An introduction to

Profinite Grothendieck-Teichmüller theory

MIT, Cambridge, Massachusetts

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Lecture 3A

Braided tensor categories and geometry

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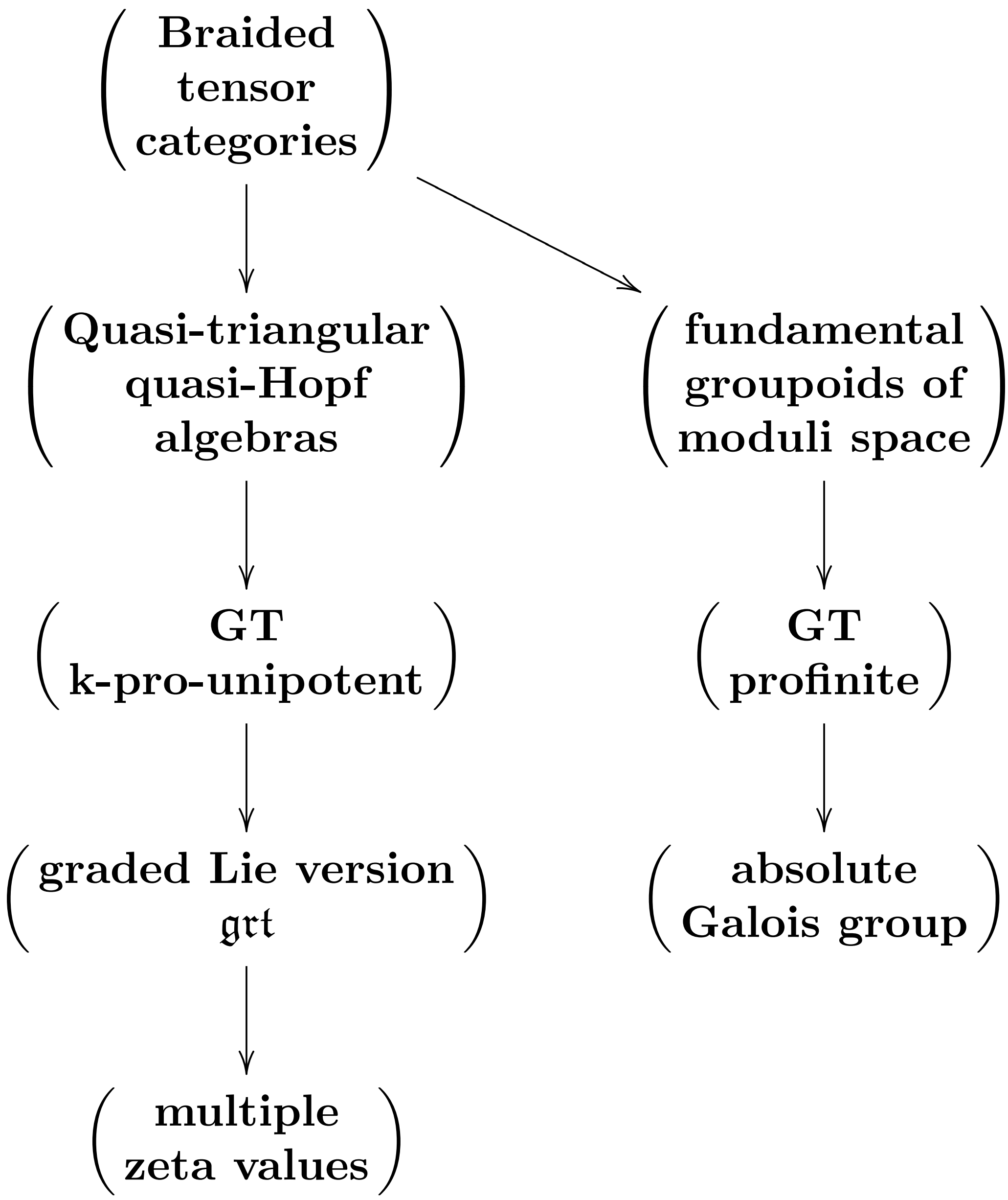
Roles and avatars of $\widehat{GT}$

The relation structure of the different objects we're considering in these lectures can be schematized as follows:



§3A.1. Braided tensor categories

Definition. Let $\mathcal{C}$ be a category and f a morphism of $\mathcal{C}$. A *tensor product* on $\mathcal{C}$ is a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Definition. An *associativity constraint* on a category with tensor product is a set of isomorphisms

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

for every triple of objects (U, V, W), such that for every triple (f, g, h) of morphisms of $\mathcal{C}$ with

$$\begin{cases} f : U \to U' \\ g : V \to V' \\ h : W \to W' \end{cases},$$

the following diagram commutes

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ \downarrow{\scriptstyle (f\otimes g)\otimes h} & & \downarrow{\scriptstyle f\otimes(g\otimes h)} \\ (U' \otimes V') \otimes W' & \xrightarrow{a_{U',V',W'}} & U' \otimes (V' \otimes W'). \end{array}$$

An associativity constraint *satisfies the pentagon axiom* if for every quadruple of objects (U, V, W, X) of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
((U \otimes V) \otimes W) \otimes X & \xrightarrow{a_{U,V,W} \otimes \mathrm{id}_X} & (U \otimes (V \otimes W)) \otimes X \\
\downarrow{\scriptstyle a_{U \otimes V, W, X}} & & \downarrow{\scriptstyle a_{U, V \otimes W, X}} \\
(U \otimes V) \otimes (W \otimes X) & & \\
\downarrow{\scriptstyle a_{U,V,W \otimes X}} & & \\
U \otimes (V \otimes (W \otimes X)) & \xleftarrow{\mathrm{id}_U \otimes a_{V,W,X}} & U \otimes ((V \otimes W) \otimes X).
\end{array}$$

Definition. A *monoidal tensor category* is a category $\mathcal{C}$ with a tensor product with an associativity constraint satisfying the pentagon axiom*.

