Special loci in moduli spaces of curves

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Abstract

Let S be a topological surface of genus g with n marked points, and let φ be a finite-order element of the mapping class group of S. We study the *special locus* associated to φ in the moduli space $\mathcal{M}(S)$ of Riemann surfaces of topological type (g, n); this is the set of points in $\mathcal{M}(S)$ corresponding to Riemann surfaces admitting φ as an automorphism. Another definition of the special locus is that it is the image on $\mathcal{M}(S)$ of the points in the Teichmüller space $\mathcal{T}(S)$ fixed by φ under the natural action of the mapping class group on $\mathcal{T}(S)$. We completely describe all special loci in the moduli spaces of small type $(0, 4)$, $(0, 5)$, $(1, 1)$ and $(1, 2)$, and also of the general genus zero spaces $(0, n)$, including determining their fields of moduli. Then, based on results of Harvey et. al., we show how the (normalization of the) special locus of φ in $\mathcal{M}(S)$ provides a finite covering of the moduli space of the topological quotient S/φ, and give conditions on φ for this covering to be as close as possible to an isomorphism. Finally, we translate these results in terms of the mapping class groups and show that when the conditions on φ are satisfied, we obtain a homomorphism between mapping class groups which has geometric and arithmetic significance, and that in genus zero, these two conditions are always satisfied. We end with two explicit examples of such homomorphisms, one in genus zero and one in genus one.

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Contents

1. Introduction

§1.1. Overview

Let $\mathcal{M}_{g,n}$ denote the moduli space of Riemann surfaces of genus g with n ordered marked points. By permuting the marked points on the Riemann surfaces, the permutation group S_n acts naturally on this space; the moduli space $\mathcal{M}_{g,[n]} = \mathcal{M}_{g,n}/S_n$ classifies the Riemann surfaces of genus g with n unordered marked points. It is sometimes useful to consider 'partially ordered' moduli spaces, i.e. quotients of $\mathcal{M}_{g,n}$ by subgroups of S_n.

The main goal of this article is to study *special loci* on moduli spaces. Topologically, the moduli spaces of curves are *orbifolds*; in fact, the moduli space $\mathcal{M}_{g,n}$ (resp. $\mathcal{M}_{g,[n]}$) is a quotient of a contractible space of complex dimension $3g-3+n$, the Teichmüller space $\mathcal{T}_{g,n}$, by the action of a discrete group called the *mapping class group* $\Gamma_{g,n}$ (resp. $\Gamma_{g,[n]}$). If S denotes a topological surface of genus g with n marked points, then $\Gamma_{g,n}$ (resp. $\Gamma_{g,[n]}$) is exactly the group of orientation-preserving diffeomorphisms fixing (resp. permuting) the marked points of S, up to those isotopic to the identity. The mapping class groups act properly discontinuously on the Teichmüller space, but not always freely; some points of Teichmüller space have isotropy groups of finite order inside the mapping class group, and conversely, every finite-order subgroup of the mapping class group fixes some point on Teichmüller space. The quotient of a simply connected space by a group acting in this way is called a topological *orbifold*, and the groups themselves are called *orbifold fundamental groups* (cf. [HN] for an introduction to these groups). The images in moduli space of the points with non-trivial isotropy in Teichmüller space are called *special orbifold points.* If φ is an element of finite order in the mapping class group, then we consider the set of points in Teichmüller space fixed by φ; the image of this set in the quotient moduli space is called the *special locus of* φ. This article is essentially devoted to studying these special loci, and the morphisms between moduli spaces and the corresponding homomorphisms between their fundamental groups which can be deduced from them. The main observations are as follows.

Harvey showed that in a suitable quotient of $\mathcal{M}_{g,n}$, the normalization of the special locus of a finite-order element φ in $\Gamma_{g,[n]}$ naturally gives a finite covering of the moduli space of the topological quotient of S by the action of a finite-order diffeomorphism lifting the diffeomorphism class φ (cf. §4.1). We give two conditions on φ which ensure that the special locus of φ is actually isomorphic to the moduli space of S/φ up to a trivial orbifold structure. We then translate these results in terms of fundamental groups and show that when the two corresponding conditions are satisfied, we obtain interesting *special homomorphisms* between mapping class groups. We also show that when S is of genus 0, these two conditions are always fulfilled.

§1.2. Outline of the article

The outline of the paper is as follows. In §2, we recall the basic facts about Teichmüller space and moduli spaces of curves (cf. [M] for a beautiful introduction) and mapping class groups. We recall that the mapping class group of type (g,n) has three different descriptions, namely as the group of diffeomorphisms of a topological surface S of type (g,n) up to isotopy, the orbifold fundamental group of the moduli space of type (g,n) (§2.1) and the group of special outer automorphisms of $\pi_1(S)$ (§2.2). Then we concentrate on the genus zero moduli spaces and mapping class groups, and give several propositions showing how to pass explicitly between these three descriptions (§2.3).

The following section, §3, is devoted to explicitly examining the details of the structure of the genus zero ordered and unordered moduli spaces. Working over the complex numbers (i.e. topologically rather than algebraically), we review well-known features such as their mapping class groups, their stable compactifications, their topological tangential base points, their orbifold structures, paths given by standard Dehn twists, their fundamental groups, the points of special automorphism group on the ordered moduli space, and the special loci on the unordered spaces. We first consider the one-dimensional spaces $(0,4)$ (§3.1) and $(1,1)$ (§3.2), then the two-dimensional spaces $(0,5)$ (§3.3) and $(1,2)$ (§3.4). Finally, in §3.5, we give an explicit description of the special loci in genus zero moduli spaces with any number of marked points and determine their fields of definition.

In §4, we continue to investigate special loci in moduli spaces associated to finite cyclic subgroups $\langle\varphi\rangle$ in the mapping class groups. In §4.1, we compare a special locus in the moduli space of a topological surface S of type (g,n) to the moduli space of the topological quotient S/φ, showing (based on a theorem due to Harvey et. al.) that the normalization of the first provides a finite covering of the second. We give two conditions on φ which ensure that this finite covering is actually an isomorphism, up to a trivial orbifold structure due to the automorphism associated to φ at every point of the special locus. In §4.2, we translate the finite covering into a homomorphism of the associated fundamental groups, and translate the two conditions on φ into *splitting* and *surjectivity* conditions on this homomorphism. When these two conditions are fulfilled, we show that we obtain new and interesting *special homomorphisms* between mapping class groups, which have geometric and arithmetic significance. In §4.3, we prove that the two conditions are satisfied whenever φ is a finite-order element of a genus zero mapping class group $\Gamma_{0,[n]}$. This means that *all special loci in genus zero moduli space corresponding to cyclic subgroups of the mapping class groups are themselves moduli spaces.* The last two sections are devoted to two examples of the splitting and surjectivity conditions and explicit determination of the corresponding special homomorphisms.

§1.3. Connections with Galois theory

In this section, we give a very brief sketch of the connection between the special loci in moduli space and the associated special homomorphisms between mapping class groups, and the wider world of Galois and Grothendieck-Teichmüller theory. Although we do not return to this topic within the paper, it should provide an understanding of the motivation behind the results.

The arithmetic significance of homomorphisms between mapping class groups coming from topological manipulations leading to geometric morphisms between the moduli spaces is the following. The moduli spaces themselves are defined over $\mathbb{Q}$. Let S and T be topological surfaces and let $\mathcal{M}(S)$ and $\mathcal{M}(T)$ denote their associated moduli spaces (with or without allowing permutation of the marked points). If we have a geometric morphism $f : \mathcal{M}(S) \to \mathcal{M}(T)$ which is defined over $\mathbb{Q}$, then up to inner automorphisms, the following diagram commutes for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$:

$$\begin{tikzcd} \widehat{\pi}_1(\mathcal{M}(S)) \arrow[r, "f_*"] \arrow[d, "\sigma"'] & \widehat{\pi}_1(\mathcal{M}(T)) \arrow[d, "\sigma"'] \\ \widehat{\pi}_1(\mathcal{M}(S)) \arrow[r, "f_*"] & \widehat{\pi}_1(\mathcal{M}(T)), \end{tikzcd} \tag{1.3.1}$$

where the $\widehat{\pi}_1$ are the algebraic (profinite) fundamental groups of the moduli spaces, which are equipped with a canonical outer $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action.

Let the *tower* $\mathcal{T}$ consist of the groups $\widehat{\pi}_1(\mathcal{M}(S))$ for all topological types S, equipped with homomorphisms f_* coming from geometric morphisms f between the moduli spaces which are defined over $\mathbb{Q}$. We can define the *special outer automorphism group of* $\mathcal{T}$ to be the collection of tuples $(\phi_S)_S$ where ϕ_S is a special outer automorphism of $\widehat{\pi}_1(\mathcal{M}(S))$, i.e. an outer automorphism preserving conjugacy classes of inertia generators, and such that the tuple makes all diagrams (1.3.1) commute for every f_* in the tower. It is then clear that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ injects into the automorphism group of $\mathcal{T}$, and it is an open question (due to Grothendieck) whether this automorphism group, of which various versions have been explicitly computed, according to the precise collection of homomorphisms f_* which are included in the tower, and which is generically known as the Grothendieck-Teichmüller group, may actually be equal to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The more $\mathbb{Q}$-homomorphisms are included in the tower $\mathcal{T}$, the closer the corresponding automorphism group (Grothendieck-Teichmüller group) will be to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Until now, the Teichmüller tower has been equipped with homomorphisms coming from morphisms between moduli spaces coming from erasing marked points on the topological surfaces, and including subsurfaces into surfaces by cutting along simple closed loops. As a natural sequel to the present article, the author hopes to compute the new Grothendieck-Teichmüller group associated to the tower equipped with the *special homomorphisms* as well as these.

2. Moduli spaces of curves

§2.1. Teichmüller space and the mapping class group

Let S be a topological surface of genus g equipped with n ordered marked points $x_1, \ldots, x_n$ (we say that S is of type (g, n)). A Riemann surface X of genus g with n ordered marked points $y_1, \ldots, y_n$ is said to be *marked* if we choose a diffeomorphism $\Phi : S \to X$ such that $\Phi(x_i) = y_i$ for $1 \le i \le n$. Two marked Riemann surfaces X (with marked points $y_1, \ldots, y_n$ and marking Φ) and X' (with marked points $y'_1, \ldots, y'_n$ and marking Φ') are said to be *isomorphic* if there exists an isomorphism $\alpha : X \to X'$ of Riemann surfaces with $\alpha(y_i) = y'_i$ for $1 \le i \le n$, and a diffeomorphism $h : S \to S$ with $h(x_i) = x_i$ for $1 \le i \le n$, which is isotopic to the identity, such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "h"'] & X \arrow[d, "\alpha"] \\ S \arrow[r, "\Phi'"] & X'. \end{tikzcd} \tag{2.1.1}$$

The *Teichmüller space* $\mathcal{T}_{g,n}$ is the set of isomorphism classes of marked Riemann surfaces of type (g, n). It is well-known that the Teichmüller space forms a contractible complex analytic space of dimension $3g - 3 + n$.

Let S be a topological surface of type (g, n) as above, and set

$$\Gamma_{g,[n]} = \mathrm{Diff}^+([S])/\mathrm{Diff}^0(S),$$

where $\mathrm{Diff}^+([S])$ denotes the group of orientation-preserving diffeomorphisms of S which permute the marked points, and $\mathrm{Diff}^0(S)$ is the group of those which are isotopic to the identity. The group $\Gamma_{g,[n]}$ is known as the *full mapping class group*. We also define the *pure mapping class group*, or *pure subgroup of the full mapping class group*, by setting

$$\Gamma_{g,n} = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S),$$

where $\mathrm{Diff}^+(S)$ is the subgroup of $\mathrm{Diff}^+([S])$ consisting of diffeomorphisms which fix each marked point.

The mapping class group $\Gamma_{g,[n]}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$. Let us show how. To begin with, if $\psi \in \Gamma_{g,[n]}$, let ψ' denote a lifting of ψ to a diffeomorphism of S. Then ψ' maps the marked Riemann surface (Φ, X) to $(\Phi \circ \psi', X)$. Now we show explicitly that elements of $\Gamma_{g,[n]}$ act on isomorphism classes of marked Riemann surfaces. First, we show that if (Φ, X) and (Φ', X') are isomorphic marked Riemann surfaces and ψ' is a diffeomorphism of S, then the images $(\Phi\psi', X)$ and $(\Phi'\psi', X')$ are isomorphic. To see this, let α be as in (2.1.1). Then the diagram

$$\begin{tikzcd} S \arrow[r, "\psi'"] \arrow[d, "\psi'^{-1}h\psi'"'] & S \arrow[r, "\Phi"] \arrow[d, "h"'] & X \arrow[d, "\alpha"] \\ S \arrow[r, "\psi'"] & S \arrow[r, "\Phi'"] & X' \end{tikzcd}$$

commutes, i.e.

$$\begin{tikzcd} S \arrow[r, "\Phi\psi'"] \arrow[d, "\psi'^{-1}h\psi'"'] & X \arrow[d, "\alpha"] \\ S \arrow[r, "\Phi'\psi'"] & X' \end{tikzcd}$$

commutes, and $\psi'^{-1}h\psi'$ is isotopic to the identity since the diffeomorphisms isotopic to the identity form a normal subgroup of the group of diffeomorphisms. Next, we show that two equivalent diffeomorphisms of S take a marked Riemann surface (Φ, X) to two isomorphic marked Riemann surfaces. Let h be a diffeomorphism of S which is isotopic to the identity; then $(\Phi \circ \psi', X)$ is isomorphic to $(\Phi \circ \psi' \circ h, X)$, since we have

$$\begin{tikzcd} S \arrow[r, "\Phi\circ\psi'"] \arrow[d, "h^{-1}"'] & X \arrow[d, "\mathrm{id}"] \\ S \arrow[r, "\Phi\circ\psi'\circ h"] & X. \end{tikzcd}$$

This shows that the mapping class group (equivalence classes of diffeomorphisms of S) acts on the Teichmüller space $\mathcal{T}_{g,n}$ (isomorphism classes of marked Riemann surfaces).

It is well-known that this action of $\Gamma_{g,[n]}$ on $\mathcal{T}_{g,n}$ is *properly discontinuous*. This means that for any compact subset K of $\mathcal{T}_{g,n}$, there are at most finitely many elements γ in the mapping class group such that $\gamma(K) \cap K \neq \emptyset$. Note that this fact implies that the stabilizer in the mapping class group of any point $x \in \mathcal{T}_{g,n}$ is a finite group, since if K is a small compact neighborhood of x, then $\gamma(K) \cap K \neq \emptyset$ for every γ in the stabilizer of x.

