality and the denominators than the right-hand operator $Ad_{ari}(invpal)$ considered directly.

Proof of (iii): swap(A) is ∗circ-neutral. Let $b \in W_{\mathfrak{krv}}$, and assume that b is of homogeneous degree n. Let $B = ma(b)$. Then by Corollary 4.20, $swap(B)$ is circ-constant, and even circ-neutral if n is even.

We need to show that $swap \cdot \Xi(B) = swap \cdot Ad_{ari}(invpal) \cdot pari(B)$ is ∗circ-neutral. To do this, we use (59) with $N = pari(B)$, and in fact show the result on the left-hand side, which is equal to

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari \cdot swap(B)$$

(noting that *pari* commutes with *swap*). We prove that this mould is ∗circ-neutral in three steps:

- First we show that the operator $\overline{ganit}(poc) \cdot pari$ changes a circ-constant mould into one that is circ-neutral (Proposition 4.22).
- Secondly, we show that the operator $Ad_{\overline{ari}}(invpil)$ preserves the property of circ-neutrality (Proposition 4.25).
- Finally, we show that if M is a mould that is not circ-constant but only ∗circ-constant, and if M_0 is the (unique) constant-valued mould such that $M + M_0$ is circ-constant, then $Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari(M) + M_0$ is circ-neutral, which says that $Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari(M)$ is ∗circ-neutral (a paragraph after Proposition 4.25).

Proposition 4.22. *Fix $n \geq 1$, and let $M \in \overline{ARI}(\mathcal{F}_{\rm pol})$ be a circ-constant (cf. Definition 3.33) polynomial-valued mould of homogeneous degree n. Then $\overline{ganit}(poc) \cdot pari(M)$ is circ-neutral.*

When $n = 1$, the homogeneous weight $n = 1$ means that M is concentrated in depth 1 where it has constant value $M(v_1) = c$. Thus the condition of being circ-constant is automatically satisfied. Direct computation shows that

$$\overline{ganit}(poc) \cdot pari(M)(v_1, \ldots, v_r) = \frac{-c}{(v_1 - v_2)(v_2 - v_3) \cdots (v_{r-1} - v_r)}.$$

Summing the images of this mould under powers of the circ-operator and putting them over the common denominator $(v_r - v_1)(v_1 - v_2) \cdots (v_{r-1} - v_r)$ shows that this mould is circ-neutral. For the remainder of the proof, we assume that $n > 1$.

Notation for the proof of Proposition 4.22. Let $\mathbf{v} = (v_1, \ldots, v_r)$, and let $\mathbf{W_v}$ be the set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s$ as in (50). We always denote consecutive chunks $\mathbf{b}_i$ in the form of tuples $(v_k, v_{k+1}, \ldots, v_{k+l})$, but they can also be considered simply as subsets of $\{v_1, \ldots, v_r\}$.

For any decomposition $d_{\mathbf{v}}$, we let its $\mathbf{b}$-part be the subset $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_s$ of $\{v_1, \ldots, v_r\}$ and its $\mathbf{a}$-part be the subset $\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_s$; the $\mathbf{a}$-part is equal to the complement of the $\mathbf{b}$-part in $\{v_1, \ldots, v_r\}$. We write $|\mathbf{a}|$ for the number of letters in the $\mathbf{a}$-part, i.e. $|\mathbf{a}| = |\mathbf{a}_1| + \cdots + |\mathbf{a}_s|$.

Set

$$\mathbf{W} = \coprod_{1 \leq i \leq r} \mathbf{W}_{\sigma_r^i(\mathbf{v})}, \tag{123}$$

where the $\sigma_r^i(\mathbf{v})$ are the cyclic permutations of $\mathbf{v} = (v_1, \ldots, v_r)$. For a fixed subset $\mathbf{b} \subsetneq \{v_1, \ldots, v_r\}$, let $\mathbf{W^b}$ denote the subset of decompositions in $\mathbf{W}$ having $\mathbf{b}$-part equal to $\mathbf{b}$; thus we have

$$\mathbf{W} = \coprod_{\mathbf{b} \subsetneq \{v_1,\dots,v_r\}} \mathbf{W}^{\mathbf{b}}. \tag{124}$$

Let $\mathbf{w} = (v_{i+1},\dots,v_r,v_1,\dots,v_i)$ be any cyclic permutation of $\mathbf{v} = (v_1,\dots,v_r)$; let $d_{\mathbf{w}} = \mathbf{a_1 b_1} \cdots \mathbf{a_s b_s}$ be a decomposition of $\mathbf{w}$, and let $\mathbf{b} = \mathbf{b_1} \cup \cdots \cup \mathbf{b_s}$ be its $\mathbf{b}$-part. We will list all the elements of $\mathbf{W}^{\mathbf{b}}$, i.e. all decompositions of all cyclic permutations of $\mathbf{v}$ having $\mathbf{b}$-part equal to $\mathbf{b}$. Let $\mathbf{a} = \mathbf{a_1} \cup \cdots \cup \mathbf{a_s}$ be the $\mathbf{a}$-part of $d_{\mathbf{w}}$. Then there exists a decomposition of a cyclic permutation of $\mathbf{v}$ having $\mathbf{b}$-part equal to $\mathbf{b}$ if and only if the cyclic permutation begins with a letter $v_k \in \mathbf{a}$; for such a cyclic permutation, there is exactly one decomposition with $\mathbf{b}$-part $\mathbf{b}$, obtained by cyclically shifting the pieces of the decomposition $d_{\mathbf{w}}$.

Example 4.23. Let $\mathbf{w} = (v_3,v_4,v_5,v_6,v_7,v_1,v_2)$ and consider the decomposition

$$\mathbf{w} = \mathbf{a_1 b_1 a_2 b_2 a_3} = (v_3,v_4)(v_5)(v_6)(v_7,v_1)(v_2).$$

Then $\mathbf{b} = \{v_1,v_5,v_7\}$ and $\mathbf{a} = \{v_2,v_3,v_4,v_6\}$. The only cyclic permutations of $\mathbf{v} = (v_1,\dots,v_7)$ admitting the $\mathbf{b}$-part $\{v_1,v_5,v_7\}$ are the ones starting with $v_k \in \mathbf{a}$, and for each one, there is a unique decomposition determined by $\mathbf{b}$:

$$\begin{cases} (v_2,v_3,v_4)(v_5)(v_6)(v_7,v_1) \\ (v_3,v_4)(v_5)(v_6)(v_7,v_1)(v_2) \\ (v_4)(v_5)(v_6)(v_7,v_1)(v_2,v_3) \\ (v_6)(v_7,v_1)(v_2,v_3,v_4)(v_5). \end{cases}$$

The set $\mathbf{W}^{\mathbf{b}}$ consists of these four decompositions.

Let the *ordered* $\mathbf{a}$-*part* of a decomposition $d_{\mathbf{w}}$ of a cyclic permutation $\mathbf{w}$ of $\mathbf{v} = (v_1,\dots,v_r)$ be the word $\mathbf{a_1} \cdots \mathbf{a_s}$ of the decomposition $d_{\mathbf{w}}$. Then by the above, there are exactly $|\mathbf{a}|$ decompositions in $\mathbf{W}^{\mathbf{b}}$, and their ordered $\mathbf{a}$-parts are given by

$$\{\sigma^j_{|\mathbf{a}|}(\mathbf{a_1} \cdots \mathbf{a_s}) \,|\, j = 0,\dots,|\mathbf{a}|-1\} \tag{125}$$

i.e. the cyclic permutations of the letters of $\mathbf{a_1} \cdots \mathbf{a_s}$.

