An introduction to

Profinite Grothendieck-Teichmüller theory

MIT, Cambridge, Massachusetts

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Lecture 3B

Grothendieck-Teichmüller theory

in higher genus

Outline

Lecture 1A: Geometric Galois actions

Lecture 1B: The Grothendieck-Teichmüller group

Lecture 3A: Braided tensor categories and geometry

Lecture 3B: G-T theory in higher genus

Grothendieck in his *Esquisse* made some suggestions about how to study the absolute Galois group geometrically:

• we should try to characterize $G_{\mathbb{Q}}$ as a group of automorphisms of algebraic fundamental groups of moduli spaces respecting certain Galois-like group-theoretic properties such as *preserving inertia*, and also *respecting morphisms between the moduli spaces*,

• we should explain the action of $G_{\mathbb{Q}}$ (or Grothendieck-Teichmüller-type groups) on these fundamental groups as a sort of "lego" game corresponding to the fact that topological surfaces can be cut into smaller pieces.

• we should understand the 2-level principle, which refers to the fact that the automorphism groups of the fundamental groups of moduli spaces of dimension ≤ 2 should also act on all the fundamental groups of the higher dimensional moduli spaces (as we saw for $\widehat{GT}$ in the genus zero situation).

All of these suggestions came to fruition when we tried to pass to higher genus curves.

We could not prove that $\widehat{GT}$ acts on the algebraic fundamental groups of moduli spaces of higher genus curves. But we were able to add just one relation, coming from the 2-dimensional moduli space $M_{1,2}$ to obtain a subgroup of $\widehat{GT}$ that does.

§3B.1. Higher genus moduli spaces, mapping class groups and morphisms between them

The morphisms between moduli spaces that Grothendieck mentioned are those coming from topological operations on the curves themselves:

- **erasing or adding marked points**
- **cutting the curves along simple closed loops.**

We will first look at the morphisms these operations give between moduli spaces and between their fundamental groups, the mapping class groups.

Let $\Sigma_{g,n}^m$ denote a topological surface with genus $g \geq 0$, $n \geq 0$ punctures, and $m \geq 0$ boundary components.

The associated *pure mapping class group* $\Gamma_{g,n}^m$ is the group of classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}^m$ fixing the boundaries pointwise, modulo those which are isotopic to the identity fixing the boundaries pointwise. As we saw in the genus zero case, this are identified with the fundamental group of the moduli space of Riemann surfaces of type (g, n, m).

Also, as in the genus zero case, the group $\Gamma_{g,n}^m$ is generated by Dehn twists a along simple closed curves α on $\Sigma_{g,n}^m$.

An explicit presentation of the mapping class groups $\Gamma_{g,n}^m$ is known, the generators being the Dehn twists.

We define the following *braid relations*:

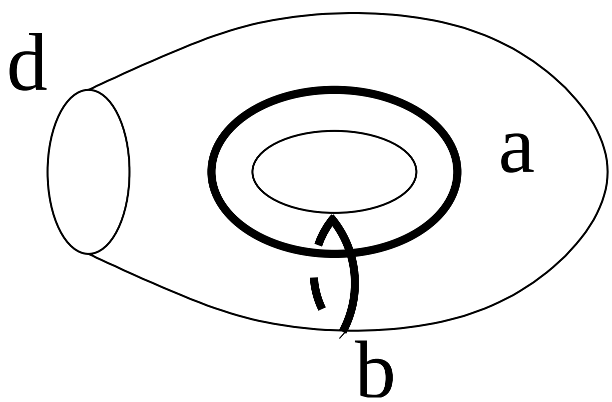
$$(C) \qquad ab = ba \ \text{ if } \ |\alpha \cap \beta| = 0$$

$$(B) \qquad aba = bab \ \text{ if } \ |\alpha \cap \beta| = 1$$

The *doughnut relation*, taking place on a subsurface Σ' of Σ of type $(1, i, j)$ with $i + j = 1$, is given by

$$(D) \qquad (aba)^4 = d$$

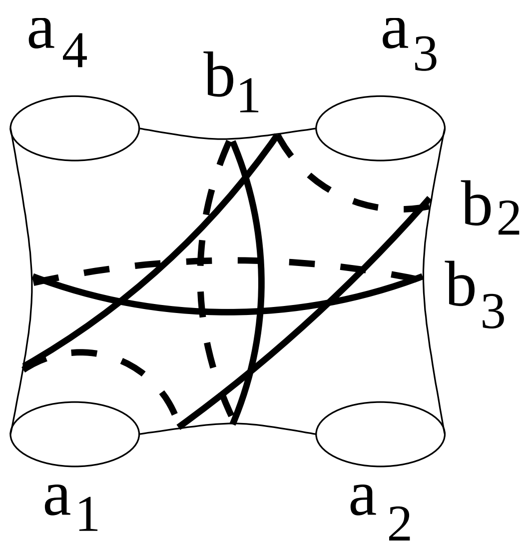
where δ is the boundary loop of Σ' (so it may just surround a puncture), and α and β are as in the figure below.



Finally, the *lantern relation*, taking place on a subsurface Σ' of Σ of type $(0, i, j)$ with $i + j = 4$, is given by

$$a_1 a_2 a_3 a_4 = b_1 b_2 b_3 \tag{L}$$

where the α_i are loops on Σ bounding Σ' (we allow α_i to be a loop surrounding a puncture, so that $a_i = 1$) and β_1, β_2 and β_3 are the loops in the interior of Σ' shown in the figure below.



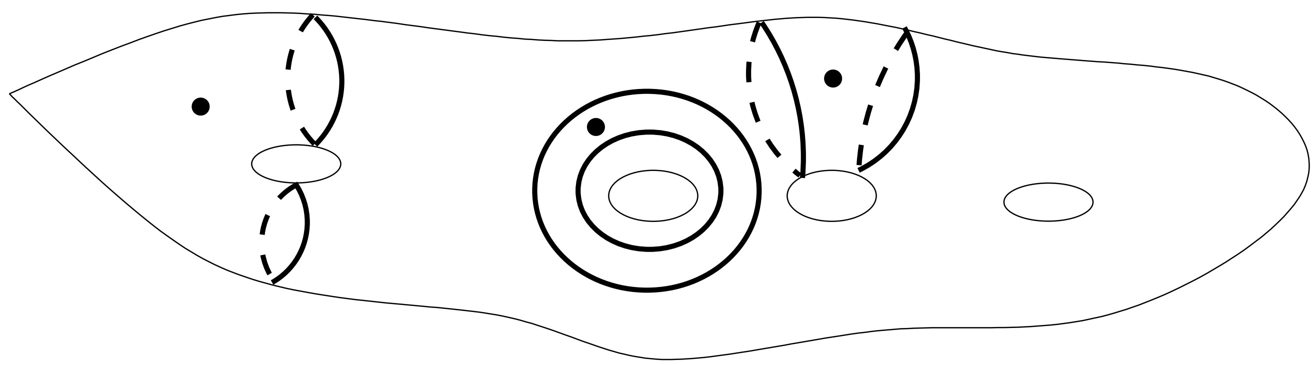
Presentation Theorem. *A presentation for the pure mapping class group $\Gamma^m_{g,n}$ for every $g, n, m \geq 0$ is given by taking all Dehn twists along simple closed loops as generators and imposing all the relations (C), (B), (D) and (L).*

