An introduction to

Profinite Grothendieck-Teichmüller theory

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Lecture 1B

The Grothendieck-Teichmüller group

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§1B.1. Background for definition: braid groups

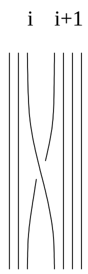
Let us recall the definitions of the Artin braid groups and the mapping class groups.

Definition. Let B_n denote the Artin braid group on n strands generated by $\sigma_1, \ldots, \sigma_{n-1}$, with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

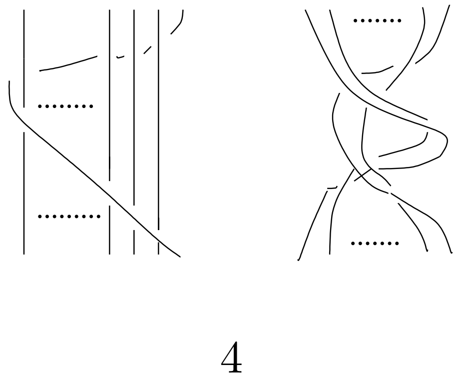
and

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2.$$



Let $\Gamma_{0,[n]}$ be the quotient of B_n by the relations

$$\sigma_{n-1} \cdots \sigma_1^2 \cdots \sigma_{n-1} = 1 \quad \text{and} \quad (\sigma_1 \cdots \sigma_{n-1})^n = 1.$$



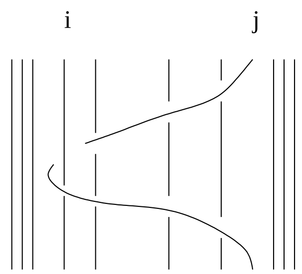
Let K_n and $\Gamma_{0,n}$ denote the *pure* subgroups of B_n and $\Gamma_{0,[n]}$, i.e. kernels of the natural surjections

$$B_n \twoheadrightarrow S_n, \qquad \Gamma_{0,n} \twoheadrightarrow S_n$$

obtained by considering only the permutations of the braid-ends.

The pure subgroups are generated by the braids

$$x_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, \quad 1 \leq i < j \leq n.$$



In particular, a presentation of $\Gamma_{0,5}$ is given by generators $x_{i,i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$, and relations

$$x_{i,i+1} x_{j,j+1} = x_{j,j+1} x_{i,i+1} \text{ if } |i-j| \geq 2$$

and

$$x_{51} x_{23}^{-1} x_{12} x_{34}^{-1} x_{23} x_{45}^{-1} x_{34} x_{51}^{-1} x_{45} x_{12}^{-1} = 1.$$

Moduli spaces of genus zero curves

Let $\Sigma_{g,n}$ denote a topological surface of genus g with n fixed marked points.

The group $\Gamma_{0,n}$ is the group of diffeotopies of the topological surface $\Sigma_{0,n}$ that preserve the marked points. The group $\Gamma_{0,[n]}$ is the group of diffeotopies of $\Sigma_{0,n}$ that permute the marked points.

As we saw in the previous lecture, we have an isomorphism

$$\Gamma_{0,n} \simeq \pi_1(M_{0,n})$$

where $M_{0,n}$ is the moduli space of genus zero Riemann surfaces with n ordered marked points. We also have

$$\Gamma_{0,[n]} \simeq \pi_1(M_{0,[n]})$$

where $M_{0,[n]} = M_{0,n}/S_n$ is the moduli space of genus zero Riemann surfaces with n unordered marked points; indeed, $\Gamma_{0,[n]}$ fits into the short exact sequence

$$1 \to \Gamma_{0,n} \to \Gamma_{0,[n]} \to S_n \to 1.$$

Each point of $M_{0,n}$ represents an isomorphism class of Riemann spheres with n marked points; the isomorphisms are given by $\mathrm{PSL}_2(\mathbb{C})$, so there is a unique representative of each isomorphism class determined by giving the marked points in the form $(0, 1, \infty, x_1, \ldots, x_{n-3})$.

