Grothendieck-Teichmüller Lie theory

and multiple zeta values

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Lecture 4B

Motivic multizetas and Brown's theorem

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Definition

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§0. Necessary background

Hopf algebra of framed mixed Tate motives

Recall that $\mathfrak{fr} = \mathrm{Lie}[s_3, s_5, \ldots]$, so that $\mathcal{U}\mathfrak{fr}$ is the polynomial ring $\mathbb{Q}[s_3, s_5, \ldots]$ with coproduct given by $\Delta(s_i) = s_i \otimes 1 + 1 \otimes s_i$. Let $\mathcal{F}' = \mathcal{U}\mathfrak{fr}^*$, so $\mathcal{F}'$ is equal to $\mathbb{Q}[f_3, f_5, \ldots]$ as a vector space, but its ring generators are monomials, and it is equipped with the shuffle product and the deconcatenation coproduct

$$\Delta(w) = \sum_{uv=w} u \otimes v.$$

As before, we write $\mathcal{A}_{MTM}$ for the graded Hopf algebra of framed mixed Tate motives (isomorphic to the Hopf algebra of affine functions of U over $\mathbb{Q}$; equipped with the shuffle product). We have the non-canonical Hopf algebra duality isomorphism

$$\mathcal{A}_{MTM} \simeq \mathcal{F}'. \tag{0.1}$$

Define a slightly larger Hopf algebra by adding in a formal variable f_2 that commutes with $\mathcal{A}_{MTM}$ (it plays a special role analogous to the role of $\zeta(2)$). We define slightly larger Hopf algebras

$$\begin{aligned} \mathcal{H}_{MTM} &= \mathcal{A}_{MTM} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2] \\ \mathcal{F} &= \mathcal{F}' \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]. \end{aligned}$$

The coproducts Δ on $\mathcal{A}_{MTM}$ and $\mathcal{F}'$ extend to coactions

$$\begin{aligned} \Delta :& \mathcal{H}_{MTM} \to \mathcal{A}_{MTM} \otimes \mathcal{H}_{MTM} \\ \Delta :& \mathcal{F} \to \mathcal{F}' \otimes \mathcal{F} \end{aligned} \tag{0.2}$$

by setting $\Delta(f_2) = 1 \otimes f_2$. In analogy with (0.1), we also have non-canonical isomorphisms

$$\mathcal{H}_{MTM} \simeq \mathcal{F}. \tag{0.3}$$

Recall that $\dim \mathcal{F}_n = d_n$ where the d_n are defined by $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_n = d_{n-2} + d_{n-3}$, or equivalently by the generating series $1/(1 - t^2 - t^3)$.

Hopf algebra of motivic multizeta values

Let $\mathcal{A} \subset \mathcal{A}_{MTM}$ be the Hopf algebra generated by the framed mixed Tate motives called *motivic multizeta values*; recall that we have two notations for these:

$$\zeta^m(k_1, \ldots, k_r) = I^m(\epsilon_0; \epsilon_1, \ldots, \epsilon_n; \epsilon_{n+1}),$$

for $0 \leq n$ but $n \neq 2$, where

$$(\epsilon_1, \ldots, \epsilon_n) = \big(1, \underbrace{0, \ldots, 0}_{k_r-1}, \ldots, 1, \underbrace{0, \ldots, 0}_{k_2-1}, 1, \underbrace{0, \ldots, 0}_{k_1-1}\big)$$

The motivic multizeta values satisfy some known properties, of which we specifically list a some that will be used later on:

(1) $I^m(\epsilon_0; \epsilon_1) = 1$ and $I^m(\epsilon_0; \epsilon_1; \epsilon_2) = 0$ for all $\epsilon_0, \epsilon_1, \epsilon_2 \in \{0, 1\}$;

(2) $I^m(\epsilon_0; \epsilon_1, \ldots, \epsilon_n; \epsilon_{n+1}) = 0$ if $\epsilon_0 = \epsilon_{n+1}$ and $n \geq 1$;

(3) $I^m(0; \epsilon_1, \ldots, \epsilon_n; 1) = (-1)^n I^m(1; \epsilon_n, \ldots, \epsilon_1; 0)$;

(4) $I^m(0; \epsilon_1, \ldots, \epsilon_n; \epsilon_{n+1}) = I^m(0; 1 - \epsilon_n, \ldots, 1 - \epsilon_1; 1)$;

(5) the shuffle product formula

$$I^m(x; \epsilon_1, \ldots, \epsilon_r; y) I^m(x; \epsilon_{r+1}, \ldots, \epsilon_s; y) =$$

$$\sum_\sigma I^m\big(x; \sigma\big((\epsilon_1, \ldots, \epsilon_r), (\epsilon_{r+1}, \ldots, \epsilon_s)\big); y\big),$$

where the sum is over the shuffle permutations σ of of $\epsilon_1, \ldots, \epsilon_s$, i.e. permutation such that $\sigma(\epsilon_1) < \cdots < \sigma(\epsilon_r)$ and $\sigma(\epsilon_{r+1}) < \cdots < \sigma(\epsilon_s)$.

The reason we have avoided the case $n = 2$ is that Goncharov-Manin's definition of motivic multizetas yields 0 when $n = 2$. To deal combinatorially with the zeta value for $n = 2$, we add a formal symbol to $\mathcal{A}$ which commutes with all of $\mathcal{A}$,and unhesitatingly denote it by $\zeta^m(2) = I^m(0; 1, 0; 1)$. By (1)-(4), this also formally defines the symbols $I^m(a; b, c; d)$ for $a, b, c, d \in \{0, 1\}$. We set

$$\mathcal{H} = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^m(2)].$$

Then we have a surjective period map $\mathcal{H} \to$ *real multizeta values* mapping

$$\zeta^m(k_1, \ldots, k_r) = I^m(\epsilon_0; \epsilon_1, \ldots, \epsilon_n; \epsilon_{n+1}) \to \zeta(k_1, \ldots, k_r) \in \mathbb{R};$$

recall that this real multizeta value is given by the period integral

$$\zeta(k_1, \ldots, k_r) = (-1)^r \int_{0<t_1<\cdots<t_n<1} \frac{dt_1}{t_1 - \epsilon_1} \cdots \frac{dt_n}{t_n - \epsilon_n}.$$

Notation: recall that $\mathcal{F}' \simeq (\mathcal{U}\mathfrak{fr})^*$, so $\mathfrak{fr}^* = \mathcal{F}'/(\mathcal{F}'_{>0})^2$.

- Write $\mathcal{L} = \mathcal{A}_{>0}/\big(\mathcal{A}_{>0}\big)^2$;
- let π denote the surjection $\mathcal{A}_{>0} \to \mathcal{L}$.
- Write $\zeta_n = \pi\big(\zeta^m(n)\big) \in \mathcal{L}$.

Remark. Any choice of isomorphism $\mathcal{A}_{MTM} \xrightarrow{\sim} \mathcal{F}'$ induces an isomorphism $\mathcal{H}_{MTM} \xrightarrow{\sim} \mathcal{F}$, and thus an inclusion

$$\mathcal{H} \hookrightarrow \mathcal{F}$$

due to the inclusion $\mathcal{H} \subset \mathcal{H}_{MTM}$. The goal of the game is to calculate the dimensions of the graded parts $\mathcal{H}_n$ to prove that they are equal to those of $\mathcal{F}'$.

Brown's strategy is to calculate the dimensions of $\mathcal{H}_n^{2,3}$ where $\mathcal{H}^{2,3}$ is the subspace of $\mathcal{H}$ generated by $\zeta^m(k_1, \ldots, k_r)$ having only 2's and 3's as arguments, and shows that this already has dimension equal to that of $\mathcal{F}$.

For this, it's sufficient to show linear independence of the $\zeta^m(2, ..., 3, ...)$ since there is exactly the right number of them: indeed, the number d_n of these multizetas in weight n is given by $d_n = d_{n-2} + d_{n-3}$, and we have $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ as for the dimensions of $\mathcal{F}$.

§1. Statement of Brown's main theorem and corollaries

Hoffman's famous conjecture on real multizetas states that

Conjecture: *The real multiple zeta values* $\zeta(k_1, \ldots, k_r)$ *with* $k_i \in \{2,3\}$ *are all linearly independent and those with* $k_1 + \cdots + k_r = n$ *generate* $\mathcal{Z}_n$*, which is thus* d_n*-dimensional.*

Francis Brown gave the proof of this result in the case of motivic multizeta values.

