Grothendieck-Teichmüller Lie theory

and multiple zeta values

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MIT, November 5-9, 2012

Lecture 2A

The graded Grothendieck-Teichmüller Lie algebra

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The graded Grothendieck-Teichmüller Lie algebra

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Definition

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B) $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the Deligne-Ihara Lie algebra DI_p

Definition

Ihara's theorem: injection $DI_p \hookrightarrow \mathfrak{grt} \otimes \mathbb{Q}_p$

C) Kashiwara-Vergne Lie algebra

The Kashiwara-Vergne problem (Alekseev-Torossian)

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Definitions, properties of MTM and motivic multizetas

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Corollaries: $DI_p = \mathfrak{fr} \otimes \mathbb{Q}_p$ and $\mathfrak{fr} \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds}$.

The graded Grothendieck-Teichmüller Lie algebra

Let $\mathrm{Lie}\, P_n$ be the **n-strand braid Lie algebra** given by

Generators: x_{ij}, $1 \leq i < j \leq 5$

Relations: $[x_{ij}, x_{ij} + x_{ik} + x_{jk}] = 0$

$$\sum_{j \neq i} x_{ij} = 0$$

$$[x_{ij}, x_{kl}] = 0 \text{ if } \{i,j\} \cap \{k,l\} = \emptyset$$

In particular, $\mathrm{Lie}\, P_4$ is free on x_{12}, x_{23} and $\mathrm{Lie}\, P_5$ is generated by $x_{12}, x_{23}, x_{34}, x_{45}, x_{15}$.

Definition. The Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ is given by

$$\begin{aligned} \mathfrak{grt} = \Big\{ f \in & \mathrm{Lie}_{\geq 3}[x,y] \,|\, \text{(I)} \quad f(x,y) + f(y,x) = 0 \\ & \text{(II)} \quad f(x,y) + f(z,x) + f(y,z) = 0 \text{ if } x+y+z=0 \\ & \text{(III)} \quad f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12}) \\ & \qquad + f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0 \quad \text{in} \quad \mathrm{Lie}\, P_5 \Big\} \end{aligned}$$

Every $f \in \mathrm{Lie}[x,y]$ yields a derivation D_f of $\mathrm{Lie}[x,y]$:

$$D_f(x) = 0, \quad D_f(y) = [y, f].$$

We can put a Lie bracket on $\mathrm{Lie}[x,y]$ called the Poisson or Ihara bracket via

$$\{f, g\} = [f, g] + D_f(g) - D_g(f),$$

corresponding to bracketing derivations:

$$[D_f, D_g] = D_{\{f,g\}}.$$

Theorem 1. (Ihara) $\mathfrak{grt}$ *forms a Lie algebra under the Poisson bracket.*

Definition. A **special derivation** D of $\mathrm{Lie}\, P_n$ is a derivation such that for $1 \leq i < j \leq n$, there exists $t_{ij} \in \mathrm{Lie}\, P_n$ such that $D(x_{ij}) = [x_{ij}, t_{ij}]$. Special derivations (also special outer derivations) form a graded Lie algebra. S_n acts on $\mathrm{Lie}\, P_n$ by permuting the indices of the x_{ij}. Let $\mathcal{D}_n$ denote the Lie algebra of S_n-invariant special outer derivations of $\mathrm{Lie}\, P_n$. (S_n-invariant means invariant under the action of S_n on special outer derivations induced by $\sigma \circ D \circ \sigma^{-1}$.)

Theorem 2. (Ihara) *Let* $\mathfrak{grt}_0$ *be defined by relations (I) and (II) only. Then* $\mathcal{D}_4 = \mathfrak{grt}_0$ *and* $\mathcal{D}_n = \mathfrak{grt}$ *for all* $n \geq 5$.

Remarks. (1) In other words, only relation (III) is needed to ensure that elements of $\mathfrak{grt}_0$ extend to $\mathcal{D}_5$, and then they automatically also act on $\mathcal{D}_n$ for all n since in fact the $\mathcal{D}_n$ are all equal for $n \geq 5$.

(2) This is the exact Lie analog of the profinite result showing that

$$\begin{cases} \widehat{GT}_0 \simeq \mathrm{Out}^*_{S_4}(\widehat{\Gamma}_{0,4}) \\ \widehat{GT} = \mathrm{Out}^*_{S_5}(\widehat{\Gamma}_{0,5}) = \mathrm{Out}^*_{S_n}(\widehat{\Gamma}_{0,n}). \end{cases}$$

Theorem 3. (Furusho) *If $f \in \mathrm{Lie}[x, y]$ satisfies relation (III), then it satisfies (I) and (II); thus $\mathfrak{grt}$ needs only one defining relation.*

Remark. The implication (III) $\Rightarrow$ (I) is trivial by pulling out one strand of the braid, and was observed for both $\mathfrak{grt}$ and $\widehat{GT}$ by Furusho in 1999. But his new implication (III) $\Rightarrow$ (II) led to a lot of research trying to prove the same result for $\widehat{GT}$. This would be a very useful result, as it would make it possible to compute the image of $\widehat{GT}$ in finite quotients using the coboundary expressions of elements of $\widehat{GT}$. Then we could compare those images on examples directly with the Galois image.

