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On a new version of the Grothendieck-Teichmüller group

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Abstract – In this note we introduce a certain subgroup $\mathbb{I}\Gamma$ of the Grothendieck-Teichmüller group $\widehat{GT}$, obtained by adding two new relations to the definition of $\widehat{GT}$. We show that $\mathbb{I}\Gamma$ gives an automorphism group of the profinite completions of certain surface mapping class groups with geometric compatibility conditions, and that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is embedded into $\mathbb{I}\Gamma$.

Sur une nouvelle version du groupe de Grothendieck-Teichmüller

Résumé – Nous montrons qu'un certain sous-groupe $\mathbb{I}\Gamma$ du groupe de Grothendieck-Teichmüller $\widehat{GT}$, obtenu en ajoutant à la définition de $\widehat{GT}$ deux nouvelles relations, fournit un groupe d'automorphismes de groupes modulaires de Teichmüller profinis associés à des surfaces de genre quelconque. Ces automorphismes respectent des relations de nature géométrique entre les groupes modulaires, et de plus le groupe de Galois absolu $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ s'injecte dans $\mathbb{I}\Gamma$ de manière naturelle.

Version française abrégée – Soit Σ une surface compacte éventuellement à bord, munie d'un nombre fini de points marqués. On écrit $\Sigma = \Sigma_{g,n}$ dans le cas d'une surface de genre g sans bord à n points marqués. Soit $\Gamma_{g,[n]}$ le groupe modulaire ("mapping class group") de $\Sigma_{g,n}$, c'est-à-dire le groupe des classes de difféomorphismes permutant les points marqués, modulo isotopie, et notons $\Gamma_{g,n} = \Gamma(\Sigma_{g,n})$ son sous-groupe coloré formé des classes ne permutant pas les points marqués. On définit la généralisation $\Gamma(\Sigma)$ de $\Gamma_{g,n}$ à tout Σ comme le groupe des classes de difféomorphismes ne permutant ni les points marqués ni les composantes de bord, modulo isotopie fixant le bord.

Le groupe $\Gamma_{0,[n]}$ est engendré par les générateurs usuels $\sigma_1, \ldots, \sigma_{n-1}$ du groupe de tresses planes à n brins, et le sous-groupe coloré $\Gamma_{0,n}$ par les éléments $x_{ij} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$

pour $1 \leq i < j \leq n$. On convient que $x_{ji} = x_{ij}$. On note $\widehat{G}$ le complété profini d'un groupe G. En particulier, le groupe $\widehat{\Gamma}_{0,4}$ est prolibre, topologiquement engendré par x_{12} et x_{23}; on l'identifiera au sous-groupe de $\widehat{\Gamma}_{0,5}$ engendré par les éléments de mêmes noms. Notons que comme générateurs de $\widehat{\Gamma}_{0,5}$, on peut prendre 5 éléments $x_{i,i+1}$ (les indices étant entendus modulo 5). Si ϕ est un homomorphisme de $\widehat{\Gamma}_{0,4}$ dans un groupe (profini) G donné par $\phi(x_{12}) = a$, $\phi(x_{23}) = b$, nous écrirons $\phi(f) = f(a,b)$ pour $f \in \widehat{\Gamma}_{0,4}$. Enfin on note $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Définition 1. – Soit $\underline{\widehat{GT}}$ l'ensemble formé des couples $F = (\lambda, f) \in \widehat{\mathbb{Z}}^* \times [\widehat{\Gamma}_{0,4}, \widehat{\Gamma}_{0,4}]$ vérifiant les trois relations suivantes:

(I) $f(x_{23}, x_{12})f(x_{12}, x_{23}) = 1$,

(II) $f(x_{12}, x_{23})x_{12}^m f(x'_{13}, x_{12}){x'_{13}}^m f(x_{23}, x'_{13})x_{23}^m = 1$ avec $m = (\lambda-1)/2$ et $x'_{13} = (x_{12}x_{23})^{-1}$,

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$.

Un élément $F \in \underline{\widehat{GT}}$ induit un endomorphisme de $\widehat{\Gamma}_{0,4}$ donné par $F(x_{12}) = x_{12}^{\lambda}$ et $F(x_{23}) = f^{-1}x_{23}^{\lambda}f$, et la composition de ces endomorphismes fait de $\underline{\widehat{GT}}$ un monoïde. Le groupe de Grothendieck-Teichmüller $\widehat{GT}$, introduit par Drinfeld dans [1], est défini comme les éléments inversibles de $\underline{\widehat{GT}}$. On a une injection $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ obtenue à partir de l'identification de $\widehat{\Gamma}_{0,4}$ avec le groupe fondamental algébrique $\pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$, basé au point base tangentiel '$\overrightarrow{01}$' (Belyi, Drinfeld; voir Ihara [5]).

