Grothendieck-Teichmüller Lie theory

and multiple zeta values

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Lecture 4A

The "absolute Galois" Lie algebra,

the Kashiwara-Vergne Lie algebra,

mixed Tate motives,

and motivic multizeta values

§1. The graded Grothendieck-Teichmüller Lie algebra

Definition

Characterization as derivations of braid Lie algebras

H. Furusho's single-relation theorem

Modular forms and Bernoulli numbers

§2. Connections with other parts of mathematics

A) Multiple zeta values ("double shuffle")

Definition

Furusho's theorem: injection $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$

B) $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the Deligne-Ihara Lie algebra DI_p

Definition

Ihara's theorem: injection $DI_p \hookrightarrow \mathfrak{grt} \otimes \mathbb{Q}_p$

C) Kashiwara-Vergne Lie algebra

The Kashiwara-Vergne problem (Alekseev-Torossian)

Schneps' theorem: injection $\mathfrak{grt} \hookrightarrow \mathfrak{kv}$

D) Mixed Tate motives and motivic multizetas

Definitions, properties of MTM and motivic multizetas

The fundamental (free) Lie algebra $\mathfrak{fr}$ of MTM

Brown's theorem: motivic multizetas generate MTM

Corollaries: $DI_p = \mathfrak{fr} \otimes \mathbb{Q}_p$ and $\mathfrak{fr} \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds}$.

§**2.B.** $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ **and Deligne-Ihara Lie algebra DI**

Let $\pi = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$, $\pi^{(\ell)}$ denote the pro-ℓ completion.

Let $\left(\pi^{(\ell)}\right)^i$ denote the groups of the descending central sequence

$$\left(\pi^{(\ell)}\right)^0 = \pi^{(\ell)}, \quad \left(\pi^{(\ell)}\right)^i = \left(\pi^{(\ell)}, \left(\pi^{(\ell)}\right)^{i-1}\right).$$

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and set

$$G_{\mathbb{Q}}^i = \mathrm{Ker}\left(G_{\mathbb{Q}} \to \mathrm{Out}\left(\pi^{(\ell)}/\left(\pi^{(\ell)}\right)^i\right)\right).$$

This is an ascending filtration

$$G_{\mathbb{Q}}^0 \subset G_{\mathbb{Q}}^1 \subset G_{\mathbb{Q}}^2 \subset \cdots$$

We have

$$G_{\mathbb{Q}}^0 = G_{\mathbb{Q}}, \quad G_{\mathbb{Q}}^1 = \{\sigma \in G_{\mathbb{Q}} \mid \chi_\ell(\sigma) = 1\}$$

The successive quotients $G^i_{\mathbb{Q}}/G^{i-1}_{\mathbb{Q}}$ are $\mathbb{Z}_\ell$-modules. The **Deligne-Ihara Lie algebra**

$$\Big(\bigoplus_{i\geq 1} G^i_{\mathbb{Q}}/G^{i-1}_{\mathbb{Q}}\Big)$$

is a Lie algebra over $\mathbb{Z}_\ell$ with Lie bracket coming from commutators $\sigma\tau\sigma^{-1}\tau^{-1}$ in $G_{\mathbb{Q}}$. We write DI_ℓ for the tensor product of this Lie algebra with $\mathbb{Q}_\ell$.

Theorem 10. There is an injection

$$DI_\ell \hookrightarrow grt \otimes \mathbb{Q}_\ell$$

coming from the injection of profinite groups from Grothendieck-Teichmüller theory

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}. \qquad (*)$$

Remarks. (1) This conjecture is much *weaker* than the conjecture that (*) is an isomorphism.

(2) The (*) isomorphism would mean that knowledge of certain "combinatorial" ($\widehat{GT}$) properties of the action of $G_{\mathbb{Q}}$ on π_1's of moduli spaces of curves determines $G_{\mathbb{Q}}$ completely.

(3) The Lie algebra conjecture means that knowledge of certain analogous "combinatorial" properties (being *stable derivations* of Lie$[x, y]$ =Lie π_1 of $\mathbb{P}^1 - \{0, 1, \infty\}$) is equivalent to knowledge of the Galois action on the pro-ℓ π_1 of $\mathbb{P}^1 - \{0, 1, \infty\}$.