The *unordered moduli space* $\mathcal{M}_{g,[n]}$ is realized as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,[n]}$. This is tantamount to forgetting the marking, so the points of $\mathcal{M}_{g,[n]}$ correspond to isomorphism classes of Riemann surfaces of genus g with n unordered marked points. Similarly, the *ordered moduli space* $\mathcal{M}_{g,n}$ is the quotient of $\mathcal{T}_{g,n}$ by the pure subgroup $\Gamma_{g,n}$ of $\Gamma_{g,[n]}$, and its points correspond to isomorphism classes of Riemann surfaces of genus g with n ordered marked points. Because the Teichmüller space is topologically just a ball, and the moduli space is the quotient of Teichmüller space by the proper discontinuous action of a discrete group, the moduli spaces are *topological orbifolds* (if the group acted *freely*, as is actually the case for the pure genus zero groups $\Gamma_{0,n}$, or whenever n is sufficiently large with respect to g, they would be simply *ordinary topological manifolds*). When an orbifold arises in this manner, as a quotient of a simply-connected topological space by a discrete group acting properly discontinuously, it is called a *good orbifold*, and the discrete group is called the *orbifold fundamental group*. Thus we have

$$\Gamma_{g,[n]} = \pi_1^{\text{orbifold}}(\mathcal{M}_{g,[n]}) \ \text{ and } \ \Gamma_{g,n} = \pi_1^{\text{orbifold}}(\mathcal{M}_{g,n}). \tag{2.1.2}$$

The key to studying such orbifolds is the study of the (finite) isotropy subgroups of the fundamental group, i.e. the subgroups which fix points of the simply connected space.

In the case of the moduli space, these isotropy subgroups have a particular geometric meaning. Namely, the isotropy subgroup of a point of moduli space (i.e. an isomorphism class of Riemann surfaces) inside the mapping class group is exactly the automorphism group of the Riemann surface associated to the point. The main focus of this article is the set of elements φ of finite order inside the mapping class groups (particularly in genus zero), and the corresponding *special loci*, i.e. the set of points on the moduli space having an automorphism which, topologically, corresponds to φ.

It is easy to give an explicit description of the genus zero moduli spaces. Indeed, an isomorphism class of spheres with n ordered marked points is an orbit of n-tuples of points up to the action of $\mathrm{PSL}_2(\mathbb{C})$. This means that we can choose a unique representative of each class with the first three points fixed at three given values, usually taken to be 0, 1 and ∞. Thus, points of the ordered moduli space $\mathcal{M}_{0,n}$ are in bijection with n-tuples $(0, 1, \infty, x_4, \ldots, x_n)$ where the x_i are distinct from 0, 1, ∞ and each other, which gives

$$\mathcal{M}_{0,n} \simeq \left(\mathbb{P}^1 - \{0,1,\infty\}\right)^{n-3} - \Delta \tag{2.1.3}$$

where Δ denotes the multidiagonal of points with $x_i = x_j$. The unordered moduli space is the quotient of this space by the action of S_n.

§2.2. A second definition of the mapping class group

As before, let S be a topological surface of type (g, n), and consider its fundamental group given by generators and relations as

$$\pi_{g,n} = \pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid \prod_{i=1}^{g}(a_i, b_i)c_1 \cdots c_n = 1\rangle. \tag{2.2.1}$$

Here, the loops $c_1, \ldots, c_n$ correspond to loops around each of the ordered marked points of S, and they are the generators of the inertia subgroups in $\pi_1(S)$.

For an element $c \in \pi_1(S)$, let $\langle c \rangle$ denote the conjugacy class of c. Recall that outer automorphisms act on conjugacy classes. Define the group of "inertia-preserving" outer automorphisms $\mathrm{Out}^*([\pi_{g,n}])$ to be the group

$$\mathrm{Out}^*([\pi_{g,n}]) = \{\psi \in \mathrm{Out}(\pi_{g,n}) \mid \exists \sigma \in S_n \text{ such that } \psi(\langle c_i \rangle) = \langle c_{\sigma(i)} \rangle \ \text{ for } \ 1 \le i \le n\}.$$

The notation with square brackets [ ] indicates that the inertia subgroups can be permuted, mimicking the notation $\Gamma_{g,[n]}$ when the marked points can be permuted. We have a natural homomorphism

$$\begin{aligned} \mathrm{Out}^*([\pi_{g,n}]) &\to S_n \\ \psi &\mapsto \sigma, \end{aligned}$$

and we let $\mathrm{Out}^*(\pi_{g,n})$ be the kernel of this homomorphism; thus it is the group of *pure* automorphisms ψ, i.e. automorphisms such that $\psi(\langle c_i \rangle) = \langle c_i \rangle$ for $1 \le i \le n$.

This definition affords a new, and very useful, definition of the mapping class group, as attested in the following well-known theorem.

Theorem 2.2.1. *Let S be a topological surface of type (g, n). Then*

$$\mathrm{Out}^*([\pi_{g,n}]) \simeq \Gamma_{g,[n]} \quad \textit{and} \quad \mathrm{Out}^*(\pi_{g,n}) \simeq \Gamma_{g,n}.$$

This is a classical result (cf. [Mac]), so we do not give the complete proof here. Let us simply indicate the general wherefore of it, by giving an explicit description of the homomorphism

$$\Gamma_{g,[n]} \to \mathrm{Out}^*([\pi_{g,n}]).$$

It is easy to see that an element $\psi \in \Gamma_{g,n}$ can be lifted to a diffeomorphism ψ' which fixes the base point of $\pi_{g,n}$ on $S - \{x_1, \dots, x_n\}$, and which thus induces an automorphism of $\pi_{g,n}$ simply by acting on S. If ψ'' is another lifting of ψ also fixing the base point, then $\psi'' \cdot {\psi'}^{-1}$ is isotopic to the identity and fixes the base point of $\pi_{g,n}$, so it acts by an inner automorphism on $\pi_{g,n}$. Thus one obtains a well-defined map from $\Gamma_{g,n}$ to $\mathrm{Out}(\pi_{g,n})$, whose image actually lies in $\mathrm{Out}^*([\pi_{g,n}])$ since each c_i can be represented by a loop $\tilde{c}_i$ around x_i which consists of a path γ_i from the base point nearly to x_i, followed by a tiny circle around x_i and then γ_i^{-1}; a diffeomorphism necessarily maps this loop $\tilde{c}_i$ to a conjugate of a power of $\tilde{c}_j$. It is straightforward to check that we obtain a group homomorphism.

§2.3. Genus zero mapping class groups

In this section, we consider a certain generating system for the mapping class groups $\Gamma_{g,[n]}$ and $\Gamma_{g,n}$, by elements known as *Dehn twists*. Then we consider the case $g = 0$ and show how the Dehn twists allow us to make the isomorphisms

$$\Gamma_{0,[n]} \simeq \pi_1^{\rm orbifold}(\mathcal{M}_{0,[n]}) \;\text{ and }\; \Gamma_{0,n} \simeq \mathrm{Out}^*(\pi_{0,n})$$

explicit, and to show that the genus zero mapping class groups are closely related to the Artin braid groups. Recall that for $n \geq 2$, the *Artin braid group* B_n is generated by $n-1$ generators denoted $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

and

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i-j| \geq 2.$$

Set $y_1 = z_1 = 1$, and for $2 \leq i \leq n$, set $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$ and $z_i = (\sigma_1 \ldots \sigma_{i-1})^i$. It is known that the center of the group B_n is cyclic, generated by z_n. For $1 \leq i < j \leq n$, we write $x_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$; in particular $x_{i,i+1} = \sigma_i^2$ for $1 \leq i < n$. These elements generate the subgroup of *pure* braids, i.e. braids each of whose strands ends up in the same position it started from.

Definition. Let S be a topological surface of type (g,n) and let γ be a simple closed loop on S passing through either zero or two marked points. We define a certain diffeomorphism of S associated to the loop γ, in the following way. First, cut out a neighborhood of γ in S; the neighborhood of a simple closed curve has the form of a cylinder with the curve itself as a sort of "belt". Parametrize the cylinder by parameters (y,θ) with $y \in [-1,1]$ and $\theta \in [0,2\pi)$, in such a way that $y = 0$ corresponds to the simple closed loop γ, such that if the loop passes through two marked points, they lie at $(0,0)$ and $(0,\pi)$. Then, define the Dehn twist diffeomorphism on this cylinder by $(y,\theta) \mapsto \big(y, \theta + \pi(y+1)\big)$. This diffeomorphism acts like the identity on the boundaries of the cylinder, and we extend it to the whole of S by the identity.

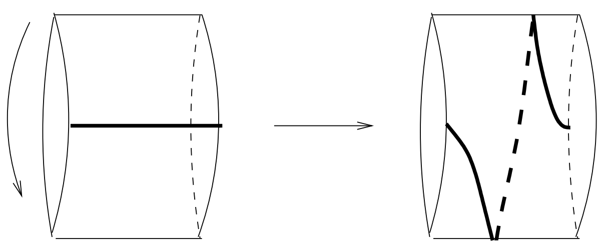


Figure 2.1. A Dehn twist

All of the following propositions are standard results (cf. [B]).

Proposition 2.3.1. (Dehn) *The pure mapping class groups $\Gamma_{g,n}$ are generated by Dehn twists along simple closed loops passing through* 0 *marked points, and the full mapping class groups are generated by Dehn twists along simple closed loops passing through* 0 *or* 2 *marked points.*

Proposition 2.3.2. *Let S be a sphere with marked points $x_1, \ldots, x_n$ with $n \geq 5$, which can be considered (topologically) as lying on a line. For $1 \leq i \leq n-1$, let $\sigma_i \in \Gamma_{0,[n]}$ denote the Dehn twist along the simple closed loop γ_i passing through the neighboring points x_i and x_{i+1}. Then $\Gamma_{0,[n]}$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$.*

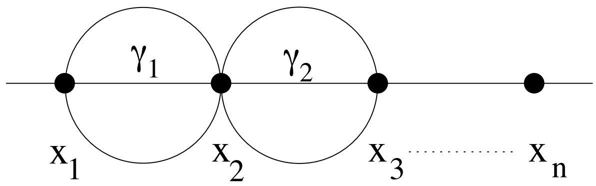


Figure 2.2. The loops γ_1 and γ_2.

Proposition 2.3.3. *Let $n \geq 5$, and recall from (2.1.3) that*

$$\mathcal{M}_{0,n} \simeq \left(\mathbb{P}^1 - \{0, 1, \infty\}\right)^{n-3} - \Delta,$$

each point of $\mathcal{M}_{0,n}$ being given by a unique representative of the form

$$\left(x_1 = 0, x_2 = 1, x_3 = \infty, x_4, \ldots, x_n\right).$$

Fix a base point $X = (0, 1, \infty, x_4, \ldots, x_n)$ such that $x_4 < x_5 < \ldots < x_n < 0$ are ordered real numbers. The Dehn twists σ_i based at X correspond to paths (not loops) on $\mathcal{M}_{0,n}$, since they permute the marked points; these paths become loops on $\mathcal{M}_{0,[n]}$. On $\mathcal{M}_{0,n}$, these paths can be explicitly parametrized by

$$\sigma_i \mapsto \Big(x_1, \ldots, x_{i-1}, f_i(t), g_{i+1}(t), x_{i+2}, \ldots, x_n\Big),$$

with $t \in [0, 1]$, where (because $x_3 = \infty$) we have

$$f_i(t) = \begin{cases} \dfrac{x_i - x_{i+1}}{2} e^{\pi i t} + \dfrac{x_i + x_{i+1}}{2} & i \neq 2, 3 \\ 1 - i\left(\dfrac{t}{1-t}\right) & i = 2 \\ x_4 - i\left(\dfrac{1-t}{t}\right) & i = 3 \end{cases}$$

and

$$g_{i+1}(t) = \begin{cases} \dfrac{x_{i+1}-x_i}{2}e^{\pi it} + \dfrac{x_i+x_{i+1}}{2} & i \neq 3,4 \\ 1+i(\dfrac{1-t}{t}) & i=3 \\ x_4+i(\dfrac{t}{1-t}) & i=4. \end{cases}$$

Proposition 2.3.4. *Let $c_1,\ldots,c_n$ with $c_1\ldots c_n = 1$ be standard generators of $\pi_{0,n}$ as in (2.2.1). The isomorphism $\Gamma_{0,[n]} \simeq \mathrm{Out}^*([\pi_{0,n}])$ of theorem 2.2.1 of $\pi_{0,n}$ associates the following automorphism of $\pi_{0,n}$ to the Dehn twist $\sigma_i \in \Gamma_{0,[n]}$ for $1 \leq i \leq n-1$:*

$$c_i \mapsto c_{i+1}, \quad c_{i+1} = c_{i+1}^{-1}c_ic_{i+1}, \quad c_j \mapsto c_j \ \text{ for } \ j \neq i, i+1.$$

Finally, the following proposition relates the genus zero mapping class group $\Gamma_{0,[n]}$ with the generators $\sigma_1,\ldots,\sigma_{n-1}$ to the Artin braid group B_n.

Proposition 2.3.5. *The genus zero mapping class group $\Gamma_{0,[n]}$ (resp. the pure subgroup $\Gamma_{0,n}$) is isomorphic to the quotient of the Artin braid group B_n (resp. the pure Artin braid group K_n) by the relations $z_n = 1$ and $y_n = 1$, where z_n and y_n are as at the beginning of this section.*

3. Geometry and special loci of small moduli spaces

In this section, we turn our attention to the explicit example of moduli spaces in dimension 1 and 2, as well as the higher-dimensional genus zero moduli spaces. For these spaces, we review the notions of mapping class groups, stable compactifications, topological tangential base points, orbifold structures, paths given by standard Dehn twists, fundamental groups, and most importantly, their points of special automorphism groups and the special loci they form.

In order to give a simple topological description of the *stable compactification* of the ordered moduli space $\mathcal{M}_{g,n}$ introduced by Deligne and Mumford, we need to introduce *pants decompositions* of a topological surface S of type (g,n). It is known that the maximal number of disjoint simple closed loops which can be placed on S is $3g-3+n$; such a collection cuts S into $2g-2+n$ disjoint *pairs of pants*, i.e. spheres with three holes or punctures. A *pants decomposition* is an equivalence class of such unions of circles modulo the action of the pure mapping class group $\Gamma_{g,n}$ of S.