Proof of Proposition 4.22. Let $c = \big(M(v_1)\,|\,v_1^{n-1}\big)$, and let $N = pari(M)$ (where *pari* is defined in Definition 3.2), so that N is a polynomial mould of fixed homogeneous degree n, with $N(v_1) = -cv_1^{n-1}$. Since M is circ-constant for c, we have

$$N(v_1,\dots,v_r) + \cdots + N(v_r,v_1,...,v_{r-1}) = (-1)^r c \sum_{\substack{e_1+\cdots+e_r=n-r \\ e_i \geq 0}} v_1^{e_1} \cdots v_r^{e_r}. \tag{126}$$

By the explicit formula (51), we have

$$\big(\overline{ganit}(poc) \cdot N\big)(v_1,\dots,v_r) = \sum_{\mathbf{W_v}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) N(\mathbf{a}_1 \cdots \mathbf{a}_s), \tag{127}$$

where we recall that the sum over $\mathbf{W_v}$ is the sum over all decompositions $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s$ of $\mathbf{v} = (v_1,\dots,v_r)$ as in (50). Now, summing over the set $\mathbf{W}$ which is the union of such decompositions not only for $\mathbf{v}$ but also for all the cyclic permutations of $\mathbf{v}$ (cf. (123)), we have

$$\begin{aligned}\sum_{i=0}^{r-1}\big(\overline{ganit}(poc)\cdot N\big)\big(\sigma_r^i(\mathbf{v})\big) &= \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s\in\mathbf{W}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\ &= \sum_{\mathbf{b}\subsetneq\{v_1,\ldots,v_r\}}\ \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s\in\mathbf{W}^{\mathbf{b}}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\ &= \sum_{\substack{\mathbf{b}\subsetneq\{v_1,\ldots,v_r\}\\ \mathbf{b}=\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s}} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)\sum_{j=0}^{|\mathbf{a}|-1} N\big(\sigma^j_{|\mathbf{a}|}(\mathbf{a}_1\cdots\mathbf{a}_s)\big)\\ &= c\sum_{\substack{\mathbf{b}\subsetneq\{v_1,\ldots,v_r\}\\ \mathbf{b}=\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s}} (-1)^{|\mathbf{a}|}poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s)\sum_{\substack{e_1+\cdots+e_{|\mathbf{a}|}=n-|\mathbf{a}|\\ e_j\geq 0}} v_{i_1}^{e_1}\cdots v_{i_{|\mathbf{a}|}}^{e_{|\mathbf{a}|}}, \qquad (128)\end{aligned}$$

where the disjoint consecutive chunks $\mathbf{b}_1,\ldots,\mathbf{b}_s$ in the sum run over the possible $\mathbf{b}$-parts of decompositions as in (50). Here, the first equality is the definition of $\overline{ganit}(poc)$, the second equality follows directly from (124), the third follows directly from (125), and the last equality from (126) using the notation $\mathbf{a}_1\cup\cdots\cup\mathbf{a}_s = \{v_{i_1},\ldots,v_{i_{|\mathbf{a}|}}\}$ for the subset $\mathbf{a}$ of $\{v_1,\ldots,v_r\}$.

If $c=0$, i.e. if M is a circ-neutral mould, the expression (128) is trivially equal to zero in all depths $r>1$, proving Proposition 4.22 in the case where M is circ-neutral. In order to deal with the case where M is circ-constant for a value $c\neq 0$, we use a trick and subtract off a known mould that is also circ-constant for c.

For $A\subset\{1,\ldots,n\}$, let S_d^A denote the sum of all monomials of degree d in the letters v_i, $i\in A$. In particular, if $d=0$ we set $S_d^A=1$ and if $d<0$ we set $S_d^A=0$.

Lemma 4.24. *For $n>1$ and any constant c, let T_c^n be the homogeneous polynomial mould of degree n (cf. Definition 3.33) defined by $T_c^n(\emptyset)=0$ and*

$$T_c^n(v_1,\ldots,v_r) = \frac{c}{r}S_{n-r}^{\{1,\ldots,r\}}$$

for $r>1$; in particular we have $T_c^n(v_1,\ldots,v_n)=\frac{1}{n}$ and $T_c^n(v_1,\ldots,v_r)=0$ when $r>n$. Then T_c^n is circ-constant and $\overline{ganit}(poc)\cdot pari(T_c^n)$ is circ-neutral.

The proof of this lemma is surprisingly long and technical, so we have relegated it to Appendix A. Using the result, we can now finish the proof of Proposition 4.22. Indeed we have $M(v_1)=T_c^n(v_1)=cv_1^{n-1}$, so the mould $M-T_c^n$ is circ-neutral and thus $\overline{ganit}(poc)\cdot pari(M-T_c^n)$ is also circ-neutral. But Lemma 4.24 shows that $\overline{ganit}(poc)\cdot pari(T_c^n)$ is itself circ-neutral, so we have

$$\sum_{i=1}^{r} circ^i\big(\overline{ganit}(poc)\cdot pari(M)\big) = \sum_{i=1}^{r} circ^i\big(\overline{ganit}(poc)\cdot pari(T_c^n)\big) = 0$$

and thus $\overline{ganit}(poc)\cdot pari(M)$ is also circ-neutral, completing the proof of Proposition 4.22. □

We now proceed to the second step, showing that the operator $Ad_{\overline{ari}}(invpil)$ preserves circ-neutrality.

Proposition 4.25. *If* $M \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})$ *is circ-neutral then* $Ad_{\overline{ari}}(invpil) \cdot M$ *is also circ-neutral.*

Proof. By (55), we have

$$Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big) = \sum_{n\geq 0} \frac{(-1)^n}{n!}\big(ad_{\overline{ari}}(lopil)\big)^n. \tag{129}$$

The definition of *lopil* in (53) shows that *lopil* is trivially circ-neutral. Thus, since M is circ-neutral (as explained in Example 3.34), $ad_{\overline{ari}}(lopil)\cdot M = \overline{ari}(lopil, M)$ is also circ-neutral by Proposition 3.41, and successively so are all the terms $ad_{\overline{ari}}(lopil)^n(M)$. Thus $Ad_{\overline{ari}}(invpil) \cdot M$ is circ-neutral. □

Proof of (iii) (continued): Finally, we now assume that $B \in ARI(\mathcal{F}_{\mathrm{ser}})_{al+tsen*circconst}$. In particular, $swap(B)$ is a $*$circ-constant polynomial-valued mould in $\overline{ARI}(\mathcal{F}_{\mathrm{ser}})$ of homogeneous degree n. Let B_0 be the (unique) constant-valued mould such that $swap(B) + B_0$ is circ-constant. Then by Propositions 4.22 and 4.25, the mould

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari\big(swap(B) + B_0\big)$$

is circ-neutral. This mould breaks up as the sum

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari(swap(B)) + Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari(B_0),$$

but the operator $Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)$ preserves constant-valued moulds (cf. [S2], Lemma 4.6.2 for the proof). Thus the mould

$$\begin{aligned} Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari\big(swap(B) + B_0\big) &= \\ Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot pari(swap(B)) &+ B_0 \end{aligned}$$

is circ-neutral, or equivalently,

$$Ad_{\overline{ari}}(invpil) \cdot ganit(poc) \cdot pari(swap(B))$$

is $*$circ-neutral. However, using the fact that *pari* trivially commutes with *swap* and also the fact that by Proposition 4.21 (which we recall relies on Écalle's assertion (113)) $Ad_{ari}(invpal) \cdot pari(B)$ is push-invariant, we can apply (59) to find that

$$\begin{aligned} Ad_{\overline{ari}}(invpil) \cdot ganit(poc) \cdot swap\big(pari(B)\big) &= swap \cdot Ad_{ari}(invpal) \cdot pari(B) \\ &= swap \cdot \Xi(B). \end{aligned}$$

Thus $swap \cdot \Xi(B)$ is $*$circ-neutral, which concludes the proof of (iii). □

Proof of (iv): $A \in ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}$. For any mould $A \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})$, let us use the notation

$$\overline{\Delta}(A)(v_1, \ldots, v_r) := v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r\, A(v_1, \ldots, v_r),$$

so that we have

$$swap\big(\Delta(A)\big) = \overline{\Delta}\big(swap(A)\big). \tag{130}$$

We write $\overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$ for the subspace of moulds $M \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})$ such that $\overline{\Delta}(M)$ is polynomial. By (130), we see that $M \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$ means that $swap(M) \in ARI(\mathcal{F}_{\mathrm{Lau}})^{\Delta}$.

