An introduction to

Profinite Grothendieck-Teichmüller theory

MIT, Cambridge, Massachusetts

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Lecture 1A

Geometric Galois actions

Outline

Lecture 1A: Geometric Galois actions

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In this introduction to Grothendieck-Teichmüller theory, we will discuss Grothendieck's idea, expressed in his *Esquisse d'un Programme* (1983) of studying $G_{\mathbb{Q}}$, the absolute Galois group over $\mathbb{Q}$, via its action on geometric and topological objects (curves, fundamental groups, dessins) such as rather than algebraic numbers.

The principal object of interest in much of the *Esquisse* is the profinite free group $\widehat{F}_2$ on two generators, identified with the algebraic fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, and the canonical outer action of $G_{\mathbb{Q}}$ on this group.

§1A.1. Galois groups and fundamental groups

In sections §2 and §3 of the *Esquisse*, Grothendieck considers the action of $G_{\mathbb{Q}}$ on two types of objects:

• dessins d'enfants; these are graphs embedded into topological surfaces, whose faces are all cells.

• diffeotopies (really "pro-diffeotopies") of topological surfaces. (The group of diffeotopies is the group of oriented diffeomorphisms modulo those isotopic to the identity.)

Recall that the *profinite completion* of a group is given by the inverse limit of the system of all its finite quotients:

$$\widehat{G} = \varprojlim G/N$$

where N runs through the finite index normal subgroups of G.

The term "pro-diffeotopy" refers to elements of the profinite completion of the group of diffeotopies of a topological surface.

The two kinds of actions are really the same: they both come from the following situation.

If X is an algebraic variety defined over $\mathbb{Q}$, let $\pi_1(X)$ denote its topological fundamental group and $\widehat{\pi}_1(X)$ its algebraic fundamental group, which is the profinite completion of the topological one. Then there is a **canonical outer action**

$$G_{\mathbb{Q}} \to \mathrm{Out}\big(\widehat{\pi}_1(X)\big). \tag{1}$$

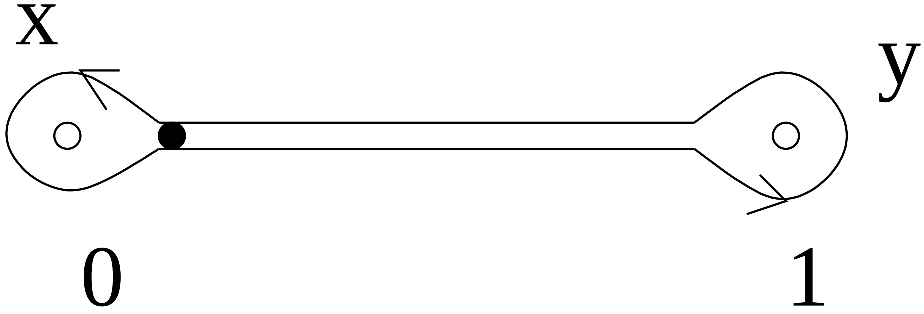
A characteristic feature of the Galois action on an algebraic π_1 is that it **preserves conjugacy classes of inertia groups**.

Here is where that outer Galois action comes from:

$$
\begin{array}{cccccc}
 & \widetilde{X} & \widetilde{\mathbb{C}(X)} & & \widetilde{\overline{\mathbb{Q}}(X)} & \\
 & | & | & & | & \\
\pi_1(X) & Y & \mathbb{C}(Y) & \widehat{\pi}_1(X) & \overline{\mathbb{Q}}(Y) & \widehat{\pi}_1(X) \\
 & | & | & & | & \\
 & X & \mathbb{C}(X) & & \overline{\mathbb{Q}}(X) & \\
 & & & & | & G_{\mathbb{Q}} \\
 & & & & \mathbb{Q}(X) &
\end{array}
$$

§1A.2. The case $\mathbb{P}^1 - \{0,1,\infty\}$

Let $X = \mathbb{P}^1 - \{0,1,\infty\}$, so that the topological π_1 is F_2, the free group on two generators, which we write $\langle x, y, z \mid xyz = 1\rangle$, identifying x, y and z with loops around 0, 1 and ∞ respectively.



We saw in §1 that we have a canonical homomorphism

$$G_{\mathbb{Q}} \to \mathrm{Out}\big(\widehat{\pi}_1(\mathbb{P}^1 - \{0,1,\infty\}\big)$$

i.e.

$$G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2).$$

The inertia groups are $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$, so since $G_{\mathbb{Q}}$ preserves inertia, we know that for each $\sigma \in G_{\mathbb{Q}}$, there exist $\alpha, \beta, \lambda \in \widehat{\mathbb{Z}}^*$ and $f,\, g \in \widehat{F}_2$ such that

$$\begin{cases} \sigma(x) = x^{\alpha} \\ \sigma(y) = g^{-1} y^{\beta} g \\ \sigma(z) = h^{-1} z^{\lambda} h \end{cases} \tag{$*$}$$

is a lifting of the canonical outer action of σ on $\widehat{F}_2$.