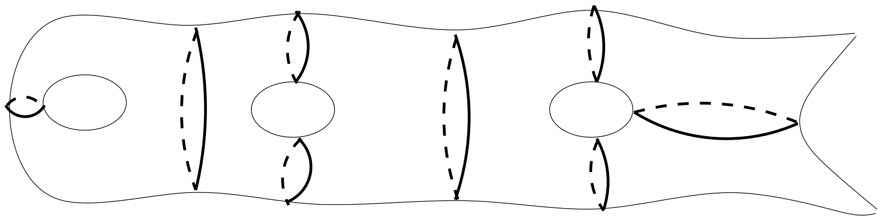


Figure 3.1. A pants decomposition of S of type $(3,2)$.

These pants decompositions define the underlying topological surfaces of *degenerate stable curves*, which are those curves of type S for which one or more of the loops of some pants decomposition are pinched to a point. Such curves can be equipped with analytic structure, but they do not belong to the moduli space $\mathcal{M}_{g,n}$. Adding all of them to $\mathcal{M}_{g,n}$ forms the *stable (or Deligne-Mumford) compactification* $\overline{\mathcal{M}}_{g,n}$. In particular, the points of *maximal degeneration* in $\overline{\mathcal{M}}_{g,n}$ are those where all loops of some pants decomposition are pinched to points; the analytic structure which can be put on such a degenerate curve is unique (as it is just a union of thrice-punctured Riemann spheres), so that there is exactly one such point in $\overline{\mathcal{M}}_{g,n}$ for each pants decomposition. The *divisor at infinity* $\mathcal{D}^{\infty}_{g,n}$ is the difference $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$.

This procedure gives a natural *stratification* of the compactification $\overline{\mathcal{M}}_{g,n}$. Each stratum is given by specifying (i) a pants decomposition and (ii) which of its loops are pinched to zero. The open stratum is $\mathcal{M}_{g,n}$, where no loops are pinched to zero. Pinching loops cuts S into a union of smaller surfaces $\bigcup S_{g',n'}$, and the corresponding stratum in $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the product of the corresponding $\mathcal{M}_{g',n'}$. If a maximal number of loops, i.e. $3g-3+n$ loops forming a pants decomposition, are pinched to zero, we obtain a point of the dimension 0 stratum of $\overline{\mathcal{M}}_{g,n}$, which is the union of the points of *maximal degeneration.*

Roughly – and purely topologically – speaking, *tangential base points* are simply connected regions of $\mathcal{M}_{g,n}$ in the neighborhood of the divisor at infinity $\mathcal{D}^{\infty}_{g,n}$. The most frequently considered tangential base points are in the neighborhood of points of maximal degeneration. One takes a neighborhood of a point x of maximal degeneration in the compactified moduli space and then considers the intersection of this neighborhood with the uncompactified moduli space $\mathcal{M}_{g,n}$. This intersection forms a topological region in $\mathcal{M}_{g,n}$ which can be cut into simply connected pieces; these are the topological tangential base points. It is extremely important to note, however, that the most interesting characteristic of these base points which are "infinitely close to infinity" is that they have a natural algebraic structure for which they can be considered as points in the modular varieties which are "defined over $\mathbb{Q}$" [N1,N2]. Thus the absolute Galois group has a canonical outer action on the algebraic fundamental groups of moduli space based at these tangential base points.

Before proceeding to the examination of the genus zero moduli spaces, we recall some important facts about orbifolds and their fundamental groups. For our purposes, it is enough to consider a complex orbifold M obtained by quotienting a simply connected space by a group G acting properly discontinuously. Then G is by definition the orbifold fundamental group of M.

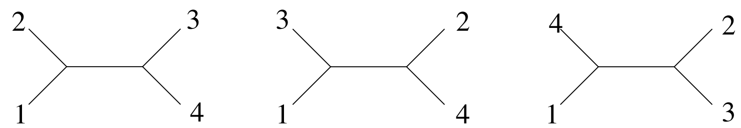
The delicate fact about an orbifold fundamental group like G is that *it can be identified with a group of loops on the orbifold only if the chosen base point is not an orbifold point, i.e. has no isotropy in the fundamental group.* If the base point is an orbifold point, then there will be non-trivial elements of G which will give trivial paths on M; these are in some sense automorphisms of the special points, i.e. as paths they remain "at the point" but they "do something to the point", so they are not trivial.

§3.1. Genus zero, four marked points

§3.1.1. The ordered moduli space $\mathcal{M}_{0,4}$

The mapping class group $\Gamma_{0,4}$. This group is free on two generators, namely the twists σ_1^2 and σ_2^2 along loops surrounding the first and second, resp. second and third marked points. It has no torsion.

The stable compactification of $\mathcal{M}_{0,4}$. Let us list the stable curves of type $(0,4)$. For this, we consider the possible pants decompositions on the topological sphere $S_{0,4}$ with four marked points. Such pants decompositions consist of a single loop (up to the action of $\Gamma_{0,4}$), so there are only three possibilities depending on whether this loop separates the first and second points from the the third and fourth, or the first and third from the second and fourth, or the first and fourth from the second and third. They are schematically represented by graphs



If S is now a Riemann sphere with marked points $(0,1,\infty,\lambda)$, let γ denote a geodesic simple closed loop on S separating the four points 0, 1, ∞ and λ into two packets of two. Then λ is paired with one of 0, 1 or ∞. Modifying the analytic structure of S by pinching the γ to a point (i.e. decreasing its length) means that λ eventually becomes identified with the point it is paired with. Thus, the three points of maximal degeneration (and the only degenerate points) correspond to the three degenerate spheres $(0,\lambda,1,\infty)$ with $\lambda \in \{0,1,\infty\}$.

The stable compactification $\overline{\mathcal{M}}_{0,4}$ consists of $\mathcal{M}_{0,4}$ together with the divisor at infinity, so here it comes down to simply adding the three points 0, 1 and ∞ to $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$, i.e. $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$.

Topological tangential base points on $\mathcal{M}_{0,4}$. The neighborhood of each of the three points of maximal degeneration in $\overline{\mathcal{M}}_{0,4}$ is homeomorphic to a disk, and its intersection with $\mathcal{M}_{0,4}$ to a pointed disk (i.e. a disk with a point removed). It is necessary to take two separate pieces of each disk in order to obtain simply connected regions which can serve as base points for fundamental groups. A useful convention for doing so is to consider only the intersection of the *real locus* of each disk with $\mathcal{M}_{0,4}$; one naturally obtains six small segments of the real line on $\mathbb{P}^1\mathbb{C} - \{0, 1, \infty\}$ neighboring 0, 1 and ∞. They are usually denoted $\overrightarrow{01}$, $\overrightarrow{10}$, $\overrightarrow{0\infty}$, $\overrightarrow{\infty 0}$, $\overrightarrow{1\infty}$, $\overrightarrow{\infty 1}$; we write $\widehat{B}_{0,4}$ for the set of the six.

Orbifold structure. By (2.1.3), the moduli space $\mathcal{M}_{0,4}$ of Riemann spheres with four marked points is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$. The pure mapping class group $\Gamma_{0,4}$ acts not only properly and discontinuously but also freely on the Teichmüller space $\mathcal{T}_{0,4}$, so $\mathcal{M}_{0,4}$ is just an ordinary topological manifold. This holds for all the genus zero ordered moduli spaces.

Dehn twists on $S_{0,4}$. Consider a topological sphere S with four ordered marked points x_1, x_2, x_3, x_4, which we can consider as lying in a row. Recall that since the Dehn twists σ_1, σ_2 and σ_3 permute these marked points, they correspond to paths on the ordered moduli space $\mathcal{M}_{0,4}$ (starting at some chosen base point), which descend to loops on the unordered space $\mathcal{M}_{0,[4]}$. In §3.1.2, we will discuss how to represent these paths on $\mathcal{M}_{0,4}$. In the present section, we restrict ourselves to the loops on $\mathcal{M}_{0,4}$ corresponding to the two generators σ_1^2 and σ_2^2 of the pure mapping class group $\Gamma_{0,4}$, which is free, and as we saw above, acts freely on the upper half-plane.

These elements are Dehn twists along the loops α_1 and α_2 shown in the following figure.

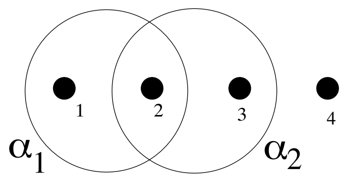


Figure 3.2. The Dehn twists σ_1^2 and σ_2^2, generators of $\Gamma_{0,4}$.

Let us show how to parametrize loops on the moduli space

$$\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\},$$

much as in proposition 2.3.3, except that we work with the σ_i^2 instead of the σ_i (working with the σ_i themselves when $n = 4$ is more delicate, which is why it is not included in proposition 2.3.3; see §3.1.2 for details.)

Let us fix $\overrightarrow{\infty 0}$ for our choice of base point; it is represented on $\mathcal{M}_{0,4}$ by a simply-connected region of points of the form $(0, 1, \infty, x)$ with $x \in (-\infty, -A)$ for some very large positive number A. The Dehn twist σ_i^2 can be parametrized as a movement of marked points by

$$(-\frac{1}{2}e^{2\pi it} + \frac{1}{2}, \frac{1}{2}e^{2\pi it} + \frac{1}{2}, \infty, x)$$

for $t \in [0, 1]$, which by the transformation $e^{-2\pi it}(z + \frac{1}{2}e^{2\pi it} - \frac{1}{2})$ we bring to the standard form

$$\Big(0, 1, \infty, e^{-2\pi it}(x + \frac{1}{2}e^{2\pi it} - \frac{1}{2})\Big).$$

Thus, on the moduli space $\mathcal{M}_{0,4}$ parametrized by the fourth component, we have the loop σ_1^2 in Figure 3.3 below, starting at the base point $\overrightarrow{\infty 0}$. Similarly, σ_2^2 can be parametrized by

$$\Big(0, 1 - i(\frac{(2t-1)^2 - 1}{2t - 1}), 1 + i(\frac{2t - 1}{(2t-1)^2 - 1}), x\Big),$$

which can be transformed to

$$\Big(0, 1, \infty, \frac{x}{x - 1 - i\big(\frac{2t-1}{(2t-1)^2-1}\big)} \cdot \frac{-i\big(\frac{(2t-1)^2-1}{2t-1}\big) - i\big(\frac{2t-1}{(2t-1)^2-1}\big)}{1 - i\big(\frac{(2t-1)^2-1}{2t-1}\big)}\Big),$$

which describes the loop σ_2^2 of Figure 3.3.

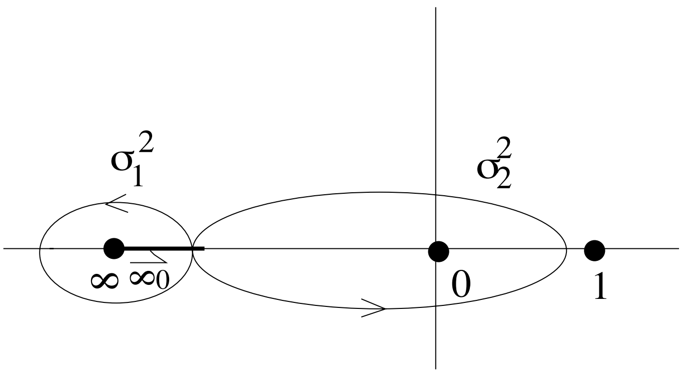


Figure 3.3. The Dehn twists σ_1^2 and σ_2^2 as paths on $\mathcal{M}_{0,4}$.

Fundamental group. The fundamental group of $\mathcal{M}_{0,4}$ is isomorphic to $\Gamma_{0,4}$, which is a free group on two generators σ_1^2 and σ_2^2, which acts freely on the Teichmüller space $\mathcal{T}_{0,4}$. The fundamental group $\pi_1(\mathcal{M}_{0,4}; \overrightarrow{\infty 0})$ is generated by the two loops shown in Figure 3.3. If we do a similar computation, using the base point $\overrightarrow{01}$ and a standard form $(0, x, 1, \infty)$

for representatives of the points of $\mathcal{M}_{0,4}$, we find the more standard identification shown in the following figure.

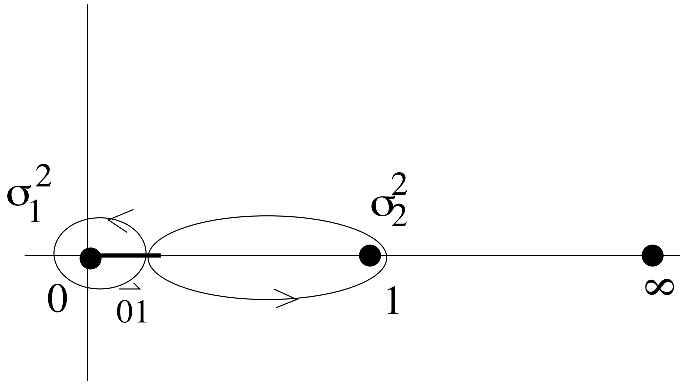


Figure 3.4. The generators of $\pi_1(\mathcal{M}_{0,4}, \overrightarrow{01})$.

Points with special automorphism group. A special feature of Riemann surfaces with ordered marked points is that it can happen that a permutation of the points can be realized as an automorphism of the surface, for instance the rotation of a sphere having n marked points on its equator through an angle of $2\pi/n$. Such points are not orbifold points on the ordered moduli space, but they are preimages of orbifold points on the unordered moduli space, since they have less than $n!$ preimages under the action of S_n. Let us determine all such points in $\mathcal{M}_{0,4}$. To begin with, we see that the Klein 4-subgroup of S_4 fixes each and every point of $\mathcal{M}_{0,4}$. Indeed, if $(0,1,\infty,x)$ is a point, then the action of say $(12)(34)$ on it takes it to $(1,0,x,\infty)$, and then the transformation by the isomorphism $z \mapsto (xz-x)/(z-x)$ brings it back to $(0,1,\infty,x)$, so it is the same point on moduli space, and this also holds for $(13)(24)$ and $(14)(23)$. Thus, every point of $\mathcal{M}_{0,4}$ has automorphism group at least isomorphic to the Klein 4-group; this is the generic automorphism group. Five points of $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0,1,\infty\}$ have special automorphism groups. The first three are those given by representatives $(\tau,0,1,\infty)$ with $\tau \in \{1/2, 2, -1\}$, which apart from being fixed by the Klein 4-group are also fixed by the permutations (23), (34) and (24) respectively, forming three different dihedral groups of order 8. These automorphism groups can be identified with the automorphism group of the octahedron, by identifying the octahedron with the sphere with four marked points around the equator and a north and a south pole; the automorphism group is generated by a rotation of order 4 around the north-south axis and a north-south flip of order 2. The two remaining points with special automorphism group are the points $(\tau,0,1,\infty)$ with $\tau = exp(\pm 2\pi i/6)$, which are each fixed by the permutation group $\langle (123) \rangle$ as well as the Klein 4-group, forming a group of order 12 isomorphic to the alternating group A_4, realized as the automorphism group of the tetrahedron, by identifying the tetrahedron with the sphere with three marked points around the equator and one at the north pole.