We will again use Écalle's assertion (113) and the equality (59); this time we will study the left-hand side of (59) to track the denominators that appear in the right-hand side. By (59), if B is a polynomial-valued mould satisfying the twisted senary relation, and if $A = \Xi(B) = Ad_{ari}(invpal) \cdot pari(B)$ is push-invariant, then A lies in $ARI(\mathcal{F}_{\rm Lau})^{\Delta}$ if and only if

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in \overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}. \tag{131}$$

The identity (59) applies in the situation where $B \in ARI_{al+tsen*circconst}(\mathcal{F}_{\rm ser})$ and $A = \Xi(B)$, since we proved in (i) that A is indeed push-invariant. We will prove that $A \in ARI(\mathcal{F}_{\rm Lau})^{\Delta}$ by studying the denominators that arise in (131), produced first by applying $\overline{ganit}(poc)$ and then by applying $Ad_{\overline{ari}}(invpil)$. Let us first show that the denominators introduced by applying $\overline{ganit}(poc)$ to a polynomial-valued mould are at worst of the form $(v_1 - v_2) \cdots (v_{r-1} - v_r)$.

Lemma 4.26 ([B, Proposition 4.38])**.** *Let $M \in \overline{ARI}(\mathcal{F}_{\rm ser})$. Then*

$$\overline{ganit}(poc) \cdot M \in \overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}$$

and in fact $(v_1 - v_2) \cdots (v_{r-1} - v_r)$ is a common denominator for all terms arising in $\big(\overline{ganit}(poc) \cdot M\big)(v_1, \ldots, v_r)$ for all M.

Proof. The explicit expression for $\overline{ganit}(Q)$ given in (51) shows that the only denominators that can occur in $\overline{ganit}(poc) \cdot M$ come from the factors

$$poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) \tag{132}$$

for all decompositions $d_{\mathbf{v}} = \mathbf{a}_1 \mathbf{b}_1 \cdots \mathbf{a}_s \mathbf{b}_2$ of $\mathbf{v} = (v_1, \ldots, v_r)$ into chunks as in (50), and

$$\lfloor \mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1})$$

(for $k > 1$) as in (52). Since poc is defined as in (56), the only factors that can appear in (132) are $(v_l - v_{l-1})$ where v_l is a letter in one of $\mathbf{b}_i$, and these factors appear in each term with multiplicity one. Since the sum ranges over all possible decompositions, the only letter of $\mathbf{v}$ that never belongs to any $\mathbf{b}_i$ is v_1; all the other factors $(v_i - v_{i-1})$ appear. Thus $(v_1 - v_2)(v_2 - v_3) \cdots (v_{r-1} - v_r)$ is a common denominator for all the terms in the sum defining $\overline{ganit}(poc) \cdot M$, which proves the lemma. □

Lemma 4.27. *Let $M, N \in \overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}_{*circneut}$ be two moulds. Then $\overline{ari}(M, N)$ also lies in $ARI(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}$.*

Proof. We begin by showing that if $M \in \overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}_{*circneut}$ then for all $r > 1$, $\overline{\Delta}(M)$ satisfies the identity

$$\overline{\Delta}(M)(0, v_2, \ldots, v_r) = \overline{\Delta}(M)(v_2, \ldots, v_r, 0). \tag{133}$$

In fixed depth $r > 1$, the $*$circ-neutrality of M means that there exists a constant c_r such that $M(v_1, \ldots, v_r) + c_r$ is circ-neutral, i.e.

$$M(v_1, \ldots, v_r) + M(v_2, \ldots, v_r, v_1) + \cdots + M(v_r, v_1, \ldots, v_{r-1}) + rc_r = 0. \tag{134}$$

Writing

$$M(v_1, \ldots, v_r) = \frac{\overline{\Delta}(M)(v_1, \ldots, v_r)}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)v_r}$$

allows us to rewrite (134) in terms of $\overline{\Delta}(M)$, as

$$\begin{aligned}&\frac{\overline{\Delta}(M)(v_1,\dots,v_r)}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}+\frac{\overline{\Delta}(M)(v_2,\dots,v_r,v_1)}{v_2(v_2-v_3)\cdots(v_r-v_1)v_1}\\ &+\cdots+\frac{\overline{\Delta}(M)(v_r,v_1,\dots,v_{r-1})}{v_r(v_r-v_1)\cdots(v_{r-2}-v_{r-1})v_{r-1}}+rc_r=0.\end{aligned} \tag{135}$$

Only the first two terms of this sum have a pole at $v_1=0$, so multiplying (135) by v_1 and then setting $v_1=0$ leaves only the first two terms, which become

$$\frac{\overline{\Delta}(M)(0,v_2,\dots,v_r)}{-v_2(v_2-v_3)\cdots(v_{r-1}-v_r)v_r}+\frac{\overline{\Delta}(M)(v_2,\dots,v_r,0)}{v_2(v_2-v_3)\cdots(v_r)}=0. \tag{136}$$

Since the two terms in (136) have the same denominator with opposite signs, this is equivalent to the desired identity (133).

We can now proceed to the proof of Lemma 4.27. By additivity, it is enough to consider the case where M and N are moulds concentrated in depths $r\geq 1$ and $s\geq 1$ respectively.

In §4.7 of [S2] (proof of Theorem 4.7.1, see also §4.3.4 of Baumard's thesis or Prop. 5.1 of [BS]) the explicit expressions for $\overline{\Delta}\big(\overline{arit}(M)\cdot N\big)$, $\overline{\Delta}\big(\overline{arit}(N)\cdot M\big)$ and $\overline{\Delta}\big(lu(M,N)\big)$ are calculated, so as to examine the poles of each term. It is shown there that

(i) $\overline{\Delta}\big(\overline{arit}(M)\cdot N\big)$ has potential poles (linear factors in the denominator) only at v_i-v_{i+r+1} $(1\leq i\leq s-1)$ and v_i-v_{i+r} $(1\leq i\leq s)$, and at $v_{r+1}=0$ and $v_s=0$;

(ii) The residues at the poles $v_i=v_{i+r+1}$ are equal to zero with only the hypothesis $M,N\in\overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}$;

(iii) The residues at the poles $v_i=v_{i+r}$ are equal to zero thanks to (133);

(iv) $\overline{\Delta}\big(\overline{arit}(M)\cdot N\big)$ does have simple poles at $v_{r+1}=0$ and $v_s=0$, with residues

$$\overline{\Delta}(N)(0,v_{r+2},\dots,v_{r+s})\overline{\Delta}(M)(v_1,\dots,v_r)\quad\text{at } v_{r+1}=0 \tag{137}$$

and

$$\overline{\Delta}(N)(v_1,\dots,v_{s-1},0)\overline{\Delta}(M)(v_{s+1},\dots,v_{r+s})\quad\text{at } v_s=0. \tag{138}$$

The symmetric statements hold for $\overline{\Delta}\big(arit(N)\cdot M\big)$, which thus has simple poles only at $v_r=0$ and $v_{s+1}=0$ with residues

$$\overline{\Delta}(M)(0,v_{s+2},\dots,v_{r+s})\overline{\Delta}(N)(v_1,\dots,v_s)\quad\text{at } v_{s+1}=0 \tag{139}$$

and

$$\overline{\Delta}(M)(v_1,\dots,v_{r-1},0)\overline{\Delta}(N)(v_{r+1},\dots,v_{r+s})\quad\text{at } v_r=0. \tag{140}$$