* Fairly large part of the definition left out, with further axioms.

Definition. A Let $\mathcal{C}$ be a tensor category. A *commutativity constraint* on $\mathcal{C}$ is a collection of isomorphisms $c_{U,V} : U \otimes V \to V \otimes U$ for every pair (U, V) of objects of $\mathcal{C}$, such that for every pair of morphisms (f, g) of $\mathcal{C}$ with

$$\begin{cases} f : V \to V' \\ g : W \to W' \end{cases}$$

the following diagram commutes

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V'. \end{array}$$

The commutativity constraint is said to *satisfy the two hexagon axioms* if the two following diagrams commute for every triple (U, V, W) of objects of $\mathcal{C}$:

$$\begin{array}{ccccc}
(U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) & \xrightarrow{c_{U,V \otimes W}} & (V \otimes W) \otimes U \\
\downarrow{\scriptstyle c_{U,V} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle a_{V,W,U}} \\
(V \otimes U) \otimes W & \xrightarrow{a_{V,U,W}} & V \otimes (U \otimes W) & \xrightarrow{\mathrm{id} \otimes c_{U,W}} & V \otimes (W \otimes U)
\end{array}$$

and

$$\begin{array}{ccccc}
(U \otimes V) \otimes W & \xrightarrow{c_{U \otimes V,W}} & W \otimes (U \otimes V) & \xrightarrow{a^{-1}_{W,U,V}} & (W \otimes U) \otimes V \\
\uparrow{\scriptstyle a^{-1}_{U,V,W}} & & & & \uparrow{\scriptstyle c_{U,W} \otimes \mathrm{id}_V} \\
U \otimes (V \otimes W) & \xrightarrow{\mathrm{id}_U \otimes c_{V,W}} & U \otimes (W \otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U \otimes W) \otimes V.
\end{array}$$

Definition. A *braided tensor category* is a category with

- a tensor product
- an associativity constraint
- a commutativity constraint,

satisfying the pentagon and hexagon axioms.

§3A.2. The braided tensor category of trivalent trees

In this section we construct an actual example of a combinatorial braided tensor category $\mathcal{C}$. It is a groupoid, i.e. all morphisms are isomorphisms.

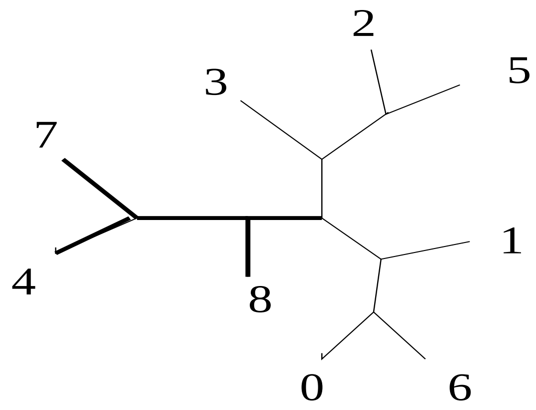
* The set of *objects* of $\mathcal{C}$ is the set of trivalent trees equipped with:

(i) a cyclic order on the edges coming out of each vertex;

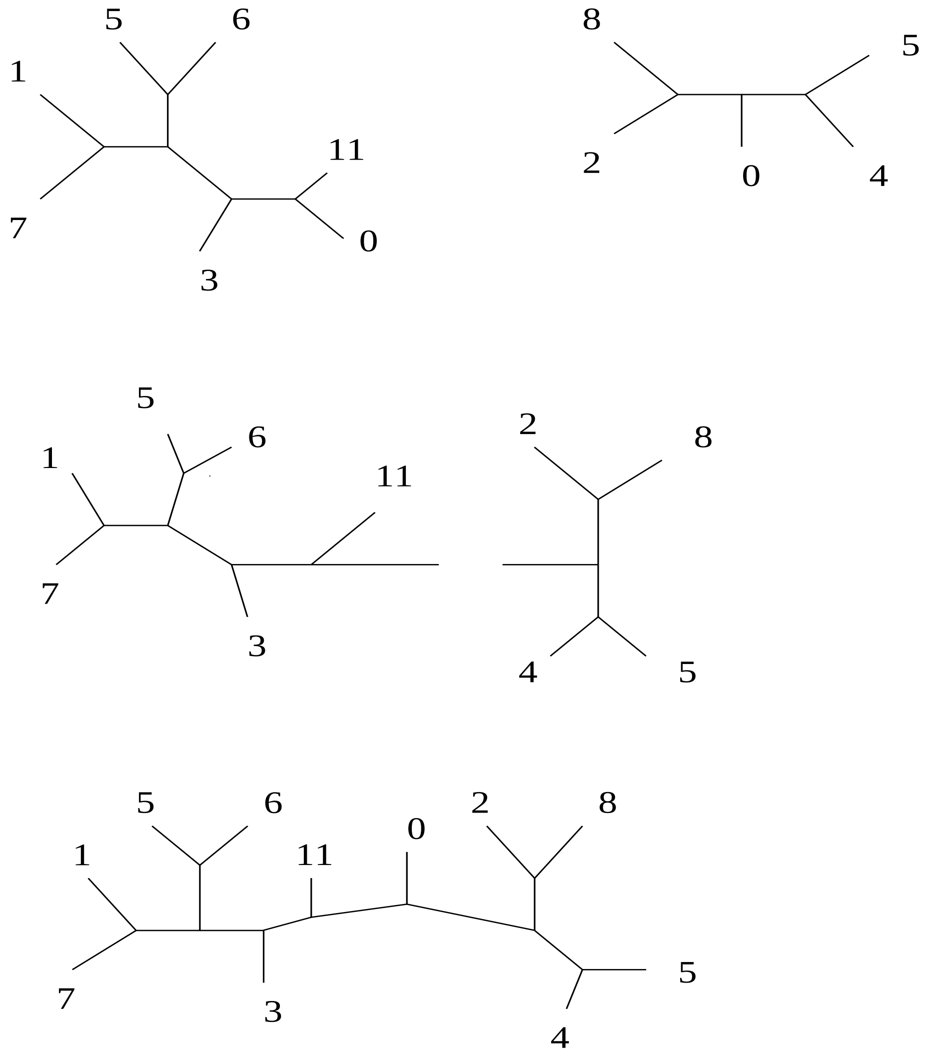
(ii) a numbering on the tails of the following type: one tail is distinguished and numbered 0, and every other tail is indexed with a strictly positive integer.