On montre dans [7] que pour tout $F = (\lambda, f) \in \widehat{GT}$, il existe un unique $g \in \widehat{\Gamma}_{0,4}$ tel que $f(x_{12}, x_{23}) = g(x_{23}, x_{12})^{-1}g(x_{12}, x_{23})$. Par analogie avec les calculs de [8-9] dans le cas galoisien, on est conduit à introduire deux relations supplémentaires d'un type nouveau, qui suggèrent une nouvelle version du groupe de Grothendieck-Teichmüller:

Définition 2. – Soit σ_i, $1 \leq i \leq 4$, les générateurs du groupe $\Gamma_{0,[5]}$. Soit $\mathbb{I}\Gamma \subset \widehat{GT}$ l'ensemble des $F = (\lambda, f)$ tels que

(III′) $g(x_{45}, x_{51})f(x_{12}, x_{23})f(x_{34}, x_{45}) = f(\sigma_1\sigma_3, \sigma_2^2)$;

(IV) $f(\sigma_1, \sigma_2^4) = \sigma_2^{8\rho_2(F)}f(\sigma_1^2, \sigma_2^2)\sigma_1^{4\rho_2(F)}(\sigma_1\sigma_2)^{-6\rho_2(F)}$,

où $\rho_2(F) \in \widehat{\mathbb{Z}}$ est défini par $g(x,y) \equiv (xy)^{-\rho_2(F)}$ dans $\widehat{\Gamma}_{0,4}^{\mathrm{ab}} \simeq \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$.

La dénomination (III′) vient de ce que ladite relation implique la relation (III) de la définition 1. Soit en effet η l'automorphisme de $\widehat{\Gamma}_{0,[5]}$ donné par $\sigma_1 \mapsto \sigma_3$, $\sigma_2 \mapsto \sigma_2$, $\sigma_3 \mapsto \sigma_1$, $\sigma_4 \mapsto \sigma_{15}$, qui est donné par une symétrie par rapport à un axe, en considérant cinq points équidistants sur un cercle. Cet automorphisme fixe $f(\sigma_1\sigma_3, \sigma_2^2)$, de sorte que si G désigne le membre de gauche de (III′), on a l'identité $\eta(G)^{-1}G = 1$. On remarque alors que $\eta(G)^{-1}G$ n'est autre que le membre de gauche de (III).

Soit $\mathcal{P}$ la "découpe en pantalons" de $\Sigma_{g,1}$ définie par les lacets δ_i, δ'_i et ϵ_i de la figure 1 ($1 \leq i \leq g$; voir version anglaise); autrement dit $\Sigma_{g,1} = \bigcup_{i=1}^{2g-1} P_i$ où les P_i sont les "pantalons" (sphères à trois trous) délimités par ces lacets.

Théorème Principal. – *(i)* $\mathbb{I}\Gamma$ *est un sous-groupe de* $\widehat{GT}$ *qui contient l'image de* $G_{\mathbb{Q}}$ *dans* $\widehat{GT}$.

(ii) Il existe une action naturelle et fidèle du groupe $\widehat{GT}$ *sur* $\widehat{\Gamma}_{g,1} = \widehat{\Gamma}(\Sigma_{g,1})$.

(iii) (a) Cette action de $\widehat{GT}$ *préserve l'image de* $\widehat{\Gamma}(P_i) \to \widehat{\Gamma}(\Sigma_{g,1})$ *pour tous les* P_i *de* $\mathcal{P}$*;*

(b) L'action induite du sous-groupe $\mathbb{I}\Gamma$ *préserve l'image de* $\widehat{\Gamma}(P_i \cup P_j) \to \widehat{\Gamma}(\Sigma_{g,1})$ *pour tous les* P_i, P_j *de* $\mathcal{P}$*,* $i \neq j$.

En accord avec l'*Esquisse d'un programme* de Grothendieck ([3]), les résultats ci-dessus suggèrent l'existence d'une "tour de Teichmüller" construite avec la collection des groupes $\widehat{\Gamma}_{g,n}$ pour $2-2g-n<0$, reliés entre eux par des homomorphismes "provenant de la géométrie", et telle que le groupe des automorphismes de cette tour (automorphismes des groupes respectant les homomorphismes qui les relient) soit précisément $\mathbb{\Gamma}$. Notons enfin que nous laissons ouverte la question de savoir si les groupes $\mathbb{\Gamma}$ et $\widehat{GT}$ sont effectivement différents.