Thus, the genus zero moduli spaces have a simple geometric structure:

$$M_{0,n} \simeq (\mathbb{P}^1)^{n-3} - \Delta,$$

where Δ is the "fat" diagonal consisting of all lines $x_i = x_j$.

They are all manifolds; indeed, $\Gamma_{0,n}$ has no torsion, so no fixed points on Teichmüller space.

Examples. We have

$$\begin{cases} M_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\} \\ M_{0,5} \simeq (\mathbb{P}^1)^2 - \{x = y\}. \end{cases}$$

We can visualize braids in $\Gamma_{0,n}$ as points moving on the Riemann sphere.

We also have

$$M_{0,[n]} \simeq M_{0,n}/S_n.$$

Note that topologically, $M_{0,n}$ is a manifold whereas $M_{0,[n]}$ is an orbifold since the S_n-action always has fixed points.

Examples. The point $(0, 1, \infty, \frac{1}{2})$ is fixed by the transposition $(1, 2)$. Indeed, $(1, 0, \infty, \frac{1}{2})$ is brought back to the standard representative by the automorphism $z \mapsto 1 - z$ in $\mathrm{PSL}_2(\mathbb{C})$, which fixes $\frac{1}{2}$.

More simply, $\mathbb{P}^1 - \{1, \zeta, \ldots, \zeta^{n-1}\}$ where $\zeta^n = 1$ is fixed under the action of the n-cycle. Thus, the $M_{0,[n]}$ are all orbifolds. In general the $M_{g,n}$ are orbifolds for $g > 1$ and small n, but manifolds for $n > 2g + 3$.

§1B.2. $\widehat{GT}$: definition and main properties

For any discrete group G, let $\hat{G}$ denote its profinite completion and $\hat{G}'$ the derived group of $\hat{G}$. Let F_2 denote the free group on two generators x and y.

For any profinite group homomorphism

$$\widehat{F}_2 \to G: \quad x \mapsto a, \quad y \mapsto b$$

let $f(a,b) \in G$ denote the image of $f \in \widehat{F}_2$.

Definition. [**Drinfeld,1991**] Let $\widehat{GT}_0$ be the set of pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ such that $x \mapsto x^\lambda, \quad y \mapsto f^{-1}y^\lambda f$ extends to an automorphism of $\widehat{F}_2$, and (λ, f) satisfies

(I) $f(x,y)f(y,x) = 1$

(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$.

Let $\widehat{GT}$ be the subset of $\widehat{GT}_0$ whose elements additionally satisfy the **pentagon relation** in $\widehat{\Gamma}_{0,5}$:

(III) $f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51}) = 1.$

Rephrased definition of $\widehat{GT}$

Let θ be the automorphism of F_2 and $\hat{F}_2$ defined by $\theta(x) = y$ and $\theta(y) = x$.

Let ω be the automorphism of F_2 and $\hat{F}_2$ defined by $\omega(x) = y$ and $\omega(y) = (xy)^{-1}$.

Let ρ be the automorphism of $\Gamma_{0,5}$ and $\hat{\Gamma}_{0,5}$ given by $\rho(x_{i,i+1}) = x_{i+3,i+4}$.

Then $\widehat{GT}_0$ is the set of pairs $(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$ such that

$$x \mapsto x^\lambda, \quad y \mapsto f^{-1}yf$$

extends to an automorphism of $\widehat{F}_2$ and

(I) $\theta(f)f = 1$,

(II) $\omega^2(fx^m)\omega(fx^m)fx^m = 1$ where $m = (\lambda - 1)/2$.

Similarly, $\widehat{GT}$ is the subset of elements $(\lambda, f) \in \widehat{GT}_0$ satisfying the **pentagon relation**

(III) $\rho^4(\tilde{f})\rho^3(\tilde{f})\rho^2(\tilde{f})\rho(\tilde{f})\tilde{f} = 1$ in $\widehat{\Gamma}_{0,5}$, where $\tilde{f} = f(x_{12}, x_{23})$.