The n-tailed trees are called $(n-1)$-objects of $\mathcal{C}$. The tail numbered 0 is the *distinguished edge*, and its trivalent vertex is *distinguished* too. Note that this definition automatically equips trees with a *cyclic order on the indices of tails*.

Let a *branch* denote the part of a tree determined by a given vertex and a given edge coming out of it.



* The *tensor product* of two trees T and T' is given by gluing together their 0 tails, adding a new 0 tail, and equipping the new trivalent vertex with the cyclic order 0, T, T'. The tensor product of an n-object with an m-object is an $(n+m)$-object.

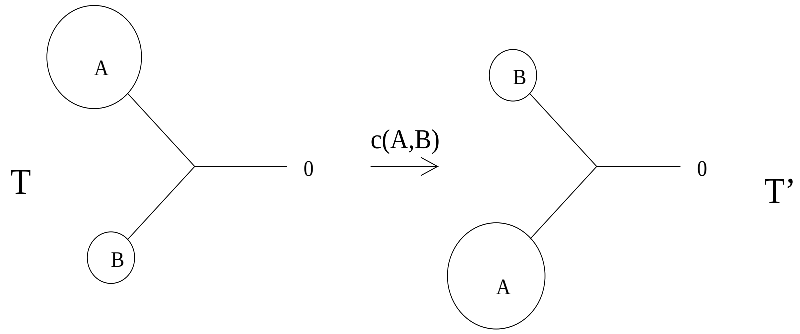


Remark. Every n-object of $\mathcal{C}$ for $n \geq 1$ decomposes uniquely into a tensor product of n not necessarily distinct 1-objects.

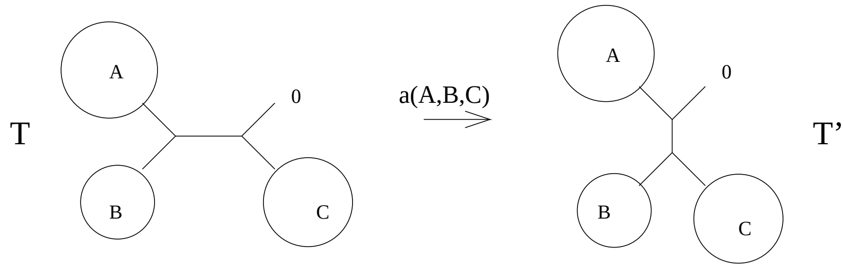
* There are three types of *elementary morphisms* in $\mathcal{C}$; all morphisms are obtained by taking compositions and tensor products of these.

Identity morphisms. Every object A of $\mathcal{C}$ has an identity morphism id_A.

Commutativity morphisms. A tree T is changed to another tree T' by flatly exchanging the two branches coming out of the distinguished vertex.



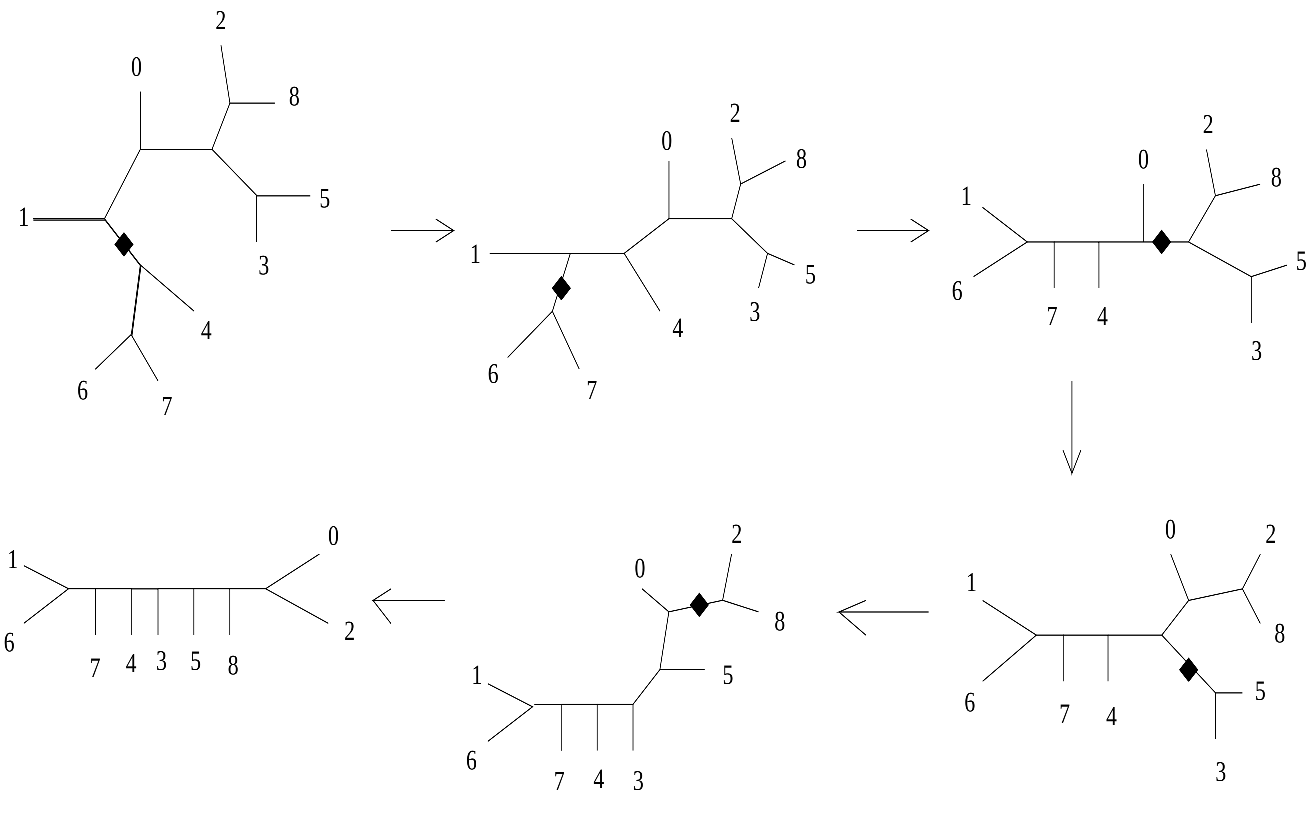
Associativity morphisms. If T has at least three vertices, then let C' and C denote the two branches coming out of the distinguished vertex, so that $T = C' \otimes C$. The associativity morphism is only defined on T if C' has more than one edge, so that $C' = A \otimes B$. Then $T = (A \otimes B) \otimes C$, and the associativity morphism changes it to $T' = A \otimes (B \otimes C)$.