In $\widehat{F}_2^{\rm ab} = \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$, this means that $x^{\alpha} y^{\beta} z^{\lambda} = 1$, which is only possible if $\alpha = \beta = \lambda$. Suppose $g \equiv x^{\delta} y^{\epsilon}$ in $\widehat{F}_2^{\rm ab}$, and set $f = y^{-\epsilon} g x^{\delta}$. Then

$$\begin{cases} \sigma(x) = x^{\lambda} \\ \sigma(y) = f^{-1} y^{\lambda} f \end{cases}$$

is the unique lifting of the outer action of σ to an automorphism of type (*) such that the element conjugating y^{λ} lies in $\widehat{F}_2'$. (Indeed, $\widehat{F}_2'$ can be viewed as the group of elements of $\widehat{F}_2$ that are in the kernel of $x \mapsto 1$ and also in the kernel of $y \mapsto 1$.)

We have obtained a map

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \times \widehat{F}_2'.$$

This map is NOT a group homomorphism. It corresponds to associating to $\sigma \in G_{\mathbb{Q}}$ the automorphism $F_\sigma \in \mathrm{Aut}(\widehat{F}_2)$ associated to the pair $(\lambda_\sigma, f_\sigma)$ such that $x \mapsto x^{\lambda_\sigma}$, $y \mapsto f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma$.

It follows from Belyi's theorem (every algebraic curve over $\overline{\mathbb{Q}}$ can be realized as a finite cover of $\mathbb{P}^1 - \{0, 1, \infty\}$) that it is injective. It is an isomorphism onto its image if that image is equipped with the multiplication corresponding to composition of automorphisms, as follows.

If σ, $\tau \in G_{\mathbb{Q}}$, the product $\sigma \cdot \tau$ corresponds to applying first the automorphism τ, then σ, so we get

$$x \overset{\tau}{\mapsto} x^{\lambda_\tau} \overset{\sigma}{\mapsto} x^{\lambda_\sigma \lambda_\tau}$$

$$y \overset{\tau}{\mapsto} f_\tau^{-1} y^{\lambda_\tau} f_\tau \overset{\sigma}{\mapsto} F_\sigma(f_\tau)^{-1} f_\sigma^{-1} y^{\lambda_\sigma \lambda_\tau} f_\sigma F_\sigma(f_\tau).$$

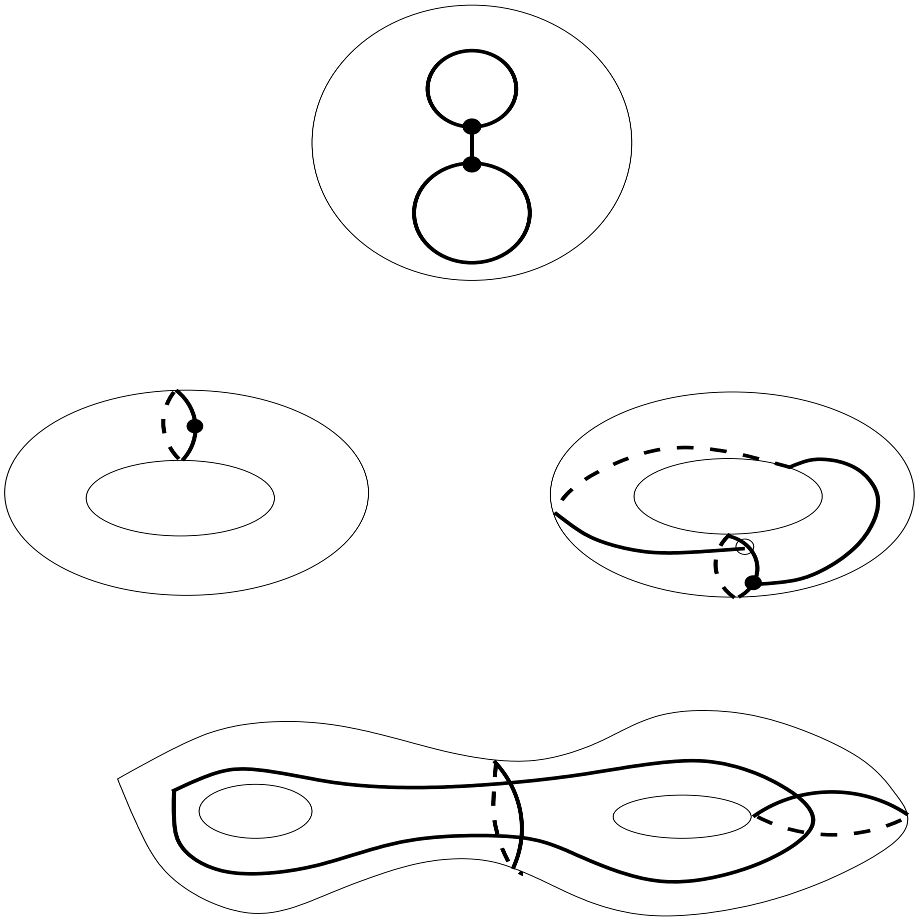
In other words, the pair corresponding to $\sigma \cdot \tau$ is