Since we have

$$\overline{\Delta}\big(\overline{ari}(M,N)\big)=\overline{\Delta}\big(\overline{arit}(N)\cdot M\big)-\overline{\Delta}\big(\overline{arit}(M)\cdot N\big)+\overline{\Delta}\big(lu(M,N)\big), \tag{141}$$

we can show that $\overline{ari}(M,N)\in\overline{ARI}(\mathcal{F}_{\rm Lau})^{\bar{\Delta}}$ by showing that $\overline{\Delta}\big(\overline{ari}(M,N)\big)$ is polynomial, i.e. that it has no poles, by showing firstly that $\overline{\Delta}\big(lu(M,N)\big)$ has no poles outside of $v_r=0$, $v_s=0$, $v_{r+1}=0$, $v_{s+1}=0$, and secondly that the residues at these poles cancel out exactly with those of $\overline{\Delta}\big(\overline{arit}(N)\cdot M - \overline{arit}(M)\cdot N\big)$. We write out $\overline{\Delta}\big(lu(M,N)\big)$ explicitly as

$$\begin{aligned}
&\overline{\Delta}\big(lu(M,N)\big)(v_1,\ldots,v_{r+s})\\
&= v_1(v_1-v_2)\cdots(v_{r+s-1}-v_{r+s})v_{r+s}\\
&\quad\cdot\big(M(v_1,\ldots,v_r)N(v_{r+1},\ldots,v_{r+s}) - N(v_1,\ldots,v_s)M(v_{s+1},\ldots,v_{r+s})\big)\\
&= \frac{v_r-v_{r+1}}{v_rv_{r+1}}\overline{\Delta}(M)(v_1,\ldots,v_r)\overline{\Delta}(N)(v_{r+1},\ldots,v_{r+s})\\
&\quad - \frac{v_s-v_{s+1}}{v_sv_{s+1}}\overline{\Delta}(N)(v_1,\ldots,v_s)\overline{\Delta}(M)(v_{s+1},\ldots,v_{r+s}).
\end{aligned}$$

This shows that the only poles are at $v_r = 0$, $v_s = 0$, $v_{r+1} = 0$, $v_{s+1} = 0$. Multiplying by v_{r+1} and setting $v_{r+1} = 0$, we find that the residue at $v_{r+1} = 0$ comes from the first line only and is equal to (137) which is the residue at $v_{r+1} = 0$ in $\overline{\Delta}\big(\overline{arit}(M)\cdot N\big)$. Thus these poles cancel out in (141). Similarly, the pole at $v_s = 0$ has residue equal to (138), the pole at $v_{s+1} = 0$ has residue equal to the negative of (139), and the pole at $v_r = 0$ has residue equal to the negative of (140). So all these poles cancel out in (141), and we conclude that $\overline{\Delta}\big(\overline{ari}(M,N)\big) \in \overline{ARI}(\mathcal{F}_{\mathrm{ser}})$, i.e. $\overline{ari}(M,N) \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$, as desired. □

Corollary 4.28. *If $P \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})$ is a circ-neutral mould lying in $\overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$, then we also have*

$$Ad_{\overline{ari}}(invpil)\cdot P \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}. \tag{142}$$

Proof. The mould $lopil \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})$ given in (53) is circ-neutral (as explained in Example 3.34) and by its defining equation (53), $lopil$ lies in $\overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$. Therefore by Lemma 4.27, we have $\overline{ari}(lopil,P) \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$. Furthermore, by Proposition 3.41, $\overline{ari}(lopil,P)$ is also circ-neutral. Then, applying Lemma 4.27 successively shows that $ad_{\overline{ari}}(lopil)^n(P)$ is a circ-neutral mould lying in $\overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$ for all $n \geq 1$. Since $Ad_{\overline{ari}}(invpil)\cdot P$ is obtained by summing these terms by (129), we obtain (142). □

<u>*Proof of (iv) (continued):*</u> We can now complete the proof of (iv) of Proposition 4.21. Recall that $B \in ARI(\mathcal{F}_{\mathrm{ser}})_{al+tsen*circconst}$ and $A = \Xi(B)$. We may assume that B is homogeneous of fixed degree $n \geq 3$. Then there exists a constant mould B_0 such that $swap(B + B_0)$ is circ-constant. By Lemma 4.26, we have

$$\overline{ganit}(poc)\cdot swap\cdot pari(B + B_0) \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}. \tag{143}$$

Since $swap$ and $pari$ commute, the mould in (143) is equal to

$$\overline{ganit}(poc)\cdot pari\cdot swap(B + B_0) \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}. \tag{144}$$

By Proposition 4.22, this mould is circ-neutral, and by (144) it lies in $\overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}$. Therefore we can apply Corollary 4.28 with $P = \overline{ganit}(poc)\cdot swap\cdot pari(B + B_0)$ to conclude that

$$Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)\cdot swap\cdot pari(B + B_0) \in \overline{ARI}(\mathcal{F}_{\mathrm{Lau}})^{\overline{\Delta}}.$$

Observe that this mould is equal to

$$Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)\cdot swap\cdot pari(B)(u_1,\ldots,u_n) + (-1)^n B_0(u_1,\ldots,u_n)$$

since $pari{\cdot}swap(B_0)(u_1,\ldots,u_n) = (-1)^n B_0(u_1,\ldots,u_n)$ and $Ad_{ari}(invpil){\cdot}\overline{ganit}(poc)$ preserves constant moulds (cf. [S2], Lemma 4.6.2) which follows from $\overline{ganit}(pic)^{-1} = \overline{ganit}(poc)$ in (58). Thus

$$Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)\cdot swap\cdot pari(B)\in\overline{ARI}(\mathcal{F}_{\rm Lau})^{\overline{\Delta}}.$$

Swapping this mould and applying (59) with $N = pari(B)$, we finally find that

$$Ad_{ari}(invpal)\cdot pari(B)=\Xi(B)\in ARI(\mathcal{F}_{\rm Lau})^{\Delta},$$

which completes the proof of (iv), and therefore the proof of Proposition 4.21, and thus Step 3 of the proof of Theorem 1.18.

4.2.4. *Step 4: Composing with* Δ. The final step in the proof is very easy; it consists simply of composing the injective map (122)

$$\Xi: ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}\to ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}$$

with the injective map Δ to finally obtain a map

$$\Delta\circ\Xi: ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst}\to \Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut})$$

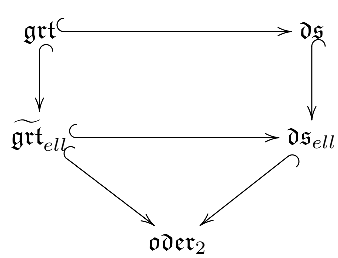
As explained just before paragraph 4.2.1, since by definition the left-hand space is the image of the space $W_{\mathfrak{krv}}$ under the mould map ma, where $W_{\mathfrak{krv}}$ is isomorphic to $\mathfrak{krv}$, and the right-hand space is just the image of $\mathfrak{krv}_{ell}$ under the mould map ma (see diagram (98)), we thus obtain an injective map

$$\mathfrak{krv}\to\mathfrak{krv}_{ell}$$

as shown by the sequence of vertical maps in the right-hand side of the diagram (147) below. This completes the proof of Theorem 1.18. $\square$

4.3. **Relations with elliptic Grothendieck-Teichmüller and double shuffle Lie algebras (Theorem 1.20).** The final result in this paper is the proof of Theorem 1.20. In fact, this result is simply a consequence of putting together the results of the previous sections with known results. Indeed, the commutativity of the diagram

(145)



where $Ad_{ari}(invpal):\mathfrak{ds}\to\mathfrak{ds}_{ell}$ is the right-hand vertical map is shown in [S3].