Brown's Dimension Theorem. *For all* $n \geq 0$*, we have*

$$\dim \mathcal{H}_n = d_n,$$

and a basis for $\mathcal{H}_n$ *is given by* $\zeta^m(k_1, \ldots, k_r)$ *where* $k_1 + \cdots + k_r = n$ *and* $k_i \in \{2,3\}$.

Set $\mathcal{F}' = \mathbb{Q}\langle f_3, f_5, \ldots \rangle$ and $\mathcal{F} = \mathcal{F}' \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]$ as before.

Corollary 1. *The subalgebra* $\mathcal{H}$ *of* $\mathcal{H}_{MTM}$ *is in fact equal to all of* $\mathcal{H}_{MTM}$*. Thus, there exist (non-canonical) Hopf algebra isomorphisms:*

$$\phi : \mathcal{H} \xrightarrow{\sim} \mathcal{F}.$$

Corollary 2. *The Deligne-Ihara Lie algebra* DI_ℓ *is isomorphic to* $\mathfrak{fr}\otimes\mathbb{Q}_\ell$*.*

Proof. *By Brown's theorem,* $\mathfrak{nmz}\simeq\mathfrak{fr}$*, so it acts on the fundamental Lie algebras of all mixed Tate motives, in particular* $\mathrm{Lie}\,P_5$*, associated to the motive* $M_{0,5}$*. Thus we have the commutative diagram:*

$$\begin{tikzcd}
DI\otimes\mathbb{Q}_\ell \arrow[r, hook] & \mathfrak{grt}\otimes\mathbb{Q}_\ell \simeq & Der^*\mathrm{Lie}\,P_5\otimes\mathbb{Q}_ell \\
\mathfrak{fr}\otimes\mathbb{Q}_\ell \simeq \arrow[u, two heads] & \mathfrak{nmz}\otimes\mathbb{Q}_\ell \arrow[u, hook] \arrow[ur, hook] \arrow[r, hook] & {}
\end{tikzcd}$$

But Brown's theorem and Ihara's theorem show that $\mathfrak{fr}\simeq\mathfrak{nmz}\hookrightarrow\mathfrak{grt}$*, and thus the commutative diagram shows that* $\mathfrak{fr}\twoheadrightarrow DI\otimes\mathbb{Q}_\ell\hookrightarrow\mathfrak{grt}\otimes\mathbb{Q}_\ell$ *can't have any kernel, so* $\mathfrak{fr}\otimes\mathbb{Q}_\ell\simeq\mathfrak{grt}\otimes\mathbb{Q}_\ell$*.* $\diamondsuit$

Proof of Brown's Dimension Theorem.

Before giving the proof, let us explain its structure in three main steps.

Step 1 (§3). *The multizeta values $\zeta^m(2,\dots,2,3,2,\dots,2)$ with a single 3 play an important role: using a theorem of Zagier for real multizeta values, Brown shows that*

$$\zeta^m(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b}) \equiv c_{2^a32^b}\,\zeta^m(2r+1) \quad \text{mod products} \tag{1.1}$$

for $r=a+b+1$, for a rational constant $c_{2^a32^b}$ given by

$$c_{2^a32^b} = 2(-1)^r\left[\binom{2r}{2a+2} - (1-2^{-2r})\binom{2r}{2b+1}\right].$$

He also shows the simple, technical but useful result that, setting

$$\zeta_1^m(\underbrace{2,\dots,2}_{r}) = I^m(0;0,\underbrace{1,0,\dots,1,0}_{r};1),$$

we have

$$\zeta_1^m(\underbrace{2,\dots,2}_{r}) = I^m(0;0,\underbrace{1,0,\dots,1,0}_{r};1) = -2\sum_{a=0}^{r-1}\zeta^m(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{r-1-a}),$$

and concludes from this that

$$\zeta_1^m(\underbrace{2,\dots,2}_{r}) \equiv c_{12^r}\,\zeta^m(2r+1) \ \ \text{mod products},$$

for a rational constant c_{12^r} given by

$$c_{12^r} = -2\sum_{a=0}^{r-1} c_{2^a32^{r-1-a}}.$$

Step 2 (§§2,4). *Brown defines the derivation* D_{2r+1} *as the bigraded part of biweight* $(2r+1, n-2r-1)$ *of Goncharov's coproduct with the left-hand factor projected down from* $\mathcal{A}$ *to* $\mathcal{L}$*, i.e. mod products (this derivation is actually used already in the proof of the result of step 1). He restricts this map to the subspace* $\mathcal{H}^{2,3}$ *of motivic multizetas generated by* $\zeta^m(w)$ *where* w *is a word in only the letters 2 and 3. He equips* $\mathcal{H}^{2,3}$ *with an increasing level filtration* F *according to how many 3's are in* w*, so that* $F^\ell\mathcal{H}^{2,3}$ *is generated by* $\zeta^m(w)$ *where* w *is a word in 2's and 3's with at most* ℓ *3's. Then he shows that the map* D_{2r+1} *passes to a map*

$$gr_\ell^F\mathcal{H}_N^{2,3} \to \mathcal{L}_{2r+1} \otimes gr_{\ell-1}^F\mathcal{H}_{N-2r-1}^{2,3} \tag{1.2}$$

He then shows the key fact that, letting ζ_{2r+1} *denote the image of* $\zeta^m(2r+1)$ *in* $\mathcal{L}$*, the image in the left-hand factor lies in* $\mathbb{Q}\zeta_{2r+1} \subset \mathcal{L}_{2r+1}$.

This is proved as follows: when subsequences are chosen from w *which leave a quotient factor having exactly level* $\ell-1$*, the subsequences can only either be of level 1 exactly, or be of the form* $001010\cdots101$*, yielding left-hand factors that are either* $\zeta^m(w')$ *with* w' *a word of level 1, or* $I^m(0;01\cdots010;1)$ *which is exactly* $\zeta_1^m(2,\ldots,2)$*. Then (1.1) shows that both these left-hand factors are scalar multiples of* $\zeta_{2r+1} \in \mathcal{L}_{2r+1}$*, and their coefficients can be computed explicitly via* D_{2r+1}.

Finally, Brown defines the map

$$\partial_{N,\ell} : gr_\ell^F\mathcal{H}_N^{2,3} \to \bigoplus_{r=1}^{[\frac{N-1}{2}]} gr_{\ell-1}^F\mathcal{H}_{N-2r-1}^{2,3}$$

by composing the maps in (1.2) with $\zeta_{2r+1} \mapsto 1$ *and then summing them over* r*: the image of* $\zeta^m(w)$ *under* $\partial_{N,\ell}$ *is thus a linear combination of* $\zeta^m(w')$ *of weight* $< N$ *and level exactly* $\ell-1$ *with coefficients which are explicitly computable linear combinations of the numbers of the form* $c_{2^a32^b}$ *and* c_{12^a}.

Step 3 (§5). *Let $W_{\ell,n}$ be the set of words containing ℓ 3's and n 2's, so of weight $N = 3\ell + 2n$. Then by definition, the (images in the associated graded of the) $\zeta^m(w)$ form a spanning set for $gr_\ell^F \mathcal{H}_N^{2,3}$.*

Let $\mathbf{W} = \coprod_{m=0}^n W_{\ell-1,m}$. Then

- $|W_{\ell,n}| = |\mathbf{W}|$ *and*
- *the elements of $\mathbf{W}$ form a spanning set for $\oplus_{r=1}^{[\frac{N-1}{2}]} gr_{\ell-1}^F \mathcal{H}_{N-2r-1}^{2,3}$.*

Index the columns of a square matrix $M_{N,\ell}$ by $w \in W_{\ell,n}$ and the rows by $w' \in \mathbf{W}$; define the matrix by $(M_{N,\ell})_{w',w} = f_{w,w'}$ where the entries are defined by

$$\partial_{N,\ell}\big(\zeta^m(w)\big) = \sum_{w' \in \mathbf{W}} f_{w,w'} \zeta^m(w'), \qquad \text{for all } w \in W_{\ell,n},$$

with the $f_{w,w'}$ coming directly from the explicit calculation of $\partial_{N,\ell}$ using the D_{2r+1}. They are all explicit linear combinations of the $c_{2^a 3 2^b}$ and c_{12^r}.