Special loci. Special loci are orbifold points, and there are none on $\mathcal{M}_{0,4}$, which has no orbifold structure. As remarked above, the points of special automorphism group given above determine where the special loci will lie on the unordered moduli space $\mathcal{M}_{0,[4]}$.

§3.1.2. The unordered moduli space $\mathcal{M}_{0,[4]}$

The mapping class group $\Gamma_{0,[4]}$. This group is generated by Dehn twists σ_i for $i = 1, 2, 3$, subject to the relations $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$, $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3 = 1$ and $(\sigma_1\sigma_2\sigma_3)^4 = 1$. These relations imply the following:

$$\begin{cases} \sigma_1^2 = \sigma_3^2 \\ (\sigma_1\sigma_2)^3 = 1 \\ (\sigma_1\sigma_2\sigma_1)^2 = 1. \end{cases}$$

To prove the second relation, we just use the braid relations to show that

$$(\sigma_1\sigma_2)^3(\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3) = (\sigma_1\sigma_2\sigma_3)^4 = 1,$$

so $(\sigma_1\sigma_2)^3 = 1$. Then, by the braid relations, we have $(\sigma_1\sigma_2)^3 = \sigma_1^2\sigma_2\sigma_1^2\sigma_2$, so $\sigma_1^2 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}$. But the relation $\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3 = 1$ shows that $\sigma_3^2 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}$, so $\sigma_1^2 = \sigma_3^2$, which gives the first relation. For the third, using only $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, we see that

$$(\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^3 = 1.$$

The torsion elements in $\Gamma_{0,[4]}$ are as follows. There is only one conjugacy class of elements of order 4, namely the class of $\sigma_1\sigma_2\sigma_3$. There is only one conjugacy class of elements of order 3, namely that of $\sigma_1\sigma_2$. There are two conjugacy classes of order 2, namely that of $(\sigma_1\sigma_2\sigma_3)^2$ and that of $\sigma_1\sigma_2\sigma_1$. (Note that this contradicts the statement of the Corollary on p. 508 of [HM]; their corollary is however valid for $n \geq 5$.)

The stable compactification of $\mathcal{M}_{0,[4]}$. Degenerate stable curves with four unordered marked points must correspond to the unique trivalent tree with four unnumbered tails, so there is only one such point. Indeed, this corresponds to the fact that under the morphism $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\} \to \mathcal{M}_{0,[4]}$ by quotienting by the action of S_4, the three points $0, 1, \infty$ all pass to a single point on $\mathcal{M}_{0,[4]}$.

Topological tangential base points on $\mathcal{M}_{0,[4]}$. The neighborhood of the missing point is a disk, so dividing it into two simply connected regions, there are two tangential base points at the maximally degenerate point. Note that only one of these is the image of all six tangential base points on $\mathcal{M}_{0,4}$.

Orbifold structure. Let S be a sphere with four marked points x_1, x_2, x_3 and x_4. By definition, the unordered moduli space $\mathcal{M}_{0,[4]}$ is the quotient of the Teichmüller space $\mathcal{T}_{0,4}$ by the action of the full mapping class group $\Gamma_{0,[4]}$, which we recall (Figure 2.1) is generated by Dehn twists σ_i for $i = 1, 2, 3$ along loops passing through the marked points x_i and x_{i+1} respectively. As in the previous section, we have an isomorphism

$$\Gamma_{0,[4]} \xrightarrow{\sim} \pi_1(\mathcal{M}_{0,[4]}, \overrightarrow{\infty 0}).$$

However, there is a fundamental difference between the ordered moduli space $\mathcal{M}_{0,4}$ and the unordered space $\mathcal{M}_{0,[4]} = \mathcal{M}_{0,4}/S_4$, due to the orbifold structure of $\mathcal{M}_{0,[4]}$ which arises from the fact that $\Gamma_{0,[4]}$ does not act freely on $\mathcal{T}_{0,4}$. Equivalently, since $\Gamma_{0,4}$ acts freely on $\mathcal{T}_{0,4}$ and we have an exact sequence

$$1 \to \Gamma_{0,4} \hookrightarrow \Gamma_{0,[4]} \to S_4 \to 1,$$

the orbifold structure arises because S_4 does not act freely on $\mathcal{M}_{0,4}$.

We saw in the discussion of orbifold fundamental groups given just before §3.1.1 that such a group can be identified with a group of loops if the base point is not an orbifold point, i.e. has trivial isotropy. However, in the case of $\mathcal{M}_{0,[4]}$, every point has a non-trivial isotropy subgroup in $\Gamma_{0,[4]}$. Indeed, letting $\sigma_{41} = \sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}$, $\sigma'_{24} = \sigma_3^{-1}\sigma_2\sigma_3$ and $\sigma_{13} = \sigma_2\sigma_1\sigma_2^{-1}$, the three elements

$$a = \sigma_1\sigma_3^{-1}, \quad b = \sigma_{13}{\sigma'}_{24}^{-1} \quad \text{and} \quad c = \sigma_{41}\sigma_2^{-1}$$

all fix every point of $\mathcal{T}_{0,4}$. Let us show that these elements are all of order 2, and commute in $\Gamma_{0,[4]}$. For the first one, we use the fact that $\sigma_1^2 = \sigma_3^2$ in $\Gamma_{0,[4]}$, which we saw in the paragraph on the mapping class group above. Then, it follows directly from writing their expressions that the other two elements are of order two, since in fact $b = \sigma_2 a\sigma_2$ and $c = \sigma_3\sigma_2 a\sigma_2^{-1}\sigma_3^{-1}$. Now we show that $\langle a, b, c\rangle$ is a Klein 4-subgroup of $\Gamma_{0,[4]}$. We first show that $ab = c$. In fact, using only the braid relations, we check that

$$ab = \sigma_1\sigma_3^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} = \sigma_1^2\sigma_3^{-2}\sigma_3\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} = \sigma_1^2\sigma_3^{-2}c,$$

and this is equal to c since $\sigma_1^2 = \sigma_3^2$. Thus, we have $ab = c$, so $abab = 1$ since c is of order 2, so $ab = b^{-1}a^{-1} = ba$ since a and b are of order 2, so a and b commute. This shows that $\langle a, b, c\rangle$ is a Klein 4-subgroup.

Now we can give a complete description of the orbifold structure of $\mathcal{M}_{0,[4]}$. Every point of $\mathcal{M}_{0,[4]}$ is an "orbifold point" in the sense that it has non-trivial isotropy in $\Gamma_{0,[4]}$. The special (non-generic) orbifold points are the images of the points with special automorphism group given in $\mathcal{M}_{0,4}$. The three points $x = 1/2, 2, -1$ on $\mathcal{M}_{0,4}$ with dihedral automorphism group of order 8 pass to a single point on $\mathcal{M}_{0,[4]}$, and the two points $j, \overline{j}$ on $\mathcal{M}_{0,4}$ with special automorphism group of order 12 (isomorphic to A_4) pass to a single point on $\mathcal{M}_{0,[4]}$. Topologically, $\mathcal{M}_{0,[4]}$ is a sphere with one missing point at infinity; its orbifold structure is given by these two special points and the generic isotropy group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ at each point.

This space is not dissimilar to the moduli space of elliptic curves, given as the quotient of the Teichmüller space $\mathcal{T}_{1,1}$, which is again the Poincaré upper half-plane, by the action of $\mathrm{SL}_2(\mathbb{Z})$; it looks like a sphere with one missing point and two "special" points, and there is a non-trivial isotropy group at every point, only it is just $\mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Indeed, if $\mathcal{M}'_{1,1}$ denotes the reduced orbifold of elliptic curves, we have an orbifold isomorphism $\mathcal{M}'_{1,1} \simeq \mathcal{M}_{0,4}/S_3$, both of these orbifolds having π_1 isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$.

Dehn twists on $S_{0,[4]}$. By the discussion of the orbifold fundamental group given just before §3.1.1, we see that since there is a non-trivial isotropy group at each point of $\mathcal{M}_{0,[4]}$, if we attempt to identify the generating elements $\sigma_1, \sigma_2, \sigma_3$ of $\Gamma_{0,[4]}$ with loops on the moduli space $\mathcal{M}_{0,[4]}$ (or paths on $\mathcal{M}_{0,4}$) based at an arbitrary non-special point, we will not be able to distinguish σ_1 from σ_3. Let us show this explicitly, working on $\mathcal{M}_{0,4}$ (parametrized by the single variable x), with the base point $\overrightarrow{\infty 0}$.

We first parametrize the twists σ_1, σ_2 and σ_3, shown in the following figure, directly on the sphere, as in proposition 2.3.3.

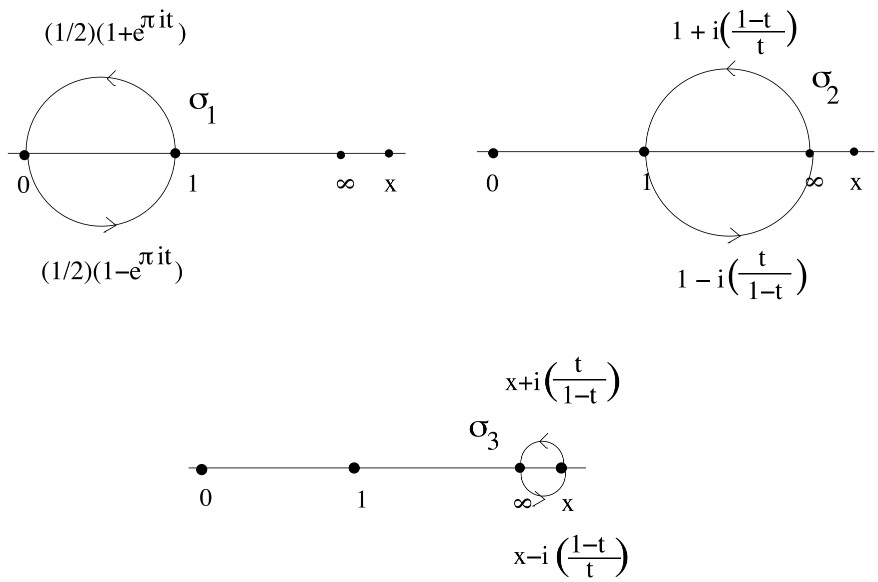


Figure 3.5. The Dehn twists σ_1, σ_2, σ_3

We obtain

$$\begin{cases} \sigma_1 = \left(-\frac{1}{2}e^{\pi it}+\frac{1}{2}, \frac{1}{2}e^{\pi it}+\frac{1}{2}, \infty, x\right) \\ \sigma_2 = \Big(0, 1-i\big(\frac{t}{1-t}\big), 1+i\big(\frac{1-t}{t}\big), x\Big) \\ \sigma_3 = \Big(0, 1, x-i\big(\frac{1-t}{t}\big), x+i\big(\frac{t}{1-t}\big)\Big). \end{cases}$$

Bringing these parametrizations back to the standard form $\big(0,1,\infty,f(t)\big)$, we find

$$\begin{cases} \sigma_1 = \Big(0,1,\infty, e^{-\pi it}\big(x+\frac{1}{2}e^{\pi it}-\frac{1}{2}\big)\Big) \\ \sigma_2 = \Big(0,1,\infty, \frac{x}{tx-t-(1-t)i}\cdot\frac{-i(2t^2-2t+1)}{1-t-it}\Big) \\ \sigma_3 = \Big(0,1,\infty, \frac{\big((1-t)x+it\big)\big(t-tx+(1-t)i\big)}{it^2+i(1-t)^2}\Big). \end{cases}$$

These three paths are shown in Figure 3.6 below, where it can be seen that σ_1 and σ_3 give rise to homotopic paths.

Fundamental group. In Figure 3.6, we show the three Dehn twists σ_1, σ_2 and σ_3 as paths on $\mathcal{M}_{0,4}$, starting from the tangential base point $\overrightarrow{\infty 0}$. They are shown in bold lines; σ_1 is the tiny half-circle from the base point $\overrightarrow{\infty 0}$ to the base point $\overrightarrow{\infty 1}$ in the northern hemisphere, σ_3 is indistinguishable from it (as we saw, $\sigma_1\sigma_3^{-1}$ is "invisible" as a path), and σ_2 is the large path from $\overrightarrow{\infty 0}$ to $\overrightarrow{10}$.

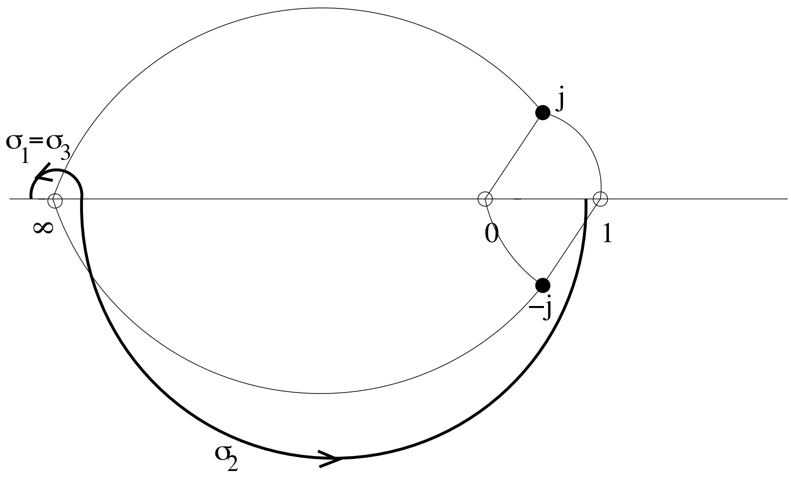


Figure 3.6. The paths σ_1, σ_2 and σ_3 on $\mathcal{M}_{0,4}$.