Let $b = b(x,y)\in\mathfrak{ds}$. By (10), the injective map $\mathfrak{ds}\hookrightarrow\mathfrak{krv}$ sends b to the derivation of Lie$[x,y]$ given by $y\mapsto\hat b(x,y) := b(-x-y,-y)$ and $[x,y]\mapsto 0$ (which determines the value of the derivation on x uniquely). If $b(x,y)\in\mathfrak{ds}$, then $b(x,-y)$ lies in $W_{\mathfrak{krv}}$ and $b(z,-y)$ lies in $V_{\mathfrak{krv}}$, so this map unpacks to

$$\mathfrak{ds}\xrightarrow{y\mapsto -y}W_{\mathfrak{krv}}\xrightarrow{x\mapsto z}V_{\mathfrak{krv}}\longrightarrow\mathfrak{krv},$$

where the last map comes from (38). We can thus construct a commutative square

$$
\begin{array}{ccc}
\mathfrak{ds} & \longrightarrow & \mathfrak{krv} \\
\downarrow & & \downarrow \\
\mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell}
\end{array}
\tag{146}
$$

given in detail by

$$
\begin{array}{ccc}
\mathfrak{ds} & \xhookrightarrow{y\mapsto -y} & W_{\mathfrak{krv}} \simeq \mathfrak{krv} \\
\downarrow{\scriptstyle ma} & & \downarrow{\scriptstyle ma} \\
ARI(\mathcal{F}_{\rm ser})_{\underline{al}*\underline{il}} & \xhookrightarrow{pari} & ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \\
\downarrow{\scriptstyle Ad_{ari}(invpal)} & & \downarrow{\scriptstyle Ad_{ari}(invpal)\circ pari} \\
ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{\underline{al}*\underline{al}} & \hookrightarrow & ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut} \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
\Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{\underline{al}*\underline{al}}) & \hookrightarrow & \Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}) \\
\downarrow{\scriptstyle ma^{-1}} & & \downarrow{\scriptstyle ma^{-1}} \\
\mathfrak{ds}_{ell} & \hookrightarrow & \mathfrak{krv}_{ell}.
\end{array}
\tag{147}
$$

The second line of this diagram is the direct mould translation of the top line, as the left-hand space is exactly $ma(\mathfrak{ds})$, the right-hand space is $ma(W_{\mathfrak{krv}})$ by (116), and the map *pari* restricted to Lie series is nothing other than $y \mapsto -y$. The proof of the vertical morphism

$$Ad_{ari}(invpal) : ARI(\mathcal{F}_{\rm ser})_{\underline{al}*\underline{il}} \to ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{\underline{al}*\underline{al}}$$

has two parts: the fact that $Ad_{ari}(invpal)$ maps $ARI(\mathcal{F}_{\rm Lau})_{\underline{al}*\underline{il}}$ to $ARI(\mathcal{F}_{\rm Lau})_{\underline{al}*\underline{al}}$ is one of the fundamental results of Écalle's mould theory, and follows directly from Écalle's fundamental identity (57) (see [S2], Theorem 4.6.1), while the fact that restricted to $ARI(\mathcal{F}_{\rm ser})_{\underline{al}*\underline{il}}$, the operator $Ad_{ari}(invpal)$ produces denominators at worst Δ was proved in [B], Thm. 4.35. The vertical morphism

$$Ad_{ari}(invpal) \circ pari : ARI(\mathcal{F}_{\rm ser})_{al+tsen*circconst} \hookrightarrow ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut}$$

follows directly from our assumption (113). Since *pari* is an involution, the square formed by the second and third lines of the diagram commutes, where the horizontal inclusion of the third line comes from Corollary 3.46. The fourth horizontal inclusion is obtained simply by applying Δ to the third line. Finally, the last line of the diagram comes from the definitions $ma(\mathfrak{ds}_{ell}) = \Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{\underline{al}*\underline{al}})$ ([S3]) and $ma(\mathfrak{krv}_{ell}) = \Delta(ARI(\mathcal{F}_{\rm Lau})^{\Delta}_{al+push*circneut})$ by Definition 4.6.

This diagram shows that the diagram (145) above can be completed by the diagram (146) to the commutative diagram of Theorem 1.20. □

Appendix A. Proof of Lemma 4.24

Let us recall the statement of the technical lemma 4.24. Recall that for $A \subset \{v_1, \ldots, v_r\}$, we let M_d^A denote the set of all monomials of degree d in the letters of A, and S_d^A the sum of all monomials in M_d^A. Recall from the notation given between the statement of Proposition 4.22 and its proof that for any cyclic permutation permutation $(v_i, \ldots, v_r, v_1, \ldots, v_{i-1})$ of $(v_1, \ldots, v_r)$ and any decomposition of it as $\mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s$ for some $s \geq 1$ (where none of these subsets is allowed to be empty except for possibly $\mathbf{b}_s$), we write $\mathbf{b} = \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_s$ and call this the $\mathbf{b}$-part of the decomposition. Recall also the notation $\mathbf{W}$, $\mathbf{W}^{\mathbf{b}}$ etc. given just after the statement of Proposition 4.22. We will also use the following further notation: for each $0 \leq i \leq r$, let $\mathcal{B}_i$ denote the set of all $\mathbf{b}$-parts (occurring in the sum in (149)) that contain v_i but not $v_{i+1}, \ldots, v_r$; in other words, a given $\mathbf{b}$-part $\mathbf{b} = \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_s$ lies in $\mathcal{B}_i$ if and only if i is the largest index such that v_i occurs in $\mathbf{b}$.

The following are examples for $i = 0, 1, 2$.

Example A.1. We have $\mathcal{B}_0 = \{\emptyset\}$, since a $\mathbf{b}$-part that contains no v_i can only be the empty set; empty $\mathbf{b}$-parts arise in the trivial decompositions

$$\sigma_r^j(\mathbf{v}) = (v_{j+1}, \ldots, v_r, v_1, \ldots, v_j) = \mathbf{a}_1$$

for $0 \leq j \leq r-1$. The set $\mathcal{B}_1$ contains only the single element $\mathbf{b} = (v_1)$, and corresponds to the decompositions

$$\sigma_r^j(\mathbf{v}) = (v_{j+1}, \ldots, v_r, v_1, \ldots, v_j) = (v_{j+1}, \ldots, v_r)(v_1)(v_2, \ldots, v_j) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2$$

for $1 \leq j \leq r-1$ (in fact just $\mathbf{a}_1\mathbf{b}_1$ for $j = 1$). The set $\mathcal{B}_2$ contains two different $\mathbf{b}$-parts, namely (v_2) and (v_1, v_2). The $\mathbf{b}$-part (v_2) occurs in the decompositions

$$\begin{cases} (v_1, \ldots, v_r) = (v_1)(v_2)(v_3, \ldots, v_r) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \\ (v_3, \ldots, v_r, v_1, v_2) = (v_3, \ldots, v_r, v_1)(v_2) = \mathbf{a}_1\mathbf{b}_1 \\ (v_{j+1}, \ldots, v_r, v_1, \ldots, v_j) = (v_{j+1}, \ldots, v_r, v_1)(v_2)(v_3, \ldots, v_j) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \quad \text{for } 3 \leq j \leq r-1. \end{cases}$$

The $\mathbf{b}$-part (v_1, v_2) occurs in the decompositions

$$\begin{cases} (v_3, \ldots, v_r)(v_1, v_2) = \mathbf{a}_1\mathbf{b}_1 \\ (v_j, \ldots, v_r)(v_1, v_2)(v_3, \ldots, v_{j-1}) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \quad \text{for } 4 \leq j \leq r \end{cases}$$

Indeed, for $1 \leq i \leq r-1$, the set $\mathcal{B}_i$ is simply in bijection with the set of all subsets $B \subset \{1, \ldots, i-1\}$, by associating B to the $\mathbf{b}$-part $\{v_j | j \in B\} \cup \{v_i\}$); when $i = r$, $\mathcal{B}_r$ is in bijection with the set of all strict subsets of $\{1, \ldots, r-1\}$.