The key result is to prove that the matrix $M_{N,\ell}$ is invertible (§6), *using the explicit knowledge of the form of the matrix entries and of the rational numbers $c_{2^a 3 2^b}$ and c_{12^a}.*

This accomplished, Brown deduces the linear independence theorem at once by induction. Indeed, for level 0 the multizeta values are linearly independent (as there is only one such value for even N, none for odd N). Suppose they are linearly independent for level $\ell - 1$, but that there is a non-trivial linear combination in level ℓ, inducing a non-trivial linear relation $L = 0$ in $gr_\ell^F \mathcal{H}_N^{2,3}$ between multizetas of level exactly ℓ. Let V denote the vector in the spanning set $\{\zeta^m(w) | w \in W_{\ell,n}\}$ whose entries are the coefficients of the linear combination L. Then because $M_{N,\ell}$ is invertible, the vector $W = M_{N,\ell}(V)$ is a non-zero vector, yielding a linear combination L_W with non-zero coefficients of the spanning set $\{\zeta^m(w') | w' \in \mathbf{W}\}$. But the linear combination L_W is equal to 0 in $gr_\ell^F \mathcal{H}_N^{2,3}$ since it is the image under $\partial_{N,\ell}$ of a linear combination $L = 0$ in level ℓ. This means that the non-zero vector W gives a non-trivial linear relation between multizeta values of level $\ell - 1$, contradicting the induction hypothesis, which concludes the proof.

§2. Brown's essential tool: the D_r operator

Goncharov's coproduct. *The Hopf algebra $\mathcal{A}_{MTM}$ is naturally equipped with a coproduct Δ. Goncharov computed the explicit expression of this coproduct restricted to the subalgebra $\mathcal{A}$. On elements of $\mathcal{A}$, it is given as a sum over subsets $S = \{s_1, \ldots, s_r\} \subset \{1, \ldots, n\}$, where to each S we also associate the set of "intervals"*

$$I_1 = \{1, \ldots, s_1 - 1\},\ I_2 = \{s_1 + 1, \ldots, s_2 - 1\}, \ldots, I_{r+1} = \{s_r + 1, \ldots, n\},$$

it being understood that $S = \emptyset$ and $I_j = \emptyset$ are possible. Writing $E_S = (\epsilon_{s_1}, \ldots, \epsilon_{s_r})$ when $S = \{s_1, \ldots, s_r\}$, we have

$$\Delta\big(I^m(0; \epsilon_1, \ldots, \epsilon_n; 1)\big) = \sum_S \Big(\prod_j I^m(E_{I_j})\Big) \otimes I^m(0; E_S; 1).$$

We can break Δ up into a sum

$$\Delta = \sum_{r,n-r} \Delta_{r,n-r},$$

where $\Delta_{n,r-n}$ is the sum over just the terms in which the right-hand factor is of weight $r-n$ (so $(|S| = r-n$ for motivic multizetas).

Example. *Applying the explicit formula for Δ directly to*

$$\zeta^m(2r+1) = I^m(0; 1, \underbrace{0, \ldots, 0}_{2r}; 1),$$

we find that almost all the terms have a factor of I^m that starts and ends in 0, so vanishes. The only subsets S giving non-vanishing terms are $S = \{1, \underbrace{0, \ldots, 0}_{2r}\}$ with no intervals, and $S = \emptyset$ with a single interval given by $I = \{1, \underbrace{0, \ldots, 0}_{2r}\}$. Thus $\zeta^m(2r+1)$ is primitive for Δ, i.e.

$$\Delta\big(\zeta^m(2r+1)\big) = \zeta^m(2r+1) \otimes 1 + 1 \otimes \zeta^m(2r+1).$$

In particular, this implies that we can choose Hopf algebra isomorphisms $\phi : \mathcal{H}_{MTM} \to \mathcal{F}$ such that $\phi\big(\zeta^m(2r+1)\big) = f_{2r+1}$. These are called normalized isomorphisms.

Definition. *The most important tool in Brown's proof is the operator D_r which is obtained by composing $\Delta_{r,n-r}$ with the projection of the left-hand factor from $\mathcal{A}_{MTM}$ to its quotient modulo constants and products.*

When calculating on elements of $\mathcal{A}$, this is $\Delta_{r,n-r}$ composed with the projection $\pi : \mathcal{A} \to \mathcal{L}$ on the left-hand factor. Here, because the projection kills products, only subsets S with a single interval of length r remain in the sum. So for every odd $r \geq 3$, we can write the action of the operator on all of $\mathcal{H}$ as $D_r : \mathcal{H} \to \mathcal{L} \otimes \mathcal{H}$ as

$$D_r\big(I^m(\epsilon_0; \epsilon_1, \ldots, \epsilon_n; \epsilon_{n+1})\big) =$$

$$\sum_{p=0}^{n-r} \pi\big(I^m(\epsilon_p; \epsilon_{p+1}, \ldots, \epsilon_{p+r}; \epsilon_{p+r+1})\big) \otimes I^m(\epsilon_0; \epsilon_1, \ldots, \epsilon_p, \epsilon_{p+r+1}, \ldots, \epsilon_n; \epsilon_{n+1}). \tag{2.1}$$

The operator D_r is a derivation in the sense that

$$D_r(\zeta_1\zeta_2) = (1 \otimes \zeta_1)D_r(\zeta_2) + (1 \otimes \zeta_2)D_r(\zeta_1). \tag{2.2}$$

Small lemma. *If an element $z \in \mathcal{H}_{MTM}$ lies in the kernel of D_s for each odd $s < r$, then it is a rational multiple of $\zeta^m(2r+1)$.*

Example of computation with D_{2r+1}.

Let $w = (\underbrace{2, \ldots, 2}_{a}, 3, \underbrace{2, \ldots, 2}_{b})$, and let $2r + 1 < wt(\zeta) = 2(a + b) + 3$. Then we have

$$D_{2r+1}(\zeta^m(\underbrace{2, \ldots, 2}_{a}, 3, \underbrace{2, \ldots, 2}_{b})) = \pi(\xi^r_{a,b}) \otimes \zeta^m(\underbrace{2, \ldots, 2}_{a+b+1-r}), \tag{2.3}$$

where

$$\xi^r_{a,b} = \sum_{\substack{\alpha \leq a, \beta \leq b \\ \alpha+\beta+1=r}} \big(\zeta^m(\underbrace{2, \ldots, 2}_{\alpha}, 3, \underbrace{2, \ldots, 2}_{\beta}) - \zeta^m(\underbrace{2, \ldots, 2}_{\beta}, 3, \underbrace{2, \ldots, 2}_{\alpha})\big) +$$

$$\big(\mathbb{I}(b \geq r) - \mathbb{I}(a \geq r)\big)\zeta^m_1(\underbrace{2, \ldots, 2}_{r}). \tag{2.4}$$

Proof slide (for computation): skip if necessary.

To see this, we have to consider all possible subsequences of length $2r+1$ *of*

$$(0;\underbrace{10\ldots10}_{a}100\underbrace{10\ldots10}_{b}).$$

There are four kinds of such subsequences: those in which the 100 *isn't contained at all, those in which only part of the* 100 *appears, those with 100 on the right edge, so that the ";" appears between the two zeros, and finally those which contain 100 otherwise than on the right edge.*

If they don't intersect it, then they necessarily start and end in the same symbol 1 *or* 0*, so they give a zero factor on the left-hand side of the tensor product. If they intersect part of it or contain it at the right end, then either they start and end with the same symbol so don't count, or they are of one of the forms*

$$\begin{cases} 01\cdots01 & \textit{with the right-hand 1 being the first of 100} \\ 10\cdots1010 & \textit{with the right-hand 10 being the first 2 of 100} \\ 10\cdots10100 & \textit{with the right-hand 100 being the 3 of 100} \\ 01\cdots01 & \textit{with the left-hand 0 being the last of 100} \\ 001\cdots01 & \textit{with the left-hand 00 being the last 2 of 100.} \end{cases}$$

Of these, the first, second and fourth have an even number of letters, so in fact we have only to consider

$$\begin{cases} 10\cdots10100 & \textit{with the right-hand 100 being the 3 of 100} \\ 00101\cdots01 & \textit{with the left-hand 00 being the last 2 of 100.} \end{cases}$$

These give the terms

$$\begin{cases} \zeta^m(1;0\cdots1010;0) = -\zeta^m(0;0101\cdots0;1) = \zeta_1^m(\underbrace{2,\ldots,2}_{\alpha}) & \textit{for } \alpha\le a \\ \zeta^m(0;0101\cdots0;1) = \zeta_1^m(\underbrace{2,\ldots,2}_{\beta}) & \textit{for } \beta\le b, \end{cases}$$

which gives the formula for $\xi^r_{a,b}$.