The thin lines show the real axis, the line segment from 0 to j, and the curves which are the six images of this line segment under the group of automorphisms of $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$, generated by $z \mapsto 1-z$ and $z \mapsto 1/(1-z)$. These curves divide $\mathcal{M}_{0,4}$ into six regions, each of which is a fundamental domain for the action of S_4 on $\mathcal{M}_{0,4}$.

In order to represent σ_1 and σ_2 as loops on $\mathcal{M}_{0,[4]}$, we first show, in Figures 3.7 and 3.8, the six paths σ_1 and σ_2 starting from the six tangential base points, obtained as the images of the σ_1 and σ_2 in Figure 3.6 under the automorphism group of $\mathcal{M}_{0,4}$.

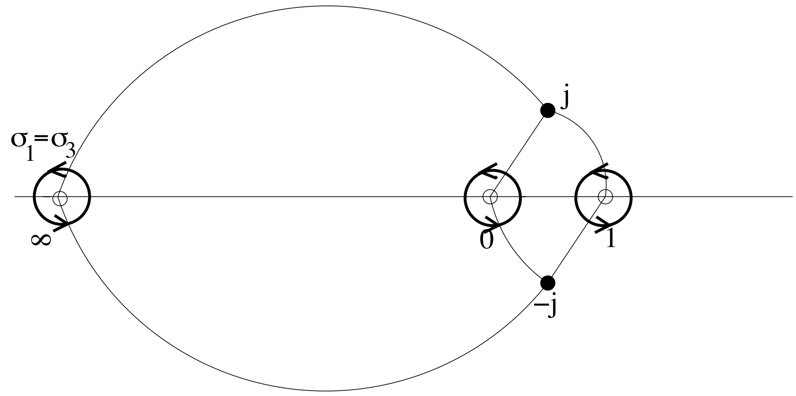


Figure 3.7. The six paths σ_1 on $\mathcal{M}_{0,4}$.

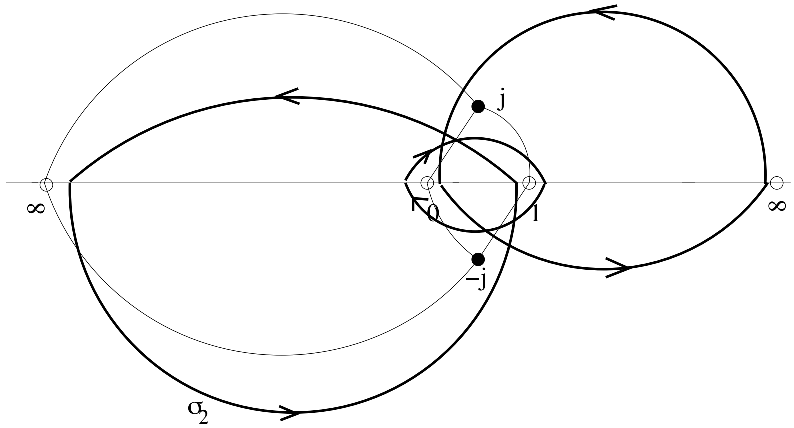


Figure 3.8. The six paths σ_2 on $\mathcal{M}_{0,4}$.

Now, to see the loops on $\mathcal{M}_{0,[4]}$, we select one fundamental domain, and use only the trace of the paths σ_1 and σ_2 lying inside it. We find

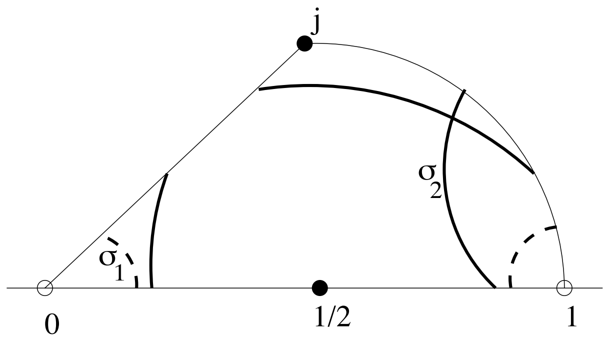


Figure 3.9. The paths σ_1 and σ_2 in a fundamental domain.

where σ_1 is represented with dashed curves. Now, we make $\mathcal{M}_{0,[4]}$ by identifying the two edges of this domain coming out of the point j, and folding the real segment $(0,1)$ in half, pinching at $1/2$ and gluing it together. This reveals the orbifold structure of $\mathcal{M}_{0,[4]}$ as a sphere with one hold and two "pinched" orbifold points coming from j and $1/2$, and the paths σ_1 and σ_2 pass to the loops shown in the following figure.

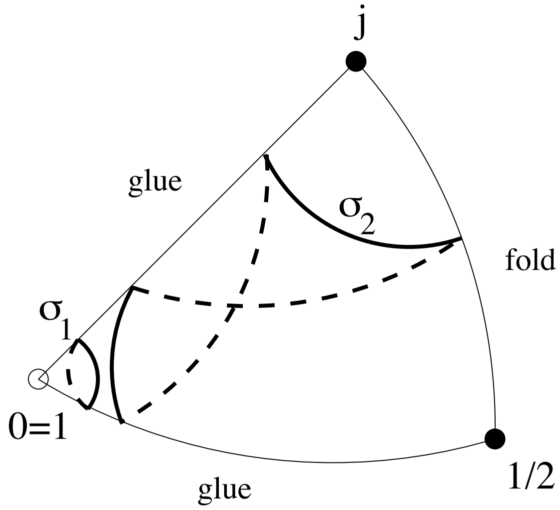


Figure 3.10. The loops σ_1 and σ_2 on $\mathcal{M}_{0,[4]}$.

Finally, we note how much more natural it is to consider the finite order loops $\sigma_1\sigma_2$ and $\sigma_1\sigma_2\sigma_1$ as generators of the fundamental group of $\mathcal{M}_{0,[4]}$; indeed, these loops, shown in the following figure, are just the finite-order loops surrounding the two special orbifold points.

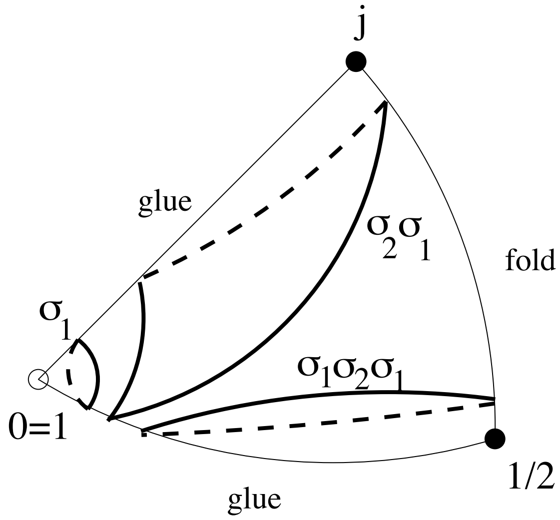


Figure 3.11. The loops σ_1, $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1$ on $\mathcal{M}_{0,[4]}$.

Special loci. The whole of $\mathcal{M}_{0,[4]}$ is a special locus in the sense that every point is an orbifold point of group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, this group being identified with the Klein 4-subgroup of S_4. Apart from these generic automorphism groups, as we saw, the points on $\mathcal{M}_{0,[4]}$ having special automorphism group are the two points which are the images of the points $\{1/2, 2, -1\}$ and $\{j, \bar{j}\}$ in $\mathcal{M}_{0,4}$, and these two points have associated isotropy group D_4 and A_4 respectively.

§**3.2. Genus one, one marked point**

As a topological space, the moduli space $\mathcal{M}_{1,1}$ of one-pointed tori (i.e. elliptic curves) is homeomorphic to $\mathbb{C}$. This is because it is realizable as the quotient of the Poincaré upper half-plane $\mathcal{H}$ by the action of $\mathrm{SL}_2(\mathbb{Z})$ given by $z \mapsto (az+b)/(cz+d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. A fundamental domain for this action has the familiar shape shown in figure 3.12 (with $j = \exp(2\pi i/6)$). As usual, the domain in this figure is rolled up with the two vertical edges identified and the arc of the boundary circle from j^2 to i identified with the arc from i to j. This makes the fundamental domain appear rather like a sock, with a toe at j and a pointed heel at i.

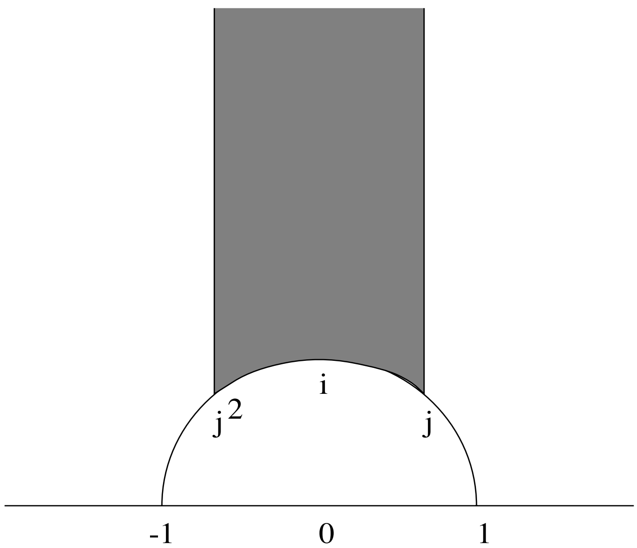


Figure 3.12. The moduli space $\mathcal{M}_{1,1}$.

Mapping class group. The mapping class group $\Gamma_{1,1}$ is isomorphic to the quotient $\mathrm{Diff}(S_{1,1})/\mathrm{Diff}^o(S_{1,1})$, which in turn is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$. It is generated by the two Dehn twists a and b along the loops α and β in the figure below, which can be identified with the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, satisfying the relations $aba = bab$ and $(ab)^3 = 1$. Taking aba and ba as generators gives the familiar free generation of $\mathrm{PSL}_2(\mathbb{Z})$ by an element of order 2 and one of order 3.

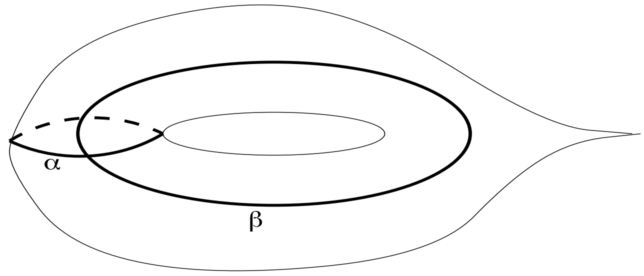


Figure 3.13. Generators of the mapping class group $\Gamma_{1,1}$.

However, there is an interesting subtlety involving the identification of $\mathcal{M}_{1,1}$ with the quotient of the Teichmüller space $\mathcal{T}_{1,1}$ by the mapping class group. Namely, the Teichmüller space $\mathcal{T}_{1,1}$ is the Poincaré upper half-plane $\mathcal{H}$, and we saw above that $\mathcal{M}_{1,1}$ is the quotient of $\mathcal{T}_{1,1}$ by $\mathrm{SL}_2(\mathbb{Z})$. Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathcal{T}_{1,1}$, the quotients $\mathcal{T}_{1,1}/\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{T}_{1,1}/\mathrm{PSL}_2(\mathbb{Z})$ are identical as topological spaces, but they are not identical as orbifolds. The fundamental group of $\mathcal{M}_{1,1}$ is actually isomorphic to $\mathrm{SL}_2(\mathbb{Z})$, and the homomorphism

$$\pi_1(\mathcal{M}_{g,n}) \to \Gamma_{g,n}$$

which is usually an isomorphism has a kernel in this exceptional case.

The stable compactification of $\overline{\mathcal{M}}_{1,1}$. It is obtained by adding the missing point to $\mathcal{M}_{1,1}$, so we have $\overline{\mathcal{M}}_{1,1} \simeq \mathbb{P}^1\mathbb{C}$ as topological spaces.

Topological tangential base points on $\mathcal{M}_{1,1}$. The intersection of the neighborhood of the maximally degenerate point ∞ on $\overline{\mathcal{M}}_{1,1}$ with the space $\mathcal{M}_{1,1}$ is a pointed disk, so as before there must be two tangential base points. We take the two half-segments $i\lambda$ and $1/2 + i\lambda$ where $\lambda \in (\Lambda, \infty)$ for some $\Lambda >> 0$, and we denote them by $\overrightarrow{\infty i}$ and $\overrightarrow{\infty j}$ respectively in analogy with the notation $\overrightarrow{01}$ (i.e. small segments from ∞ towards i and from ∞ towards i).

Orbifold structure. The space $\mathcal{M}_{1,1}$ is the orbifold $\mathcal{T}_{1,1}/\mathrm{SL}_2(\mathbb{Z})$. Under the action of $\mathrm{SL}_2(\mathbb{Z})$, every point of $\mathcal{T}_{1,1}$ is fixed by one element of $\mathrm{SL}_2(\mathbb{Z})$, namely the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and the two points i and j are fixed by subgroups isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ respectively. So the orbifold $\mathcal{M}_{1,1}$ considered in this way has a non-trivial automorphism of order 2 at every point and two additional special points. The automorphism group of each point is equal to the automorphism group of the isomorphism class of elliptic curves associated to that point.

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ fixes each point of $\mathcal{H}$, there is a natural surjection

$$\mathcal{M}_{1,1} \simeq \mathcal{T}_{1,1}/\mathrm{SL}_2(\mathbb{Z}) \to \mathcal{T}_{1,1}/\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathcal{M}_{0,[4]} \tag{3.2.1}$$

which consists in forgetting the automorphism of order 2 at each point.

Dehn twists on $S_{1,1}$. Let us consider the tangential base point $\overrightarrow{i\infty}$ on $\mathcal{M}_{1,1}$; it is the preimage under (3.2.1) of the tangential base point on $\mathcal{M}_{0,[4]}$ which is itself the image of the six standard tangential base points on $\mathbb{P}^1 - \{0,1,\infty\} = \mathcal{M}_{0,4}$. We will determine the paths on $\mathcal{M}_{1,1}$, based at $\overrightarrow{i\infty}$, corresponding to the Dehn twists a and b along the loops α and β of figure 3.13, and also the paths corresponding to aba and ba. To do this, since (3.2.1) is a one-to-one map, it suffices to study the paths corresponding to σ_1, σ_2, $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1$ on $\mathcal{M}_{0,[4]}$. The first two of these are shown in figure 3.10 and the second two in figure 3.11.