Unfortunately, the attempts actually ended up giving some heuristic arguments why in fact $\widehat{GT}$ may simply not satisfy the implication (III) $\Rightarrow$ (II). This is maybe the first important result about $\mathfrak{grt}$ that seems to have a chance of actually being false for $\widehat{GT}$, so a significant difference between the profinite and Lie situations.

Sketch of proof. Write $f_{ijk} = f(x_{ij}, x_{jk})$, so the pentagon is $f_{123} + f_{345} + f_{512} + f_{234} + f_{451} = 0$. Any permutation of $\{1, 2, 3, 4, 5\}$ also yields a valid pentagon, and Furusho thus adds up four pentagons differing by permutations to get

$$\big(f_{123}+f_{345}+f_{512}+f_{234}+f_{451}\big)+\big(f_{431}+f_{125}+f_{543}+f_{312}+f_{254}\big)$$

$$+\big(f_{542}+f_{231}+f_{154}+f_{423}+f_{315}\big)+\big(f_{134}+f_{425}+f_{513}+f_{342}+f_{251}\big) = 0.$$

Simplifying by $f_{ijk} = -f_{kji}$ leaves only twelve terms:

$$\big(f_{123}+f_{231}+f_{312}\big)+\big(f_{512}+f_{125}+f_{251}\big)$$
$$+\big(f_{234}+f_{423}+f_{342}\big)+\big(f_{254}+f_{542}+f_{425}\big)=0.$$

This is in fact a sum of four permuted copies of relation (II). (Furusho claims to have thought of this proof while watching a soccer match and admiring the way pentagons surround hexagons on the ball.) Furusho then uses a cute trick to show that if this sum of four is zero, then each one must be zero.

The trick consists in noting that the remaining sum of four triangles can be written in the four variables $x_{12}, x_{23}, x_{24}, x_{25}$. Indeed, the first triangle contains x_{12}, x_{13}, x_{23}, but since $x_{12}+x_{13}+x_{23}$ commutes with all three, bracketing any expression with x_{13} is equal to bracketing it with $-x_{12}-x_{23}$, so x_{13} can be eliminated from the first triangle, which can be written as a polynomial $R(x_{12}, x_{23})$. Similarly, the second can be expressed in x_{12}, x_{25}, the third in x_{23}, x_{24} and the fourth in x_{24}, x_{25}. The four variables sum to zero.

Furusho then considers each of the four triangles under the mapping $x_{12}\mapsto x$, $x_{23}\mapsto y$, $x_{24}\mapsto x$ and $x_{25}\mapsto -2x-y$, obtaining $R(x,y)$, $R(x,y)$, $R(x,-2x-y)$, $R(x,-2x-y)$.

Thus the sum of four relations maps to $2R(x,y)+2R(x,-2x-y)$. But Furusho shows that for a Lie element of this form, we have $R(x,-2x-y)=R(2x,y)=2R(x,y)$, so we see that $R(x,y)=0$, i.e. f satisfies relation (II). QED

Modular forms and Bernoulli numbers in $\mathfrak{grt}$

The Lie algebra $\mathfrak{grt}$ is weight-graded (i.e. by degree), and it has a decreasing filtration by depth (=smallest number of y's in any monomial).

It is easy to calculate elements of $\mathfrak{grt}$ in small weight. In weights 3, 5, 7 and 9 $\mathfrak{grt}_n$ is 1-dimensional; in weights 4 and 6 it is zero. Let f_n be a generator for $n = 3, 5, 7, 9$, normalized to have integer coefficients. We have

$$f_3 = [x, [x, y]] - [[x, y], y].$$

Ihara discovered a surprising identity:

$$2\{f_3, f_9\} - 27\{f_5, f_7\} \equiv 0 \mod 691.$$

This polynomial also has no terms of depth < 4.

Exploration of the latter fact led to the following theorems.

Theorem 4. (Ihara,Takao) *For even $n \geq 12$, the dimension of the space of linear combinations*

$$\sum_{i=1}^{[\frac{n-4}{4}]} a_i \{f_{2i+1}, f_{n-2i-1}\} \qquad (*)$$

having no terms in depth < 4 is equal to

$$\dim S_k\big(\mathrm{SL}_2(\mathbb{Z})\big) = [\frac{n-4}{4}] - [\frac{n-2}{6}].$$

Definition. Let n be even ≥ 12 and let $g(z)$ be a cusp form of weight n for $\mathrm{SL}_2(\mathbb{Z})$. The **reduced period polynomial** of g is the polynomial

$$= \int_0^{+i\infty} g(z)(z - X)^{n-2} dz$$

minus its $(X^{n-2} - 1)$-factor.