§3. Zagier's theorem

We start this section with a technical lemma, then give the statement of Zagier's theorem identifying

$$\zeta(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b})$$

as a rational multiple of $\zeta^m(2a+2b+3)$ *plus products with explicit coefficients. Then we show how Brown lifts Zagier's formula to the motivic multizeta values.*

Let

$$\zeta_1^m(k_1,\dots,k_r) = I^m(0;0,\underbrace{1,0,\dots,0}_{k_1},\underbrace{1,0,\dots,0}_{k_2},\dots,\underbrace{1,0,\dots,0}_{k_r};1).$$

Lemma 1. *We have*

$$\begin{aligned}\zeta_1^m(\underbrace{2,\dots,2}_{r+1}) &= -2\sum_{i_1+\cdots+i_r=1}\zeta^m(2+i_1,\cdots,2+i_r)\\ &= -2\sum_{a=0}^{r}\zeta^m(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{r-a}).\end{aligned}$$

Proof slide (for Lemma 1): skip if necessary

Proof. *We can just directly prove the general formula*

$$\zeta_k^m(k_1,\ldots,k_r) =$$

$$(-1)^k \sum_{i_1+\cdots+i_r=k} \binom{k_1+i_1-1}{i_1}\cdots\binom{k_r+i_r-1}{i_r}\zeta^m(k_1+i_1,\cdots,k_r+i_r).$$

This follows directly from the standard regularization formula due to Furusho (which we write from right to left to agree with Francis, so that convergent words v start with y and end with x):

$$\zeta(x^b v y^a) = \sum_{r=0}^{a}\sum_{s=0}^{b}(-1)^{a+b}\zeta\big(conv(x^s\cdot x^{b-s}vy^{a-r}\cdot y^r)\big)$$

(where the $\cdot$ denotes the shuffle product). From this formula we deduce the simpler one, dealing only with the x's on the left (π_y means projection onto the words starting in y):

$$\zeta(x^b v y^a) = \sum_{s=0}^{b}(-1)^b\zeta\big(\pi_y(x^s\cdot x^{b-s}vy^a)\big).$$

But this simpler version, which expresses $\zeta(w)$ for w starting in x in terms of $\zeta(w')$ for words w' which start in y but are allowed to also end in y, has a non-zero term only when $b=s$, so we get

$$\zeta(x^b v y^a) = (-1)^s\zeta\big(\pi_y(x^s\cdot vy^a)\big).$$

The coefficients in the formula in the statement give the exact number of shuffles of x^s with vy^a that will actually give rise to the same word starting in y. ◇

Example slide (for Lemma 1): skip if necessary

Example 1: *By lemma 1,*

$$
\begin{aligned}
\zeta_2(2,1) &= \zeta(xxyxy) \\
&= \sum_{(i_1,i_2)\in\{(0,2),(1,1),(2,0)\}} \binom{2+i_1-1}{i_1}\binom{1+i_2-1}{i_2}\zeta^m(2+i_1,1+i_2) \\
&= \binom{1}{0}\binom{2}{2}\zeta^m(2,3)+\binom{2}{1}\binom{1}{1}\zeta^m(3,2)+\binom{3}{2}\binom{0}{0}\zeta^m(4,1) \\
&= \zeta^m(2,3)+2\zeta^m(3,2)+3\zeta^m(4,1).
\end{aligned}
$$

By Furusho (for formal zetas, a fortiori for motivic zetas)

$$
\begin{aligned}
\zeta(xxyxy) &= \zeta\big(\pi_y(x^2\cdot yxy)\big) \\
&= \zeta\big(\pi_y(XXyxy+XyXxy+XyxXy+XyxyX+yXXxy+yXxXy \\
&\qquad\qquad +yXxyX+yxXXy+yxXyX+yxyXX)\big) \\
&= \zeta(yXXxy+yXxXy+yXxyX+yxXXy+yxXyX+yxyXX) \\
&= 3\zeta(4,1)+2\zeta(3,2)+\zeta(2,3).
\end{aligned}
$$

Example 2. *We have*

$$
\zeta_1^m(\underbrace{2,\dots,2}_{r}) = \zeta\big(x(yx)^r\big) = -\zeta\big(\pi_y(x\cdot yx\cdots yx)\big)
$$

$$
= -2\zeta(3,2,\dots,2)-2\zeta(2,3,\dots,2)-\cdots-2\zeta(2,\dots,2,3). \tag{3.1}
$$

The following theorem was key to the proof of Brown's result. The form of the identity was discovered and proved directly by Brown in the motivic case, using the proof given in his theorem below. Zagier was then able to identify the actual coefficients in the real case using some analytic methods, and Brown's proof then shows that the same coefficients still work in the motivic case.

Theorem. *(Zagier) The real ζ values satisfy the identity*

$$\zeta(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = \sum_{s=1}^{r} \alpha_s^{a,b}\zeta(2s+1)\zeta(\underbrace{2,\ldots,2}_{r-s}), \qquad (3.2)$$

where for $1 \le s \le r = 2a+2b+3$, we have

$$\alpha_s^{a,b} = 2(-1)^s\left[\binom{2s}{2a+2} - (1-2^{-2s})\binom{2s}{2b+1}\right]. \qquad (3.3)$$

Theorem. *(Brown) The equality (3.2) holds for motivic multiple zeta values, namely we have*

$$\begin{aligned}\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) &= \sum_{s=1}^{r} \alpha_s^{a,b}\zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s})\\ &= \alpha_r^{a,b}\zeta^m(2r+1) + \sum_{s=1}^{r-1} \alpha_s^{a,b}\zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s}),\end{aligned} \qquad (3.4).$$

Brown uses the notation $\alpha_r^{a,b} = c_{2^a32^b}$.

First proof slide (for $\zeta(2,\ldots,2,3,2,\ldots,2)$): skip if necessary

Proof. *We will prove this by induction on the weight r. It is obvious for $r=1$, where the identity reduces to $\zeta^m(3)=\zeta^m(3)$. Assume now that it holds for all elements of weight $< r = 2a+2b+3$. The strategy is to apply D_{2s+1} for $1 \le s < r$ to both sides of the (3.4) and compare the results.*

Left-hand term. By (2.3), we have

$$D_{2s+1}\big(\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})\big) = \pi(\xi^s_{a,b}) \otimes \zeta^m(\underbrace{2,\ldots,2}_{r-s}), \tag{3.5}$$

where $\xi^s_{a,b}$ is of weight lower than $2a+2b+3$. By (2.4), the elements $\xi^s_{a,b}$ are linear combinations of ζ^m's with only 2's and one 3 in weight $2s+1$, and of $\zeta^m_1(2,\ldots,2)$. But by lemma 1 above, we see that $\zeta^m_1(2,\ldots,2)$ is also a linear combination of ζ^m's with only 2's and a single 3, so all of $\xi^s_{a,b}$ is a linear combination of such terms, and thus, by induction, $\xi^s_{a,b}$ has the form (3.2). In particular, there exists a constant such that $\xi^s_{a,b} \equiv \beta^{a,b}_s \zeta^m(2s+1)$ modulo products, so (3.5) becomes

$$D_{2s+1}\big(\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b})\big) = \beta^{a,b}_s \zeta_{2s+1} \otimes \zeta^m(\underbrace{2,\ldots,2}_{r-s}) \tag{3.6}$$

for $1 \le s < r$, where ζ_{2s+1} denotes the image of $\zeta^m(n)$ in $\mathcal{L}$, i.e. mod products.