$$\big(\lambda_\sigma \lambda_\tau, f_\sigma F_\sigma(f_\tau)\big).$$

§1A.3. Galois action on dessins d'enfants

Definition. A *dessin d'enfant* is a triple $X_0 \subset X_1 \subset X_2$ where X_0 is a finite set of points on a compact topological surface X_2 of genus g, and X_1 is a set of edges connecting the vertices such that $X_2 \setminus X_1$ is a disjoint union of open cells (simply connected regions) of X_2.

A dessin is defined up to isotopy on the surface, and we also require it to be *bicolorable*, i.e. we want to be able to color the vertices in two colors, black and white, in such a way that all neighbors of every vertex of a given color are of the opposite color.



WHICH ONES ARE DESSINS?

We have bijections between the following sets

- $\{$dessins d'enfant$\}$
- $\{$finite covers of $\mathbb{P}^1$ unramified outside $\{0, 1, \infty\}$ $\}$, known as *Belyi covers*
- $\{$finite etale covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ $\}$
- $\{$conj. classes of subgroups of finite index of $\widehat{F_2}\}$
- transitive subgroups of S_n generated by 2 elements (for all n), up to conjugacy
- $\{$finite degree extensions of $\mathbb{C}(T)$ unramified outside $(T), (T-1), (1/T)\}$
- $\{$finite degree extensions of $\overline{\mathbb{Q}}(T)$ unramified outside $(T), (T-1), (1/T)$ $\}$

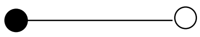
The first bijection is given by associating to a Belyi cover

$$\beta : X \to \mathbb{P}^1$$

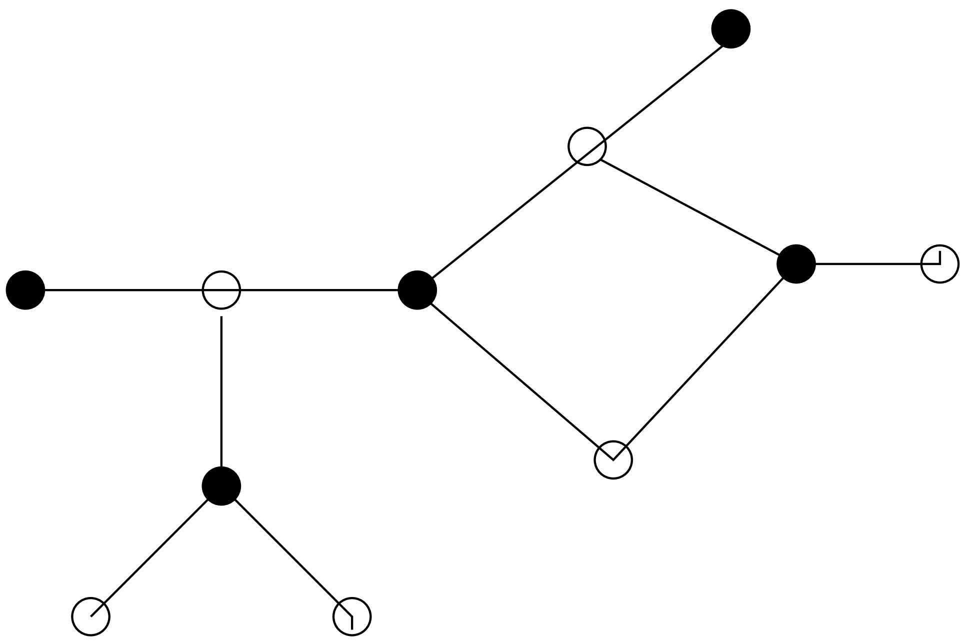
the preimage $\beta^{-1}([0,1])$ of the segment $[0,1]$ in $\mathbb{P}^1$ (automatically bicolorable). The second and third bijections are basic facts about Riemann surfaces and topological covers. The fourth is the geometric Galois correspondence. The fifth is obtained by identifying the permutations corresponding to x and y of the fiber over an unramified point of the cover. The last ones are just the function field analogues (with Lefschetz' theorem).