* *Relations in* $\mathcal{C}$. The pentagon and hexagon axioms. If two sequences of a and c morphisms go from T to T', they are identified in $Hom(T, T')$ if and only if one can be obtained from the other by a finite number of substitutions from the pentagon and hexagon diagrams.

Proposition. *If T and T' are two objects of $\mathcal{C}$ having the same indices on the tails in the same cyclic order, then there exists a unique morphism in $\mathcal{C}$ built only of identity and associativity morphisms, taking T to T'.* (You can do it in many ways, but the pentagon assures that they are all equivalent as morphisms.)

Proof by picture:

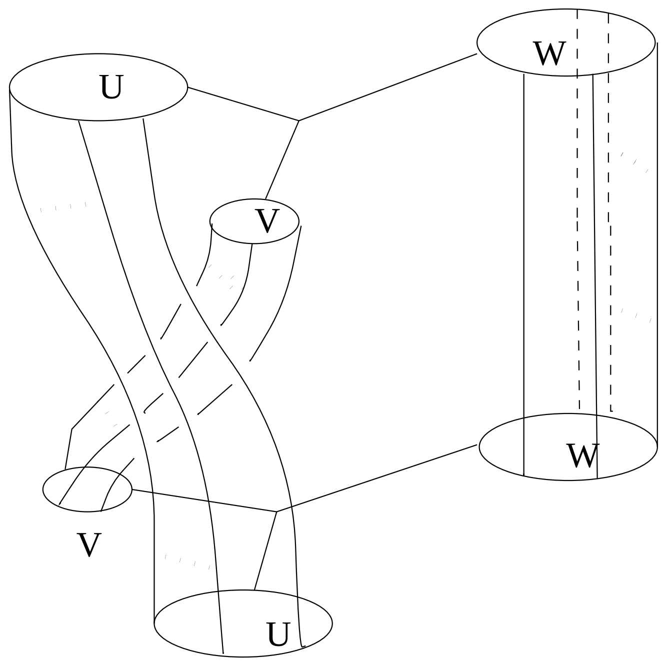


QED

The way to visualize the morphisms of $\mathcal{C}$ is by visualizing objects of $\mathcal{C}$ as having a strand hanging from each tail except 0: then

• commutativity morphisms correspond to crossing two packets of strands whenever the two packets of strands correspond to branches meeting at a vertex;

• associativity morphisms don't braid any strands but change the internal form of the tree. By the Proposition, this enables one to get any two adjacent packets of tails into adjacent branches, for instance in order to apply the next commutativity morphism.

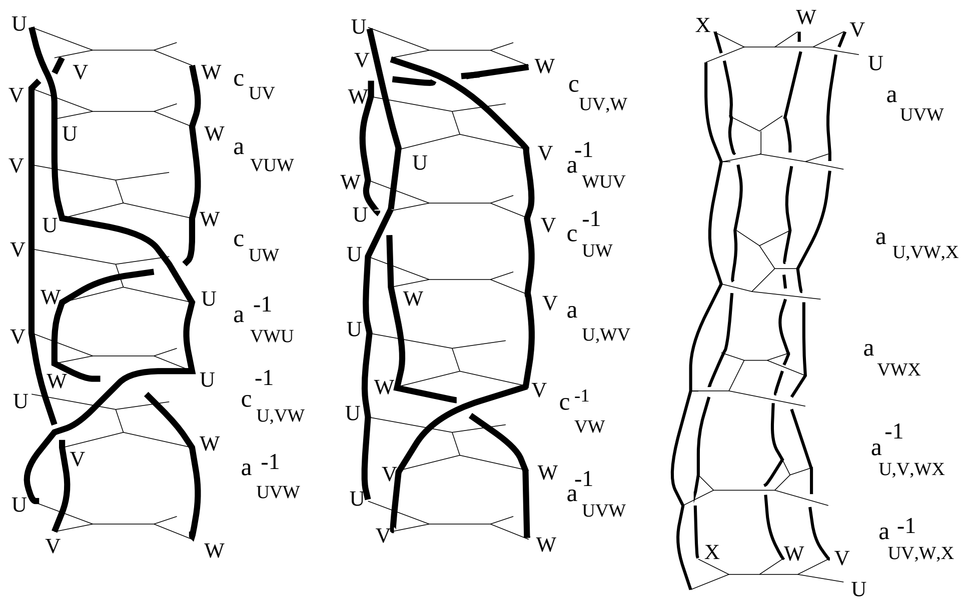


Proposition. *Let T be an n-object of $\mathcal{C}$ with distinct indices. Then*

$$\mathrm{Hom}(T,T) \simeq K_n$$

where K_n is the pure Artin braid group on n strands.