Lemma 4.24. *For $n > 1$ and any constant $c \neq 0$, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1, \ldots, v_r) = \frac{c}{r} S_{n-r}^{\{v_1, \ldots, v_r\}}.$$

Then T_c^n is circ-constant and $\overline{ganit}(poc) \cdot pari(T_c^n)$ is circ-neutral.

Proof. The mould T_c^n is trivially circ-constant for the value c. For the rest of this proof we set $c = 1$ and $T^n = T_1^n$; it suffices to multiply all identities in the proof below by the constant c to prove the general case.

Let $N = pari(T^n)$. In order to show that $\overline{ganit}(poc) \cdot N$ is circ-neutral, we start by recalling from the beginning of the proof of Proposition 4.22 that for each $r > 1$, the cyclic sum

$$\overline{ganit}(poc) \cdot N(v_1,\dots,v_r) + \cdots + \overline{ganit}(poc) \cdot N(v_r, v_1,\dots,v_{r-1}) \tag{148}$$

is equal to the expression (128)

$$\sum_{\substack{\mathbf{b}\subsetneq\{v_1,\dots,v_r\}\\ \mathbf{b}=\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s}} (-1)^{|\mathbf{a}|} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s) S^{\mathbf{a}}_{n-|\mathbf{a}|}, \tag{149}$$

where the sum runs over all the distinct $\mathbf{b}$-parts that can arise from decomposing the cyclic permutations $\sigma_r^i(\mathbf{v}) = (v_{i+1},\dots,v_r,v_1,\dots,v_i)$ into chunks $\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s$ in which only $\mathbf{b}_s$ can be empty (so in particular $\mathbf{b}=\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s$ cannot be the full set $\{v_1,\dots,v_r\}$). For each term of the sum, $\mathbf{a}$ denotes the subset of $\{v_1,\dots,v_r\}$ which is the complement of the $\mathbf{b}$-part $\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s$.

To prove the Lemma, we will show that (149) is equal to zero for all $r>1$ by breaking up the sum into simpler parts that can be expressed explicitly. To do this, we observe that every $\mathbf{b}$-part corresponding to a decomposition of a cyclic permutation $\sigma_r^j(\mathbf{v})$ lies in a unique $\mathcal{B}_i$. Therefore if we set

$$R_i^r := \sum_{\mathbf{b}\in\mathcal{B}_i} (-1)^{|\mathbf{a}|} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s) S^{\mathbf{a}}_{n-|\mathbf{a}|} \tag{150}$$

for $0\le i\le r$, we can write the sum (149) as

$$\sum_{\substack{\mathbf{b}\subsetneq\{v_1,\dots,v_r\}\\ \mathbf{b}=\mathbf{b}_1\cup\cdots\cup\mathbf{b}_s}} (-1)^{|\mathbf{a}|} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s) S^{\mathbf{a}}_{n-|\mathbf{a}|} = R_0^r + \cdots + R_r^r. \tag{151}$$

We have

$$R_0^r = (-1)^r S^{\{v_1,\dots,v_r\}}_{n-r} = \sum_{j=0}^{r-1} N\big(\sigma_r^j(\mathbf{v})\big), \tag{152}$$

where the first equality comes from (150) and the second from the fact that $N = pari(T^n)$ and T^n is circ-constant. For R_1^r, the only possible $\mathbf{b}$-part is (v_1) and we have

$$R_1^r = \frac{(-1)^r}{v_1 - v_r} S^{\{v_2,\dots,v_r\}}_{n-r+1}. \tag{153}$$

Let us now consider R_i^r for $i>1$. Consider any decomposition

$$\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s \tag{154}$$

of any cyclic permutation of $(v_1,\dots,v_r)$. For any non-empty chunk $\mathbf{b}_j$ of this decomposition, write $\mathbf{b}_j = (v_{k+1}, v_{k+2},\dots,v_l)$, with indices k and l considered mod r from 1 to r (for example $\mathbf{b}_j = (v_{r-1}, v_r, v_1)$ with $k=r-2$ and $l=1$). Then by the definition of *poc* (cf. (56)), we have

$$\begin{aligned} poc(\lfloor\mathbf{b}_j) &= poc(v_{k+1}-v_k, v_{k+2}-v_k,\dots,v_l-v_k)\\ &= \frac{-1}{(v_{k+1}-v_k)(v_{k+1}-v_{k+2})\cdots(v_{l-1}-v_l)}\\ &= \prod_{v_m\in\mathbf{b}_j} \frac{1}{(v_{m-1}-v_m)}, \end{aligned}$$

again with indices m mod r with values from 1 to r. Thus, writing as usual $\mathbf{a}$ for the $\mathbf{a}$-part of a decomposition as in (154), (150) can be written

$$R_i^r = \sum_{\mathbf{b}' \subseteq \{v_1,\ldots,v_{i-1}\}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1} - v_j)} \tag{155}$$

for $1 \le i \le r-1$, where $\mathbf{b}'$ runs over all subsets of $\{v_1, \ldots, v_{i-1}\}$ so $\mathbf{b} = \mathbf{b}' \cup \{v_i\}$ runs over the elements of $\mathcal{B}_i$, and for $i = r$ we have

$$R_r^r = \sum_{\mathbf{b}' \subsetneq \{v_1,\ldots,v_{r-1}\}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1} - v_j)}. \tag{156}$$

We will use these explicit expressions in the proofs of the following two claims, which will allow us to easily conclude the proof of Lemma 4.24. The proofs of the claims are given farther below.

Claim 1. *(i) For $i = 1$, we have*

$$R_1^r = \frac{(-1)^{r-1} S_{n-r+1}^{\{v_2,\ldots,v_r\}}}{(v_r - v_1)}. \tag{157}$$

(ii) For $2 \le i \le r-1$, we have

$$R_i^r = \frac{(-1)^{r-i} S_{n-r+i}^{\{v_{i-1},v_{i+1},\ldots,v_{r-1}\}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{158}$$

(iii) For $i = r$, we have

$$R_r^r = \frac{v_{r-1}^{n-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})}. \tag{159}$$

Claim 2. *For $0 \le i \le r-1$ we have*

$$R_0^r + \cdots + R_i^r = \frac{(-1)^{r-i} S_{n-r+i}^{\{v_i,\ldots,v_{r-1}\}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{160}$$

Accepting these two claims, we observe that by (160) when $i = r-1$, we have

$$R_0^r + \cdots + R_{r-1}^r = \frac{-v_{r-1}^{n-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})}, \tag{161}$$

but this equal to $-R_r^r$ by Claim 1 (iii). This shows that we have $R_0^r + \cdots + R_r^r = 0$. Thus the left-hand side of (151) is equal to zero. But this left-hand side is equal to (149), which in turn is equal to (148), showing that $\overline{ganit}(poc) \cdot N$ is circ-neutral. This completes the proof of Lemma 4.24. □

It remains only to prove Claims 1 and 2. We begin by proving Claim 2 using Claim 1, then give an elementary (but surprisingly complicated) proof of Claim 1.

Proof of Claim 2. We first note the following trivial but useful identity. Recall the notation $V_m = \{v_1, \ldots, v_m\}$ for $1 \le m \le r$. Let $A' \subsetneq V_r$, let $v_j \notin A'$, and let $A = A' \cup \{v_j\}$: then we have the useful identity

$$S_{d+1}^{A'} + v_j S_d^A = S_{d+1}^A. \tag{162}$$

Indeed, the first term is the sum of all monomials of degree $d+1$ in the elements of A', and the second is the sum of all monomials of degree $d+1$ in the letters of A that contain v_j, so their sum forms the sum of all monomials of degree $d+1$ in the letters of A.