Right-hand term. We compute the image under D_{2s+1} of the right-hand term of (3.4). Firstly, since the $\zeta^m(2i+1)$ are primitive for Goncharov's Δ, it follows from the definition of D_{2s+1} as part of Δ that

$$D_{2s+1}\big(\zeta^m(2i+1)\big) = \begin{cases} \zeta^m(2s+1)\otimes 1 \in \mathcal{L}\otimes\mathcal{H} & \textit{if } s=i \\ 0 & \textit{otherwise.}\end{cases}$$

Then, using this and (2.2), we compute

$$\begin{aligned} D_{2s+1}\big(\zeta^m(2i+1)\zeta^m(\underbrace{2,\ldots,2}_{r-i})\big) &= \big(1\otimes\zeta^m(2i+1)\big)D_{2s+1}\big(\zeta^m(\underbrace{2,\ldots,2}_{r-i})\big) \\ &\quad + \big(1\otimes\zeta^m(\underbrace{2,\ldots,2}_{r-i})\big)D_{2s+1}\big(\zeta^m(2i+1)\big)\big) \\ &= \begin{cases} \zeta^m(2s+1)\otimes\zeta^m(\underbrace{2,\ldots,2}_{r-s}) & \textit{if } s=i \\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

Second proof slide for $\zeta(2,\ldots,2,3,2,\ldots,2)$: skip if necessary

Indeed, $D_{2r+1}\big(\zeta^m(\underbrace{2,\ldots,2})\big)=0$ automatically, because $\zeta^m(\underbrace{2,\ldots,2})=I^m(0;1010\cdots10;1)$ and any odd-length subsequence of $01\cdots01$ must start and end in the same symbol, so be equal to 0 by 1) of the definition of motivic multizeta values (cf. §0).

Thus, we have now shown that the LHS and the RHS of (3.4) have equal images under D_{2s+1} for $1\le s<r$. Thus by the remark after the lemma in §2, the difference between them is a rational multiple of $\zeta^m(2r+1)$. So we have shown that an equation of the form (3.4) holds, with all coefficients $\alpha_s^{a,b}$ on the right-hand side except for $s=r$ equal to those in Zagier's theorem. In other words, we have

$$\zeta^m(\underbrace{2,\ldots,2}_a,3,\underbrace{2,\ldots,2}_b)-\sum_{s=1}^{r-1}\alpha_s^{a,b}\zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s})=\beta_r^{a,b}\zeta^m(2r+1),\tag{3.7}$$

so that projecting down to the real multizeta values, we have

$$\zeta(\underbrace{2,\ldots,2}_a,3,\underbrace{2,\ldots,2}_b)-\sum_{s=1}^{r-1}\alpha_s^{a,b}\zeta(2s+1)\zeta(\underbrace{2,\ldots,2}_{r-s})=\beta_r^{a,b}\zeta(2r+1),\tag{3.8}$$

and by (3.2), we have

$$\zeta(\underbrace{2,\ldots,2}_a,3,\underbrace{2,\ldots,2}_b)-\sum_{s=1}^{r-1}\alpha_s^{a,b}\zeta(2s+1)\zeta(\underbrace{2,\ldots,2}_{r-s})=\alpha_r^{a,b}\zeta(2r+1).\tag{3.9}$$

Comparing (3.8) and (3.9), since it is known that $\zeta(2r+1)\neq 0\in\mathbb{R}$ (although not much else is known about $\zeta(2r+1)$), we find that $\beta_r^{a,b}=\alpha_r^{a,b}$, concluding the proof. $\diamondsuit$

Notation. *Following Brown, let us rewrite*

$$\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = \sum_{s=1}^{r} \alpha_s^{a,b} \zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s})$$

as

$$\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) = \sum_{s=1}^{r-1} \alpha_s^{a,b} \zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s}) + c_{2^a32^b}\zeta^m(2r+1),$$

i.e. $c_{2^a32^b}$ *is a new notation for* $\alpha_r^{a,b}$*, with* $r = a+b+1$*. By (3.3), we have*

$$c_{2^a32^b} = 2(-1)^r \left[\binom{2r}{2a+2} - (1-2^{-2r}) \binom{2r}{2b+1} \right].$$

Since by lemma 1, $\zeta_1^m(\underbrace{2,\ldots,2}_{a})$ *is also a linear combination of* $\zeta^m(w)$ *with* w *having many 2's and a single 3, (3.4) shows that* $\zeta_1^m(\underbrace{2,\ldots,2}_{r})$ *can also be written in the form (3.4); we write*

$$\zeta_1^m(\underbrace{2,\ldots,2}_{r}) = \sum_{s=1}^{r-1} \beta_s \zeta^m(2s+1)\zeta^m(\underbrace{2,\ldots,2}_{r-s}) + c_{12^r}\zeta^m(2r+1).$$

Thanks to lemma 1, we have

$$c_{12^r} = -2\sum_{a=0}^{r-1} c_{2^a32^{r-1-a}}. \tag{3.10}$$

§4. Brown's 2-3 subspace $\mathcal{H}^{2,3} \subset \mathcal{H}$ and its filtration

Let $\mathcal{H}^{2,3}$ denote the sub-Hopf algebra of $\mathcal{H}$ generated by $\zeta^m(k_1,\ldots,k_r)$ with $k_i \in \{2,3\}$. This subalgebra has a natural filtration

$$F^\ell\mathcal{H}^{2,3} = \langle \zeta^m(w) \,|\, \deg_3 w \leq \ell \rangle,$$

where w is a "word" in 2's and 3's. So $F^\ell\mathcal{H}^{2,3}$ contains all $\zeta^m(w)$ with words w having $1 \leq j \leq \ell$ 3's, i.e.

$$F^\ell\mathcal{H}^{2,3} \subset F^{\ell+1}\mathcal{H}^{2,3}.$$

Let π denote the surjection $\pi : \mathcal{A} \twoheadrightarrow \mathcal{L}$ as usual, and $\zeta_{2r+1} = \pi\big(\zeta^m(2r+1)\big)$ as before.

Proposition.

(i) We have

$$D_{2r+1}(F^\ell\mathcal{H}^{2,3}) \subset \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} F_{\ell-1}\mathcal{H}^{2,3},$$

i.e. the right-hand factor has strictly less than ℓ 3's in it.

(ii) We have

$$gr_\ell^F\, D_{2r+1} : gr_\ell^F\, \mathcal{H}^{2,3} \to \mathcal{L}_{2r+1} \otimes_{\mathbb{Q}} gr_{\ell-1}^F\, \mathcal{H}^{2,3},$$

i.e. if the map in (i) is composed with quotienting the right-hand factor of the RHS of (i) by $F^{\ell-2}\mathcal{H}^{2,3}$, then $F^{\ell-1}\mathcal{H}^{2,3}$ is in the kernel.

(iii) We have

$$gr_\ell^F\, D_{2r+1} : gr_\ell^F\, \mathcal{H}^{2,3} \to \mathbb{Q}\zeta_{2r+1} \otimes_{\mathbb{Q}} gr_{\ell-1}^F\, \mathcal{H}^{2,3},$$

i.e. the left-hand factor is nothing but a multiple of ζ_{2r+1}.

First proof slide (for Proposition): skip if necessary

Proof. *From §2, we have the formula*

$$D_{2r+1}\big(I^m(\epsilon_0;\epsilon_1,\ldots,\epsilon_n;\epsilon_{n+1})\big) =$$

$$\sum_{p=0}^{n-2r-1} \pi\big(I^m(\epsilon_p;\epsilon_{p+1},\ldots,\epsilon_{p+2r+1};\epsilon_{p+2r+2})\big)\otimes I^m(\epsilon_0;\epsilon_1,\ldots,\epsilon_p,\epsilon_{p+2r+2},\ldots,\epsilon_n;\epsilon_{n+1})$$

So for (i), we have only to see what happens when a consecutive subsequence is removed from a sequence of 10's and 100's (coming from a sequence of 2's and 3's). If the removed subsequence is of length 1, or starts and ends with 1 or starts and ends with 0, the corresponding term in (3.1) is 0 because of the left-hand factor. If the removed subsequence starts with 1 and ends with 0 or vice versa, then it is impossible to ever have a string of more than two consecutive zeros in the quotient sequence (i.e. the remaining part after removal of the subsequence), so the quotient sequence is still a sequence of 10's and 100's (i.e. 2's and 3's). Furthermore, the only way to take a subsequence out of a sequence of 10's and 100's without decreasing the number of 100's is to remove a subsequence of the form 1010...10 or 0101...01, but these must have even length and we are dealing with the case of subsequences of length $2r+1$. *This proves (i).*

For (ii), consider the map Ψ *given by composing*

$$D_{2r+1} : F^\ell\mathcal{H}^{2,3} \to \mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} F^{\ell-1}\mathcal{H}^{2,3}$$

with

$$\mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} F^{\ell-1}\mathcal{H}^{2,3} \to \mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} F^{\ell-1}\mathcal{H}^{2,3}/F^{\ell-2}\mathcal{H}^{2,3} = \mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} gr^F_{\ell-1}\mathcal{H}^{2,3}.$$

If $x\in F^{\ell-1}\mathcal{H}^{2,3}\subset F^\ell\mathcal{H}^{2,3}$, *then* $D_{2r+1}(x)\in\mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} F^{\ell-2}\mathcal{H}^{2,3}$ *by (i), so* x *is in the kernel of the composition of maps* Ψ, *which thus factors through the quotient* $F^\ell\mathcal{H}^{2,3}/F^{\ell-1}\mathcal{H}^{2,3}$. *Thus we have defined a map*

$$gr^F_\ell D_{2r+1} : gr^F_\ell\mathcal{H}^{2,3} \to \mathcal{L}_{2r+1}\otimes_{\mathbb{Q}} gr^F_{\ell-1}\mathcal{H}^{2,3}.$$

This proves (ii).