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1. Introduction. – Let Σ denote a compact oriented topological surface of genus g, possibly with boundary, equipped with n marked points; we write $\Sigma = \Sigma_{g,n}$ when there is no boundary component. Let $\Gamma_{g,[n]}$ denote the mapping class group of $\Sigma_{g,n}$, and $\Gamma_{g,n} = \Gamma(\Sigma_{g,n})$ its pure subgroup. For any Σ, let $\Gamma(\Sigma)$ denote the pure mapping class group of Σ.

Let $\widehat{GT}$ be the Grothendieck-Teichmüller group defined by Drinfel'd in [1] (and recalled in Définition 1 above). Then it is known that $\widehat{GT}$ acts naturally and faithfully on the profinite groups $\widehat{\Gamma}_{0,n}$ and $\widehat{\Gamma}_{0,[n]}$, respecting certain homomorphisms between these groups. The goal of this Note is to extend this result to some higher genus mapping class groups and homomorphisms between them. In doing so, we are led to add two new relations to $\widehat{GT}$, forming the subgroup $\mathbb{\Gamma}$ of $\widehat{GT}$ given in Définition 2 above. Our main result is as follows. Let $\mathcal{P}$ denote the pants decomposition of $\Sigma_{g,1}$ cut out by the loops δ_i, δ_i' and ϵ_i of figure 1 below, for $1 \le i \le g$.

Main Theorem. – *(i)* $\mathbb{\Gamma}$ *forms a subgroup of* $\widehat{GT}$ *and contains the image of* $G_{\mathbb{Q}}$ *in* $\widehat{GT}$.

(ii) $\widehat{GT}$ *acts naturally and faithfully on* $\widehat{\Gamma}_{g,1} = \widehat{\Gamma}(\Sigma_{g,1})$.

(iii) (a) $\widehat{GT}$ *preserves the image of* $\widehat{\Gamma}(P_i) \to \widehat{\Gamma}(\Sigma_{g,1})$ *for all* P_i *in* $\mathcal{P}$*;*

(b) $\mathbb{\Gamma}$ *preserves the image of* $\widehat{\Gamma}(P_i \cup P_j) \to \widehat{\Gamma}(\Sigma_{g,1})$ *for all* P_i, P_j *in* $\mathcal{P}$.

Figure 1

2. Main Theorem (ii): $\widehat{GT}$ **is an automorphism group of** $\widehat{\Gamma}_{g,1}$**.** – The following theorem gives a $\widehat{GT}$-action on $\Gamma_{g,1}$ extending the known $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action given in [8].

Theorem 2.1. – *Each element* $F = (\lambda, f) \in \widehat{GT}$ *acts on* $\widehat{\Gamma}_{g,1}$ *via*

$$F(d_2) = d_2^{\lambda} \quad \text{and} \quad F(a_i) = f(a_i^2, y_i)\, a_i^{\lambda} f(y_i, a_i^2) \quad \text{for} \quad 1 \le i \le 2g.$$

Here $y_1 = 1$ *(so* $F(a_1) = a_1^{\lambda}$*) and* $y_i = a_{i-1}a_{i-2}\cdots a_1 a_1 \cdots a_{i-2}a_{i-1}$.

The proof of theorem 2.1 is an involved computation using the defining relations of $\widehat{GT}$ and the presentation of the mapping class group $\Gamma_{g,1}$ given in the following proposition. Note that we use Roman letters for Dehn twists along loops given by Greek letters; in particular the generators given in the proposition are Dehn twists along the corresponding loops in figure 1.

Proposition 2.2. – *Let $g > 1$. The following is a presentation for the mapping class group $\Gamma_{g,1}$.*