We will prove Claim 2 by induction on i. The base case $i=0$ is given by (152). Now let $1 \le i \le r-1$ and assume (160) up to $i-1$. Then by the induction hypothesis and Claim 1, we have

$$R_0^r + \cdots + R_i^r = (R_0^r + \cdots + R_{i-1}^r) + R_i^r$$

$$= \frac{(-1)^{r-i+1} S_{n-r+i-1}^{\{v_{i-1},\ldots,v_{r-1}\}}}{(v_r-v_1)(v_1-v_2)\cdots(v_{i-2}-v_{i-1})} + \frac{(-1)^{r-i} S_{n-r+i}^{\{v_{i-1},v_{i+1},\ldots,v_{r-1}\}}}{(v_r-v_1)(v_1-v_2)\cdots(v_{i-1}-v_i)}$$

$$= \frac{(-1)^{r-i+1}\Big(v_{i-1} S_{n-r+i-1}^{\{v_{i-1},\ldots,v_{r-1}\}} - v_i S_{n-r+i-1}^{\{v_{i-1},\ldots,v_{r-1}\}} - S_{n-r+i}^{\{v_{i-1},v_{i+1},\ldots,v_{r-1}\}}\Big)}{(v_r-v_1)(v_1-v_2)\cdots(v_{i-1}-v_i)}.$$

By (162), the second term and third terms in the numerator sum to $-S_{n-r+i}^{\{v_{i-1},\ldots,v_{r-1}\}}$, so the numerator becomes $(-1)^{r-i+1}\Big(v_{i-1} S_{n-r+i-1}^{\{v_{i-1},\ldots,v_{r-1}\}} - S_{n-r+i}^{\{v_{i-1},\ldots,v_{r-1}\}}\Big)$ which again by (162) sums to $(-1)^{r-i}\Big(S_{n-r+i}^{\{v_i,\ldots,v_{r-1}\}}\Big)$. Thus we have

$$R_0^r + \cdots + R_i^r = \frac{(-1)^{r-i} S_{n-r+i}^{\{v_i,\ldots,v_{r-1}\}}}{(v_r-v_1)(v_1-v_2)\cdots(v_{i-1}-v_i)},$$

which proves Claim 2. □

Finally, we proceed to the proof of Claim 1.

Proof of Claim 1. (i) When $i=1$ we have $\mathcal{B}_1 = \{v_1\}$, so here there is only one term in the sum (155) corresponding to $\mathbf{b}' = \emptyset$, $\mathbf{b} = \{v_1\}$, $\mathbf{a} = \{v_2,\ldots,v_r\}$, $|\mathbf{a}| = r-1$, so that (155) for $i=1$ comes down to (157).

(ii) Let $V_m = \{v_1,\ldots,v_m\}$ for $1 \le m \le r$. Fix a value of i with $2 \le i \le r-1$. Recall that R_i^r was given in (155) as the sum

$$R_i^r = \sum_{\mathbf{b}' \subseteq V_{i-1}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1}-v_j)}. \tag{163}$$

where $\mathbf{b} = \mathbf{b}' \cup \{v_i\}$ and $\mathbf{a}$ is the complement of $\mathbf{b}$ in V_r.

Multiplying R_i^r by the common denominator $(v_r-v_1)\cdots(v_{i-1}-v_i)$ and setting $v_0 = v_r$ as usual, we rewrite (163) as

$$\prod_{j=1}^{i} (v_{j-1}-v_j) R_i^r = \sum_{\mathbf{b}' \subseteq V_{i-1}} (-1)^{|\mathbf{a}|} \prod_{v_j \in V_{i-1} \setminus \mathbf{b}'} (v_{j-1}-v_j) S_{n-|\mathbf{a}|}^{\mathbf{a}}. \tag{164}$$

To conclude the proof, we need one more claim.

Claim 3. *For each pair i,k with $1<i<r$ and $1\le k\le i-1$, define the polynomial Q^i_k by*

$$\sum_{v_1,\dots,v_k\notin B'\subset V_{i-1}}(-1)^{r-|B'|+k-1}\Bigg(\prod_{v_j\in V_{i-1}\setminus(B'\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)S^{V_r\setminus(B'\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B'|+k+1}$$

Let Q^i_0 denote the right-hand side of (164). *Then we have*

$$Q^i_0=Q^i_1=Q^i_2=\cdots=Q^i_{i-1}.$$

Proof of claim 3. Let us show that for $0\le k\le i-2$ we have $Q^i_k=Q^i_{k+1}$. We break the expression for Q^i_k into the terms with $v_{k+1}\in B'$ and those with $v_{k+1}\notin B'$, writing Q^i_k as

$$\sum_{\substack{v_1,\dots,v_k\notin B'\subseteq V_{i-1}\\ v_{k+1}\in B'}}(-1)^{r-|B'|+k-1}\Bigg(\prod_{v_j\in V_{i-1}\setminus(B'\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)S^{V_r\setminus(B'\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B'|+k+1}$$

$$+\sum_{v_1,\dots,v_{k+1}\notin B'\subseteq V_{i-1}}(-1)^{r-|B'|+k-1}\Bigg(\prod_{v_j\in V_{i-1}\setminus(B'\cup\{v_1,\dots,v_k\})}(v_{j-1}-v_j)\Bigg)S^{V_r\setminus(B'\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B'|+k+1}.$$

We will use an analogous decomposition for Q^i_0, breaking it up as

$$\prod_{j=1}^{i}(v_{j-1}-v_j)R^r_i=\sum_{v_1\in B'\subseteq V_{i-1}}(-1)^{|A|}\prod_{v_j\in V_{i-1}\setminus B'}(v_{j-1}-v_j)S^A_{n-|A|}$$
$$+\sum_{v_1\notin B'\subseteq V_{i-1}}(-1)^{|A|}\prod_{v_j\in V_{i-1}\setminus B'}(v_{j-1}-v_j)S^A_{n-|A|},$$

where $B'=\mathbf{b}'$ and $A=\mathbf{a}$ (to harmonize the notation with the Q^i_k), and the v_0 that occurs in the second line is equal to v_r as in (164).

Now fix $k\in\{0,\dots,i-2\}$, and write $B'':=B'\setminus\{v_{k+1}\}$ in the first line of the decomposition of Q^i_k, and simply replace the notation B' by B'' in the second line, obtaining

$$\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k}\Bigg(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_{k+1},v_i,v_r\})}_{n-r+|B''|+k+2}$$

$$+\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k-1}(v_k-v_{k+1})\times$$
$$\Bigg(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+1},$$

except when $k=0$, where (using $|A|=r-|B''|-1$), the S-factors are $S^{V_r\setminus(B''\cup\{v_1,v_i\}}_{n-r+|B''|+1}$ in the first line and $S^{V_r\setminus(B''\cup\{v_i\})}_{n-r+|B''|+1}$ in the second.