Erasing marked points

The maps between topological surfaces obtained by erasing marked points give morphisms between the moduli spaces and the associated fundamental groups (mapping class groups).

If the n-th point is erased from a Riemann surface with n ordered marked points, we get a Riemann surface with $n-1$ ordered marked points, giving a morphism from $M_{g,n}$ to $M_{g,n-1}$. *This map is defined over* $\mathbb{Q}$ *and therefore respected by the Galois action.*

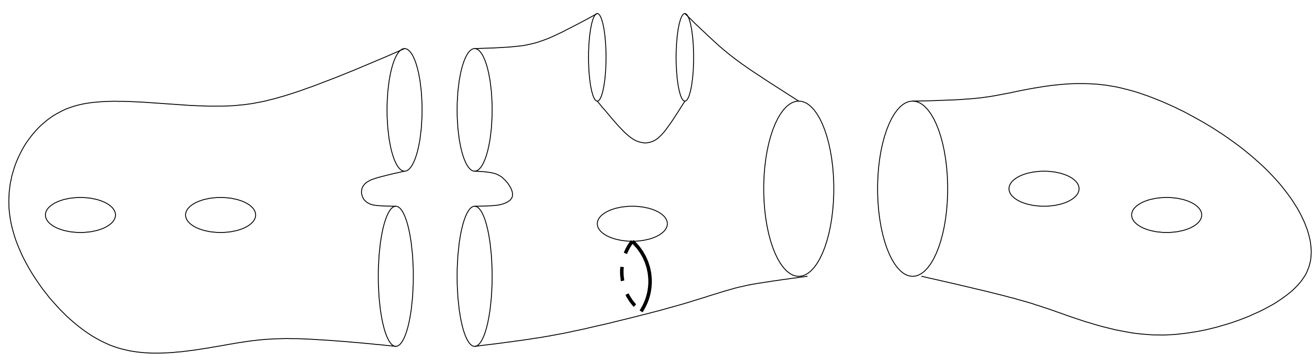
For the fundamental groups, we obtain a quotient map that is easily understood on the Dehn-twist generators: they become equal if they differ only by the marked point that is erased.



Cutting out subsurfaces along simple closed loops

The map between topological surface that we call "subsurface-inclusion", corresponding to a subsurface cut out of a larger surface along disjoint simple closed loops, is harder to understand on the moduli spaces; it involves mapping the smaller moduli space to the divisor at infinity in the Deligne-Mumford compactification of the larger one. *This map is, however, defined over $\mathbb{Q}$ and therefore respected by the Galois action.*

However, the corresponding homomorphism of mapping class groups is very simple; a Dehn twist on the subsurface maps to the same one on the larger surface.



§3B.2. Higher genus Grothendieck-Teichmüller group

We start with a more topological formulation of the definition of $\widehat{GT}$.

Definition. Let $\widehat{GT}^1$ be the set of elements $(\lambda, f) \in \widehat{GT}$ with $\lambda = 1$. For simplicity, we only work with this group from now on.

We can reformulate the usual three relations in $\widehat{GT}^1$ using Dehn twists on topological surfaces as follows.

(I) $f(a_2^2, a_1^2)f(a_1^2, a_2^2) = 1$ in $\widehat{\Gamma}_1^1$, where α_1 and α_2 are as in figure 1(a);

(II) $f(b_3, b_1)f(b_2, b_3)f(b_1, b_2) = 1$ in $\widehat{\Gamma}_0^4$, where β_1, β_2 and β_3 are as in figure 1(b);

(III) $f(b_3, b_4)f(b_5, b_1)f(b_2, b_3)f(b_4, b_5)f(b_1, b_2) = 1$ in $\widehat{\Gamma}_0^5$, where the β_i are as in figure 1(c).

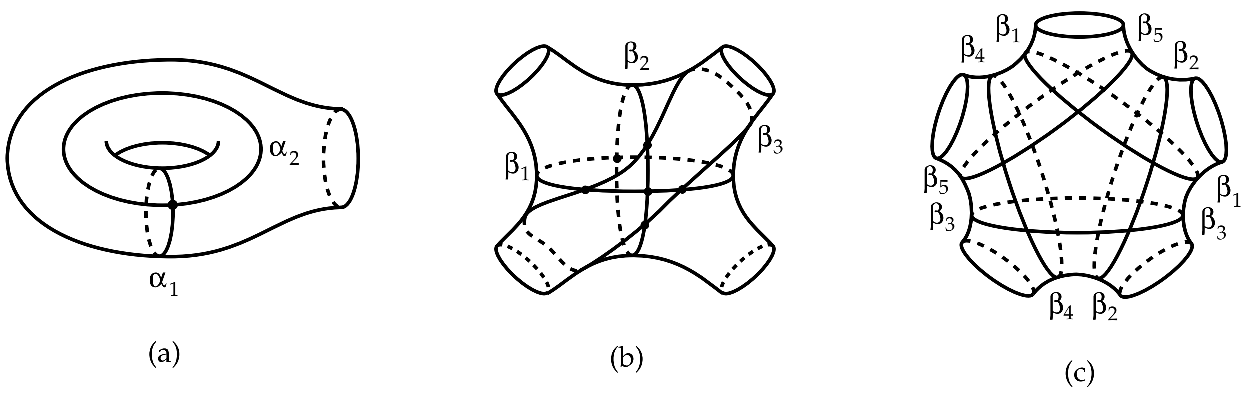


Figure 1

Let $\mathbb{I}\Gamma$ be the subset of $\widehat{GT}^1$ consisting of the elements $f \in \widehat{F}_2'$ satisfying the following additional relation, taking place in $\widehat{\Gamma}_1^2$, where the loops α_i and ϵ_i are as in figure 2.