Theorem 5. (S,2006) *Normalize the f_n so that the coefficient of $x^{n-1}y$ is equal to 1. Then a linear combination of the form*

$$\sum_{i=1}^{[\frac{n-4}{4}]} a_i \{f_{2i+1}, f_{n-2i-1}\}$$

has no terms in depth < 4 if and only if the associated polynomial

$$P(X) = \sum_{i=1}^{[\frac{n-4}{4}]} a_i (X^{n-2i-2} - X^{2i})$$

is the reduced period polynomial of a cusp form of weight n for $\mathrm{SL}_2(\mathbb{Z})$.

Example. The modified period polynomial of the weight 12 Ramanujan $\Delta(z)$ is

$$(X^8 - X^2) - 3(X^6 - X^4).$$

Ihara's polynomial rewritten for normalized f_n becomes:

$$\{f_3, f_9\} - 3\{f_5, f_7\} \text{ has no terms of depth } < 4.$$

The appearance of the numerators of Bernoulli numbers (such as 691 in weight 12) is still mysterious!

Such congruences occur in higher weight, but in a less obvious manner since when $\dim \mathfrak{grt}_n > 1$ (i.e. when $n > 10$), there is no single choice of a generator f_n, so one has to ask whether there exist choices that work?

For example, we have $\dim \mathfrak{grt}_{11} = 2$ and $\dim \mathfrak{grt}_{13} = 3$. There exists a single (reduced) period polynomial in weight 16, given by

$$2(X^12 - X^2) - 7(X^10 - X^4) + 11(X^8 - X^6).$$

Fix a choice of f_{11} and f_{13}. Then all choices are given by $f_{11} + a\{f_3, \{f_3, f_5\}\}$ and $f_{13} + b\{f_5, \{f_3, f_5\}\} + c\{f_3, \{f_3, f_7\}\}$. There are an infinite number of choices for the parameters (a, b, c) to make the Bernoulli congruence

$$2\{f_3, f_{13}\} - 7\{f_5, f_{11}\} + 11\{f_7, f_9\} \equiv 0 \mod 3617$$

hold, but not all choices work. We call these elements arising from period polynomials "modular elements" in $\mathfrak{grt}$.

Open question: When do such congruences occur?

Numerical exploration gave rise to another, similar phenomenon.

Observation (in small weight): Let $P(X,Y)$ be a homogenized period polynomial in weight n (degree $n-2$), let M denote the associated modular element for some choice of f_n, and let T denote the "truncated" modular element, meaning that we keep only the terms of the form $x^a y x^b y^3$. Write

$$T = \sum_{(a,b)} c_{a,b} x^a y x^b y^3$$

and

$$Q(X,Y) = \sum_{(a,b)} c_{a,b} X^{a+1} Y^{b+1}$$

for commutative variables X, Y. Then there exist choices of f_n for which

$$Q(X,Y) = k\, P(X,Y)$$

and $k \equiv 0$ modulo a prime dividing the numerator of the Bernoulli number B_n. This phenomenon seems to occur at the same time as the Ihara congruences, and the constant seems to be divisible by the same Bernoulli numerator.

At least we have a conjectural answer about when this happens, if not yet about why.

Conjecture. (S) *Let $n \geq 12$ be even and*

- *let p be a prime dividing $\frac{B_n}{n}$*

- *let $g(z)$ be a cusp form g of weight n for $\mathrm{SL}_2(\mathbb{Z})$ which is an eigenform for the Hecke operators*

- *let $P(X,Y) = \sum_{i=1}^{\frac{n-4}{2}} a_i X^{n-2-2i} Y^{2i}$ denote the modified period polynomial of g.*

- *let M denote the associated modular element*

$$\sum_{i=1}^{[\frac{n-4}{4}]} a_i \{f_{2i+1}, f_{n-2i-1}\}$$

- *let $Q(X,Y)$ denote the associated truncated polynomial*

- *let $G_n(z)$ denote the Eisenstein series of weight n.*

Then the following are equivalent:

(1) $g(z) \equiv G_n(z) \mod p$.

(2) there exists a choice of depth 1 elements $f_j \in ds$ for odd $j \geq 3$ such that $M \equiv 0$ moduli p.

(3) there exists a choice of depth 1 elements $f_j \in ds$ for odd $j \geq 3$ such that the truncated polynomial $Q(X,Y)$ associated to M satisfies

$$Q(X,Y) = k\,P(X,Y)$$

with $k \equiv 0$ mod p.