Multiplication law on $\widehat{GT}$

By composition, we can put a multiplication law on the pairs $F = (\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ that induce automorphisms of $\widehat{F}_2$ via

$$F(x) = x^\lambda, \quad F(y) = f^{-1} y^\lambda f.$$

The composition law for two such automorphisms is given by

$$(\lambda, f)(\mu, g) = \big(\lambda\mu, f\, F(g)\big).$$

The first main theorem concerning $\widehat{GT}$ is the following non-trivial result:

Theorem 1. *$\widehat{GT}_0$ and $\widehat{GT}$ are profinite groups.*

Indeed, it is not obvious why the composition of two automorphisms satisfying the three relations will also satisfy them; similarly for the inverse of an automorphism.

$\widehat{GT}_0$ as automorphism group

Let $F = (\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$ and assume that

$$x \mapsto x^\lambda, \quad y \mapsto f^{-1} y^\lambda f$$

gives an automorphism of $\hat{F}_2$.

Identify $\hat{F}_2$ as a subgroup of $\hat{B}_3$ by setting $x = \sigma_1^2$ and $y = \sigma_2^2$.

Lemma 1. *The automorphism $F = (\lambda, f)$ of $\widehat{F}_2$ extends to an automorphism of $\hat{B}_3$ via*

$$\sigma_1 \mapsto \sigma_1^\lambda, \quad \sigma_2 \mapsto f^{-1} \sigma_2^\lambda f$$

if and only if $(\lambda, f) \in \widehat{GT}_0$.

The proof of this lemma consists in an easy direct computation of the constraints imposed on the couple (λ, f) by the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and the fact that $(\sigma_1\sigma_2)^3$ is central in $\hat{B}_3$.

Theorem 1 a). $\widehat{GT}_0$ is a profinite group.

Proof. By Lemma 1, two pairs (λ, f) and (μ, g) in $\widehat{GT}_0$ give automorphisms of $\hat{B}_3$, so their composition $(\lambda\mu, fF(g))$ also gives an automorphism of $\hat{B}_3$. This composition maps σ_1 to $\sigma_1^{\lambda\mu}$ and σ_2 to a conjugate of $\sigma_2^{\lambda\mu}$, so by Lemma 1 it also lies in $\widehat{GT}_0$. The automorphism $F = (\lambda, f)$ also has an inverse F^{-1} which is given by $\big(\lambda^{-1}, F^{-1}(f^{-1})\big)$, cf. the multiplication law

$$(\lambda, f)(\mu, g) = (\lambda\mu, f\,F(g)).$$

Let $\mathrm{Out}^*(\widehat{\Gamma}_{g,n})$ denote the group of outer automorphisms of $\widehat{\Gamma}_{g,n}$ that preserve inertia in the sense of the Galois action, i.e. *preserving conjugacy classes of inertia groups.*

In genus zero, the inertia generators are the pure braid generators x_{ij} of $\Gamma_{0,n}$, which correspond precisely to loops around the "missing divisors" at infinity (those that are added in the stable compactification). The inertia generators of $\Gamma_{0,[n]}$ are the σ_i. The braid x_{ij} corresponds to a Dehn twist along a loop on the sphere surrounding only the points i and j.

It is easy to check that the action of $F = (\lambda, f) \in \widehat{GT}_0$ raises the elements $\sigma_2\sigma_1^2\sigma_2$ and $(\sigma_1\sigma_2)^3$ to the λ-th power, so the $\widehat{GT}_0$-action on $\hat{B}_3$ passes to the quotient $\widehat{\Gamma}_{0,[4]}$, and from Theorem 1 a) we obtain:

Theorem 2 a). $\widehat{GT}_0 \simeq \mathrm{Out}^*(\Gamma_{0,[4]})$.

To see that $\widehat{GT}_0$ is a profinite group, it suffices to note that it is the inverse limit of its own images in the (finite) automorphism groups of the quotients $\hat{F}_2/N$, where N runs over the characteristic subgroups of finite index of $\hat{F}_2$.

$\widehat{GT}$ as automorphism group

Let $c \in \Gamma_{0,[n]}$ denote the n-cycle $c = \sigma_{n-1} \cdots \sigma_1$. The groups $\Gamma_{0,[n]}$ are generated by σ_1 and c because $c^{-1}\sigma_i c = \sigma_{i+1}$.