Generators: $a_1, a_2, ..., a_{2g}, d_2$

Relations: (A) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *for* $i = 1, \ldots, 2g-1$
$d_2 a_4 d_2 = a_4 d_2 a_4$
a_i *commutes with* a_j *for* $|i-j| \geq 2$
d_2 *commutes with* a_j *for* $j \neq 4$;
(B) $d_2 d_2' = (a_1 a_2 a_3)^4$;
(C) (only needed for $g \geq 3$*)* $d_3 a_6 d_3 = a_6 d_3 a_6$;
(D) (only needed for $g \geq 4$*)* d_2' *commutes with* $t_2^{-1} t_3^{-1} t_1^{-1} t_2^{-1} d_2 t_2 t_1 t_3 t_2$,

where $t_1 := a_2 a_1 a_3 a_2$, $t_2 := a_4 a_3 a_5 a_4$, $t_3 := a_6 a_5 a_7 a_6$, $g_1 := t_2^{-1} d_2 t_2$, $g_2 := t_1^{-1} t_2^{-1} d_2 t_2 t_1$, $d_3 := g_2 g_1 d_2 a_1^{-1} a_3^{-1}\ a_5^{-1}$, $y_5 := a_4 a_3 a_2 a_1^2 a_2 a_3 a_4$, $d_2' = y_5^{-1} d_2 y_5$.

Proof. This result can be deduced from Wajnryb's presentation in [10], by observing that his relation (C) is used in his equations (18) and (30), and only to obtain relations (D) and (C) above respectively. □

3. Main Theorem (i): There is an injection $G_{\mathbb{Q}} \hookrightarrow \mathbb{I}\Gamma$. – To begin with, we indicate why $\mathbb{I}\Gamma$ is a subgroup of $\widehat{GT}$, i.e. why the conditions (III′) and (IV) are closed under the multiplication of $\widehat{GT}$. For (III′), the statement follows by a direct calculation using the multiplication laws for f and g. As for (IV), we use the fact that ρ_2 gives a 1-cocycle $\widehat{GT} \to \hat{\mathbb{Z}}$. More precisely, if $F = (\lambda, f)$ and $F' = (\lambda', f')$ are elements of $\widehat{GT}$, one has $\rho_2(FF') = \rho_2(F) + \lambda\rho_2(F')$. Using this it is not difficult to see that (IV) is a closed condition under the multiplication of $\widehat{GT}$.

Let us now consider the injection of the absolute Galois group. To any $\sigma \in G_{\mathbb{Q}}$, we associate as in [5] an element $F_\sigma = (\lambda_\sigma, f_\sigma) \in \widehat{GT}$; in particular $\lambda_\sigma = \chi(\sigma)$ where χ denotes the cyclotomic character. In this section, composition is written from left to right.

Lemma 3.1. – *The cocycle ρ_2 coincides with a Kummer cocycle on the elements of $G_{\mathbb{Q}}$. Namely, for each $\sigma \in G_{\mathbb{Q}}$, $\rho_2(\sigma) = \rho_2(F_\sigma)$ is given for $n \geq 1$ by $\sigma(\sqrt[n]{2}) = \exp(\frac{2\pi i}{n})^{\rho_2(\sigma)} \sqrt[n]{2}$, where $\sqrt[n]{2}$ lies in $\mathbb{R}_+$.*

Proof. Let r be the path along the real axis on $\mathbb{P}^1_t(\mathbb{C})$ from the tangential base point $\overrightarrow{01}$ to $t = 1/2$. Defining $g_\sigma(x, y)$ by $\sigma(r) = g_\sigma(x, y)^{-1} r$ (here x, y are standard loops running around $0, 1 \in \mathbb{P}^1$ respectively) as in [7], we have $f_\sigma(x, y) = g_\sigma(y, x)^{-1} g_\sigma(x, y)$. The lemma follows from observing how the algebraic functions $t^{1/n}$ and $(1-t)^{1/n}$ are analytically continued from $\overrightarrow{01}$ to $1/2$ along the two (pro)paths r and $\sigma(r)$. This argument is a variant of Ihara [5] Prop.1.5 on f_σ. □

In order to prove that there is an injection $G_{\mathbb{Q}} \hookrightarrow \mathbb{I}\Gamma$, it suffices to show that the image F_σ of $\sigma \in G_{\mathbb{Q}}$ via $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ satisfies (III′) and (IV). The validity of condition (IV) was proved in [9] Th.4.16, so that we have only to show that (III′) holds for the F_σ. Let $M_{g,n}$ denote the moduli stack of smooth n-pointed curves of genus g. In [8], a tangential base point on $M_{g,1}$ is constructed by linearly patching g copies of the Tate elliptic curve $\mathrm{Tate}(q^2)$ of level 2, and it defines a Galois action on $\widehat{\Gamma}_{g,1}$ such that

$$d_2 \mapsto d_2^{\lambda_\sigma} \quad \text{and} \quad a_i \mapsto w^{-4\rho_2(\sigma)} f_\sigma(a_i^2, w_i) a_i^{\lambda_\sigma} f_\sigma(w_i, a_i^2) w^{4\rho_2(\sigma)} \quad \text{for} \quad 1 \le i \le 2g, \quad \sigma \in G_{\mathbb{Q}},$$