Theorem 11. (Hain-Matsumoto) *The Lie algebra DI_ℓ is generated by one generator σ_n in each odd weight $n \geq 3$ ("dual to Soulé character κ_m", see below).*

Questions: (1) Ihara asked whether the $\mathbb{Q}_\ell$-Lie algebras DI_ℓ are *motivic* (independent of ℓ), i.e. all given by tensor products with $\mathbb{Q}_\ell$ of a $\mathbb{Q}$-Lie algebra.

(2) A long standing conjecture asked about the *freeness* of the Hain-Matsumoto generators.

Both questions were solved recently, as consequences of Francis Brown's result on motivic multiple zetas (see next lecture). Thus we now have the following result:

Theorem 12. (Brown) *For each ℓ, DI_ℓ is freely generated by the Hain-Matsumoto generators. Thus, letting*

$$\mathfrak{fr} \simeq \mathrm{Lie}[s_3, s_5, s_7, \ldots]$$

be the free $\mathbb{Q}$-Lie algebra with one generator in each odd weight ≥ 3, we have

$$DI_\ell \simeq \mathfrak{fr} \otimes \mathbb{Q}_\ell.$$

Supplement

Fix a prime ℓ, let $K = \mathbb{Q}(\mu_{\ell^\infty})$, and M the maximal extension of K unramified outside ℓ. Let L denote the subextension of the maximal abelian extension of K generated by all ℓ-powerth roots of $\zeta - 1$ for all $\zeta \in \mu_{\ell^\infty} - \{1\}$.

For each odd $m \geq 1$, the Soulé character

$$\kappa_m : \mathrm{Gal}(L/K) \to \mathbb{Z}_\ell(m)$$

is defined by

$$\left(\left(\prod_{\substack{a \pmod{\ell^n} \\ (a,\ell)=1}} (\zeta_n^a - 1)^{\langle a^{m-1} \rangle}\right)^{1/\ell^n}\right)^{\sigma - 1} = \zeta_n^{\kappa_m(\sigma_m)}$$

for all $n \geq 1$ (where $\langle a^{m-1} \rangle$ is the representative of a^{m-1} between 0 and ℓ^n).

Let M_m denote the fixed field of $G_{\mathbb{Q}}^m$. Then

$$G_{\mathbb{Q}}^m / G_{\mathbb{Q}}^{m+1} \simeq \mathrm{Gal}(M_m/M_{m-1}).$$

The Soulé character κ_m induces a homomorphism $\mathrm{Gal}(M_m \cap L/K) \to \mathbb{Z}_\ell$ and thus a homomorphism $\mathrm{Gal}(M_m/K) \to \mathbb{Z}_\ell$; it can be shown that it passes to a non-trivial homomorphism $\mathrm{Gal}(M_m/M_{m-1}) \to \mathbb{Z}_\ell$. The "dual elements" $\sigma_m \in G_{\mathbb{Q}}^m / G_{\mathbb{Q}}^{m+1} \simeq \mathrm{Gal}(M_m/M_{m-1})$ are chosen so that $\kappa_m(\sigma_m)$ generates the image of this homomorphism.

§2.C. Kashiwara-Vergne problem (1978)

Characterize pairs $A, B \in \mathrm{Lie}[x, y]$ such that

$$x + y - ch(y, x) = (1 - e^{-\mathrm{ad}(x)})A + (e^{\mathrm{ad}(y)} - 1)B \in \mathrm{Lie}[x, y]$$

$$\mathrm{div}(A, B) = \frac{1}{2}\mathrm{tr}\big(b(x) + b(y) - b(\mathrm{ch}(x, y))\big) \in Cyc_2$$

where

- $\mathrm{ch}(x, y) = x + y + \frac{1}{2}[x, y] + \cdots =$ Campbell-Hausdorff law
- $Cyc_2 = \mathbb{Q}\langle x, y\rangle / \langle ab - ba\rangle$ (not commutative!) Cyclic means mod cyclic permutation of the letters: $xxyy = yxxy = yyxx = xyyx$, $xyxy = yxyx$.
- $\mathrm{tr} : \mathbb{Q}\langle x, y\rangle \to Cyc_2$ quotient map
- $\mathrm{div}(A, B) = \mathrm{tr}(A_x x + B_y y) \in Cyc_2$,
  where $A = A_x x + A_y y$ and $B = B_x x + B_y y$.