Lemma 2. *A pair $F = (\lambda, f) \in \hat{\mathbf{Z}}^* \times \hat{F}_2'$ gives an automorphism of $\hat{\Gamma}_{0,[5]}$ via $\sigma_1 \mapsto \sigma_1^\lambda$ and $c^2 \mapsto \tilde{f}c^2$ (with $\tilde{f} = f(x_{23}, x_{12})$) if and only if the couple lies in $\widehat{GT}$, i.e. satisfies (I), (II), (III).*

Proof. One direction: if the proposed action is an automorphism of $\Gamma_{0,[5]}$, then by squaring, it is immediate that the proposed action satisfies $c^{-1} \mapsto f(x_{23}, x_{12})f(x_{51}, x_{45})c^{-1}$. The subgroup $\langle\sigma_1, \sigma_2\rangle \subset \hat{\Gamma}_{0,[5]}$ is isomorphic to $\hat{B}_3$, and this subgroup is preserved by the proposed action since $\sigma_2 = c^{-1}\sigma_1 c$ maps to

$$f(x_{23}, x_{12})f(x_{51}, x_{45})c^{-1}\sigma_1^\lambda c f(x_{51}, x_{45})^{-1}f(x_{23}, x_{12})^{-1} = \tilde{f}\sigma_2^\lambda \tilde{f}^{-1}.$$

Thus, by Lemma 1, (λ, f) satisfies relations (I) and (II).

The automorphism must also respect $(c^2)^5 = 1$, so $(\tilde{f}c^2)^5 = 1$. But since $\rho = \mathrm{Inn}(c)$, we have

$$(c\tilde{f})^5 = c^5\rho^4(\tilde{f})\rho^3(\tilde{f})\rho^2(\tilde{f})\rho(\tilde{f})\tilde{f} = 1,$$

which is exactly relation (III) defining $\widehat{GT}$.

Other direction: we need to show that the action respects all the relations in a presentation of $\hat{\Gamma}_{0,[5]}$. It is easiest to use the presentation with generators σ_1, σ_2, σ_3 and σ_4 and compute the action of (λ, f) on the σ_i by using $c^{-i}\sigma_1 c^i = \sigma_{1+i}$.

If $F = (\lambda, f)$ is going to give an automorphism of $\widehat{\Gamma}_{0,[5]}$, then it must act on the σ_i by:

$$
\begin{aligned}
\sigma_1 &\mapsto \sigma_1^\lambda \\
\sigma_2 &\mapsto f(x_{23}, x_{12})\sigma_2^\lambda f(x_{12}, x_{23}) \\
\sigma_3 &\mapsto f(x_{34}, x_{45})\sigma_3^\lambda f(x_{45}, x_{34}) \qquad\qquad (*) \\
\sigma_4 &\mapsto \sigma_4^\lambda \\
\sigma_{51} &\mapsto f(x_{23}, x_{12}) f(x_{51}, x_{45})\sigma_{51}^\lambda f(x_{45}, x_{51}) f(x_{12}, x_{23}).
\end{aligned}
$$

Assuming now that $(\lambda, f) \in \widehat{GT}$, we check that all the defining relations of $\widehat{\Gamma}_{0,[5]}$ are respected by this action on the σ_i; this is straightforward and only one of them requires (III).

Thus we know that $(\lambda, f) \in \widehat{GT}$ yields an automorphism of $\widehat{\Gamma}_{0,[5]}$ by (*). Now we compute the image of c^2 under this automorphism and easily check that $c^2 \mapsto \tilde{f}c^2$, which concludes the proof.

As in the case of $\widehat{GT}_0$, since $\widehat{GT}$ is an automorphism group of a finitely presented profinite group, we immediately obtain the result we are seeking for:

Theorem 1 b). *$\widehat{GT}$ is a profinite group.*

As an analog of Theorem 2 a):

Theorem 2 b) [Harbater-S, 1997] *We have*

$$\widehat{GT} = \mathrm{Out}_\flat^*(\widehat{\Gamma}_{0,5}),$$

where the $\flat$ denotes the $*$-automorphisms of $\Gamma_{0,5}$ that commute with the point-permuting subgroup $S_n \subset \mathrm{Out}(\widehat{\Gamma}_{0,5})$.