The fundamental group of $\mathcal{M}_{1,1}$. Let us now consider $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ as generators of $\mathrm{SL}_2(\mathbb{Z})$ rather than of $\mathrm{PSL}_2(\mathbb{Z})$. The paths on $\mathcal{M}_{1,1}$ corresponding to the generators $aba = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $ba = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ of the fundamental group of $\mathcal{M}_{1,1}$ look the same as figure 3.11 since the only difference is the invisible orbifold structure at every point:

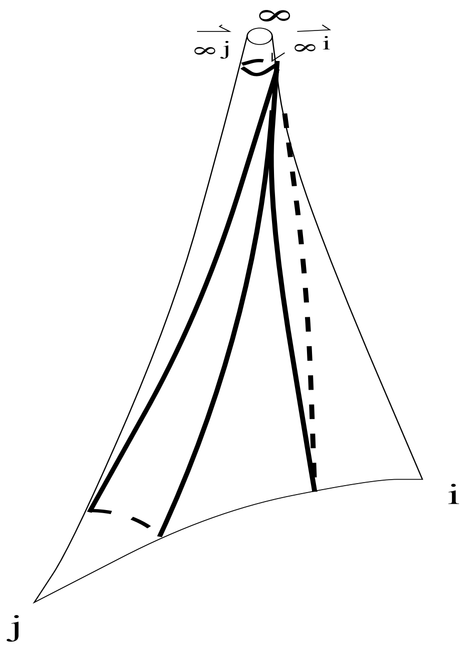


Figure 3.14. The fundamental group $\pi_1^{\mathrm{orbifold}}(\mathcal{M}_{1,1}; \overrightarrow{i\infty})$.

Note that running along the loop around j and then the loop around i gives a loop homotopic to the small ring around ∞ at the top, also that the loop around j is of order 6 and the loop around i is of order 4.

Special loci. The whole of $\mathcal{M}_{1,1}$ is the special orbifold locus of $\{\pm 1\} \in \mathrm{SL}_2(\mathbb{Z})$. On top of this, there are two isolated points which are the special loci of the finite-order elements aba and ba respectively.

§3.3. Genus zero, five marked points

§3.3.1. The ordered moduli space $\mathcal{M}_{0,5}$

The mapping class group $\Gamma_{0,5}$. As we saw in proposition 2.3.5, the full mapping class group $\Gamma_{0,[5]}$ is the quotient of the Artin braid group B_5 on 5 strands, generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, by the two relations $(\sigma_4\sigma_3\sigma_2\sigma_1)^5 = 1$ and $\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4 = 1$. There are two nice presentations for its pure subgroup $\Gamma_{0,5}$, one as a semi-direct product of a free group of rank 3 and a free group of rank two, and another which does not clearly reveal the semi-direct product structure, but is more symmetric in terms of the 5 strands. The semi-direct product presentation is given by $F_3 \rtimes F_2$ where the normal F_3 factor is generated by $x_1 = x_{12}$, $x_2 = x_{23}$, $x_3 = x_{24}$, $x_4 = x_{13}$ and $x_5 = x_{34}$, where the x_{ij} are the generators defined earlier. Then $\Gamma_{0,5} = \langle x_4, x_5\rangle \rtimes \langle x_1, x_2, x_3\rangle \simeq F_3 \rtimes F_2$, with the following relations:

$$\begin{cases} x_{13}^{-1}x_{12}x_{13} = x_{23}x_{12}x_{23}^{-1} \\ x_{13}^{-1}x_{23}x_{13} = x_{23}x_{12}x_{23}x_{12}^{-1}x_{23}^{-1} \\ x_{13}^{-1}x_{24}x_{13} = x_{23}x_{12}x_{23}^{-1}x_{12}^{-1}x_{24}x_{12}x_{23}x_{12}^{-1}x_{23}^{-1} \\ x_{34}^{-1}x_{12}x_{34} = x_{12} \\ x_{34}^{-1}x_{23}x_{34} = x_{23}x_{24}x_{23}x_{24}^{-1}x_{23}^{-1} \\ x_{34}^{-1}x_{24}x_{34} = x_{23}x_{24}x_{23}^{-1}. \end{cases}$$

For two elements a and b of a group, let $(a, b) = aba^{-1}b^{-1}$ denote their commutator. Rewriting the above presentation in the five generators x_{12}, x_{23}, x_{34}, x_{45} and x_{51} and using the relations $x_{45} = x_{12}x_{13}x_{23}$ and $x_{51} = x_{23}x_{24}x_{34}$ gives:

$$\begin{cases} (x_{12}, x_{34}) = 1 \\ (x_{23}, x_{45}) = 1 \\ (x_{34}, x_{51}) = 1 \\ (x_{45}, x_{12}) = 1 \\ (x_{51}, x_{23}) = 1 \\ x_{51}x_{23}^{-1}x_{12}x_{34}^{-1}x_{23}x_{45}^{-1}x_{34}x_{51}^{-1}x_{45}x_{12}^{-1} = 1. \end{cases}$$

Like all the pure genus zero mapping class groups, $\Gamma_{0,5}$ is torsion free.

The stable compactification of $\mathcal{M}_{0,5}$. Let S denote a sphere with 5 marked points x_1, x_2, x_3, x_4, x_5. Combinatorially, the divisor at infinity of $\mathcal{M}_{0,5}$ is obtained by adding to $\mathcal{M}_{0,5}$ all the points obtained by pinching one simple closed loop on S, given up to the action of $\Gamma_{0,5}$, to a point (this gives the strata at infinity of codimension 1) and the points obtained by simultaneously pinching two disjoint simple closed loops (given up to $\Gamma_{0,5}$), i.e. a pants decomposition, to points.

Every simple closed loop on S divides the set of marked points into two points and three points (because a loop surrounding 0 or 1 point is homotopic to a point). Thus, giving a simple closed loop on S up to the action of $\Gamma_{0,5}$ is equivalent to giving the two points

x_i and x_j it separates from the others. This means that there are $\binom{5}{2} = 10$ such classes of loops on S. Each loop corresponds to a stratum at infinity of complex codimension 1, so complex dimension 1, of $\mathcal{M}_{0,5}$. Pinching a loop surrounding points x_i and x_j to a point is like making x_i and x_j approach each other until they coalesce, so what we are doing is adding to the moduli space of isomorphism classes of 5-tuples of distinct points $(x_1, x_2, x_3, x_4, x_5)$ the 10 new sets of points with $x_i = x_j$ for $1 \leq i < j \leq 5$.

Let us examine each of these 10 sets. A single loop cuts $S_{0,5}$ into one sphere with three marked points and a boundary component, and another with two marked points and a boundary component. When the loop is pinched to zero, we obtain a sphere with four marked points and a sphere with three marked points, so the corresponding stratum at infinity is isomorphic to $\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}$. Let us show this isomorphism explicitly. Consider for example a pants decomposition whose degenerate loop encloses the points x_1 and x_5. The corresponding stratum of $\overline{\mathcal{M}}_{0,5}$ consists of isomorphism classes of spheres with five marked points $(x_1, x_2, x_3, x_4, x_1)$, and these isomorphism classes are enumerated by the representatives of the form $(\lambda, 0, 1, \infty, \lambda)$, where $\lambda \notin \{0, 1, \infty\}$ (otherwise one has a maximally degenerate point). The set of partially degenerate spheres of type $x_1 = x_5$ is thus isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$, i.e. to $\mathcal{M}_{0,4}$, so that the divisor at infinity of $\overline{\mathcal{M}}_{0,5}$ actually contains ten copies of $\mathcal{M}_{0,4}$. This completely describes the codimension one part of the divisor at infinity of $\mathcal{M}_{0,5}$.

Let us consider the codimension 2 part, corresponding to maximally degenerate points, i.e. pants decompositions. A pants decomposition on S consists of two disjoint loops, and each loop must contain two marked points, so a pants decomposition is uniquely determined by specifying two disjoint pairs of marked points among the five. This makes fifteen pants decompositions, corresponding to fifteen points of maximal degeneration. The pants decomposition given by two loops one surrounding points x_i and x_j and the other points x_k and x_l corresponds to the degenerate sphere with marked points $x_1, \ldots, x_5$ such that $x_i = x_j$ and $x_k = x_l$. These fifteen points must also be added to $\mathcal{M}_{0,5}$ to obtain the stable compactification. They can be visualized as points where the stratum $x_i = x_j$ crosses $x_k = x_l$, with $\{i, j\} \cap \{k, l\} = \emptyset$, so that in fact the stable compactification is obtained by adding ten copies of $\mathbb{P}^1$ to $\mathcal{M}_{0,5}$. Note that by (2.1.3), we have

$$\begin{aligned}\mathcal{M}_{0,5} &\simeq (\mathbb{P}^1 - \{0, 1, \infty\})^2 - \langle x = y \rangle \\ &= (\mathbb{P}^1\mathbb{C})^2 - \langle x = 0, x = 1, x = \infty, y = 0, y = 1, y = \infty, x = y \rangle,\end{aligned}$$

so that $\mathcal{M}_{0,5}$ is naturally obtained from $(\mathbb{P}^1\mathbb{C})^2$ by removing seven lines, and the stable compactification is obtained by adding ten lines to the result (it is, in fact, the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points where the diagonal meets $y = 0$, $y = 1$, $y = \infty$).

Topological tangential base points on $\overline{\mathcal{M}}_{0,5}$. Let us only consider tangential base points in the neighborhood of the fifteen maximally degenerate points of $\overline{\mathcal{M}}_{0,5}$. A detailed reference for the sketch given here is chapters 1 and 2 of [PS]. As for $\mathcal{M}_{0,4}$, we will consider the real locus of the neighborhood of each maximally degenerate point. This real locus falls naturally into four simply connected regions, in the following way. Let us consider the point of maximal degeneration given by the pants decomposition enclosing points x_1 and x_2 together and points x_4 and x_5 together, so corresponding to the isomorphism class of the sphere with five marked points $(x_2, x_2, x_3, x_4, x_4)$. The representative in standard form is given by $(0, 0, 1, \infty, \infty)$. Points of $\mathcal{M}_{0,5}$ in the neighborhood of this point correspond to isomorphism classes of spheres with five marked points $(\lambda, 0, 1, \infty, \mu)$ where λ is very near 0 and μ is very near ∞ (we are looking in $\mathcal{M}_{0,5}$ so we don't consider the partially degenerate points where $\lambda = 0$ or $\mu = \infty$). Now, the real locus of this neighborhood consists of the points corresponding to spheres with λ and μ real, and these spheres can be classed into four regions according to whether $\lambda < 0$ or $\lambda > 0$, $\mu < \infty$ or $\mu > \infty$ (this second condition is obviously to be interpreted as meaning that μ is negative and $|\mu|$ is large). Clearly these four small regions are disjoint and simply connected. There are four such regions for each of the fifteen maximally degenerate points, so we obtain a set $\widehat{B}_{0,5}$ of sixty tangential base points on $\mathcal{M}_{0,5}$, each having an automorphism of order 2.

Orbifold structure. As for all the ordered genus zero moduli spaces, the pure mapping class group $\Gamma_{0,5}$ acts freely on the Teichmüller space $\mathcal{T}_{0,5}$, so that $\mathcal{M}_{0,5}$ has no special orbifold points but is simply a topological space, with topological fundamental group $\Gamma_{0,5}$.

Dehn twists on $S_{0,5}$. Let x be a base point on $\mathcal{M}_{0,5}$. One can parametrize these twists on $S_{0,5}$ and then as loops on the moduli space, just as in Figures 3.2-3.4. We do only x_{12}, starting from the tangential base point given topologically by the set of points $(0, \epsilon, 1, 1+\epsilon, \infty)$ for real small positive values of ϵ. The Dehn twist x_{12} is parametrized on $S_{0,5}$, as after Figure 3.2, by

$$\Big(-\frac{1}{2}\epsilon e^{2\pi i t} + \frac{1}{2}\epsilon, \frac{1}{2}\epsilon e^{2\pi i t} + \frac{1}{2}\epsilon, 1, 1+\epsilon, \infty\Big),$$

which returns to standard form as

$$\Big(0, \frac{\epsilon e^{2\pi i t}}{1 + \frac{1}{2}\epsilon e^{2\pi i t} - \frac{1}{2}\epsilon}, 1, \frac{1 + \frac{1}{2}\epsilon + \frac{1}{2}\epsilon e^{2\pi i t}}{1 + \frac{1}{2}\epsilon e^{2\pi i t} - \frac{1}{2}\epsilon}, \infty\Big).$$

It is easily seen that λ, starting at ϵ, describes a small counterclockwise circle around 0, whereas μ describes a small path which is homotopic to the identity on $\mathbb{P}^1 - \{0, 1, \infty\}$.

The fundamental group of $\mathcal{M}_{0,5}$. Another way of visualizing the loop x_{12} above is that it circles around the (missing) codimension 1 stratum $\lambda = 0$ in $\mathcal{M}_{0,5}$. It is actually more revealing to describe the loops corresponding to the generators $x_{i,i+1}$ (with $i \in \mathbb{Z}/5\mathbb{Z}$) this way. Each $x_{i,i+1}$ is a twist along a loop on $S_{0,5}$ surrounding the points x_i and x_{i+1}. Assume that x is the same tangential base point as above. Consider the region on $\mathcal{M}_{0,5}$ described by points $(x_1, x_2, x_3, x_4, x_5)$ with x_i real and $x_1 < x_2 < x_3 < x_4 < x_5$ (in the obvious sense of considering the real axis as a circle with $+\infty = -\infty$). This is a real, simply connected region on $\mathcal{M}_{0,5}$ containing the tangential base point x. It is a pentagonal region in two real dimensions, with five edges bounded by the real loci of the infinite strata $x_i = x_{i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$; the five vertices are the maximally degenerate points where the infinite strata intersect. It is easy to describe the homotopy class of the loop on $\mathcal{M}_{0,5}$ corresponding to the Dehn twist $x_{i,i+1} \in \Gamma_{0,5} \simeq \pi_1(\mathcal{M}_{0,5}; x)$; it is given by composing a path γ from x nearly to the stratum $x_i = x_{i+1}$ (which is present only in the compactification of $\mathcal{M}_{0,5}$, of course, but missing from the space itself) lying in the pentagon, with a small loop around the (missing) stratum, and then with γ^{-1}. This is of course independent of the choice of γ since the pentagon is simply connected.