Going from covers to dessins and back

If $\beta : X \to \mathbb{P}^1$ is unramified over 0, 1 and ∞, then the dessin is $\beta^{-1}([0,1])$.

The degree of the cover is equal to the number of edges  of the dessin.

The points over 0 correspond to black vertices of the dessin, the points over 1 to white vertices.



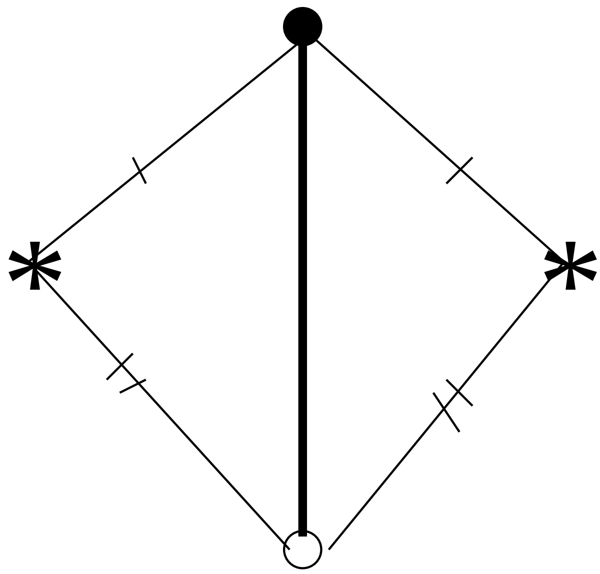
Example. Genus=0, Degree = 11

5 preimages of 0, 6 preimages of 1

2 preimages of ∞

You can visualize the cover topologically by triangulating the dessin (adding a vertex marked $\star$ in each face, and adding edges joining it up to the black and white vertices).

This paves the dessin surface with diamonds

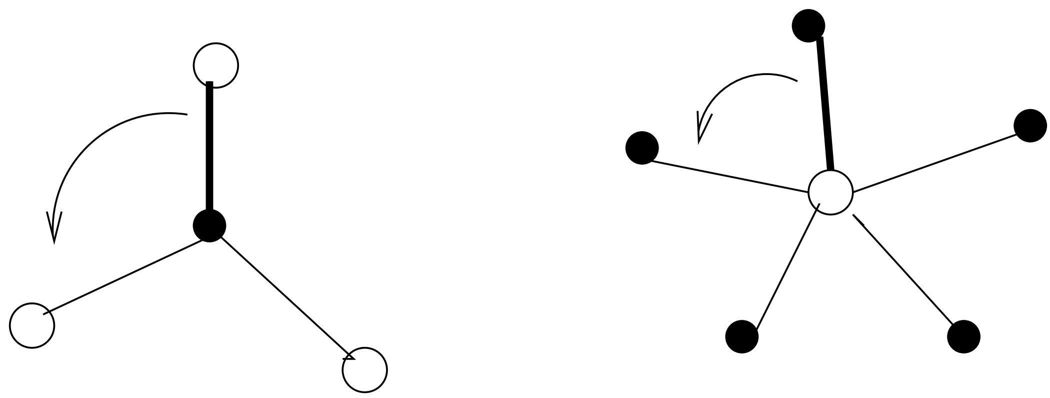


each of which contains exactly one edge of the actual dessin.

Topologically, the cover is obtained by identifying the marked pairs of edges, so the quotient is topologically a sphere with three branch points.

Putting the Riemann surface structure $\mathbb{P}^1 - \{0, 1, \infty\}$ on the quotient determines the Riemann structure on the triangulated cover.

The group F_2 acts on the set of edges of the dessin D as follows:



Pick any edge e of the dessin and let $N = \mathrm{Stab}(e)$; then N is a finite-index subgroup of $\widehat{F}_2$. The stabilizers of the different flags (=oriented edges) form a conjugacy class of finite-index subgroups in $\widehat{F}_2$, and this conjugacy class corresponds to a finite cover of $\mathbb{P}^1$, namely exactly the Belyi cover $\beta : X \to \mathbb{P}^1$.

The degree of the cover is the number of edges e, and the set of edges is in bijection with the coset space $\widehat{F}_2/N$; furthermore the action of $\widehat{F}_2$ on the edges is exactly the action on $\widehat{F}_2/N$ by right multiplication. Obviously, $\widehat{F}_2$ acts via a finite quotient, called the *monodromy group* of the dessin or the cover.