$\widehat{GT}$ acts on all braid groups

Drinfel'd showed that $\widehat{GT}$ acts on all the profinite Artin braid groups by the following simple formula: if $F = (\lambda, f) \in \widehat{GT}$, then

$$F(\sigma_1) = \sigma_1^{\lambda}, \quad F(\sigma_i) = f(\sigma_i^2, y_i)\sigma_i^{\lambda} f(y_i, \sigma_i^2),$$

where $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$.

This action passes to the quotients $\Gamma_{0,[n]}$ and their pure subgroups $\widehat{\Gamma}_{0,n}$.

The proof is very easy since all braid relations look alike, but the result is striking.

One might think that relations (I), (II) and (III) only imply that $\widehat{GT}$ acts on the braid groups with up to 5 strands, but in fact once one reaches 5, this action extends to all braid groups. Grothendieck called this **the two-level principle** and related it to a geometric/topological interpretation (see lecture 3).

A geometric way to interpret this result is that $\widehat{GT}$ is the automorphism of the **genus zero Teichmüller tower**:

$$\widehat{GT} = \mathrm{Out}^*(\mathcal{T})$$

where $\mathcal{T}$ consists of the collection of $\widehat{\Gamma}_{0,n}$ for all $n \geq 4$ equipped with the homomorphisms $d_n : \widehat{\Gamma}_{0,n} \to \widehat{\Gamma}_{0,n+1}$ obtained by doubling the first strand of a pure braid. We have $\widehat{GT} \hookrightarrow \mathrm{Out}^*(\widehat{\Gamma}_{0,n})$ for $n \geq 4$ and for each d_n, the commutative diagram

$$\begin{tikzcd} \widehat{\Gamma}_{0,n} \arrow[r, "d_n"] \arrow[d, "(\lambda,f)"'] & \widehat{\Gamma}_{0,n} \arrow[d, "(\lambda,f)"] \\ \widehat{\Gamma}_{0,n} \arrow[r, "d_n"] & \widehat{\Gamma}_{0,n}. \end{tikzcd}$$

The *two-level principle* consists in the statement that

$$\widehat{GT} = \mathrm{Out}^*(\mathcal{T}_5)$$

where $\mathcal{T}_5$ denotes the little tower of two groups $\Gamma_{0,4}$ and $\Gamma_{0,5}$ with the map $d_4 : \Gamma_{0,4} \to \Gamma_{0,5}$. The automorphism group of the two-level tower is the same as the automorphism group of the complete tower.

§1B.3. The absolute Galois group $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$