Points with special automorphism group. As in the case of $\mathcal{M}_{0,[4]}$, although there are no special orbifold points on $\mathcal{M}_{0,5}$, nevertheless determining the points of special automorphism group, i.e. those which are fixed by some permutation subgroup of S_5, will give the key to finding the orbifold points on the unordered moduli space $\mathcal{M}_{0,[5]}$.

A permutation τ acts on a point of $\mathcal{M}_{0,5}$, simply by permuting the marked points via

$$\tau\big(x_1, x_2, x_3, x_4, x_5\big) = \big(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}, x_{\tau(4)}, x_{\tau(5)}\big).$$

If the starting point is given in the form of a standard representation (with three components fixed at 0, 1 and ∞, then bringing the result of the permutation back to this form, we obtain a rational expression for the action of τ. Let us determine this rational expression for $\tau = (12)$ and $\rho_2 = (12345)$:

$$\tau\big(\lambda, 0, 1, \infty, \mu\big) = \big(0, \lambda, 1, \infty, \mu\big) \sim \big(\frac{-\lambda}{1-\lambda}, 0, 1, \infty, \frac{\mu-\lambda}{1-\lambda}\big)$$
$$\rho\big(\lambda, 0, 1, \infty, \mu\big) = \big(0, 1, \infty, \mu, \lambda\big) \sim \big(\frac{1}{\mu}, 0, 1, \infty, \frac{\lambda-1}{\lambda-\mu}\big).$$

As in the case of $\mathcal{M}_{0,4}$, the points of special automorphism group are the points which have non-trivial isotropy subgroup under the action of S_5. There are two kinds of such points, isolated ones and those lying on a stratum of real dimension 1. To compute them, one computes the fixed points of each permutation in S_5. For example, the fixed points of ρ are given by (λ, μ) with

$$\lambda = \frac{1}{\mu} \quad \text{and} \quad \mu = \frac{\lambda - 1}{\lambda - \mu},$$

so μ is a root of $\mu^3 - 2\mu + 1$. One root is $\mu = 1$, but this is excluded in $\mathcal{M}_{0,5}$, so the two solutions are

$$(\lambda, \mu) = \big(\frac{1 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{5}}{2}\big).$$

Below, we give a table containing all points of special automorphism group on $\mathcal{M}_{0,5}$. Obviously, all powers of a given permutation have the same fixed points, so that it is only necessary to consider the cyclic subgroups of S_5. Each product of two transpositions fixes a stratum of complex codimension 1, whereas each 3-cycle, each 4-cycle and each 5-cycle fixes a conjugate pair of points. We give the complete list of fifteen special loci, but only compute the conjugate pairs for one representative of each type of cycle.

(λ, μ)	fixed by
$(-\mu^2 + 2\mu, \mu)$	$\langle(12)(34)\rangle$
$(\lambda, -\lambda^2 + 2\lambda)$	$\langle(25)(34)\rangle$
$(\frac{\mu^2}{2\mu-1}, \mu)$	$\langle(14)(23)\rangle$
$(\lambda, \frac{\lambda^2}{2\lambda-1})$	$\langle(23)(45)\rangle$
(μ^2, μ)	$\langle(13)(24)\rangle$
(λ, λ^2)	$\langle(24)(35)\rangle$
$(\mu - 1, \mu)$	$\langle(13)(25)\rangle$
$(\lambda, \lambda - 1)$	$\langle(12)(35)\rangle$
$(\lambda, \frac{\lambda}{1-\lambda})$	$\langle(14)(35)\rangle$
$(\frac{\mu}{1-\mu}, \mu)$	$\langle(13)(45)\rangle$
$(\lambda, \frac{1}{2-\lambda})$	$\langle(12)(45)\rangle$
$(\frac{1}{2-\mu}, \mu)$	$\langle(14)(25)\rangle$
$(\lambda, \frac{\lambda}{\lambda-1})$	$\langle(15)(34)\rangle$
$(\lambda, 1/\lambda)$	$\langle(15)(24)\rangle$
$(\lambda, 1 - \lambda)$	$\langle(15)(23)\rangle$
$(\frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2})$, $(\frac{-1-\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2})$	$\langle(135)\rangle$
$(\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$, $(\frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$	$\langle(12345)\rangle$
$(i, -i)$, $(-i, i)$	$\langle(1234)\rangle$

Permutations having other cycle types than those which appear in this table have no fixed points on $\mathcal{M}_{0,5}$, corresponding to the fact that no Riemann surfaces of type $(0, 5)$ have corresponding automorphisms.

Let us study the fifteen special loci of dimension 1. They are all identical, since they are images of each other under the automorphism group S_5 of $\mathcal{M}_{0,5}$, so it suffices to study only one, say the locus fixed by $(15)(24)$, given by points in $\mathcal{M}_{0,5}$ of the form $(0, \lambda, 1, 1/\lambda, \infty)$.

For such a point to lie in $\mathcal{M}_{0,5}$, the five components must be distinct, which means that we must have $\lambda \notin \{0, 1, \infty, -1\}$. Thus the special locus of (15)(24) is a copy of $\mathbb{P}^1-4$ points. We know that it has exactly fifteen images under S_5, so it must have a global stabilizer of order 8, of which one element of order 2 is (15)(24), which actually fixes the locus pointwise; this stabilizer is the dihedral group of order 8 generated by (15) and (1254). The permutation (15) acts by $\lambda \mapsto -\lambda$ and (1254) acts by $\lambda \mapsto -1/\lambda$. The corresponding special locus in the unordered moduli space $\mathcal{M}_{0,[5]}$, which is really a special orbifold locus as opposed to the locus described here which is merely a locus of points having isotropy group in S_5, is isomorphic to the quotient of $\mathbb{P}^1 - \{0, 1, \infty, -1\}$ by the action of this dihedral group (see §3.3.2).

§3.3.2. The unordered moduli space $\mathcal{M}_{0,[5]}$

The mapping class group $\Gamma_{0,[5]}$. The group $\Gamma_{0,[5]}$ is generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with the usual braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$, as well as the center relation $(\sigma_1\sigma_2\sigma_3\sigma_4)^5 = 1$ and the sphere relation $\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4 = 1$.

There is only one conjugacy class each of elements of order 2, 3, 4 and 5 in $\Gamma_{0,[5]}$. Generators of these conjugacy classes are given by $(\sigma_1\sigma_2\sigma_3)^2$, $\sigma_1^2\sigma_2\sigma_3$, $\sigma_1\sigma_2\sigma_3$ and $\sigma_1\sigma_2\sigma_3\sigma_4$ respectively.

The stable compactification of $\mathcal{M}_{0,[5]}$. We saw that the divisor at infinity of the ordered moduli space $\mathcal{M}_{0,5}$ consisted of ten crossing copies of $\overline{\mathcal{M}}_{0,4}$, corresponding to the ten ways of pinching a loop surrounding two marked points on S, i.e. the ten ways of choosing two points among five. Thus, the natural action of S_5 on the compactification $\overline{\mathcal{M}}_{0,5}$ permutes these ten strata, and so the compactification of $\mathcal{M}_{0,[5]}$ is obtained by adding a single stratum at infinity to $\mathcal{M}_{0,[5]}$.

This stratum looks like the quotient of $\overline{\mathcal{M}}_{0,4}$ by the stabilizer of each stratum of $\overline{\mathcal{M}}_{0,5}$ in S_5. Since there are ten strata, the stabilizer of each one is of order 12. The stratum corresponding to the pair of points i and j is fixed *pointwise* by the transposition (ij), and stabilized globally by the copy of S_3 inside S_5 given by all permutations fixing i and j. Thus the single stratum at infinity of $\overline{\mathcal{M}}_{0,[5]}$ is isomorphic to the quotient of $\overline{\mathcal{M}}_{0,4}$ by a group $S_3 \times \{\pm 1\}$ with S_3 acting as usual and $\{\pm 1\}$ acting trivially. The fifteen points of maximal degeneration on $\overline{\mathcal{M}}_{0,5}$, three of which lie on each of the ten codimension 1 strata, are all mapped to a single point of maximal degeneration in $\overline{\mathcal{M}}_{0,[5]}$.

Orbifold structure and special loci. The determination of the orbifold structure of $\mathcal{M}_{0,[5]}$ was prepared by the determination of the points of special automorphism group in $\mathcal{M}_{0,5}$, for the orbifold points of $\mathcal{M}_{0,[5]}$ are exactly the images of these.

We saw that there were fifteen one-dimensional loci of points of special automorphism group in $\mathcal{M}_{0,5}$, each of which is the set of fixed points of one of the fifteen products of two transpositions in S_5. They are permuted by the action of S_5. Each one is globally stabilized by a D_4 subgroup of S_5 and pointwise fixed by an element of order 2. To determine the look of the corresponding one-dimensional special locus in $\mathcal{M}_{0,[5]}$, it suffices to describe the quotient of one of these loci by its automorphism group D_4. We consider the locus $(0, \lambda, 1, 1/\lambda, \infty)$, which is equal to $\mathbb{P}^1 - \{0, 1, \infty, -1\}$, is pointwise fixed by $\{1, (15)(24)\}$ and globally stabilized by $\{1, (15), (24), (15)(24), (1254), (1452), (12)(54), (14)(25)\}$, where (15) acts by $\lambda \mapsto -\lambda$ and (1254) acts by $\lambda \mapsto -1/\lambda$. The orbifold quotient can be obtained in two steps. First we quotient by the pointwise action of the subgroup $\{1, (15)(24)\}$, obtaining a space looking identical to $\mathbb{P}^1 - \{0, 1, \infty, -1\}$, but with a $\mathbb{Z}/2\mathbb{Z}$ attached to each point. Then, we quotient this space by the degree 4 map

$$\frac{1}{2}(z^2 + \frac{1}{z^2}).$$

The result is a $\mathbb{P}^1$ with 2 missing points, 1 and ∞, and one special orbifold point -1, which has only the two preimages i and $-i$. This orbifold is equipped with a $\mathbb{Z}/2\mathbb{Z}$ at each point and a $\mathbb{Z}/4\mathbb{Z}$ at the special orbifold point. This point corresponds to the points of special automorphism group $\mathbb{Z}/4\mathbb{Z}$ in $\mathcal{M}_{0,5}$ given by $(0, i, 1, -i, \infty)$ and $(0, -i, 1, i, \infty)$ (automorphism group generated by (1254)).

Let us now consider the isolated orbifold points of $\mathcal{M}_{0,[5]}$. We saw that they come from 3-cycles, 4-cycles and 5-cycles in S_5, each of which has one pair of fixed points. Let us consider the pairs of fixed points of 4-cycles. Obviously all the pairs become identified in $\mathcal{M}_{0,[5]}$. But furthermore, we just saw that the pair corresponding to one particular 4-cycle, (1254), which is given in $\mathcal{M}_{0,5}$ by $(0, i, 1, -i, \infty)$ and $(0, -i, 1, i, \infty)$ becomes a single, orbifold point on the one-dimensional special locus in $\mathcal{M}_{0,[5]}$.

Similarly, the pairs of fixed points of 3-cycles become identified in $\mathcal{M}_{0,[5]}$, so that we only need consider the image of the pair of fixed points of one 3-cycle, say the one given in §3.3.1, fixed by (135), namely

$$(0, \frac{-1+\sqrt{-3}}{2}, 1, \frac{-1-\sqrt{-3}}{2}, \infty) \text{ and } (0, \frac{-1-\sqrt{-3}}{2}, 1, \frac{-1+\sqrt{-3}}{2}, \infty).$$

These two points become identified in $\mathcal{M}_{0,[5]}$. Furthermore, as they lie on the locus $(0, \lambda, 1, 1/\lambda, \infty)$ in $\mathcal{M}_{0,5}$, their image lies in the one-dimensional special locus in $\mathcal{M}_{0,[5]}$.

The same holds for the pairs of fixed points of the 5-cycles. We consider the pair given in §3.3.1, fixed by (12345), namely

$$(0, \frac{1+\sqrt{5}}{2}, 1, \frac{-1+\sqrt{5}}{2}, \infty) \text{ and } (0, \frac{1-\sqrt{5}}{2}, 1, \frac{-1-\sqrt{5}}{2}, \infty).$$

These two points pass to a single point in $\mathcal{M}_{0,[5]}$, which also lies on the special locus.

§3.4. Genus one, two marked points

The points of the moduli space $\mathcal{M}_{1,[2]}$ are tori with two marked points determined up to translation. Because they are determined only up to translation, we can take a unique representative of each point of $\mathcal{M}_{1,[2]}$ given by an elliptic curve E equipped with two points P and Q such that $P+Q=0$. Every such curve has an involution (written $z \mapsto -z$ on the fundamental parallelogram, or $(x,y) \mapsto (x,-y)$ in terms of the points), which exchanges P and Q. Thus we see that the ordered moduli space $\mathcal{M}_{1,2}$ is pointwise the same as $\mathcal{M}_{1,[2]}$, but $\mathcal{M}_{1,[2]}$ is equipped with an additional orbifold structure coming from the presence of an involution fixing every point.

The mapping class groups $\Gamma_{1,2}$ and $\Gamma_{1,[2]}$. The pure mapping class group $\Gamma_{1,2}$ is generated by the Dehn twists σ_1, σ_2, σ_3 along the loops shown in figure 3.15 below.