Second proof slide (for Proposition): skip if necessary

Finally, for (iii), we consider the left-hand factors of elements of $gr_\ell^F\, D_{2r+1}(F^\ell\mathcal{H}$
For $w \in F^\ell\mathcal{H}^{2,3}$*, the terms of* $D_{2r+1}(w)$ *that remain in the graded situation are those where the right-hand factor contains exactly one* 00 *less than* w*, i.e. exactly* $\ell - 1$ *00's. This means that there are four possibilities for the left-hand factor:*

$$\begin{cases} I^m(0;10\ldots10010\ldots10;1) = \zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) \\ I^m(1;01\ldots0100101\ldots01;0) = -\zeta^m(\underbrace{2,\ldots,2}_{a},3,\underbrace{2,\ldots,2}_{b}) \\ I^m(0;01\ldots10;1) = \zeta_1^m(\underbrace{2,\ldots,2}_{r}) \\ I^m(1;01\ldots10;0) = -\zeta_1^m(\underbrace{2,\ldots,2}_{r}). \end{cases}$$

By lemma 1 of §3 and Brown's lifting of Zagier's theorem in §3, all of these elements project down to a scalar multiple of ζ_{2r+1} *in* $\mathcal{L}$*, which proves the result.* $\diamondsuit$

Part (iii) of the proposition shows that restricted to $\mathcal{H}^{2,3}_N$ for a fixed positive weight N,

$$gr^F_\ell D_{2r+1} : gr^F_\ell \mathcal{H}^{2,3}_N \to \mathbb{Q}\zeta_{2r+1} \otimes gr^F_\ell \mathcal{H}^{2,3}_{N-2r-1}.$$

Define

$$gr^F_\ell d_{2r+1} : gr^F_\ell \mathcal{H}^{2,3} \to gr^F_{\ell-1} \mathcal{H}^{2,3}$$

to be the composition of gr^F_ℓ with $\zeta_{2r+1} \mapsto 1$, and consider the map on $gr^F_\ell \mathcal{H}^{2,3}_N$:

$$\partial_{N,\ell} := \bigoplus_{r=1}^{[(N-1)/2]} gr^F_\ell d_{2r+1} : gr^F_\ell \mathcal{H}^{2,3}_N \to \bigoplus_{r=1}^{[(N-1)/2]} gr^F_{\ell-1} \mathcal{H}^{2,3}_{N-2r-1}. \qquad (4.1)$$

Remark. *Let $W_{\ell,n}$ denote the set of words with ℓ 3's and n 2's, and set $N = 3\ell + 2n$. Since the weight is a grading on motivic multiple zeta values, any linear relation must take place in a given weight N. If there exists any linear relation $R = 0$ between $\zeta^m(w)$ for w of weight N having only 2's and 3's, then if ℓ is the maximal level of any term appearing in R, and P is the linear combination of terms of R of level ℓ, then P lies in $F^{\ell-1}\mathcal{H}^{2,3}_N$. The fact of working in the associated graded means that $P = 0$ is a linear relation between the elements of $W_{\ell,n}$ (considered in $gr^F_\ell \mathcal{H}^{2,3}_N$). Brown's main result is that the $\zeta^m(w)$ with $w \in W_{\ell,n}$ form not just a spanning set but a basis for $gr^F_\ell \mathcal{H}^{2,3}_N$.*

A spanning set for the right-hand space in (4.1) is given by $\zeta^m(w')$ for all words $w' \in \mathbf{W} = \coprod_{m=0}^n W_{\ell-1,m}$. Following Brown's notation, we write $\partial_{N,\ell}$ for the map in (4.1).

§5. Key strategy of Brown's proof.

The first step is to show that in fact, we can write the image of $\zeta^m(w)$ *under* $\partial_{N,\ell}$ *for* $w \in W_{\ell,n}$ *explicitly as a linear combination of* $\zeta^m(w')$ *for* $w' \in \mathbf{W}$, *using the tool* D_r.

Indeed, from the expression for $D_{2r+1}\big(\zeta^m(w)\big)$, *we directly deduce the expression for* $gr_\ell^F D_{2r+1}\big(\zeta^m(w)\big) \in \mathbb{Q}\zeta_{2r+1} \otimes gr_{\ell-1}\mathcal{H}^{2,3}_{<N}$ *as in (iii) of the proposition in §4. Then, mapping* $\zeta_{2r+1} \mapsto 1$ *yields a linear combination of* $\zeta^m(w')$ *for* $w' \in \mathbf{W}$, *and we add up these linear combinations for* $1 \le r \le \left[\frac{N-1}{2}\right]$ *to obtain the image of* $\zeta^m(w)$ *under* $\partial_{N,\ell}$.

Let us write this explicitly as

$$\partial_{N,\ell}\big(\zeta^m(w)\big) = \sum_{\substack{w' \text{ level } \ell-1 \\ \text{weight} < N}} f_{w,w'}\zeta^m(w'). \tag{5.1}$$

Fundamental Remark. *The* $f_{w,w'}$ *are explicit (computable) linear combinations of the* $c_{2^a32^b}$ *and the* $c_{12^{r+1}}$. *Indeed, in the expression*

$$gr_\ell^F D_{2r+1}\big(\zeta^m(w)\big) \in \mathbb{Q}\zeta_{2r+1} \otimes gr_{\ell-1}\mathcal{H}^{2,3}_{<N}, \tag{5.2}$$

the right-hand factor is just a sum of distinct $\zeta^m(w')$ *of level* $\ell-1$, *with only* 1*'s as coefficients, so when* $\zeta_{2r+1} \mapsto 1$, *we find that* $\partial_{N,\ell}\big(\zeta^m(w)\big)$ *is equal to this sum multiplied by the coefficient of* ζ_{2r+1}. *But this coefficient is precisely the linear combination of* $c_{w'}$, *corresponding to all the ways of extracting a subsequence* w' *of length* $2r+1$ *from* w *such that the quotient sequence is of level exactly* $\ell-1$. *As we saw in the proof of (ii) of the proposition in §4, only terms* $\zeta_1^m(\underbrace{2,\ldots,2})$ *or* $\zeta^m(2^a32^b)$ *can come from such subsequences, so the coefficient of* ζ_{2r+1} *in (5.2) is a linear combination of* c_{12^a} *and* $c_{2^a32^b}$, *and can be computed explicitly for any given* N, ℓ.

Fix a weight $N = 3\ell + 2n$, *and let* $W_{\ell,n}$ *denote the set of words* w *with* ℓ *3's and* n *2's. Let* $\mathbf{W}$ *denote the set of words with* $\ell-1$ *3's and* m *2's for* $0 \le m \le n$, *i.e.* $\mathbf{W} = \coprod_{m=0}^n W_{\ell-1,m}$.

Easy Fact. *We have*

$$|W_{\ell,n}| = |\mathbf{W}| = \binom{\ell+n}{\ell}.$$

Definition. *Let* $M_{N,\ell}$ *denote the square matrix of size* $\binom{\ell+n}{\ell}$, *with columns indexed by* $w' \in \mathbf{W}$ *and rows by* $w \in W_{\ell,n}$, *given by* $(M_{N,\ell})_{w',w} = f_{w,w'}$, *so that* $M_{N,\ell}$ *acts on the* $\zeta^m(w)$ *for* $w \in W_{\ell,n}$ *as in (5.1).*

Brown's key result: The matrix $M_{N,\ell}$ is invertible for every $N \geq 3$, $\ell \geq 0$.