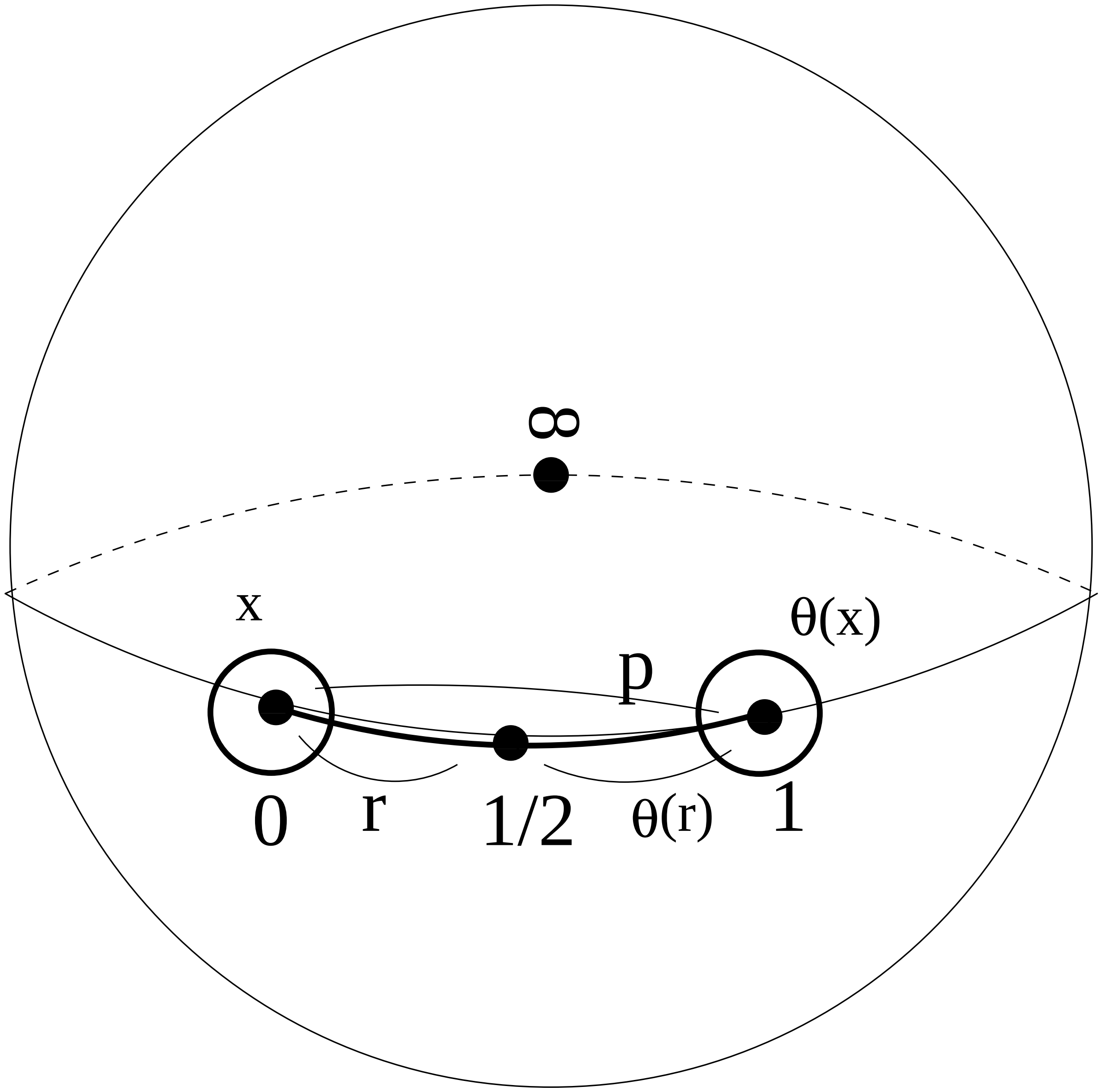
There are several ways of apprehending the fact that $\widehat{GT}$ contains the absolute Galois group. We already saw saw that an element $\sigma \in G_{\mathbb{Q}}$ can be associated to a unique pair (λ, f) such that $x \mapsto x^{\lambda}$, $y \mapsto f^{-1}y^{\lambda}f$ is a lifting of the canonical homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2)$ with F_2 identified with the space $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$. (The lifting corresponds to a choice of "tangential" base point.) The λ associated to $\sigma \in G_{\mathbb{Q}}$ is just the cyclotomic character $\lambda = \chi(\sigma) \in \widehat{\mathbb{Z}}^*$.

To prove that $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$, it remains to show that the pair $(\chi(\sigma), f_{\sigma})$ satisfies (I), (II) and (III).

A quick proof is as follows: $\widehat{GT}$ is the group of *all* automorphisms of $\widehat{\Gamma}_{0,5}$ with certain properties, namely inertia-preserving automorphisms that commute with S_5, and $G_{\mathbb{Q}}$ acts on $\widehat{\Gamma}_{0,5} \simeq \widehat{\pi}_1(M_{0,5})$ with those same properties.

However, this reasoning does not show where the relations that f_{σ} satisfies come from geometrically.

A geometric explanation. $G_{\mathbb{Q}}$ acts on the profinite fundamental groupoid of $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ based at all rational points (including "tangential" base points).



Let $\Theta(z) = 1 - z$.