The KV-problem was solved by Alekseev-Meinrenken in 2006.

Tangent KV-problem: Find pairs $A, B \in \mathrm{Lie}[x,y]$ (homogeneous of degree n, say) such that

$$[x, A] + [y, B] = 0 \in \mathrm{Lie}[x,y] \qquad (A)$$

$$div(A,B) = \mathrm{tr}\big(x^n + y^n - (x+y)^n\big) \in Cyc_2. \qquad (B)$$

Definition. Let $\widehat{kv}$ be the space of solutions.

Theorem 13. (Alekseev-Torossian) *$\widehat{kv}$ is a graded Lie algebra under the Ihara bracket, and there is an injective map*

$$\begin{aligned} grt &\hookrightarrow \widehat{kv} \\ f(x,y) &\mapsto \big(f(x,z), f(y,z)\big) \end{aligned}$$

with $x + y + z = 0$. In fact, (A) was already proven by Ihara who showed that in grt we have

$$[x, f(z,x)] + [y, f(z,y)] = 0.$$

Theorem 14. (S) *We have an injection*

$$\begin{aligned} \mathfrak{ds} &\hookrightarrow \widehat{kv} \\ f(x,y) &\mapsto \big(f(z,-x), f(z,-y)\big). \end{aligned}$$

The proof of this theorem uses a combinatorial rephrasing of the defining properties of $\widehat{kv}$, and some interesting symmetry properties of elements of $\mathfrak{ds}$.

Summary: So far, we now have the following $\mathbb{Q}$-Lie algebra injections:

$$\mathfrak{fr} \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{kv}.$$

All these are conjectured to be isomorphisms.

§2.D. Mixed Tate motives

As before, let

$$\mathfrak{fr} = \mathrm{Lie}[s_3, s_5, s_7, \ldots].$$

Then $\mathfrak{fr}$ is a pro-nilpotent Lie algebra. Let U be the associated pro-unipotent group

$$U = \exp(\mathfrak{fr}).$$

Let $\mathbb{G}_m$ act on the pro-unipotent group U according to the weight grading on $\mathfrak{fr}$ defined by $\mathrm{weight}(s_i) = -2i$.

Let MTM denote the category of mixed Tate motives unramified over $\mathbb{Z}$.

There is an equivalence of categories between MTM and the category of finite-dimensional vector spaces which are equipped with a $(U \rtimes \mathbb{G}_m)$-module structure.

The $\mathbb{G}_m$ action on such a vector space defines a grading on it.

From now on, we will think of a mixed Tate motive M as a vector space in this way, and write $gr\, M$ for the associated graded. Roughly speaking, the **geometric aspect** of motives comes from the fact that the vector spaces that turn up as mixed Tate motives are cohomology groups of schemes/stacks.

A *framing* on a mixed Tate motive M is a pair

$$\begin{cases} f : \mathbb{Q}(-n) \to gr_{2n}M \\ v : \mathbb{Q}(0) \to gr_0 M^*. \end{cases}$$

So we can think of v as being in the 0th homology group, and f as being a differential n-form.

A *framed mixed Tate motive* is an equivalence class of framed motives, where the equivalence relation is a linear map between the graded vector spaces that carries one framing to the other.

There is a **period map** from mixed Tate motives to $\mathbb{R}$ given by "integrating f over v". This is equal to the value

$$\langle v, f \rangle,$$

which is well-defined since $gr\ M$ and $gr\ M^*$ are also dual spaces.

Theorem 15. (Beilinson-Goncharov) *The framed mixed Tate motives form a Hopf algebra under the tensor product.*

The Beilinson virtuous circle

Let $\mathcal{C}$ be a Tannakian subcategory of MTM, $\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ the fiber functor.

To $\mathcal{C}$ we associate $U_L = \mathrm{Aut}(\omega)$ and $L = \mathrm{Der}(\omega)$, the automorphisms and derivations of the fiber functor.

Then L is a **Lie algebra** and $U_L = exp(\mathcal{L})$ is the pro-unipotent part of the fundamental group of the Tannakian category $\mathcal{C}$. This means that $\mathcal{C}$ is equivalent to the category of representations of $U_L \rtimes \mathbb{G}_m$.