(R) $f(e_3, a_1)f(a_2^2, a_3^2)f(e_2, e_3)f(e_1, e_2)f(a_1^2, a_2^2)f(a_3, e_1) = 1.$

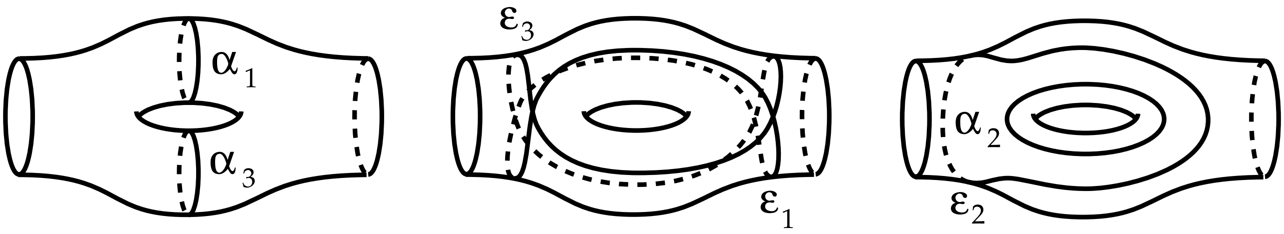


Figure 2

Our first main result concerning $\mathbb{I}\Gamma$ is the following:

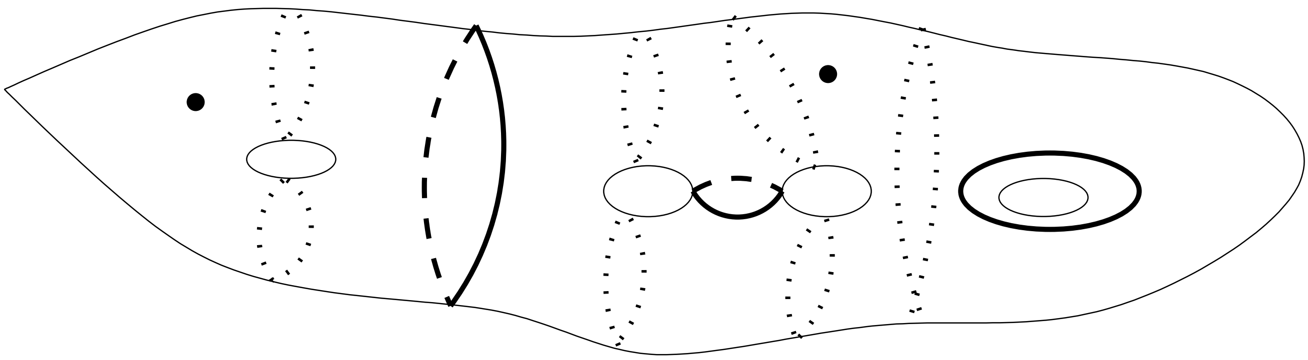
Theorem 1. $\mathbb{I}\Gamma$ *is a group under the multiplication inherited from* $\widehat{GT}$*, and we have the inclusions*

$$G_{\mathbb{Q}}^{\mathrm{ab}} \subset \mathbb{I}\Gamma \subset \widehat{GT}^1 .$$

Remark. We use $G_{\mathbb{Q}}^{\mathrm{ab}}$ only because we took $\lambda = 1$ for simplicity. It is not hard to generalize the new relation (R) to the general case, and all the results still hold.

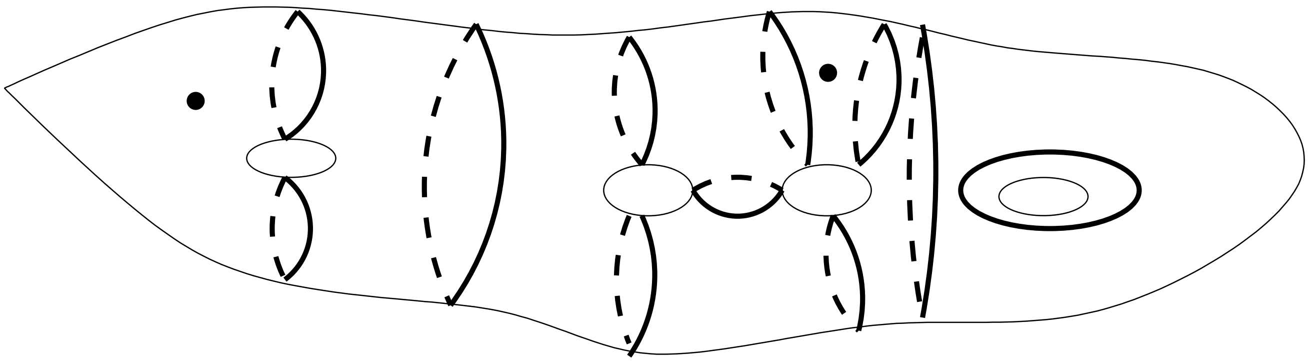
Another result in higher genus shows that not only does $\widehat{GT}$ act on all the profinite higher genus mapping class groups, but its action on the Dehn-twist generators has a "locality" property.

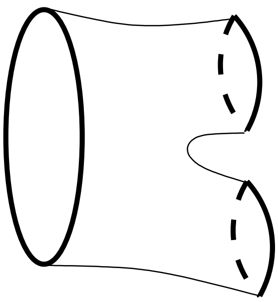
In fact, every closed loop on a topological surface has a topological neighborhood of one of only two possible types: a genus 0 subsurface with four boundary components or punctures, or else a genus 1 subsurface with one boundary component.



The "locality" property says that the action of $\widehat{GT}$ on a Dehn twist essentially only involves the Dehn twists along loops supported on the neighborhood.

If Σ is a surface of type (g, n, m), we say that P be a pants decomposition of Σ, i.e. a maximal collection of $3g-3+n$ disjoint simple closed loops; this necessarily cuts the surface into $2g-2+n$ "pants" (surfaces of type (0,3)).





The precise statement of the result is as follows.