In the fundamental groupoid of $M_{0,4}$ based at $\overrightarrow{01}$ and $\overrightarrow{10}$, we have paths x (based at $\overrightarrow{01}$, p (from $\overrightarrow{01}$ to $\overrightarrow{10}$) and $\Theta(x)$ (based at $\overrightarrow{10}$).

We also have $\Theta(p) = p^{-1}$, and for the path r from $\overrightarrow{01}$ to $1/2$, we have $\Theta(r)^{-1}r = p$.

The action of $\sigma \in G_{\mathbb{Q}}$ on x is just $\sigma(x) = x^{\lambda}$, where $\lambda = \chi(\sigma)$.

The action of σ on p is given by $\sigma(p) = pf$ for some $f \in \widehat{\pi}_1(M_{0,4}, \overrightarrow{01})$.

We have $y = p^{-1}\Theta(x)p$, so

$$\sigma(y) = f^{-1}p^{-1}\Theta(x^{\lambda})pf = f^{-1}y^{\lambda}f,$$

so this is the same as the Galois action we gave earlier on $\widehat{\pi}_1(M_{0,4}, \overrightarrow{01}) \simeq \widehat{F}_2$.

Let's show that the pair (λ, f) associated to $\sigma \in G_{\mathbb{Q}}$ satisfies defining relation (I) of $\widehat{GT}$.

For $\sigma \in G_{\mathbb{Q}}$, there also exists $g \in \widehat{\pi}_1(M_{0,4}, \overrightarrow{01})$ such that $\sigma(r) = rg$, since the endpoints of r are rational.

We have $p = \Theta(r)^{-1}r$, so

$$\sigma(p) = \Theta(rg)^{-1}rg = \Theta(g)^{-1}\Theta(r)^{-1}rg = \Theta(g)^{-1}pg$$

$$= pp^{-1}\Theta(g)^{-1}pg = pp^{-1}g(\Theta(x), \Theta(y))^{-1}pg$$

$$= pg(p^{-1}\Theta(x)p, p^{-1}\Theta(y)p)^{-1}g(x, y)$$

$$= pg(y, x)^{-1}g(x, y) = pf.$$

Therefore there exists $g \in \widehat{F}_2$ such that $f(x, y) = g(y, x)^{-1}g(x, y)$, so $f(x, y)$ certainly satisfies relation (I): $f(x, y)f(y, x) = 1$.

We show that relation (II) is satisfied similarly, by using a path from $\overrightarrow{01}$ to a 6th root of unity in $M_{0,4}$, and relation (III) is done using a path on the moduli space $M_{0,5}$. This explains geometrically why elements of $G_{\mathbb{Q}}$ satisfy (I), (II) and (III).

This gives rise to the natural question: does there exist a g such that $f(x,y) = g(y,x)^{-1}g(x,y)$ for all $(\lambda, f) \in \widehat{GT}$? And similarly for the other relations? The answer is yes.

Theorem 4. *Let $(\lambda, f) \in \widehat{GT}$, and let $m = (\lambda-1)/2$. Then there exist elements g and $h \in \hat{F}_2$ and $k \in \hat{\Gamma}_{0,5}$ such that we have the following equalities, of which the first two take place in $\hat{F}_2$ and the third in $\hat{\Gamma}_{0,5}$:*

$(I')\quad f = \theta(g)^{-1}g$

(II')

$$fx^m = \begin{cases} \omega(h)^{-1}h & \text{if } \lambda \equiv 1 \text{ mod } 3 \\ \omega(h)^{-1}\, xy\, h & \text{if } \lambda \equiv -1 \text{ mod } 3 \end{cases}$$

(III')

$$f(x_{12}, x_{23}) = \begin{cases} \rho(k)^{-1}k & \text{if } \lambda \equiv \pm 1 \text{mod } 5 \\ \rho(k)^{-1}\, x_{34}x_{51}^{-1}x_{45}x_{12}^{-1}\, k & \text{if } \lambda \equiv \pm 2 \text{ mod } 5. \end{cases}$$

Interpreting relations (I), (II) and (III) as cocycle relations as in the second definition ($\rho^4(f)\rho^3(f)\rho^2(f)\rho(f)f = 1$ etc.), the question turns into a computation of cocycles and coboundaries using non-commutative cohomology. Calculations of the non-commutative cohomology groups yields the result.