Corollary: Brown's Dimension Theorem *The motivic multiple zeta values* $\zeta^m(w)$ *with* w *a word in only 2's and 3's are linearly independent.*

Proof. *The motivic multiple zeta values are weight-graded, so any non-trivial linear relation would yield a non-trivial relation in a given weight* N, *i.e. in* $\mathcal{H}_N^{2,3}$. *In weight* N, *there is at most one word of level* 0. *Assume the theorem holds for level* $\ell - 1$, *and suppose that there exists a linear combination* R *of* $\zeta^m(w)$ *with* $w \in W_{\ell,n}$, $3\ell + 2n = N$, *such that* $R = 0$ *in* $\mathcal{H}_N^{2,3}$. *Let* P *denote the linear combination of terms of* R *of level exactly* ℓ. *Then* $P = 0$ *holds as a linear relation in* $gr_\ell^F \mathcal{H}_N^{2,3}$. *But then, if* V_P *is the vector corresponding to* P, *the entries of the vector* $W = M_{N,\ell}(V_P)$ *give coefficients – which are not all zero since* $M_{N,\ell}$ *is invertible – of a thus non-trivial linear combination of the* $\zeta^m(w')$ *which must be equal to zero, since the image of* V_P *under* $M_{N,\ell}$ *corresponds to the image of* P *under* $\partial_{N,\ell}$. *But this contradicts the induction hypothesis.* $\diamondsuit$

§6. Proof of the key invertibility result.

To conclude, we need to show the result that for all N *and* ℓ, $M_{N,\ell}$ *is invertible. Recall from §3 that for words of the form* $w = 12^r$ *or* $w = 2^a32^b$ *with* $r = a + b + 1$*, we have*

$$\zeta^m(w) \equiv c_w\, \zeta(2r+1) \mod \text{products},$$

where, setting

$$\alpha^r_{a,b} = \binom{2r}{2a+2} - (1 - 2^{-2r}) \binom{2r}{2b+1},$$

we have

$$c_w = \begin{cases} 2(-1)^r \binom{2r}{2a+2} - (1-2^{-2r})\binom{2r}{2b+1} & \text{if } w = 2^a32^b,\ r = a+b+1 \\ -2\sum_{a=0}^{r-1} c_{2^a32^{r-1-a}} & \text{if } w = 12^r. \end{cases}$$

The strategy is as follows. We noted already that the entries of $M_{N,\ell}$ *are all linear combinations of the* c_w*. Let* $M^f_{N,\ell}$ *be the matrix written in the same way, except that the* c_w *are replaced by indeterminates* C_w*. Giving a particular order to the rows and columns of* $M_{N,\ell}$*, Brown shows that* $M^f_{N,\ell}$ *is upper triangular modulo the subspace* I *generated by the* C_{12^r} *and by the differences* $C_w - C_{\tilde{w}}$ *for* w *in 2's and 3's. He then uses the two following lemmas to prove that the 2-adic valuation of the determinant of the projection* $C_w \mapsto c_w$ *of* $M^f_{N,\ell}$ *modulo* I *is non-zero. Thus the determinant of* $M_{N,\ell}$ *is non-zero. The importance of the 2-adic valuation is that the* c_w *all have at most powers of 2 in the denominator.*

Lemma 1. *Let* $w = 2^a32^b$*, and set* $\tilde{w} = 2^b32^a$*. Then*

i) $c_w - c_{\tilde{w}} \in 2\mathbb{Z}$*;*

ii) $v_2(c_{32^{a+b}}) \leq v_2(c_w) \leq 0$.

Proof. *Set $r = a + b + 1$. For i), we have*

$$c_w - c_{\tilde{w}} = 2(-1)^r \Bigg[\binom{2r}{2a+2} - (1-2^{-2r}) \binom{2r}{2b+1} - \binom{2r}{2b+2} + (1-2^{-2r}) \binom{2r}{2a+1} \Bigg] \qquad (*)$$

$$= 2(-1)^r \Bigg[\binom{2r}{2a+2} - \binom{2r}{2b+2} \Bigg]$$

where the second equality holds because

$$\binom{2r}{2a+1} = \binom{2r}{2r-2a-1} = \binom{2r}{2b+1}.$$

But (*) shows that $c_w - c_{\tilde{w}}$ is an even integer.

For ii), we have

$$\begin{aligned} v_2(c_{2^a 3 2^b}) &= v_2 \Bigg[2 \binom{2r}{2a+2} - 2(1-2^{-2r}) \binom{2r}{2b+1} \Bigg] \\ &= v_2 \Bigg[2 \binom{2r}{2a+2} - 2 \binom{2r}{2b+1} + \frac{1}{2^{2r-1}} \binom{2r}{2b+1} \Bigg] \\ &= v_2 \Bigg[\frac{1}{2^{2r-1}} \binom{2r}{2b+1} \Bigg] \\ &= 1 - 2r + v_2\Bigg(\binom{2r}{2b+1} \Bigg). \end{aligned}$$

Writing

$$\binom{2r}{2b+1} = \frac{2r}{2b+1} \binom{2r-1}{2b},$$

we obtain

$$v_2(c_{2^a 3 2^b}) = 1 - 2r + v_2(2r) + v_2(\binom{2r-1}{2b}) = 2 - 2r + v_2(r) + v_2(\binom{2r-1}{2b}).$$

Recalling that $r = a+b+1$, we have $(2r-1)-2b = 2a+2$ so this valuation is equal to

$$2 - 2r + v_2(r) + v_2(\binom{2r-1}{2a+2}),$$

which is clearly minimized when $a = 0$, proving the first inequality in ii). For the second, we note that $v_2(r) \leq r-1$, so if we show that $v_2(\binom{2r-1}{2b}) \leq r-1$, we obtain the desired

$$2 - 2r + v_2(r) + v_2(\binom{2r-1}{2b}) \leq 2 - 2r + r - 1 + r - 1 = 0. \qquad (*)$$

Let us show that

$$2^r \not| \binom{2r-1}{2b}.$$

Writing

$$\binom{2r-1}{2b} = \frac{(2r-1)\cdots(2r-2b)}{2b(2b-1)\cdots 1},$$

we see that the number obtained by dropping the odd terms in the numerator and denominator and factoring out a 2 from each even term has the same 2-adic valuation:

$$\frac{2^b(r-1)\cdots(r-b)}{2^b \cdot b(b-1)\cdots 1} = \frac{(r-1)\cdots(r-b)}{b(b-1)\cdots 1}.$$

This number is equal to $\binom{r-1}{b}$, which divides $(r-1)!$. But for the p-adic valuation of $n!$ we have the formula

$$v_p(n!) = \frac{n - S_n}{p-1}$$

where $S_n = \epsilon_0 + \cdots + \epsilon_s$, $n = \epsilon_0 + \epsilon_1 p + \cdots + \epsilon_s p^s$. For $p = 2$ and $n = r-1$, this gives $v_2((r-1)!) = r-1-S_{r-1} < r-1$, since $S_{r-1} > 0$. To summarize,

$$v_2(\binom{2r-1}{2b}) = v_2(\binom{r-1}{b}) \leq v_2\big((r-1)!\big) \leq r-1,$$

proving the second inequality (*). ♢

Lemma 2. *For a prime p, suppose that A is a square $n \times n$ matrix with entries $\epsilon_{ij} \in \mathbb{Q}$ such that*

i) $v_p(\epsilon_{ij}) \geq 1$ for all $i < j$;

ii) $v_p(\epsilon_{ii}) = \min_j\big(v_p(\epsilon_{ij})\big) \leq 0$ for all i.

Then A is invertible.

Proof. *(From Brown's remark 7.2) By i), the elements of A below the diagonal are all divisible by p, but by ii), the p-adic valuation of the diagonal elements are all ≤ 0, and this valuation is less than or equal to that of any of the elements in the corresponding column. Let $\epsilon_j = v_p(\epsilon_{jj})$ for $1 \leq j \leq n$, and let A' be the matrix defined by multiplying each column of A by $p^{-\epsilon_{jj}}$. This ensures that every element of A' is a p-adic integer, those under the diagonal are all divisible by p, and those on the diagonal are all p-units. Thus the determinant of A' mod p is non-zero, so the determinant of A' is non-zero, so the determinant of A is non-zero.* ♢

We now put the reverse lexicographic order for $3 < 2$ on the set $W_{\ell,n}$ indexing the columns of $M_{N,\ell}$ with $3\ell + 2n = N$, and also on the set $\mathbf{W}$ indexing the rows. This means that we put the words in lexicographic order for $3 < 2$, and then reverse that order. For example, for $\ell = 2$, $n = 2$ and $N = 10$, we have the lexicographical ordering 3322,3232,3223,2332,2323,2233 and so the reverse lex, corresponding to the order of the columns of $M_{N,\ell}$, is 2233,2323,2332,3223,3232,3322. The set $\mathbf{W} = \coprod_{m=0}^{n} W_{\ell-1,n}$ contains the words $\{3, 32, 322, 23, 232, 223\}$ in lex order, so the rows of $M_{N,\ell}$ are indexed by these words in the reverse order 223,232,23,322,32,3. We order the rows and columns of $M^f_{N,\ell}$ in the same way.