where $w_1 = 1, w_i = y_2 \cdots y_i$ and $w = \prod_i w_{2i}$. Collapsing the loop ϵ_{g-1} of figure 1 to a point, one cuts off a subsurface of type $\Sigma_{1,2}$, with mapping class group $\Gamma_{1,2}$ generated by the Dehn twists d_g, a_{2g}, d'_g of $\Gamma_{g,1}$ (see figure 1). The group $\Gamma_{1,2}$ is isomorphic to $B_4/\langle \omega_4 \rangle \subset \Gamma_{0,[5]}$, with standard generators $\sigma_1, \sigma_2, \sigma_3$. The aforementioned base point on $M_{g,1}$ specializes to one on $M_{1,2}$, which gives rise to the following Galois action on $\widehat{\Gamma}_{1,2} \simeq \widehat{B}_4/\langle \omega_4 \rangle$:

$$\sigma_i \mapsto \sigma_i^{\lambda_\sigma} \ (i=1,3), \quad \sigma_2 \mapsto (\sigma_1\sigma_3)^{-4\rho_2(\sigma)} f_\sigma(\sigma_2^2, \sigma_1\sigma_3) \sigma_2^{\lambda_\sigma} f_\sigma(\sigma_1\sigma_3, \sigma_2^2)(\sigma_1\sigma_3)^{4\rho_2(\sigma)} \quad (\sigma \in G_{\mathbb{Q}}).$$

On the other hand, from [9], we obtain another tangential base point on $M_{1,2}$ lying in the fibre $\mathrm{Tate}(q^2)$ over '$\frac{1}{16}\overrightarrow{01}$' on $M_{1,1}^{\text{level } 2} \approx \mathbb{P}^1 - \{0, 1, \infty\}$ which gives the following Galois action on the same group: for $\sigma \in G_{\mathbb{Q}}$, we have

$$\sigma_i \mapsto \sigma_i^{\lambda_\sigma} \ (i=1,3), \quad \sigma_2 \mapsto f_\sigma(x_{45}^2, \sigma_3) \sigma_1^{-8\rho_2(\sigma)} f_\sigma(\sigma_2^2, \sigma_1^2) \sigma_2^{\lambda_\sigma} f_\sigma(\sigma_1^2, \sigma_2^2) \sigma_1^{8\rho_2(\sigma)} f_\sigma(\sigma_3, x_{45}^2).$$

After careful comparisons of these two Galois actions and basepoints in the case of $g = 2$, by using relation (IV), we obtain

$$f_\sigma(\sigma_1\sigma_3, \sigma_2^2) f_\sigma(x_{45}, x_{34}) f_\sigma(x_{23}, x_{12}) \in \langle x_{45}, x_{51} \rangle.$$

By symmetry of $\widehat{\Gamma}_{0,[5]}$ given by $\sigma_1 \mapsto \sigma_3$, $\sigma_2 \mapsto \sigma_2$, $\sigma_3 \mapsto \sigma_1$, $\sigma_4 \mapsto \sigma_{15}$, this turns out to be $g_\sigma(x_{45}, x_{51})$, which completes the proof of Main Theorem (i). □

4. Main Theorem (iii): Compatibility with a pants-decomposition. – In Theorem 2.1, we obtained a $\widehat{GT}$-action on $\widehat{\Gamma}_{g,1}$. Now we shall prove that this $\widehat{GT}$-action has the properties of (iii) of the Main Theorem. We consider the decomposition into pants P_i of $\Sigma_{g,1}$ shown in figure 1, and write as usual d_i, d'_i, e_i for the Dehn twists along the cutting loops δ_i, δ'_i, ϵ_i. Using the generators $a_1, \ldots, a_{2g}, d_2$ of Proposition 2.2, we have $w_{2k} = d_k d'_k$, $w_{2k+1}^2 = e_k$, where by definition $w_i = (a_1 \cdots a_{i-1})^i = y_2 \cdots y_i$. The group $\Gamma(P_i)$ $i \ge 2$ is isomorphic to the product $\mathbb{Z}^3$, generated by the twists along the three loops cutting out P_i. Thus, property (iii.a) is an immediate consequence of the following

Proposition 4.1. – Let $F = (\lambda, f) \in \widehat{GT}$ act on $\widehat{\Gamma}_{g,1}$ as in Theorem 2.1. Then for $1 \le i \le g$, we have $F(d_i) = d_i^\lambda$, $F(d'_i) = d_i'^\lambda$ and $F(e_i) = e_i^\lambda$.