Now we gather the terms, writing this as

$$Q^i_k=\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k}\Bigg(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Bigg)\times$$
$$\Bigg(S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_{k+1},v_i,v_r\}))}_{n-r+|B''|+k+2}-(v_k-v_{k+1})S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+1}\Bigg),$$

or in the case $k=0$,

$$Q_0^i = \sum_{B''\subseteq V_{i-1}} (-1)^{r-|B''|}\Big(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1\})}(v_{j-1}-v_j)\Big)\times$$
$$\left(S^{V_r\setminus(B''\cup\{v_1,v_i\}))}_{n-r+|B''|+2} - (v_r-v_1)S^{V_r\setminus(B''\cup\{v_i\})}_{n-r+|B''|+1}\right),$$

We will now use (162) twice to simplify the right-hand factor. For $1\le k\le i-2$, we take $A'=V_r\setminus(B''\cup\{v_1,\ldots,v_{k-1}\}\cup\{v_{k+1},v_i,v_r\})$ and $A=A'\cup\{v_{k+1}\}$, while for $k=0$ we take $A'=V_r\setminus(B''\cup\{v_1,v_i\})$ and $A=A'\cup\{v_1\}$. We can then expand the right-hand factor as

$$S^{A'}_{n-r+|B''|+k+2} - v_k S^A_{n-r+|B''|+k+1} + v_{k+1}S^A_{n-r+|B''|+k+1}$$

(recalling that when $k=0$, $v_0=v_r$). Applying (162), this simplifies to

$$S^A_{n-r+|B''|+k+2} - v_k S^A_{n-r+|B''|+k+1}.$$

Next, since $v_k\notin B''$ we must have $v_k\in A$, so by applying (162) again we see that this simplifies to

$$S^{A\setminus\{v_k\}}_{n-r+|B''|+k+2} = S^{V_r\setminus(B''\cup\{v_1,\ldots,v_k\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+2}.$$

Thus we can rewrite Q_k^i as

$$\sum_{v_1,\ldots,v_{k+1}\notin B''\subseteq V_{i-1}} (-1)^{r-|B''|+k}\Big(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\ldots,v_{k+1}\})}(v_{j-1}-v_j)\Big)S^{V_r\setminus(B''\cup\{v_1,\ldots,v_k\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+2.}$$

But according to the definition of the polynomials Q_k^i for $1\le k\le i-1$, this is exactly equal to Q_{k+1}^i. Thus we find that $Q_0^i=Q_1^i=\cdots=Q_{i-1}^i$, completing the proof of Claim 3. ☐

It remains only to prove part (iii) of Claim 1.

(iii) In this final part we have to prove that

$$\prod_{j=1}^{r-1}(v_{j-1}-v_j)R_r^r = v_{r-1}^{n-1}. \tag{165}$$

Recall from (156) that R_r^r is given by the formula

$$R_r^r = \sum_{\mathbf{b}'\subsetneq V_{r-1}} \frac{(-1)^{|\mathbf{a}|}S^{\mathbf{a}}_{n-|\mathbf{a}|}}{\prod_{v_j\in\mathbf{b}}(v_{j-1}-v_j)},$$

where $\mathbf{b}=\mathbf{b}'\cup\{v_r\}$. Thus the common denominator of all the terms in the sum is $(v_r-v_1)(v_1-v_2)\cdots(v_{r-1}-v_r)$, and we have

$$\prod_{j=1}^{r}(v_{j-1}-v_j)R_r^r = \sum_{\mathbf{b}'\subsetneq V_{r-1}}(-1)^{r-|\mathbf{b}'|-1}\prod_{v_j\in V_{r-1}\setminus\mathbf{b}'}(v_{j-1}-v_j)\ S^{V_{r-1}\setminus\mathbf{b}'}_{n-r+|\mathbf{b}'|+1}. \tag{166}$$

Let us write $\mathbf{c}=V_{r-1}\setminus\mathbf{b}'$, so this equality can be expressed as

$$\prod_{j=1}^{r}(v_{j-1}-v_j)R_r^r = \sum_{\emptyset\ne\mathbf{c}\subseteq V_{r-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|}. \tag{167}$$

For $1 \le i \le r-1$ and $n \ge 1$, define the sum T_i^n by

$$T_i^n := \sum_{\emptyset \neq \mathbf{c} \subseteq V_i} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S_{n-|\mathbf{c}|}^{\mathbf{c}},$$

where we set $S_0^{\mathbf{c}} = 1$ and $S_m^{\mathbf{c}} = 0$ if $m < 0$. By this definition, the term T_{r-1}^n is equal to the right-hand side of (167). We will prove that

$$T_i^n = (v_i - v_r) v_i^{n-1}. \tag{168}$$

The equality (168) suffices to prove the desired result (165). Indeed, since T_{r-1}^n is equal to the right-hand side of (167), the left-hand side of (167) is equal to the right-hand side of (168) with $i = r-1$, i.e.

$$\prod_{j=1}^{r} (v_{j-1} - v_j) R_r^r = T_{r-1}^n = (v_{r-1} - v_r) v_{r-1}^{n-1}.$$

Canceling out the factor $(v_r - v_{r-1})$ from both sides yields the desired identity (165).

It remains only to prove (168). We proceed by induction on i. When $i = 1$, we have $\mathbf{c} = \{v_1\}$ and for all $n \ge 1$, we have

$$T_1^n = -(v_r - v_1) S_{n-1}^{v_1} = (v_1 - v_r) v_1^{n-1},$$

proving the base case.

Fix $n \ge 1$ and assume (168) holds for $i-1$. We break T_i^n into the sum over $\mathbf{c}$ containing v_i and $\mathbf{c}$ not containing v_i, as follows:

$$\begin{aligned} T_i^n = & \sum_{\emptyset \neq \mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S_{n-|\mathbf{c}|}^{\mathbf{c}} \\ & + \sum_{\substack{\mathbf{c} \subseteq V_{i-1} \\ \mathbf{c}' = \mathbf{c} \cup \{v_i\}}} (-1)^{|\mathbf{c}'|} (v_{i-1} - v_i) \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S_{n-|\mathbf{c}'|}^{\mathbf{c}'} \\ = & \; T_{i-1}^n + (v_{i-1} - v_i) \sum_{\mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|+1} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S_{n-|\mathbf{c}|-1}^{\mathbf{c} \cup \{v_i\}} \\ = & \; T_{i-1}^n - (v_{i-1} - v_i) v_i^{n-1} - (v_{i-1} - v_i) \sum_{\emptyset \neq \mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S_{n-|\mathbf{c}|-1}^{\mathbf{c} \cup \{v_i\}}, \end{aligned}$$

where the last line comes from separating the sum over $\mathbf{c} \subseteq V_{i-1}$ into $\mathbf{c} = \emptyset$ (giving the extra term $(v_{i-1} - v_i) v_i^{n-1}$) and the sum over $\mathbf{c} \neq \emptyset$. Since $\mathbf{c}$ does not contain v_i, we can write

$$S_{n-|\mathbf{c}|-1}^{\mathbf{c} \cup \{v_i\}} = S_{n-|\mathbf{c}|-1}^{\mathbf{c}} + v_i S_{n-|\mathbf{c}|-2}^{\mathbf{c}} + v_i^2 S_{n-|\mathbf{c}|-3}^{\mathbf{c}} + \cdots + v_i^{n-|\mathbf{c}|-2} S_1^{\mathbf{c}} + v_i^{n-|\mathbf{c}|-1}.$$

Using this, the above equality becomes

$$= T^n_{i-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)\times$$

$$\left(S^{\mathbf{c}}_{n-|\mathbf{c}|-1} + v_i S^{\mathbf{c}}_{n-|\mathbf{c}|-2} + v_i^2 S^{\mathbf{c}}_{n-|\mathbf{c}|-3} + \cdots + v_i^{n-|\mathbf{c}|-2}S^{\mathbf{c}}_1 + v_i^{n-|\mathbf{c}|-1}\right)$$

$$= T^n_{i-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\sum_{k=0}^{n-|\mathbf{c}|-1}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|-1-k}v_i^k$$

$$= T^n_{i-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|-1-k}.$$

If $n-|\mathbf{c}|-1-k<0$ then $S^{\mathbf{c}}_{n-|c|-1-k}=0$, so this is equal to

$$T^n_{i-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k T^{n-k-1}_{i-1}$$

which then by induction is equal to

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k(v_{i-1}-v_r)v_{i-1}^{n-k-2}$$

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)(v_{i-1}-v_r)\sum_{k=0}^{n-2}v_i^k v_{i-1}^{n-k-2}$$

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_r)(v_{i-1}^{n-1}-v_i^{n-1})$$

$$= (v_i-v_r)v_i^{n-1}.$$

This proves (168) and thus completes the proof of Claim 1 (iii). □

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