Proof. The way of visualizing morphisms in the tree category is the main point of the proof. The hexagon and pengagon axioms are the following braid identities:



The usual braid identities ($\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and commutation) are more complicated diagrams of a's and c's but can be deduced from the hexagons and pentagon. QED

Definition. Let an automorphism of a pure braided tensor category $\mathcal{C}$ (like the tree category) be a morphism $F : \mathcal{C} \to \mathcal{C}$ which fixes objects of $\mathcal{C}$ and respects the tensor product. Such a morphism modifies only the commutativity and associativity constraints.

Drinfel'd determined the description of such morphisms F. He found that the commutativity and associativity morphisms must be modified as follows.

For objects U, V and W of $\widehat{\mathcal{C}}$, set

$$\begin{cases} T & = (U \otimes V) \otimes W \\ T_1 & = (V \otimes U) \otimes W \\ T_2 & = U \otimes (V \otimes W) \\ T_3 & = U \otimes (W \otimes V) \\ x^T_{UV} & = c_{T_1}(V, U) c_T(U, V) \\ x^T_{VW} & = a(U, V, W)^{-1} c_{T_3}(W, V) c_{T_2}(V, W) a(U, V, W). \end{cases}$$

Then F must act via a pair $(m, f) \in \mathbb{Z} \times F_2$, by

$$F\big(c_T(U, V)\big) = c_T(U, V)(x^T_{UV})^m = (x^{T_1}_{VU})^m c_T(U, V)$$

$$F\big(a(U, V, W)\big) = a(U, V, W) f(x^T_{UV}, x^T_{VW}).$$

In order to respect the two hexagons, the pair (m, f) must satisfy

(I) $f(y, x)f(x, y) = 1$ in F_2;

(II) $f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$ in F_2, where $z = (xy)^{-1}$.

In order to satisfy the pentagon, it must satisfy

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$

in the braid group B_4, where $x_{45} = x_{12}x_{13}x_{23}$ and $x_{51} = x_{23}x_{24}x_{34}$ (actually Drinfel'd's expression was uglier, this symmetric expression is due to Ihara).

Definition. The *discrete Grothendieck-Teichmüller group GT* is the set of pairs $(\lambda, f) \in \mathbb{Z} \times F_2$ such that setting $m = (\lambda - 1)/2$, the pairs (m, f) satisfy (I), (II) and (III).

Disgruntling Lemma. $GT = \{(1, 1), (-1, 1)\}$.

Corollary. *At any rate, GT is a group.*

Drinfeld defined a quasi-triangular quasi-Hopf algebra over a field k with axioms ensuring essentially that the category of representations of a qtqH algebra is precisely a braided tensor category.

Drinfel'd then considered the case where the qtqH algebra is a *quantized universal enveloping algebra* over the ring $k[[h]]$ for a characteristic 0 field k: this is a topological quasi-triangular quasi-Hopf algebra $(A, \Delta, \epsilon, \Phi, R)$ over $k[[h]]$ such that A/hA is a universal enveloping algebra and A (as a topological $k[[h]]$-module) is isomorphic to $V[[h]]$ where V is some vector space over k.

The group of modifications of the braided tensor category in this situation is more than a discrete group; it is a k-pro-unipotent group that he denoted by $GT(k)$ and which is much bigger than $\{(1,1),(1,-1)\}$. There is a Lie algebra associated to the k-pro-unipotent group which is equal to $\mathfrak{grt}$ when $k = \mathbb{Q}$.

However, here we want to go directly to a profinite version of $\widehat{GT}$.

§3A.3. The Teichmüller groupoid and tangential base points

A *groupoid* is a category all of whose morphisms are isomorphisms. In particulier, for any object P of a groupoid, $\mathrm{Hom}(P,P)$ is a group.

If $\tilde{P}$ is a simply connected region of X containing a point P_0, we write $\pi_1(X;\tilde{P})$ for the group of homotopy classes of paths whose endpoints lie in $\tilde{P}$. This group is canonically isomorphic to $\pi_1(X;P_0)$, so from now on, we consider a "base point" to be a simply connected region, and a set of base points will be a set of disjoint simply connected regions.

Definition. The *fundamental groupoid* $\pi_1(X;A)$ of X based at a set A of base points is the set of homotopy classes of paths on X whose endpoints lie in A.

Because the space X is connected, all the groups $\mathrm{Hom}(P,P) = \pi_1(X;P)$ for $P \in A$ are isomorphic.

From now on, let $X = \mathcal{M}_{0,n}$, the moduli space of Riemann spheres with n distinct ordered marked points. Recall that