You can reconstruct the whole dessin just by knowing N (up to conjugacy):

• Edges are in bijection with $\widehat{F}_2/N$;

• orbits of $\widehat{F}_2/N$ under x are sets of "flowers" centered around black vertices (edges attached to same black vertex);

• similarly, orbits of $\widehat{F}_2/N$ under y are sets of "flowers" centered around white vertices.

Galois action on dessins

The action of $G_{\mathbb{Q}}$ on $\widehat{F}_2$ sends a finite index subgroup N to N^σ, so it sends the dessin D corresponding to N to the dessin D^σ corresponding to N^σ. The field

$$K_D = \text{fixed field of } \{\sigma \in G_{\mathbb{Q}} \mid N^\sigma = N, \text{ i.e. } D^\sigma = D\}$$

is called *the moduli field of D*.

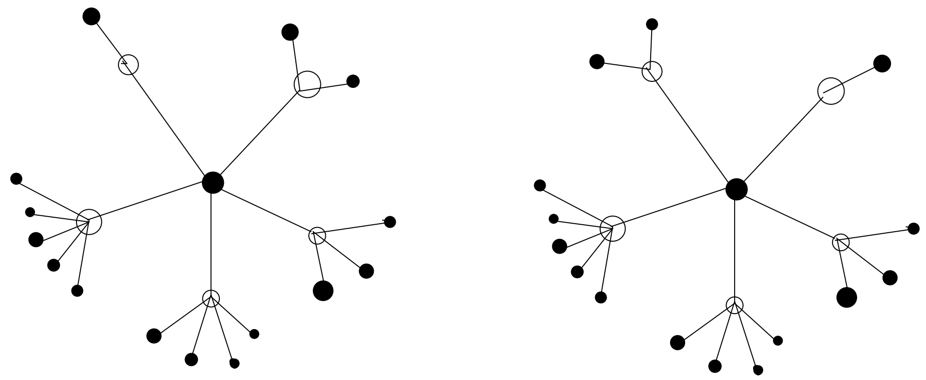
Thus, each dessin is naturally defined over a number field, and the set of dessins is naturally equipped with a Galois action.

Now, what we would like is to give a list of **combinatorial Galois invariants** of dessins, the dream being to give a list sufficient to determine Galois orbits of dessins. To start with, there are some obvious Galois invariants:

- number of edges, faces, black, white vertices
- ramification indices, i.e. valencies of black and white vertices;
- monodromy group...

All these are *geometric*, i.e. they have to do with the ramification information of the associated Belyi cover.

Example:



Every one of the preceding, geometric invariants of these two dessins is equal. There are 24 dessins having the exact same ramification indices. However, it is actually possible to EXPLICITLY COMPUTE the associated number fields and see that these two dessins are NOT Galois conjugates.

The valencies at the black vertices are $(5, 1, \cdots, 1)$ and at the white vertices $(2, 3, 4, 5, 6)$. If you take dessins with the same black valencies and various 5-tuples of white valencies, you sometimes get a Galois orbit of 24 and sometimes two Galois orbits of 12, as here.

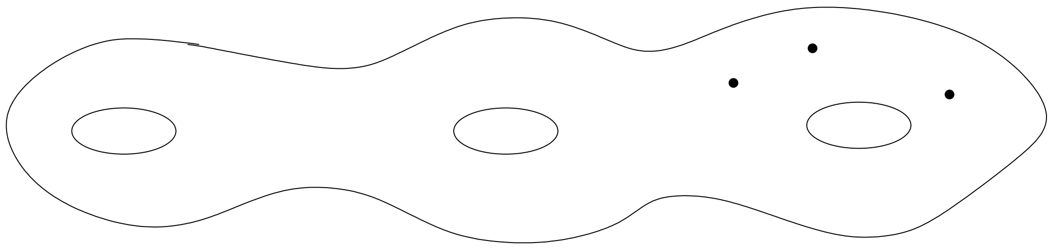
Y. Kochetkov computed many examples and noticed that the Galois orbits appeared to split the set of dessins into two parts exactly when the white valencies are (a, b, c, d, e) such that

$$abcde(a + b + c + d + e) \text{ is a square.}$$

This conjecture was generalized and proved by Leonardo Zapponi (1997), who actually came up with a NEW GALOIS INVARIANT for a large family of dessins. This Galois invariant is just a sign ± 1, but what is interesting is that it is really arithmetic, not geometric in nature.

§1A.4. Galois action on diffeotopies of topological surfaces

Now, let S be a topological surface of genus g, with n marked points.



Let $M_{g,n}$ denote the *moduli space of Riemann surfaces of type* (g, n). The points of $M_{g,n}$ are isomorphism classes of these Riemann surfaces; the moduli space can also be considered as the space of analytic structures on S up to isomorphism.