Theorem 2. *Let Σ be a surface of type (g,n,m) and let P be a pants decomposition of Σ. Then*

(i) There exists a group homomorphism

$$\psi_P : \mathbb{I}\Gamma \to \mathrm{Aut}\big(\widehat{\Gamma}_{g,n}^m\big)$$

such that setting $F_P = \psi_P(f)$, the automorphism F_P has the following "local properties":

$$\begin{cases} F_P(a) = a & \text{for all } \alpha \in P \\ F_P(b) = f(a^2,b^2)^{-1} b f(a^2,b^2) & \text{if } |\beta \cap \alpha| = 1 \text{ for some } \alpha \in P \\ & \quad \text{and } |\beta \cap \alpha'| = 0 \\ & \quad \text{for all } \alpha' \in P,\ \alpha' \neq \alpha \\ F_P(c) = f(a,c)^{-1} c f(a,c) & \text{if } |\gamma \cap \alpha| = 2_0 \text{ for some } \alpha \in P \\ & \quad \text{and } |\gamma \cap \alpha'| = 0 \\ & \quad \text{for all } \alpha' \in P,\ \alpha' \neq \alpha. \end{cases}$$

(ii) The homomorphisms $\mathbb{I}\Gamma \to \mathrm{Out}\big(\widehat{\Gamma}(\Sigma)\big)$ induced by the ψ_P for different P are all equal and give rise to a canonical homomorphism

$$\psi_{g,n}^m : \mathbb{I}\Gamma \to \mathrm{Out}(\widehat{\Gamma}_{g,n}^m) \tag{$*$}$$

for each $g,n,m \geq 0$.

We also obtain the following results showing that the $\mathbb{I}\Gamma$-action possesses the main Galois-type properties.

Theorem 3. *(i) Considering $G_{\mathbb{Q}} \subset \widehat{GT}$, the intersection*

$$G_{\mathbb{Q}} \cap \mathbb{I}\Gamma = G_{\mathbb{Q}}^{\mathrm{ab}},$$

and the homomorphism (*) restricted to $G_{\mathbb{Q}}^{\mathrm{ab}}$ is the canonical Galois homorphism

$$G_{\mathbb{Q}}^{\mathrm{ab}} \to \mathrm{Out}^*(\widehat{\Gamma}_{g,n}^m)$$

coming from the fact that $\widehat{\Gamma}_{g,n}^m$ is the algebraic fundamental group of a moduli space of curves, which is defined over $\mathbb{Q}$.

(ii) The image of the map

$$\psi_{g,n}^m : \mathbb{I}\Gamma \to \mathrm{Out}(\widehat{\Gamma}_{g,n}^m)$$

of theorem 2 actually lands in $\mathrm{Out}^*(\widehat{\Gamma}_{g,n}^m)$. Indeed, the $\mathbb{I}\Gamma$-action conjugates all Dehn twists, i.e. it is inertia-preserving.

(iii) For every point-erasing or subsurface inclusion map

$$\iota : \widehat{\Gamma}_{g,n}^m \to \widehat{\Gamma}_{g',n'}^{m'}$$

and every $F = (1, f) \in \mathbb{I}\Gamma$, the following diagram commutes:

$$\begin{array}{ccc}
\widehat{\Gamma}_{g,n}^m & \xrightarrow{\iota} & \widehat{\Gamma}_{g',n'}^{m'} \\
\downarrow F & & F \downarrow \\
\widehat{\Gamma}_{g,n}^m & \xrightarrow{\iota} & \widehat{\Gamma}_{g',n'}^{m'}.
\end{array}$$

Remarks. The proof of the theorem actually gives an explicit "lego"-procedure to calculate the action of $\mathbb{I}\Gamma$ on any Dehn twist, even for a given pants decomposition. This also gives the explicit inner automorphism relating ψ_P and ψ_Q for two pants decomposition.

This "lego"-procedure is a generalization to higher genus of the associativity moves in the braided tensor category and on the genus zero moduli space that we saw previously.

Finally, the whole theorem can be generalized to non-trivial λ and all of $G_{\mathbb{Q}}$, only ugly small terms appear in all the expressions.

§3B.3. The pants-decomposition complex

Let Σ be a topological surface of type (g, n): boundary components and marked points or punctures play the same role in our construction, so we make no difference between them.

The **pants-decomposition complex** $\mathcal{P}$ is a 2-dimensional complex defined as follows.

• the vertices of $\mathcal{P}$ are the pants decompositions of Σ (up to isotopy);

• there is a path, called an "S-move", from P to Q if Q can be obtained from P by replacing just one simple closed loop α of P by a simple closed loop β that intersects α in one point;

• there is a path, called an "A-move", from P to Q if Q can be obtained from P by replacing just one simple closed loop α of P by a simple closed loop β that intersects α in two points (with algebraic intersection equal to 0).

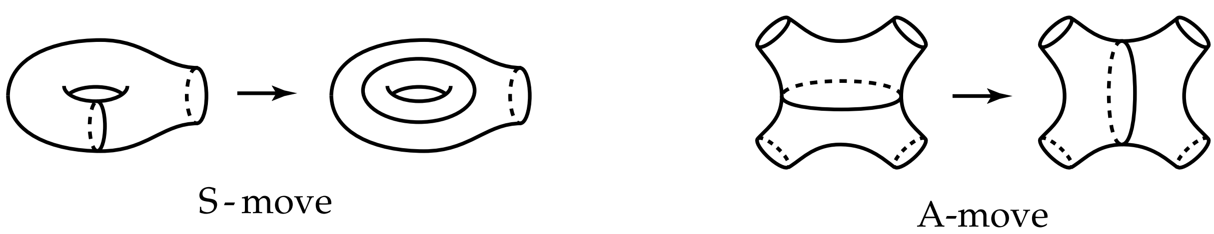


Figure 3

Note that erasing one loop from P leaves a piece of type $(0, 4)$ or $(1, 1)$ in the pants decomposition. An S-move can only be made on a loop if it lives on a type $(1, 1)$ piece, and an A-move only if it lives on a type $(0, 4)$ piece.

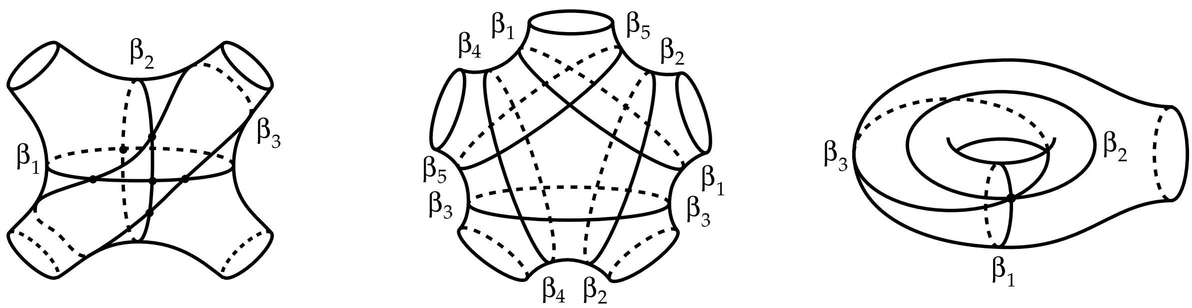
It is not hard to see that compositions of S-moves and A-moves act transitively on the set of pants decompositions of Σ; we can get from any pants decomposition to any other by a string of such moves. So the pants-decomposition complex is connected.