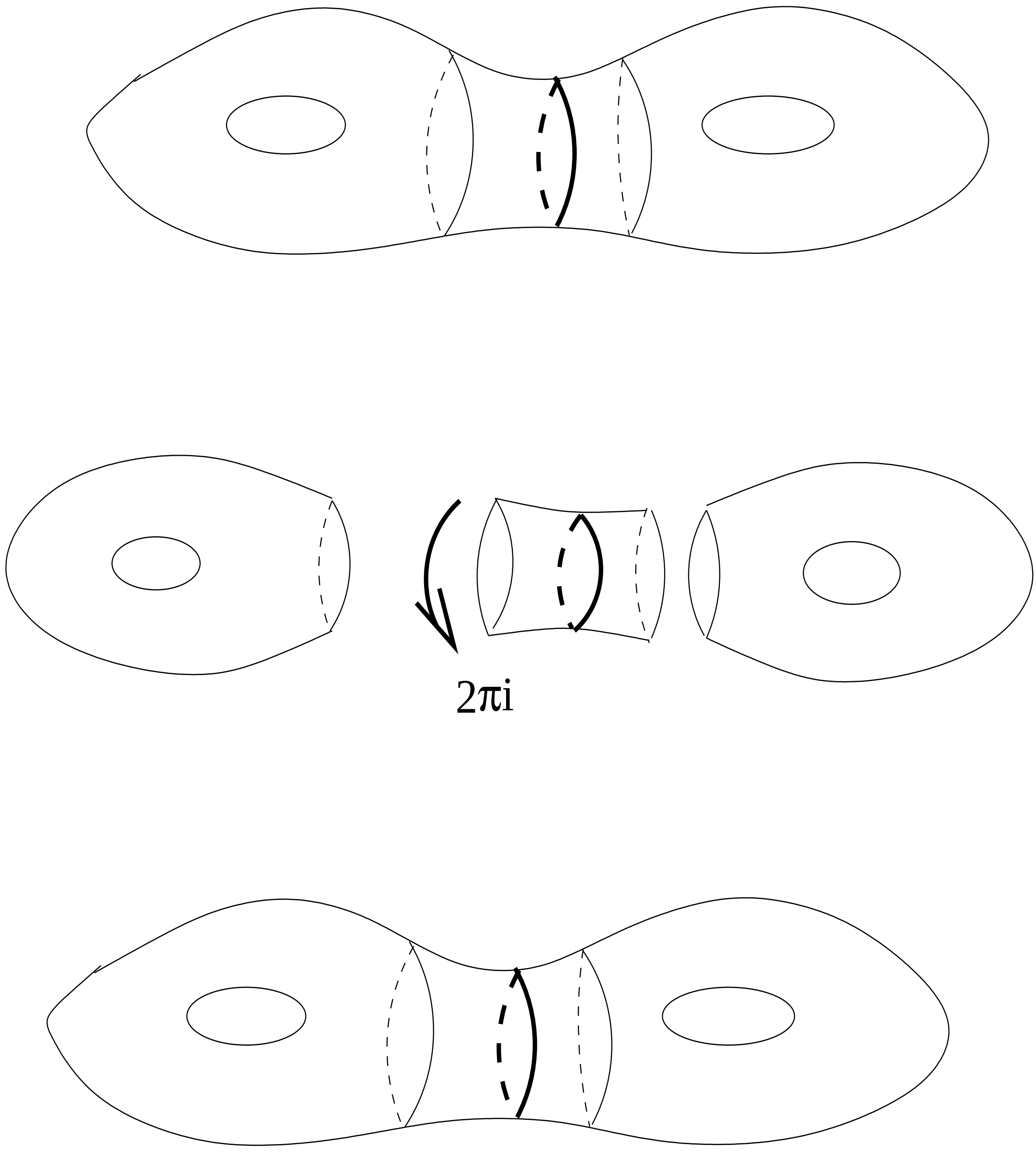
$$\mathcal{M}_{0,n} \simeq (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \Delta.$$

Recall also that we have an isomorphism

$$\pi_1(\mathcal{M}_{0,n}; x_0) \simeq \Gamma_{0,n} = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^0(\Sigma),$$

where Σ is a topological surface of type (g, n), and the right-hand group is the pure mapping class group (not permuting the marked points of Σ).

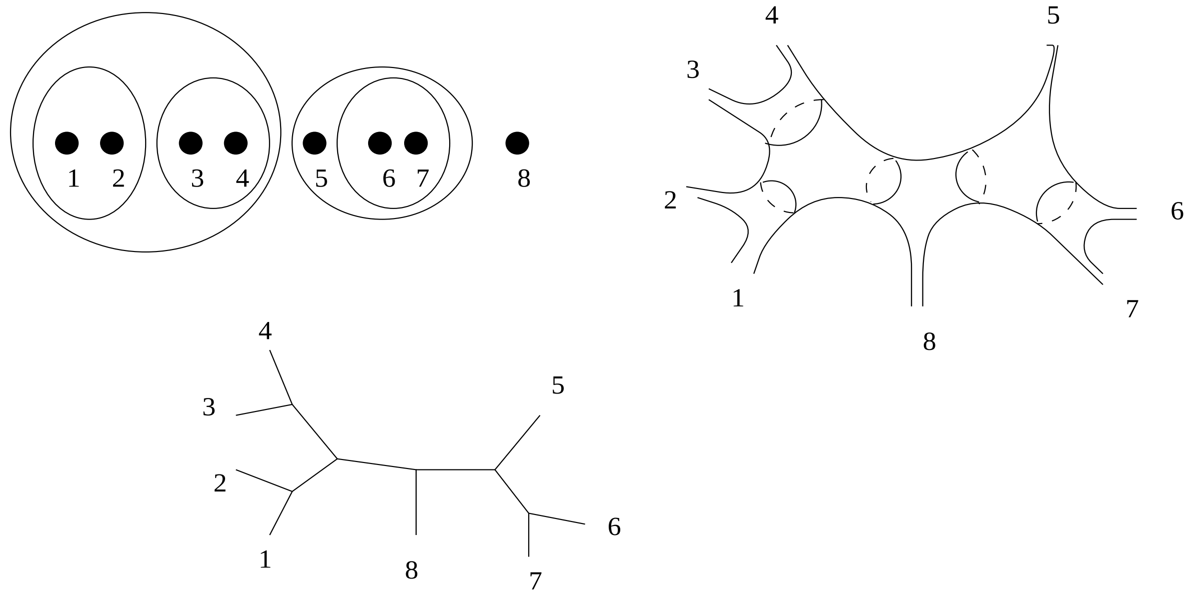
As we saw, this group is generated by the diffeomorphisms known as *Dehn twists* along simple closed loops.