Proof. Since the F-action on $\langle a_1, \ldots, a_{2g} \rangle$ is inherited from $\widehat{GT} \subset Aut(\widehat{B}_{2g+1})$, it is easy to see that $F(w_i) = w_i^\lambda$; hence that $F(e_i) = e_i^\lambda$.

Next, to consider $F(d_i)$, $F(d'_i)$, let us introduce elements $t_i := a_{2i}a_{2i-1}a_{2i+1}a_{2i}$ $(i \ge 1)$, following Wajnryb's convention [10] and extending the notation introduced in proposition 2.2. By the relation $w_{2k} = d_k d'_k$, we only have to consider the $F(d_i)$'s. Let $g_{1,i} := t_i^{-1} d_i t_i$ and $g_{2,i} := (t_2 \cdots t_i t_1 \cdots t_{i-1})^{-1} d_2 t_2 \cdots t_i t_1 \cdots t_{i-1}$ for $i \ge 2$, so that the lantern relation $g_{2,i} g_{1,i} d_i =$

$d_{i-1}a_{2i-1}a_{2i+1}d_{i+1}$ holds. We compute the following actions of $\tilde{F} := \mathrm{inn}(\prod_i f(y_{2i+1}, a_{2i+1}^2)) \circ F$ on $g_{1,i}, g_{2,i}$ $(i \geq 2)$:

$$\begin{cases} \tilde{F}(g_{1,i}) = (d_i d_i')^m f(g_{1,i}g_{1,i}', d_i d_i') g_{1,i}^\lambda f(d_i d_i', g_{1,i}g_{1,i}')(d_i d_i')^{-m}, \\ \tilde{F}(g_{2,i}) = f(g_{2,i}g_{2,i}', d_i d_i') g_{2,i}^\lambda f(d_i d_i', g_{2,i}g_{2,i}'). \end{cases}$$

Then, it is easy to see that the above lantern relations ensure inductively that $\tilde{F}(d_i) = d_i^\lambda$; hence $F(d_i) = d_i^\lambda$ for all $i \geq 2$. □

As for property (iii.b), the non-trivial cases to be checked are consecutive pairs of pants $P_i \cup P_{i+1}$. Note that $P_1 \cup P_2$, $P_{2i} \cup P_{2i+1}$ $(i \geq 1)$ are 2-holed tori, and $P_{2i+1} \cup P_{2i+2}$ $(i \geq 1)$ are 4-holed spheres. In each of these cases, the image of $\widehat{\Gamma}(P_i \cup P_{i+1})$ in $\widehat{\Gamma}_{g,1}$ is generated by $\{a_1, a_2, a_3, d_2, d_2'\}$, $\{e_i, e_{i+1}, d_{i+1}, d_{i+1}', a_{2i+2}\}$ and $\{d_{2i-1}, d_{2i-1}', d_{2i}, d_{2i}', a_{4i-1}, e_{2i-1}\}$ respectively. Thus, the following proposition, combined with Proposition 4.1, settles property (iii.b).

Proposition 4.2. – *Suppose $F = (\lambda, f) \in \mathbb{I}\Gamma$ acting on $\widehat{\Gamma}_{g,1}$ as in Theorem 2.1. Let $\rho_2 = \rho_2(F)$. Then*

$$\begin{cases} F(a_{2i}) = f(a_{2i}^2, d_i d_i') a_{2i}^\lambda f(d_i d_i', a_{2i}^2) & \text{for } 1 \leq i \leq g, \\ F(a_{2i+1}) = e_i^{-2\rho_2} f(a_{2i+1}, e_i) a_{2i+1}^\lambda f(e_i, a_{2i+1}) e_i^{2\rho_2} & \text{for } 0 \leq i \leq g-1. \end{cases}$$

Proof. Since $f(a_i^2, y_i) = f(a_i^2, w_i)$ and $w_{2k} = d_k d_k'$, the first formula follows at once. For the second, we combine relation (IV) and $w_{2k+1}^2 = e_k$ to compute

$$f(w_{2i+1}, a_{2i+1}^2) = a_{2i+1}^{-4\rho_2}(a_{2i+1}w_{2i+1})^{-4\rho_2} f(e_i, a_{2i+1}) w_{2i+1}^{4\rho_2}.$$

This completes the proof of Proposition 4.2. □

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