• the **faces** of $\mathcal{P}$ are of five types: commutativity, the A and S triangles, the A-pentagons and the mixed hexagons.

(3A) On a piece of type $(0,4)$ obtained by deleting one loop: there are loops β_1, β_2, and β_3, shown in figure 4(a), which yield a cycle of three A-moves: $\beta_1 \to \beta_2 \to \beta_3 \to \beta_1$.

(3S) On a piece of type $(1,1)$ obtained by deleting one loop: there are loops β_1, β_2, and β_2, shown in figure 4(c), which yield a cycle of three S-moves: $\beta_1 \to \beta_2 \to \beta_3 \to \beta_1$.

(5A) On a piece of type $(0,5)$ created by deleting 2 loops: there is a cycle of five A-moves involving the loops β_i shown in figure 4(b): $\{\beta_1,\beta_3\} \to \{\beta_1,\beta_4\} \to \{\beta_2,\beta_4\} \to \{\beta_2,\beta_5\} \to \{\beta_3,\beta_5\} \to \{\beta_3,\beta_1\}$.



(a) (b) (c)

(6AS) On a piece of type $(1,2)$ obtained by deleting 2 loops: there is a cycle of four A-moves and two S-moves shown in figure 5: $\{\alpha_1,\alpha_3\} \to \{\alpha_1,\varepsilon_3\} \to \{\alpha_2,\varepsilon_3\} \to \{\alpha_2,\varepsilon_3\} \to \{\alpha_2,\varepsilon_2\} \to \{\alpha_2,\varepsilon_1\} \to \{\alpha_3,\varepsilon_1\} \to \{\alpha_3,\alpha_1\}$.

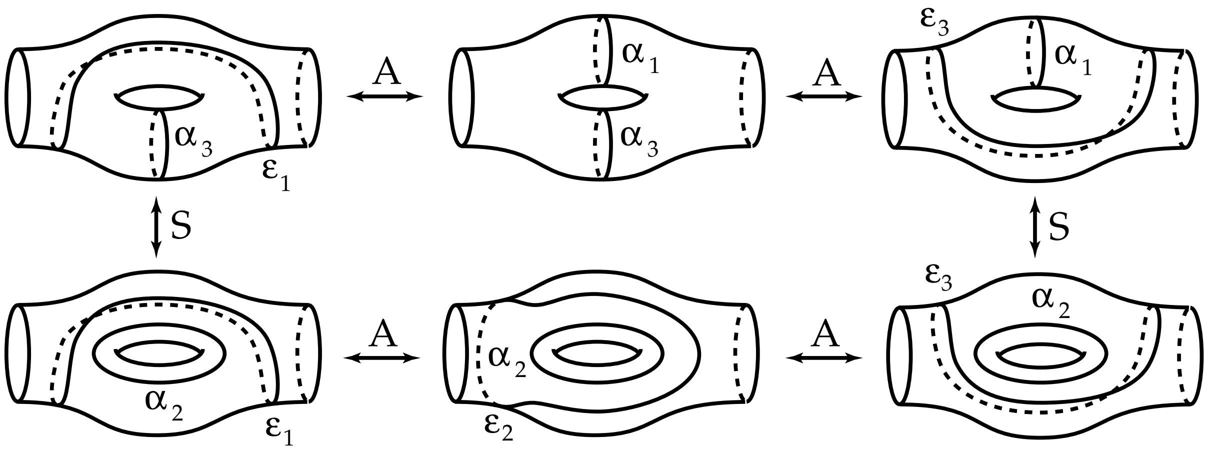


Figure 5

(C) If two moves which are either A-moves or S-moves are supported in disjoint subsurfaces of Σ, then they commute, and their commutator is a cycle of four moves. We call them *disjoint moves*.

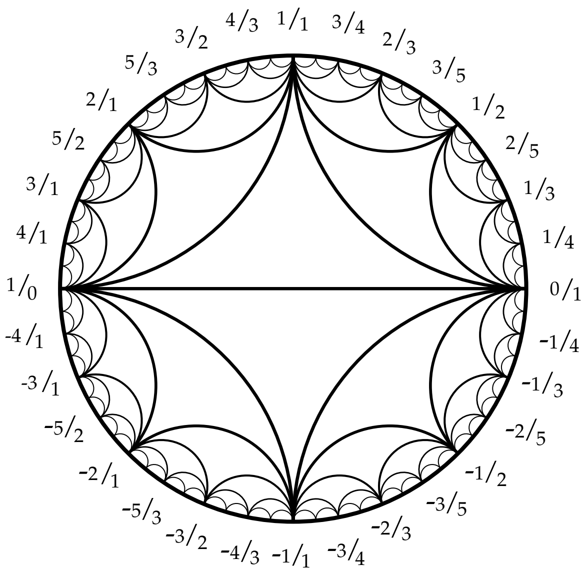
Remark. The face that the four basic relations 3A, 3S, 5A and 6AS live on surfaces of type $(0,4)$, $(1,1)$, $(0,5)$ and $(1,2)$ respectively corresponds to Grothendieck's "2-level principle": dimensions 1 and 2 are all that are needed for the theory to work for all moduli spaces.

Theorem 4. *The pants decomposition complex $\mathcal{P}$ is simply connected.*

The proof of this, due to Allen Hatcher, uses Cerf theory and sophisticated topological methods.

Remark. The theorem has the following meaning. If you have two pants decompositions P and Q, and two paths, i.e. sequences of A and S-moves from P to Q, then they are "equal" in the complex, which means that the second sequence can be deduced from the first by a finite number of insertions or deletions of the five types of cycles, together with the trivial operation of inserting or deleted a move followed by its inverse.

Example. Let Σ be of type $(1, 1)$. Then a pants decomposition consists of a single circle, only S-moves are possible, and all the faces are given by (3S) triangles. The different pants decompositions (i.e. isotopy classes of circles) are indexed by rational slopes, and $\mathcal{P}$ is equal to the Farey tesselation:



§3B.4. The $\mathbb{\Gamma}$-action on the Teichmüller tower.