§1B.4. Questions: How similar is $\widehat{GT}$ to $G_{\mathbb{Q}}$?

(1) Is the "complex conjugation" element $(-1,1)$ self-centralizing in $\widehat{GT}$? This question was asked by Y. Ihara; its answer turns out to be **yes**. A computation reduces this result to showing that the only element of $\hat{F}_2$ which is fixed under the automorphism ι given by $\iota(x) = x^{-1}$ and $\iota(y) = y^{-1}$ is the trivial element, in other words that the centralizer of ι in the semi-direct product $\hat{F}_2 \rtimes \langle\iota\rangle$ is exactly $\langle\iota\rangle$. This can be shown using the same type of non-commutative cohomology calculation as earlier.

(2) The derived subgroup $\widehat{GT}'$ of $\widehat{GT}$ is contained in the subgroup $\widehat{GT}^1$ of pairs $(\lambda, f) \in \widehat{GT}$ with $\lambda = 1$. Are these two subgroups equal? This question was also asked by Y. Ihara, and remains unsolved.

(3) Are there "p-adic" subgroups of $\widehat{GT}$ that would correspond to the p-adic Galois groups $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ (up to conjugacy) in $G_{\mathbb{Q}}$? Yves André gave an answer to this question.

It follows naturally from his definition of the p-adic "tame fundamental group" of $\mathbb{P}^1 - \{0,1,\infty\}$, which can be considered as a subgroup of the algebraic fundamental group $\widehat{\pi}_1(\mathbb{P}^1 - \{0,1,\infty\})$. One then defines $\widehat{GT}_p$ to be the subgroup of $\widehat{GT}$ consisting of automorphisms fixing the p-adic fundamental group. For this definition, André proved that $G_{\mathbb{Q}} \cap \widehat{GT}_p = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

(4) Is there any torsion in $\widehat{GT}$ apart from the elements of order 2 given by $(-1,1)$ and its conjugates? After a short discussion with Florian Pop, we were able to prove the weak result that any such torsion elements become trivial in the *pro-nilpotent quotient* of $\widehat{GT}$; no stronger result seems to be known. Let us indicate the proof of this result by showing there are any elements of order 2 with $\lambda = 1$ become trivial in the pro-nilpotent quotient of $\widehat{GT}$ (note that any torsion element must have $\lambda = \pm 1$). Suppose there exists $(1,f) \in \widehat{GT}$ such that $(1,f)(1,f) = 1$. Then $f(x,y)f\big(x, f(y,x)yf(x,y)\big) = 1$. We know that $f \in \hat{F}_2'$. Considering this equation modulo the second commutator group $\hat{F}_2'' = [\hat{F}_2, \hat{F}_2']$, we see that modulo $\hat{F}_2''$, we have $f(x,y)^2 = 1$, so since there is no torsion in this group, we must have that $f(x,y) \in \hat{F}_2''$. Working modulo $\hat{F}_2'''$ and so on, we quickly find that f lies in the intersection of all the successive commutator subgroups of $\hat{F}_2$, i.e. that the image of $f(x,y)$ in the nilpotent completion $\hat{F}_2^{\mathrm{nil}}$ is trivial.

(5) Is the outer automorphism group of $\widehat{GT}$ trivial? (F. Pop, absolutely unsolved).

(6) Is it possible to determine the finite quotients of $\widehat{GT}$? (Everyone connected with inverse Galois theory; unsolved except for the obvious remark that $(\lambda, f) \mapsto \lambda$ gives a surjection $\widehat{GT} \to \hat{\mathbb{Z}}^*$, and therefore all abelian groups occur, just as for $G_{\mathbb{Q}}$.)

(7) Is the subgroup $\widehat{GT}^1$ of pairs (λ, f) with $\lambda = 1$ profinite free? (Everyone interested in the Shafarevich conjecture; unsolved.)