2πi

• A *pants decomposition* of a topological sphere with n ordered punctures is a choice of $n-3$ disjoint simple closed loops cutting it into $n-3$ three-holed spheres, modulo the action of the pure mapping class group.

• A Riemann sphere with n marked ordered points is said to be *almost degenerate* if there exists a pants decomposition on it some of whose geodesic circles have length smaller than some small ϵ; if they all are, it is *almost maximally degenerate.* The set of almost degenerate spheres forms the *neighborhood of infinity* in $\mathcal{M}_{0,n}$; the set of almost maximally degenerate spheres forms the *neighborhood of the points of maximal degeneration.*

• A *point of maximal degeneration* is a Riemann sphere with n points, equipped with a pants decomposition the length of all of whose circles is zero; it corresponds to a point in the *stable compactification* of $\mathcal{M}_{0,n}$. Such a point is determined by a pants decomposition, and a pants decomposition is also determined by an n-tailed numbered tree.



Proposition. *The neighborhood in $\mathcal{M}_{0,n}$ of a given point of maximal degeneration is homeomorphic to D_*^{n-3}. The real part of this neighborhood (corresponding to spheres with n points defined over $\mathbb{R}$) falls naturally into 2^{n-3} simply connected regions.*

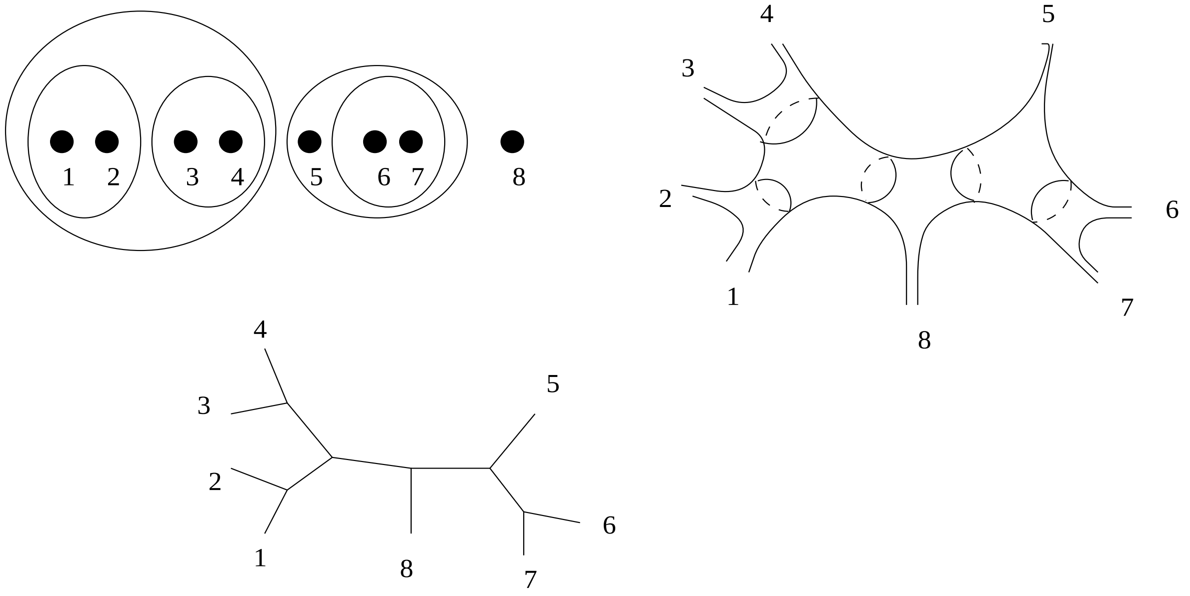
Definition. For $n \geq 4$, let $\mathcal{B}_n$ denote the union of these 2^{n-3} regions in $\mathcal{M}_{0,n}$ over each of the points of maximal degeneration in $\overline{\mathcal{M}_{0,n}}$. We call $\mathcal{B}_n$ the set of *tangential base points.* Let $\mathcal{C}_n$ denote the full subcategory of $\mathcal{C}$ whose objects are the n-tailed trees numbered $0, 1 \ldots, n-1$ (equipped with the cyclic order on the vertices).

Proposition. *The set $\mathcal{B}_n$ is in bijection with the objects of $\mathcal{C}_n$.*

Proof. As in the diagram above, the non-planar n-tailed trees indexed from 0 to $n-1$ (or from 1 to n) are in bijection with the points of maximal degeneration of $\mathcal{M}_{0,n}$. The objects of $\mathcal{C}_n$ are these trees equipped with all possible choices of cyclic order on the indices of the tails, which is equivalent to equipping them with planar embeddings. A trivalent n-tailed tree has $n-3$ inner edges, each of which can be "flipped" to give all the planar embeddings, so there are 2^{n-3} trees in $\mathcal{C}_n$ for each non-planar tree. Thus there is a bijection with $\mathcal{B}_n$.

It can be given a more precise geometric meaning by placing the tree inside a circle representing the real equator of $\mathbb{P}^1$. Place the 0 tail on the edge at ∞ and the attached trivalent vertex at the center of the circle; the two remaining branches extend towards 0 and 1 with branchings corresponding to decreasing orders of magnitude (FIGURE NEEDED HERE).

Example: The 8-tailed tree in the figure



corresponds to a near-degenerate tangential base point described by small real values of ϵ in the sphere with marked points:

$$(\infty, -\epsilon-\epsilon^2, -\epsilon+\epsilon^2, \epsilon, 1-\epsilon-\epsilon^2, 1-\epsilon+\epsilon^2, 1+\epsilon-\epsilon^2, 1+\epsilon+\epsilon^2).$$

§3A.4. The tree category and the Teichmüller groupoid

Definition. The *Teichmüller groupoid* of $\mathcal{M}_{0,n}$ is the fundamental groupoid based at the set $\mathcal{B}_n$ of tangential base points, i.e. $\pi_1(\mathcal{M}_{0,n};\mathcal{B}_n)$.