Example. We saw this above for $\mathcal{C} = MTM$, where

- $L = \mathfrak{fr}$
- $U = exp(\mathfrak{fr})$
- MTM is equivalent to $\mathrm{Rep}(U \rtimes \mathbb{G}_m)$

To the subcategory $\mathcal{C}$, we can also associate the Hopf algebra $\mathcal{A}_{\mathcal{C}}$ of equivalence classes of framed motives of $\mathcal{C}$. In our example,

- $\mathcal{A}_{MTM} \simeq \mathbb{Q}[s_3, s_5, s_7, \ldots]$ viewed as a commutative Hopf algebra generated by monomials, with multiplication given by the shuffle product.

Theorem 17. *The Hopf algebras $\mathcal{U}L$ and $\mathcal{A}_{\mathcal{C}}$ are dual.*

Equivalently, $\mathcal{A}_{\mathcal{C}}$ can be identified with the Hopf algebra of affine functions on U.

Equivalently, the Lie algebra $L = Der(\omega)$ is dual to the Lie coalgebra $\mathcal{L}$ obtained by quotienting $\mathcal{A}_{\mathcal{C}}$ by constants and products.

Motivic multizeta values

Goncharov-Manin (adapting an earlier definition by Goncharov-Deligne based on $\mathbb{P}^1 - \{0, 1, \infty\}$) defined a Tannakian subcategory MTM' of MTM of motives arising from genus zero moduli spaces of curves.

The multiple zeta motive associated to a multizeta value $\zeta(k_1, \ldots, k_r)$ is constructed roughly as follows.

Let $n = k_1 + \cdots + k_r$ and let $(\epsilon_1, \ldots, \epsilon_n)$ be the tuple of 0's and 1's associated to $(k_1, \ldots, k_r)$ as in the definition of multizeta values. Let f be the differential n-form

$$f = (-1)^r \frac{dt_1}{t_1 - \epsilon_1} \frac{dt_2}{t_2 - \epsilon_2} \cdots \frac{dt_n}{t_1 - \epsilon_n}.$$

Let v be the simplex $0 < t_1 < \cdots < t_n < 1$ on the n-dimensional moduli space $M_{0,n+3}$.

Let M be the relative cohomology group

$$H^{2n}\big(M_{0,n} - A, B - (B \cap A)\big),$$

where A is the part of the divisor at infinity where f has poles, and B is the union of the irreducible components of the divisor at infinity along which f does not have poles.

Then Goncharov-Manin show that (M, f, v) is a framed mixed Tate motive over $\mathbb{Z}$; its associated period is given by

$$\int_v f = \zeta(k_1, \ldots, k_r).$$

We use the notation

$$\zeta^m(k_1, \ldots, k_r) = I^m(0; \epsilon_n, \ldots, \epsilon_1; 1)$$

for this motivic multiple zeta; for example

$$\zeta^m(2) = I^m(0; 1, 0; 1).$$

• Let $\mathcal{MZ} = \mathcal{A}_{MTM'}$ denote the Hopf algebra of the *motivic multizeta values* $I^m(0; \epsilon_1, \ldots, \epsilon_n; 1)$, i.e. the Hopf algebra of framed mixed Tate motives in MTM'.

• Let $\mathfrak{nmz}$ be the quotient of $\mathcal{MZ}$ by the ideal generated by $\mathcal{MZ}_0$, $\mathcal{MZ}_2$ and products $\mathcal{MZ}_{>0}^2$.

Goncharov with the $\mathbb{P}^1 - \{0, 1, \infty\}$ formulation, and Soudères directly with the $M_{0,n}$ formulation (thesis) proved that the elements of $\mathcal{MZ}$ satisfy the double shuffle relations.

Thus this Hopf algebra is a quotient of the Hopf algebra $\mathcal{FZ}$ of formal multizeta values satisfying only double shuffle. But the motivic ones may satisfy other relations: it is not known.

Duality diagrams

$$
\begin{tikzcd}
 & \mathcal{FZ} \arrow[dl] \arrow[dr] & \\
\mathcal{MZ} \arrow[dr] & & \mathfrak{nfz} \arrow[dl] \\
 & \mathfrak{nmz} &
\end{tikzcd}
$$

$$\Updownarrow$$

$$
\begin{tikzcd}
 & \mathcal{U}\mathfrak{ds} & \\
\mathcal{MZ}^* \arrow[ur, hook] & & \mathfrak{nfz}^* = \mathfrak{ds} \arrow[ul, hook] \\
 & \mathfrak{nmz}^* \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
.
$$

Letting $L' = Der(\omega')$ for the fiber functor ω' of MTM', and $U' = exp(L')$, then since $MTM' \subset MTM$, we have a surjective map $U \twoheadrightarrow U'$.