A path on the moduli space is a continuous deformation of the analytic structure of the Riemann surface corresponding to a given starting point $x \in M_{g,n}$.

In particular, a loop (up to homotopy) corresponds to a diffeotopy of x, i.e. to an orientation-preserving diffeomorphism (up to isotopy).

Thus, if we define the *mapping class group* to be

$$\Gamma_{g,n} = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S),$$

then we have an isomorphism

$$\Gamma_{g,n} \simeq \pi_1(M_{g,n}).$$

Topologically, the moduli spaces $M_{g,n}$ are not always manifolds but sometimes *orbifolds*, because they are topologically obtained as the quotient of a topological ball (Teichmüller space) by the action of $\Gamma_{g,n}$, which is properly discontinuous but not always fixed-point free. The fixed points are fixed by finite subgroups of $\Gamma_{g,n}$ and correspond to *Riemann surfaces with automorphisms.* (When $g = 0$ or $n > 2g + 3$, the $M_{g,n}$ are manifolds.)

However, $\Gamma_{g,n}$ always contains a finite-index subgroup with fixed-point-free action. Thus, there is always a finite cover of $M_{g,n}$ that is a manifold.

Arithmetically, the orbifolds $M_{g,n}$ are not $\mathbb{Q}$-schemes but $\mathbb{Q}$-stacks, called *Deligne-Mumford* stacks, meaning that they have a finite cover which is a $\mathbb{Q}$-scheme. The canonical Galois homomorphism exists for stacks just as for schemes: we have

$$G_{\mathbb{Q}} \to \mathrm{Out}\big(\pi_1(M_{g,n})\big) \simeq \mathrm{Out}(\widehat{\Gamma}_{g,n}).$$

We want to know exactly how $G_{\mathbb{Q}}$ acts on diffeomorphisms of S, and we are particularly intrigued by the two following special types of diffeomorphisms, each of which forms a generating set for the mapping class groups:

- **Dehn twists**

  (G-T theory $\Rightarrow$ Galois action understood)

- **Diffeomorphisms of finite order**

  (Galois action not well understood)

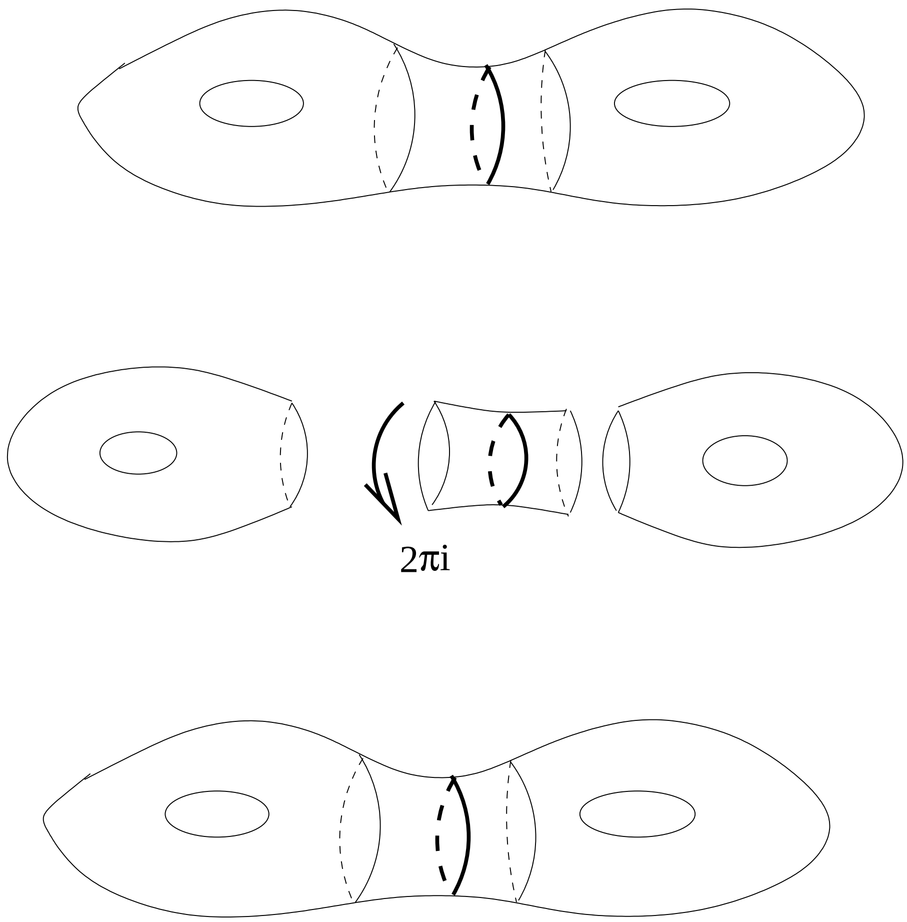
Dehn twists in $\Gamma_{g,n}$ are related to **the "divisor at infinity" of the moduli space**, i.e. the divisor that is added in the stable compactification. As paths in the fundamental group, they represent loops around irreducible components of the divisor at infinity; thus they play the role of inertia groups in the fundamental group.