Main Structure Theorem. For $n \geq 4$, there is a surjection

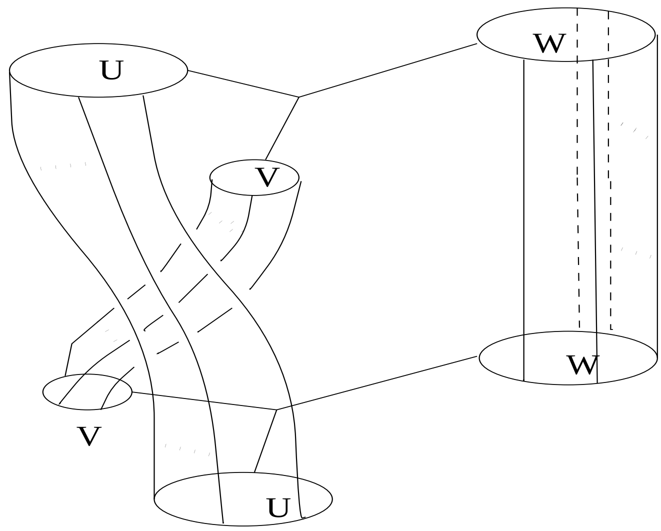
$$\mathcal{C}_n \overset{\sim}{\to} \pi_1(\mathcal{M}_{0,n};\mathcal{B}_n).$$

Proof. We describe the images of the objects and morphisms of $\mathcal{C}_n$.

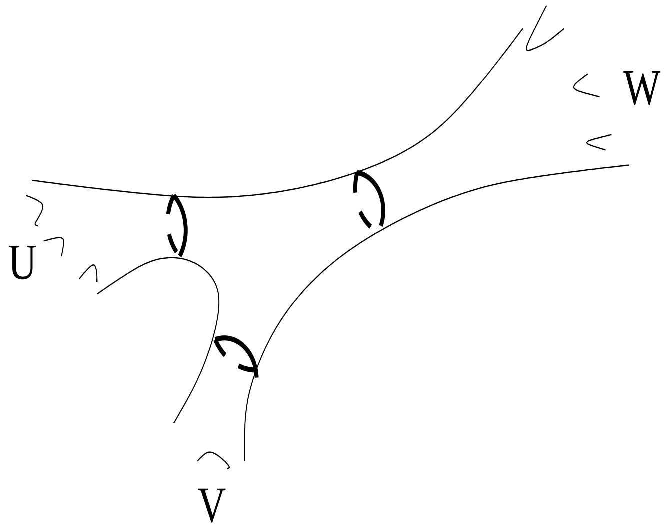
• The objects of $\mathcal{C}_n$ correspond to topological tangential base points in $\mathcal{B}_n$ as described above.

• Take an associativity morphism a of $\mathcal{C}_n$; it sends a tree T to another tree T' with the same cyclic order of the numbering. Consider the region in $\mathcal{M}_{0,n}$ of all spheres with n real marked points in the order corresponding to the order of the indices of T. This region is *simply connected* and therefore there is a unique path (up to homotopy) on $\mathcal{M}_{0,n}$ from the base point corresponding to T to that corresponding to T', which is the path in $\pi_1(\mathcal{M}_{0,n};\mathcal{B}_n)$ we associate to a. It corresponds to *sliding points along the real axis* without ever having them cross each other.

• Take a commutativity morphism in $\mathcal{C}_n$. Remember it looks like a crossing of two packets on a numbered trivalent tree, which is associated to a certain almost maximally degenerate sphere.



Draw that almost degenerate sphere with simple closed loops cutting off the branches U, V and W.



Make a half-twist along the right-hand geodesic and two inverse half-twists along the other two.

This induces a continuous deformation of the analytic structure, and therefore draws a path on moduli space ending up at the right new maximally degenerate sphere.

Finally, if $T \mapsto b \in \mathcal{B}_n$, then the map

$$\mathrm{Hom}(T,T) \simeq K_n \to \Gamma_{0,n} \simeq \pi(\mathcal{M}_{0,n}; b)$$

completes the proof of the surjective homomorphism.

§3A.5. The profinite version $\widehat{GT}$ of GT, and arithmetic

Recall the main theorem:

Theorem. *There is an injective homomorphism*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT}.$$

We saw some sketches of proofs, but here is one that welds Drinfeld's original conception with the geometry of the moduli spaces. Let $\widehat{\mathcal{C}}$ denote the profinite completion of the tree category (obtained by replacing all local groups of $\mathcal{C}$ by their profinite completions).

• $\widehat{GT}$ acts on the profinite tree category $\widehat{\mathcal{C}}$, preserving each of the subgroupoids $\widehat{\mathcal{C}}_n$ and acting on their local groups, which are the profinite Artin braid groups $\widehat{B}_n$, in an *inertia-preserving way.*

• The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the profinite Teichmüller groupoid $\widehat{\pi}_1(\mathcal{M}_{0,n};\mathcal{B}_n)$, and its action on the local groups is of course inertia-preserving since they are fundamental groups.

- The two actions are compatible with the map

$$\mathcal{C}_n \to \widehat{\pi}_1(\mathcal{M}_{0,n};\mathcal{B}_n).$$

In fact, an element $\sigma \in G_{\mathbb{Q}}$ acting on the right-hand space can be lifted via the map to the left-hand space and thus corresponds to the action of a unique $(\lambda, f) \in \widehat{GT}$.

Essentially, where Drinfeld considered the discrete GT as modifications of a braided tensor category and the k-pro-unipotent version as a generalization, classifying modifications of a quasi-triangular quasi-Hopf algebra, the profinite $\widehat{GT}$ classifies modifications of the profinite braided tensor category given geometrically by the fundamental groupoid of $M_{0,n}$ $(n \geq 5)$ based at tangential base points.