Taking the logs, we have a surjection of Lie algebras

$$\mathfrak{fr} = \log(U) \twoheadrightarrow \log(U'). \qquad (*)$$

To identify the Lie algebra $\log(U')$, we use the Beilinson circle.

Let $\mathcal{MZ}$ be the Hopf algebra of motivic multizeta values, and let $\mathfrak{nmz}$ be its quotient modulo constants and products as above.

Then we know from the Beilinson circle that

$$\log(U') = \mathfrak{nmz}^*.$$

So (*) says that we have a Lie algebra surjection $\mathfrak{fr} \twoheadrightarrow \mathfrak{nmz}^*$ and so

$$\mathcal{U}\mathfrak{fr} \twoheadrightarrow \mathcal{U}\mathfrak{nmz}^* = \mathcal{MZ}^*.$$

Because we are using graded duals, we have

$$\dim \mathcal{MZ}_n = \dim \mathcal{MZ}_n^* \text{ for all } n.$$

For all $n > 2$, we have

$$\mathcal{MZ}_n \twoheadrightarrow \mathcal{Z}_n$$

by the period map that associates a framed motive to its period. (The motivic multizeta value $\zeta^M(2) = 0$.) Let

$$\widetilde{\mathcal{MZ}} = \mathcal{MZ} \otimes \mathbb{Q}[s_2].$$

Then for $n \geq 1$ we have

$$\dim \mathcal{Z}_n \leq \dim \widetilde{\mathcal{MZ}}_n \leq \dim \left(\mathcal{U}\mathfrak{fr}_n \otimes \mathbb{Q}[s_2]\right).$$

Let $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_n = d_{n-2} + d_{n-3}$. Then

$$\sum_{n \geq 0} d_n t^n = \frac{1}{1 - t^2 - t^3};$$

these are the dimensions of $\mathcal{U}\mathfrak{fr}_n \otimes \mathbb{Q}[s_2]$.

Thus we obtain the **Zagier upper bound**:

Theorem 18: $\dim \mathcal{Z}_n \leq d_n$.

This theorem is the main result in multizeta value theory where the theory of motives seems indispensable for the proof.

It was independently proved by Goncharov, Terasoma.

Connection with double shuffle Lie algebra

Since we also have an injection $\mathfrak{nmz}^* \hookrightarrow \mathfrak{ds}$, this gives

$$\mathfrak{fr} \twoheadrightarrow \mathfrak{nmz}^* \hookrightarrow \mathfrak{ds}.$$

The above-mentioned theorem by Francis Brown (next lecture) shows that in fact $\mathfrak{fr} \simeq \mathfrak{nmz}^*$.

Every choice of a depth 1 element f_i in $\mathfrak{ds}_i$ for odd $i \geq 3$ gives a map

$$\begin{aligned} \mathfrak{fr} &\to \mathfrak{ds} \\ s_i &\mapsto f_i. \end{aligned}$$

While weaker than finding canonical depth 1 elements for $\mathfrak{ds}$ in each odd weight, the result $\mathfrak{fr} \simeq \mathfrak{nmz}^* \hookrightarrow \mathfrak{ds}$ shows that there is a canonical image of $\mathfrak{fr}$ inside $\mathfrak{ds}$: "some choices are better than others".

As we saw earlier, the map $\mathfrak{fr} \simeq \mathfrak{nmz}^* \hookrightarrow \mathfrak{ds}$ factors through

$$\mathfrak{fr} \simeq \mathfrak{nmz}^* \hookrightarrow \mathfrak{grt} \hookrightarrow \mathfrak{ds}.$$

Main conjecture. *These maps are isomorphisms.*

Remark. If true, this would show that the Hopf algebra of formal zeta values is isomorphic to the Hopf algebra of motivic multizetas, and that all relations between motivic multizetas come from double shuffle. This would solve the mystery of the "further relations" possibly satisfied by motivic zetas.

It would also provide the "sandwich" corollary $\mathfrak{fr} \simeq \mathfrak{grt}$, which is a Lie version of the Shafarevich conjecture.