Proposition. *Let C_{12^r} and $C_{2^a32^b}$ be indeterminates generating an $\mathbb{R}$-vector space, and let I be the subspace generated by the C_{12^r} and by the differences $C_{2^a32^b} - C_{2^b32^a}$. For w in the ordered set $W_{\ell,n}$, write V_w for the vector $(0, \ldots, 1, \ldots, 0)$ containing only zeros except for a 1 in the w-th place; similar let V_u denote the vector $(0, \ldots, 1, \ldots, 0)$ associated to u in the ordered set $\mathbf{W}$.*

(1) $M^f_{N,\ell}(V_w) \equiv \sum_{\substack{w=uv \\ deg_3 v=1}} C_v V_u \quad \text{modulo } I;$

(2) Modulo I, the matrix $M^f_{N,\ell}$ *is lower triangular, with only entries of the form* $C_{32^{r-1}}$ *along the diagonal (for* $r \geq 1$*), and every entry to the left of a* $C_{32^{r-1}}$ *on the diagonal of the form* $C_{2^a 32^b}$ *with* $a+b+1=r$*.*

Proof. *Notice that if the formal indeterminates are replaced by the rational numbers* c_w*, we have* $M_{N,\ell}(V_w) = \partial_{N,\ell}\big(\zeta^m(w)\big)$*, but written as a particular linear combination of the* $\zeta^m(u)$ *with* $u \in \mathbf{W}$*. We know how to calculate* $\partial_{N,\ell}$ *using the* D_{2r+1}*; the image* $\partial_{N,\ell}\big(\zeta^m(w)\big)$ *is obtained by computing* D_{2r+1} *on the symbol* $I^m(0;\ldots;1)$ *corresponding to* $\zeta^m(w)$*, projecting* $\zeta_{2r+1} \mapsto 1$ *and adding them together for* $r = 1,\ldots,\big[\frac{N-1}{2}\big]$*.*

Here, we have $w \in W_{\ell,n}$*. Note that if a sequence of odd length contains an even number of* 00*, it starts and ends in the same symbol. Thus, much as in §5, a sequence of length* $2r+1$ *from the corresponding symbol yields zero if the subsequence*

(i) contains an even number (or no) 00*, so starts and ends in the same symbol;*

(ii) contains > 2 00*, so leaves a quotient sequence of level* $< \ell - 1$*.*

The remaining sequences are those which contain exactly one 00*, and this can be placed as follows:*

(iii) 00101010101 *yielding* $I^m(0;01010;1) \otimes$ *quotient* $= C_{12^r}\zeta_{2r+1} \otimes$ *quotient*

(iv) 10101010100 *yielding* $I^m(1;01010;0) \otimes$ *quotient* $= -C_{12^r}\zeta_{2r+1} \otimes$ *quotient*

(v) 01010010101 *or* 10101001010 *yielding* $I^m(0;101001010;1) \otimes$ *quotient and* $-I^m(0;101001010;1) \otimes$ *quotient. If the selected sequence* S *is of the first type, then either it is at the end of the sequence, or it is necessarily followed in the full sequence by a* 0*, in which case the sequence* S' *obtained by dropping the initial* 0 *from* S *and adding the final zero is of the second type. Conversely, is* S *is of the second type, then it is necessarily preceded by a 0 and therefore one obtains a sequence* S' *of the first type by including this preliminary zero and dropping the last* 0 *of* S*.*

Therefore, if the selected sequence is of the first type but not at the end of the full sequence, it goes together in a pair which yields $C_{2^a 32^b} - C_{2^b 32^a}$ *(example: the sequence* 01010010101 *gives* 2322*, and its partner* 10100101010 $\mapsto I^m(1;010010101;0) = -I^m(0;101010010;1)$ *corresponds to* 2232*). If the selected sequence is of the second type, it also goes together* *in a pair so again yields* $C_{2^a 32^b} - C_{2^b 32^a}$.

Finally, if the selected sequence is of the first type but is at the end of the full sequence, we write $w = uv$ *where* v *corresponds to the selected sequence and* u *is thus the quotient sequence, so it yields the term* $C_v \zeta^m(u)$.

Thus in total, modulo I, *we have*

$$M^f_{N,\ell}(V_w) \equiv \sum_{r=1}^{[\frac{N-1}{2}]} C_v \zeta^m(u)$$

where the length of v *as a sequence of* 0 *and* 1 *is* $2r+1$ *and* v *contains exactly one* 3. *This is equivalent to (1) of the statement.*

To prove (2), let $\rho : \mathbf{W} \to W_{\ell,n}$ *denote the bijection given by* $\rho(u) = u32^{r-1}$. *Because this bijection is obviously order-preserving, the elements* $\big(M^f_{N,\ell}\big)_{u,\rho(u)}$ *are the diagonal elements of the matrix. Since by (1) we have*

$$M^f_{N,\ell}(V_w) = \sum_{\substack{uv=w \\ deg_3 v = 1}} C_v \zeta^m(u) \quad \text{modulo } I, \tag{$*$}$$

we see that $\big(M^f_{N,\ell}\big)_{u,\rho(u)} = C_v$ *where* v *is of the form* 32^{r-1}. *Now let* $u \in \mathbf{W}$ *and consider the row of* $M^f_{N,\ell}$ *indexed by* u*; the entries are the quantities* C_v *in (*) for each column corresponding to a* w *such that* $w = uv$, *i.e. if* $w = u2^a32^b$, *then* $\big(M^f_{N,\ell}\big)_{u,w} = C_{2^a32^b}$. *Since* $u2^a32^b \leq< u32^{r-1}$ *where* $a+b = r-1$, *all these entries lie to the left of the diagonal. This proves the result.* ♢

Example. *The matrix* $M^f_{N,\ell}$*:*

$$\left(\begin{array}{cccccc}
 & 2233 & 2323 & 2332 & 3223 & 3232 \\
223 & C_3 - C_{12} & & & C_{12} & \\
232 & & C_3 - C_{12} & & & C_{12} \\
23 & C_{23} - C_{32} - C_{122} & C_{23} & C_{32} & C_{32} - C_{23} + C_{122} & \\
322 & & & C_{23} - C_{32} & C_3 - C_{12} & \\
32 & & & & C_{23} - C_{122} & C_{32} \quad C_{32} - \\
3 & C_{223} - C_{322} & & & C_{322} & C_{232}
\end{array}\right.$$

and modulo I:

$$\begin{array}{cc} & \begin{matrix} 2233 & 2323 & 2332 & 3223 & 3232 & 3322 \end{matrix} \\ \begin{matrix} 223 \\ 232 \\ 23 \\ 322 \\ 32 \\ 3 \end{matrix} & \begin{pmatrix} C_3 & & & & & \\ & C_3 & & & & \\ & C_{23} & C_{32} & & & \\ & & & C_3 & & \\ & & & C_{23} & C_{32} & \\ & & & C_{322} & C_{232} & C_{322} \end{pmatrix} \end{array}$$

Theorem. *The matrix $M_{N,\ell}$ is invertible.*

Proof. *Let $\mu(C_w) = c_w$. We will show that the matrix $\mu\big(M^f_{N,\ell}\big)$ is invertible. By the formula $c_{12^r} = -2\sum c_{2^a 32^{r-1-a}}$ and by lemma 1 above, $\mu(I) \subset 2\mathbb{Z}$. Since $M^f_{N,\ell}$ is lower triangular mod I, its image under μ satisfies (i) of lemma 2. Property (ii) of lemma 2 is true for $M_{N,\ell}$ by (ii) of lemma 1. Thus by lemma 2, $M_{N,\ell}$ is invertible.* $\diamondsuit$