We're now ready to indicate the proofs of the main theorems in a series of steps. We fix Σ of type (g,n) and a pants decomposition P on Σ.

Proof of theorem 2. To each $F=(1,f)\in \mathbb{\Gamma}$, associate a "candidate" automorphism $F_P\in \mathrm{Out}^*(\widehat{\Gamma}_{g,n})$ which acts on the Dehn twists as follows:

$$\begin{cases} F_P(a)=a & \text{for all } \alpha\in P\\ F_P(b)=f(a^2,b^2)^{-1}bf(a^2,b^2) & \text{if } \alpha\mapsto\beta \text{ is an S-move}\\ F_P(c)=f(a,c)^{-1}cf(a,c) & \text{if } \alpha\mapsto\gamma \text{ is an A-move}\end{cases}$$

From this definition of the action of F_P on certain Dehn twists, we can extend the definition of the action of F_P on all Dehn twists as follows. To consider the Dehn twist d along any loop δ, we choose a pants decomposition Q containing δ and a finite sequence $M_{\alpha_r,\delta_r}\circ\cdots\circ M_{\beta_1,\delta_1}$ taking P to Q (so that δ is one of the loops δ_i of Q). We then define the action of F_P on d as:

$$\begin{aligned} F_P(d)&=\mathrm{inn}\left(\prod_{i=r}^{1} f(a_i^{\epsilon_i},d_i^{\epsilon_i})\right)(d)\\ &=f(a_1^{\epsilon_1},d_1^{\epsilon_1})^{-1}\cdots f(a_r^{\epsilon_r},d_r^{\epsilon_r})^{-1}\,d\,f(a_r^{\epsilon_r},d_r^{\epsilon_r})\cdots f(a_1^{\epsilon_1},d_1^{\epsilon_1}),\end{aligned} \tag{$*$}$$

where

$$\epsilon_i=\begin{cases}1 & \text{if } M_{\alpha_i,\delta_i} \text{ is an A-move}\\ 2 & \text{if } M_{\alpha_i,\delta_i} \text{ is an S-move.}\end{cases}$$

Step 1: *Show that this definition of $F_P(d)$ is well-defined, independent of the choice of sequence of moves from P to Q.*

The main point is that $\mathcal{P}$ is simply connected. Thus, as we saw, if we take a different sequence of moves from P to Q, it can be obtained from the first one by substitutions of (3A) and (3S)-cycles, (5A)-cycles and (6AS)-cycles.

This comes down to inserting the corresponding 3-cycles, 5-cycles and 6AS-cycles of f-terms in the expression in (*) (for example, $f(b_1, b_2)f(b_3, b_1)f(b_2, b_3)$ for a 3-cycle). But these are all equal to 1, precisely thanks to relations (I), (II), (III) and (R).

The definition of $F_P(d)$ is also independent of the choice of Q containing δ, simply because all moves involving parts of the pants decomposition away from the neighborhood of δ will commute with δ and thus disappear from the expression (*).

Step 2: *Show that this definition of F_P on all Dehn twist generators of $\widehat{\Gamma}_{g,n}$ extends to an automorphism of $\widehat{\Gamma}_{g,n}$ by checking that defining relations (C), (B), (L) and (D) are respected.*

We first show that each relation is respected by some F_Q for a suitable choice of Q.

For (C), suppose β_1 and β_2 are two disjoint loops on Σ and let Q be a pants decomposition containing both of them. Then by definition, $F_Q(b_1) = b_1$ and $F_Q(b_2) = b_2$, so they commute.

For braid relations (B), let β_1 and β_2 be loops intersecting in one point, and let Q be a pants decomposition containing β_1 and not intersecting β_2. Then the definition above shows that $F_Q(b_1) = b_1$ and $F_Q(b_2) = f(b_2^2, b_1^2)b_2 f(b_1^2, b_2^2)$. Let $\gamma = b_1(\beta_2)$. Relation (B) states that $c = b_1 b_2 b_1^{-1}$. Computing the right-hand side gives

$$\begin{aligned} F_Q(b_1)F_Q(b_2)F_Q(b_1)^{-1} &= b_1 f(b_2^2, b_1^2) b_2 f(b_1^2, b_2^2) b_1^{-1} \\ &= f(b_1 b_2^2 b_1^{-1}, b_1^2) b_1 b_2 b_1^{-1} f(b_1^2, b_1 b_2^2 b_1^{-1}) \\ &= f(c^2, b_1^2) c f(b_1^2, c^2). \end{aligned}$$

On the other hand, since the loop γ can be obtained from β_1 by a single simple move, the definition of F_Q shows that $F_Q(c) = f(c^2, b_1^2) c f(b_1^2, c^2)$, so that relation (B) is respected by F_Q.

For lantern relations (L) of the form $a_1 a_2 a_3 a_4 = b_1 b_2 b_3$ in $\widehat{\Gamma}(\Sigma)$, we let Q be a pants decomposition containing the loops α_i for $1 \leq i \leq 4$ (bounding a subsurface of type $(1, i, j)$ with $i + j = 4$) and β_1. Then $F_Q(a_i) = a_i$ and $F_Q(b_1) = b_1$, so in particular we have

$$F_Q(a_1)F_Q(a_2)F_Q(a_3)F_Q(a_4) = a_1 a_2 a_3 a_4,$$

so we just have to check that F_Q preserves $b_1 b_2 b_3$. The loop b_2 is obtained from b_1 by an A-move A_{b_1, b_2}, and b_3 is obtained from b_1 by an A-move A_{b_1, b_3}, so the definition of the action F_Q on b_2 and b_3 gives

$$F_Q(b_2) = f(b_2, b_1) b_2 f(b_1, b_2) \text{ and } F_Q(b_3) = f(b_3, b_1) b_3 f(b_1, b_3).$$

Thus

$$\begin{aligned} F_Q(b_1)F_Q(b_2)F_Q(b_3) &= b_1 f(b_2,b_1) b_2 f(b_1,b_2) f(b_3,b_1) b_3 f(b_1,b_3) \\ &= b_1 f(b_2,b_1) b_2 f(b_3,b_2) b_3 f(b_1,b_3) = a_1a_2a_3a_4 \end{aligned}$$

where the last equality is obtained by using the inverse of equation (1) and the fact that $a_1a_2a_3a_4$ is central.