Example. $M_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$; the divisor at infinity is the three points 0, 1 and ∞, the Dehn twists as paths on moduli space correspond to the loops x, y, z around 0, 1, ∞.

Finite order elements in $\Gamma_{g,n}$ are related to **special loci of the moduli space**, i.e. subsets of points consisting of curves having special automorphism groups; they play an inertia-like role in the fundamental group corresponding to controlled ramification over these special loci.

Dehn twists

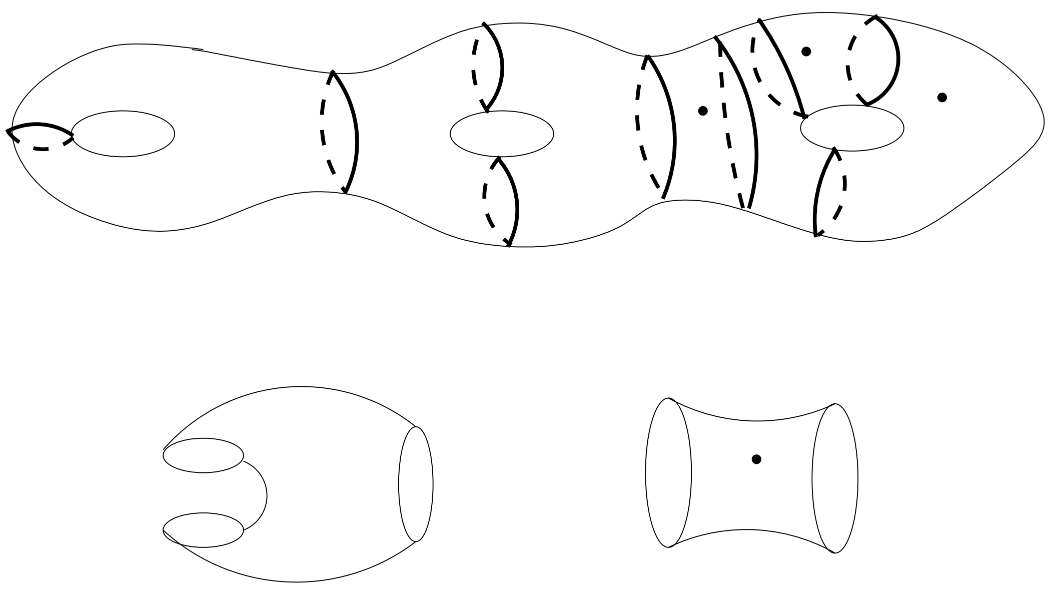
As a diffeomorphism, a Dehn twist is given by specifying a simple closed loop on S, and taking the diffeomorphism obtained as follows:



If α denotes a simple closed loop on S, then we write a for the associated Dehn twist.

Dehn proved that the mapping class group $\Gamma_{g,n}$ is generated by Dehn twists.

Let a **pants decomposition** on S be a maximal set of $3g-3+n$ disjoint simple closed loops; they cut S into "pants".

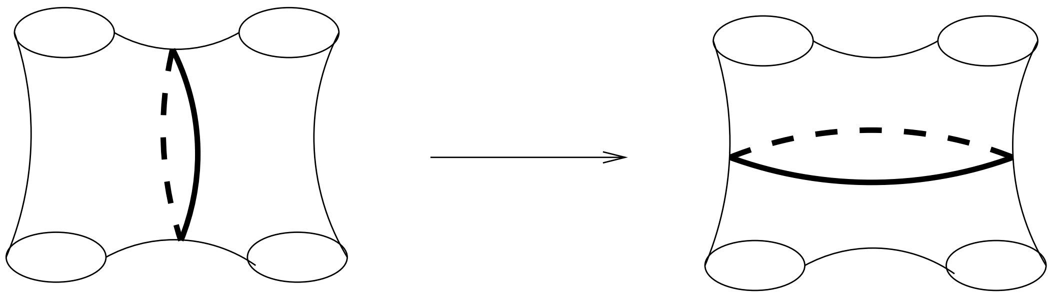


If we erase any one of these loops, then the pants decomposition becomes a decomposition into many pairs of pants and one larger piece, which is always

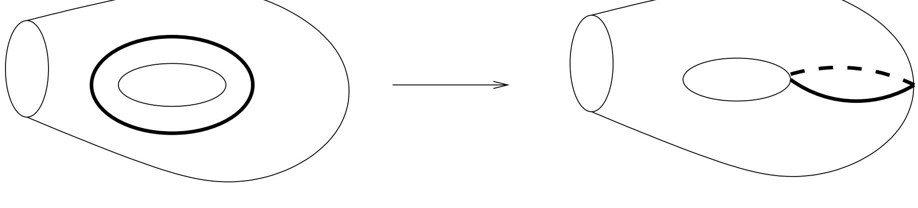
- either a genus zero piece with four boundary components
- or a genus one piece with one boundary component.