For doughnut relations (D) of the form $(a_1a_2a_1)^4 = d$, let Q be a pants decomposition containing α_1 and δ. Then $F_Q(d) = d$, $F(a_1) = a_1$ and since α_2 is obtained from α_1 by a single S-move, we have $F_Q(a_2) = f(a_2^2, a_1^2) a_2 f(a_1^2, a_2^2)$. We compute

$$\begin{aligned} F_Q(a_1)F_Q(a_2)F_Q(a_1) &= a_1 f(a_2^2, a_1^2) a_2 f(a_1^2, a_2^2) a_1 \\ &= f(a_1a_2^2a_1^{-1}, a_1^2) a_1 a_2 f(a_1^2, a_2^2) a_1 \\ &= f(a_1a_2^2a_1^{-1}, a_1^2) f(a_2^2, a_1a_2^2a_1^{-1}) a_1a_2a_1 \\ &= f(a_2^2, a_1^2) a_1a_2a_1. \end{aligned}$$

The last equality is obtained by applying relation (II), which is legitimate since setting $x = a_1^2$, $y = a_2^2$ and $z = a_1a_2^2a_1^{-1}$, we have $xyz = (a_1a_2)^3$ which commutes with x, y and z. So we have

$$F_Q(a_1)F_Q(a_2)F_Q(a_1) = f(a_2^2, a_1^2) a_1a_2a_1 = a_1a_2a_1 f(a_1^2, a_2^2)$$

(the second equality comes from passing the $a_1a_2a_1$ to the left by conjugating the arguments of f), so

$$\big(F_Q(a_1)F_Q(a_2)F_Q(a_1)\big)^2 = a_1a_2a_1 f(a_1^2, a_2^2) f(a_2^2, a_1^2) a_1a_2a_1 = (a_1a_2a_1)$$

by relation (I). A fortiori, we find that

$$\big(F_Q(a_1)F_Q(a_2)F_Q(a_1)\big)^4 = (a_1a_2a_1)^4 = d = F_Q(d).$$

Step 3: *Show that F_P respects the relations (C), (B), (L) and (D) for all P.*

For each relation, let Q be the pants decomposition in the proof above, and let Q be any pants decomposition. let $M_{\beta_r,\gamma_r} \circ \cdots \circ M_{\beta_1,\gamma_1}$ be a finite sequence of S- and A-moves taking Q to P. Let $(1, f) \in \mathbb{I}\Gamma$, and set

$$x = \prod_{i=r}^{1} f(b_i^{\epsilon_i}, c_i^{\epsilon_i}) \in \widehat{\Gamma}(\Sigma).$$

Then for all Dehn twists b, we have the equality

$$F_P(b) = \big(\mathrm{inn}(x) \circ F_Q\big)(b). \tag{$**$}$$

This proves that if F_Q respects a relation, then so does F_P for any pants decomposition P. **This completes the proof that the F_P are automorphisms of $\widehat{\Gamma}_{g,n}$ for all P.**

Step 4: $\mathbb{I}\Gamma$ *is a group.*

Consider $\mathbb{I}\Gamma$ as a subset of $\widehat{GT}^1$. We need to show that for all $f, g \in \mathbb{I}\Gamma$, the inverse f^* of f and the composition $g \cdot f$ satisfy relation (R).

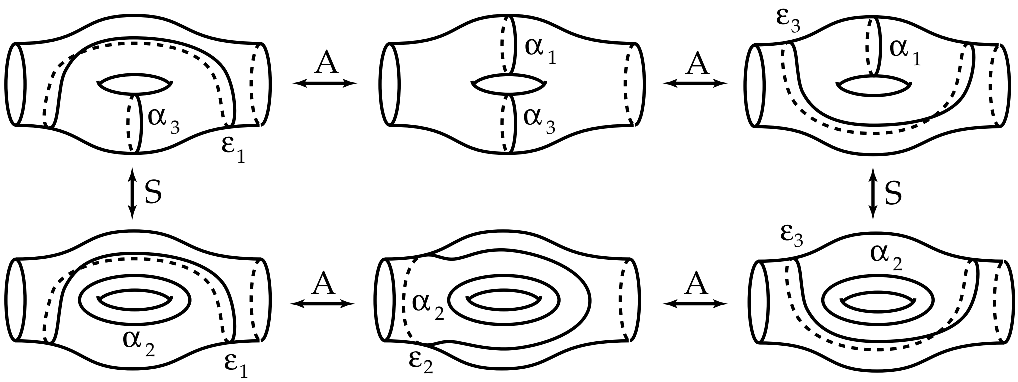
In Steps 1-3 above, we showed that for every $f \in \mathbb{I}\Gamma$, there exists a family $\big(F_P\big)_{\mathcal{P}}$ of automorphisms of $\widehat{\Gamma}(\Sigma)$, whose members are related by inner automorphisms. Let $g \in \mathbb{I}\Gamma$ and let $h = g \cdot f \in \widehat{GT}^1$. Set $H_P = G_P \circ F_P$ for every $P \in \mathbf{P}$.

We first note that even though we don't know that $h \in \mathbb{I}\Gamma$, we still have

$$H_P(b) = \big(\mathrm{inn}(x) \circ H_Q\big)(b) \tag{1}$$

just as we do for F_P, F_Q or G_P, G_Q, where x is a product of f-terms reflecting a series of S and A-moves from P to Q.

Now, consider the case where Σ is of type $(1, 2)$. The group $\Gamma_{1,2}$ is isomorphic to B_4/Z, the quotient of the Artin braid group on four strands modulo its center. Let P be the pants decomposition of Σ given by the loops α_1 and α_3 shown in figure 5, and let $Q = P$.