We call this piece the *neighborhood of the loop in the pants decomposition.*

• An A-move on a pants decomposition P is a new pants decomposition obtained from P by erasing one loop and replacing it by another one which intersects the first one in 2 points and doesn't intersect any of the others.



• An S-move on a pants decomposition P is a new pants decomposition obtained from P by erasing one loop and replacing it by another one which intersects the first one in 1 point and doesn't intersect the others.



Recall that we have an injective set map

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

$$\sigma \mapsto (\lambda, f),$$

where $\widehat{F}_2$ is the profinite free group on two generators x and y. The mapping corresponds to the fact that under

$$G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2),$$

the element σ is mapped to an inertia-preserving outer automorphism, which lifts to an automorphism of the form

$$\begin{cases} \sigma(x) = x^{\lambda} \\ \sigma(y) = f^{-1} y^{\lambda} f. \end{cases}$$

The pair (λ, f) is unique if we require $f \in \widehat{F}_2'$.

Notation: For any group homomorphism

$$\widehat{F}_2 \to G$$

$$x, y \mapsto a, b$$

we write $f(a, b)$ for the image of $f \in \widehat{F}_2$.

For example:

• under $\mathrm{id} : \widehat{F}_2 \to \widehat{F}_2$, we have $f = f(x, y)$;

• under the map $\widehat{F}_2 \to \widehat{F}_2$ exchanging the generators x and y, we have

$$f = f(x, y) \mapsto f(y, x).$$

Most results in Grothendieck-Teichmüller theory concern the Galois action on the mapping class groups. One of the main results explicitly determines the Galois action on all Dehn twists and proves that the Galois group acts *locally* on Dehn twists in the sense that this action concerns only a topological "neighborhood" of the Dehn twists.

Theorem. *Let S be a topological surface of type (g, n). For each pants decomposition P on S, there exists an injective homomorphism*

$$\rho_P : G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{\Gamma}_{g,n})$$

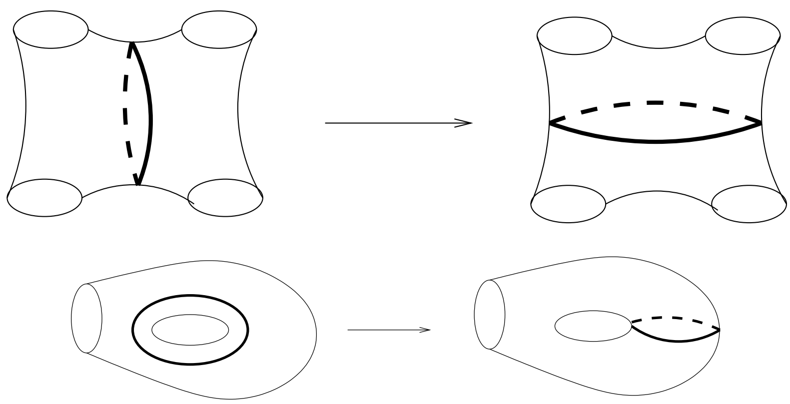
lifting the canonical homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_{g,n})$, such that:

$$\sigma(a) = a^{\lambda} \quad \textit{if} \quad \alpha \in P,$$

$$\sigma(b) = f(a,b)^{-1}\, b^{\lambda}\, f(a,b) \quad \textit{if} \quad \alpha \to \beta \textit{ is an A-move on } P,$$

$$\sigma(c) = f(a^2,c^2)^{-1}\, c^{\lambda}\, f(a^2,c^2) \quad \textit{if} \quad \alpha \to \gamma \textit{ is an S-move on } P.$$

This means that in acting on a Dehn twist a along a loop α, Galois not only conjugates the $\chi(\sigma)$-th power (we knew that – it's inertia!), but it conjugates it by a *local* element of $\widehat{\Gamma}_{g,n}$, i.e. a profinite product of Dehn twists living right on the neighborhood of the loop α.



A final remark. One of the most interesting open questions is whether there is analogous theory of the Galois action on the finite-order elements of the mapping class groups; we don't even know whether the Galois group treats them like inertia (except in $g = 0, 1$), let alone whether the action is "local" in some sense.