The sequence of moves (6AS) takes P to itself. Thus by (1), we find that

$$H_P = \operatorname{inn}\big(h(e_3, a_1)h(a_2^2, a_3^2)h(e_2, e_3)h(e_1, e_2)h(a_1^2, a_2^2)h(a_3, e_1)\big)H_P.$$

Thus the element $h(e_3, a_1)h(a_2^2, a_3^2)h(e_2, e_3)h(e_1, e_2)h(a_1^2, a_2^2)h(a_3, e_1)$ lies in the center of $\widehat{\Gamma}_{1,2}$. However, each factor of this expression belongs to the derived subgroup of $\widehat{\Gamma}_{1,2}$, and the intersection of the derived subgroup with the center (generated by the Dehn twists along the two boundary components) is trivial. **This shows that if $f, g \in \mathbb{I}\Gamma$, then $h = g \cdot f \in \mathbb{I}\Gamma$.**

It remains to show that if $f \in \mathbb{I}\Gamma$, then f^* is also in $\mathbb{I}\Gamma$. This time we consider the family $\big(F_P^{-1}\big)_{\mathbf{P}}$. We know that

$$F_P = \operatorname{inn}\big(f(b^\epsilon, c^\epsilon)\big) \circ F_Q;$$

and inverting this formula gives

$$\begin{aligned} F_P^{-1} &= F_Q^{-1} \circ \operatorname{inn}\big(f(c^\epsilon, b^\epsilon)\big) \\ &= \operatorname{inn}\big(F_Q^{-1}(f(c^\epsilon, b^\epsilon))\big) \circ F_Q^{-1} \\ &= \operatorname{inn}\big(f^*(b^\epsilon, c^\epsilon)\big) \circ F_Q^{-1}. \end{aligned}$$

Indeed, f^* is defined by $f^*(x, y)F^{-1}\big(f(x, y)\big) = 1$ with $F(x) = x$ and $F(y) = f(y, x)yf(x, y)$, so under the homomorphism $x \mapsto c$ and $y \mapsto b$, F corresponds to F_Q and we have $f^*(b^\epsilon, c^\epsilon) = F_Q^{-1}\big(f(c^\epsilon, b^\epsilon)\big)$. As for h above, using the cycle (6AS) in $\widehat{\Gamma}_{1,2}$ to bring P to $Q = P$, we find that

$$f^*(e_3, a_1)f^*(a_2^2, a_3^2)f^*(e_2, e_3)f^*(e_1, e_2)f^*(a_1^2, a_2^2)f^*(a_3, e_1) = 1.$$

This proves that $f^* \in \mathbb{I}\Gamma$. Thus, $\mathbb{I}\Gamma$ is a group.

Corollary. *By (1), the F_P are all related to each other by inner automorphisms, and thus we have a canonical group homomorphism*

$$\mathbb{I}\Gamma \to \mathrm{Out}^*(\widehat{\Gamma}_{g,n}).$$

This completes the proof of theorem 2.

Proof of theorem 3. Let us now prove theorem 3, which states a collection of Galois-type properties of $\mathbb{I}\Gamma$.

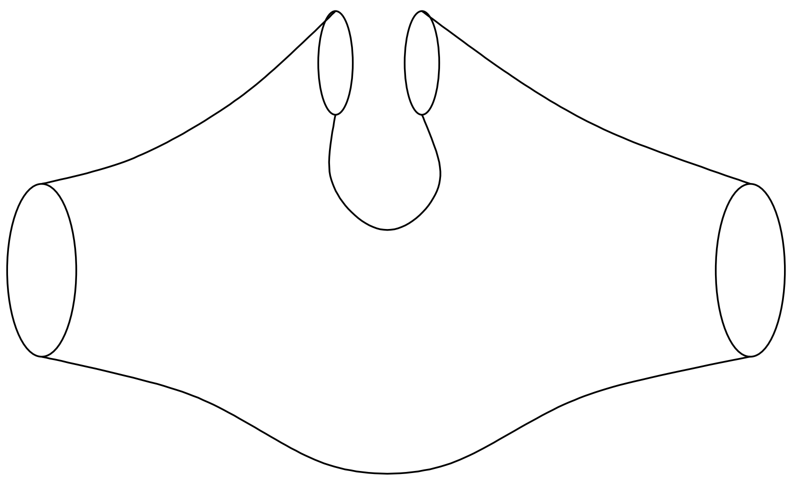
The first statement was that $\mathbb{I}\Gamma$ preserves inertia, i.e. conjugates all Dehn twists, which was proved above.

The next statement is that $F \in \mathbb{I}\Gamma$ respects the point-erasing and subsurface inclusion homomorphisms between mapping class groups. This follows essentially from the construction using pants decomposition.

For the point-erasing homomorphism, let P be a pants decomposition containing two loops α and β that become equal when the point is erased. Then since they are in the the pants decomposition, $F_P(a) = a$ and $F_P(b) = b$, so this action passes to the quotient $a = b$.

For the subsurface-inclusion homomorphism, we choose a pants decomposition P on the larger surface containing the disjoint simple closed loops cutting out the smaller surface. By the locality property of the $\mathbb{I}\Gamma$-action, the action on the subgroup corresponding to the smaller surface is exactly the action on the mapping class group of the smaller surface.

The last statement is the intersection $G_{\mathbb{Q}} \cap \mathbb{I}\Gamma = G_{\mathbb{Q}}^{\mathrm{ab}}$. For a sketch of the proof, we use the characterization of $\mathbb{I}\Gamma \subset \widehat{GT}^{1} \subset \widehat{GT}$ as being exactly those automorphisms of $\Gamma_{0,4}$ that extend to $\widehat{\Gamma}_{1,2}$ under the subsurface-inclusion morphism corresponding to;



Since we know that the group homomorphisms corresponding to subsurface inclusion come from morphisms between moduli spaces that are defined over $\mathbb{Q}$, the $G_{\mathbb{Q}}$-action on $\widehat{\Gamma}_{0,4}$ and $\widehat{\Gamma}_{1,2}$ certainly respects this morphism, so $G_{\mathbb{Q}}$ lies in $\mathbb{I}\Gamma$.

As a consequence, we obtain the precise description of the $G_{\mathbb{Q}}$-action on Dehn twists, local with respect to a pants decomposition and calculated on all Dehn twists via the lego of A and S-moves, that was given in the first lecture.

§3B.5. Final remarks

Remark 1. In the genus zero case, there are no S-moves, so the pants decomposition complex is given only by (3A) triangles and (5A) pentagons. It is exactly the same as the MacLane complex with associativity moves we saw earlier.

Remark 2. Just as we saw how the braided tensor category corresponded to a combinatorial description of the fundamental groupoid of the genus zero moduli spaces based at tangential base points, we can use the pants decomposition complex to describe the fundamental groupoid of the moduli spaces in all genera (including zero). The A- and S-moves correspond to precise paths along the divisor at infinity of the moduli space.

Remark 3. It is a very legitimate question whether $\widehat{GT}$ itself acts on the higher genus mapping class groups, or whether the fourth relation is really necessary. Although intuitively it seems necessary, who knows?

B. Enriquez has studied the higher-genus braid groups, meaning the groups of braids on higher genus surfaces. They are subgroups of the mapping class groups. Recently he showed that $\widehat{GT}$ acts on $B_{1,3}$. This is a surprising result - but it might just be an ungeneralizable fluke!