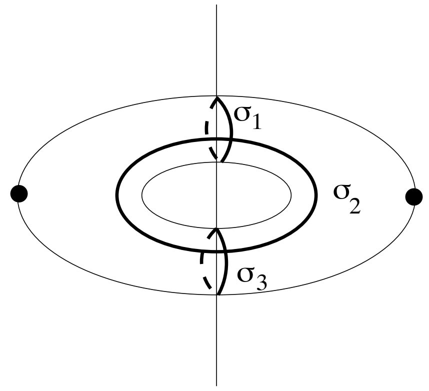


Figure 3.15. Generators of $\Gamma_{1,2}$

These twists satisfy the usual braid relations between the σ_i, as well as $(\sigma_1\sigma_2\sigma_3)^4 = 1$. Thus $\Gamma_{1,2}$ is isomorphic to the quotient of the Artin braid group B_4 modulo its center. The full mapping class group $\Gamma_{1,[2]}$ is generated by the pure mapping class group together with a single diffeomorphism class exchanging the two marked points; it is convenient to take this generator to be the rotation c around the vertical axis in figure 3.15. This rotation is of order 2 and it is obviously central since it commutes with all the σ_i. Furthermore we have $\Gamma_{1,2} \cap \langle c\rangle = \{1\}$ inside $\Gamma_{1,[2]}$ since no element of the pure mapping class group can permute the marked points. So we find that

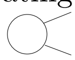
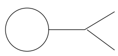
$$\Gamma_{1,[2]} \simeq \Gamma_{1,2} \times \langle c\rangle \simeq (B_4/Z) \times \mathbb{Z}/2\mathbb{Z}.$$

The ordered moduli space $\mathcal{M}_{1,2}$ can be identified with the quotient of $\mathcal{M}_{0,5}$ by the subgroup $S_4 \subset S_5$ permuting the first four marked points. Indeed, each point of the quotient $\mathcal{M}_{0,5}/S_4$ is given by an unordered quadruple of distinct points and one additional point; the quadruple determines an elliptic curve (by giving its unordered ramification points), and the fifth point can be identified with the x-coordinate of the point $P-Q$ (which is independent of the choice of P and Q up to translation). In other words, a point of $\mathcal{M}_{0,5}$ represented by $(0,1,\infty,\lambda,\mu)$ to the (isomorphism class of the) elliptic curve given by the Weierstrass equation $y^2 = x(x-1)(x-\lambda)$, equipped with any pair of marked points P and Q with $\mu = x(P-Q)$. Obviously, changing the order of the four first points gives rise to an isomorphic elliptic curve; permuting them and returning to standard form only has the effect of replacing λ by one of the six values $1/\lambda$, $1-\lambda$, $(\lambda-1)/\lambda$, $\lambda/(\lambda-1)$,$1/(1-\lambda)$, giving an isomorphic elliptic curve.

The maps $\mathcal{M}_{0,5} \to \mathcal{M}_{0,5}/S_4 \simeq \mathcal{M}_{1,2} \to \mathcal{M}_{0,5}/S_5 = \mathcal{M}_{0,[5]}$ correspond to the sequence of homomorphisms of fundamental groups

$$\Gamma_{0,5} \to \Gamma_{1,2} \to \Gamma_{0,[5]}.$$

The unordered moduli space $\mathcal{M}_{1,[2]}$ is equal to the quotient of $M_{1,2}$ by the action of the group $\langle c\rangle$, which fixes each point (acting as the involution of the corresponding elliptic curve).

The stable compactification of $\mathcal{M}_{1,2}$. There are two pants decompositions of the topological surface $S_{1,2}$ (shown in figure 3.15), one obtained by pinching two non-separating loops to points and the other obtained by pinching a separating loop and a non-separating loop. These correspond to the two possible genus 1 graphs with two tails, given by  and  . We call these two pants decompositions P_1 and P_2. The two graphs correspond to two different types of degeneration of an elliptic curve with two marked points.

The group S_5 acts on the stable compactification $\mathcal{M}_{0,5}$, as does its subgroup S_4, and the points of maximal degeneration of $\mathcal{M}_{0,5}$ are mapped to points of maximal degeneration of $\mathcal{M}_{1,2} \simeq \mathcal{M}_{0,5}/S_4$. The 3 points of maximal degeneration of $\mathcal{M}_{0,5}$ corresponding to a numbering of the graph such that the distinguished (central) tail is numbered 5 are sent to point of maximal degeneration corresponding to P_1, since those are the graphs where the four ramification points numbered 1 to 4 collide in two pairs. The 12 remaining graphs (points of maximal degeneration on $\mathcal{M}_{0,5}$) are all mapped to P_2.

The partially degenerate points are those for which one of the two loops in one of the two pants decompositions is pinched to zero. For the two loops in P_2, this gives two divisors of complex dimension 1. However, in the case of P_1, pinching one or the other of the two loops gives the same divisor, since an automorphism of the point brings one loop to the other.

The loop of P_2 enclosing the two marked points cuts $S_{1,2}$ into an $S_{1,1}$ and an $S_{0,3}$, so the corresponding divisor is a copy of $\mathcal{M}_{1,1}$. The other loop of P_2 cuts $S_{1,2}$ into an $S_{0,4}$, so the corresponding divisor is a copy of $\mathcal{M}_{0,4}$. The loops of P_1 each cut $S_{1,2}$ into an $S_{0,4}$, so there is another copy of $\mathcal{M}_{0,4}$ in the divisor at infinity of $\mathcal{M}_{1,2}$. However, this second copy is self-intersecting, as the point of maximal degeneration lies on it twice.

The stable compactification $\overline{\mathcal{M}}_{1,2}$ is obtained from the stable compactification $\overline{\mathcal{M}}_{0,5}$ by quotienting by the action of S_4. The divisor at infinity is exactly the image of the divisor at infinity of $\overline{\mathcal{M}}_{0,5}$.

Topological tangential base points on $\overline{\mathcal{M}}_{1,2}$. In order to compute the tangential base points of $\mathcal{M}_{1,2}$, it is necessary to understand the topological shape of the neighborhoods of the two points of maximal degeneration, as the images of the neighborhoods of the corresponding points in $\mathcal{M}_{0,5}$. We begin with P_2.

Let P be a point of maximal degeneration of $\overline{\mathcal{M}}_{0,5}$ lying over P_2; say P joins the pair of points 1 and 2 and the pair 4 and 5. Then P is fixed by the dihedral subgroup in S_5 generated by (12), (45) and (1425). To see what subgroup of S_4 fixes P, we take the intersection of this dihedral subgroup with S_4 and it consists only of the identity and the transposition (12). This shows that the neighborhood of P_2 in $\overline{\mathcal{M}}_{1,2}$ is isomorphic to the image of a neighborhood of P in $\overline{\mathcal{M}}_{0,5}$ quotiented by the action of the transposition. Now, the neighborhood of P in $\overline{\mathcal{M}}_{0,5}$ is the product of two disks (pointed disks if we exclude P itself); each disk contains two real segments representing the tangential base points. Retracting the pointed disks to their boundaries, their product is a torus, with four points representing the four tangential base points. The action of the transposition is like $z \mapsto z^2$ on the first disk (or circle) and the identity on the second, so it has an effect on the torus as though one cut through it to make a cylinder, rolled the cylinder over on itself twice and reglued to obtain a new torus. The four points have now been identified two by two to give two points coming from the tangential base points of $\mathcal{M}_{0,5}$, but it is necessary to have four tangential base points on the torus to cut it into simply connected regions.

To summarize, the neighborhood of P_2 in $\overline{\mathcal{M}}_{1,2}$ is still a product of two pointed disks, and therefore P_2 is flanked by four tangential base points. Two of them are images of the 48 tangential base points flanking the 12 points of maximal degeneration of $\overline{\mathcal{M}}_{0,5}$ lying over P_2, which divide into two orbits of 24 points each under the action of S_4. The other two tangential base points do not come from $\mathcal{M}_{0,5}$. This situation is analogous to the one for $\mathcal{M}_{1,1}$, where one tangential base point comes from the six tangential base points on $\mathcal{M}_{0,4}$ and another one is needed to cut the neighborhood of infinity into simply connected regions.

Now let us consider the neighborhood of P_1. It is more complicated, because if Q denotes one of the three points of maximal degeneration of $\overline{\mathcal{M}}_{0,5}$ lying over P_1, say the one corresponding to the pants decomposition identifying the points 1 and 2 and the points 3 and 4, then Q is fixed by the dihedral group of order 8 generated by (12), (34) and (1324) in S_5, and in fact this subgroup also lies in S_4. Thus, the neighborhood of P_1 is isomorphic to the neighborhood of Q quotiented by this D_4. The elements of the D_4 act on the product of two pointed disks neighboring Q by $z \mapsto z^2$ on each of the disks and by exchanging the two disks. Retracting onto the torus and performing the $z \mapsto z^2$ first sends the torus again to a torus, but where all four tangential base points have been identified. Further quotienting this torus by the order 2 element exchanging the two circles gives a Möbius band. The retraction is quite misleading here since the Möbius band is not orientable, but the neighborhood of P_2 is really the Möbius band times a small segment, which is orientable. The fundamental group of this thickened band is the same as the one of the band itself, namely $\mathbb{Z}$, so it has to be cut into two simply connected pieces. One of these pieces corresponds to the 12 tangential base points flanking the 3 maximally degenerate points of $\overline{\mathcal{M}}_{0,5}$ lying over P_1, which form a single orbit under the action of S_4.

Special loci. We saw above that the whole of the moduli space $\mathcal{M}_{1,[2]}$ is a special locus for the canonical involution c. Let us consider the moduli space $\mathcal{M}_{1,2} = \mathcal{T}_{1,2}/\Gamma_{1,2}$. There is only one special locus of dimension 1 in $\mathcal{M}_{1,2}$, corresponding to the unique conjugacy class of elements of order 2 in $\Gamma_{1,2}$, which is the conjugacy class of $(\sigma_3\sigma_2\sigma_1)^2$.

Theorem 3.4.1. *The 1-dimensional special locus of $\mathcal{M}_{1,2}$ maps bijectively to the quotient M of $\mathbb{P}^1 - \{0,1,\infty\}$ by the mapping identifying z with $1/z$, which is given by $\frac{1}{2}(z+\frac{1}{z})$.*

Proof. Let $\lambda \in \mathbb{P}^1 - \{0,1,\infty\}$, let E be the elliptic curve of equation $y^2 = x(x-1)(x-\lambda)$, and let $P \neq Q$ be two distinct points on E. Let μ be the x-coordinate of the point $P - Q$. Note that μ determines the unordered pair of points P, Q exactly up to translation. Indeed, if $P' = P + T$ and $Q' = Q + T$ then $P' - Q' = P - Q$ so $\mu = x(P' - Q') = x(P - Q)$. Conversely, if $x(P-Q) = x(P'-Q')$, then either $P' - Q' = P - Q$, in which case we have $P' - P = Q' - Q$; setting $T = P' - P$, we have $P' = T + P$ and $Q' = T + Q$, or else $P' - Q' = Q - P$, in which case $P' - Q = Q' - P$; setting $T = P' - Q$, we have $P' = T + Q$ and $Q' = T + P$. Note that $\mu \neq \infty$ ensures that $P \neq Q$.

The moduli space $\mathcal{M}_{1,2}$ can be parametrized by couples (λ, μ) with $\lambda \in \mathbb{P}^1 - \{0,1,\infty\}$ and $\mu \in \mathbb{C}$, modulo an equivalence relation for which (λ,μ) and (λ',μ') are equivalent if and only the equations $y^2 = x(x-1)(x-\lambda)$ and $y^2 = x(x-1)(x-\lambda')$ describe isomorphic curves, and μ corresponds to μ' under the isomorphism. This is explicitly given by the following equivalence relation grouping the pairs $(\lambda,\mu) \in \big(\mathbb{P}^1 - \{0,1,\infty\} \times \mathbb{C}\big)$ into the following groups of six:

$$(\lambda,\mu) \sim (1-\lambda, 1-\mu) \sim (\frac{1}{1-\lambda}, \frac{1-\mu}{1-\lambda}) \sim (\frac{1}{\lambda}, \frac{\mu}{\lambda})$$

$$\sim (1-\frac{1}{\lambda}, 1-\frac{\mu}{\lambda}) \sim \Big(\frac{\lambda}{\lambda-1}, (\frac{\lambda}{\lambda-1})(1-\frac{\mu}{\lambda})\Big).$$

The special points in $\mathcal{M}_{1,2}$ are given by the pairs $(\lambda, 0)$, $(\lambda, 1)$ and (λ,λ). They group into sextuplets of the form

$$(\lambda, 0) \sim (1-\lambda, 1) \sim (\frac{1}{1-\lambda}, \frac{1}{1-\lambda}) \sim (\frac{1}{\lambda}, 0) \sim (1-\frac{1}{\lambda}, 1) \sim \Big(\frac{\lambda}{\lambda-1}, (\frac{\lambda}{\lambda-1})\Big)$$

There is an obvious isomorphism of this subset of sextuples to M given by associating to a sextuplet the unordered pair of values $\{\lambda, 1/\lambda\}$ obtained by considering only the pair of elements in the sextuplet whose μ-component is equal to 0. $\diamondsuit$

Remark. It will be shown in §4.1 that when the special locus corresponding to a finite-order element φ of a mapping class group is normal in the moduli space, then the fundamental group of the special locus is isomorphic to the normalizer of φ. In the present case, we can take φ to be the order 2 element $(\sigma_3\sigma_2\sigma_1)^2$ of $\Gamma_{1,2}$, and its normalizer is given by

$$\mathrm{Norm}_{\Gamma_{1,2}}(\varphi) = \langle \sigma_3\sigma_2\sigma_3^{-1}, \sigma_3\sigma_1\rangle.$$

This group contains φ as a central element of order 2, and modulo φ, it is isomorphic to $\pi_1(M) = \langle \tau_1^2, \tau_2\rangle$ where τ_1 and τ_2 are standard generators of B_3/Z, the Artin braid group on three strands modulo its center.

In order to conclude the determination of the special loci in $\mathcal{M}_{1,2}$, it suffices to list the elements of finite order in $\Gamma_{1,2}$ (since we have already considered the canonical involution in $\Gamma_{1,[2]}$). But apart from the order 2 element $(\sigma_3\sigma_2\sigma_1)^2$ already studied, the only such element (up to conjugacy) is the order 4 element $\sigma_3\sigma_2\sigma_1$. The corresponding special point on $\mathcal{M}_{1,2}$ is the point on the special locus of $(\sigma_3\sigma_2\sigma_1)^2$ described above with λ-value equal to -1; this point corresponds to the elliptic curve of equation $y^2 = x^3 - x$ which has automorphism group $\mathbb{Z}/4\mathbb{Z}$ generated by $(x, y) \mapsto (-x, iy)$, equipped with marked points (∞, ∞) (the origin) and $(0, 0)$. The fundamental parallelogram of this elliptic curve in the upper half-plane $\mathcal{H}$ is the square $\langle 1, i\rangle$, and the marked points are $z = 0$ and $z = (1+i)/2$, both fixed under the rotation of the square.

§3.5. Genus zero, n marked points

In this section, we limit ourselves to investigating the points of special automorphism group and the special loci in the genus zero moduli spaces for arbitrary n. If S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma([S])$ is the class of a diffeomorphism which is simply a rotation around an axis:

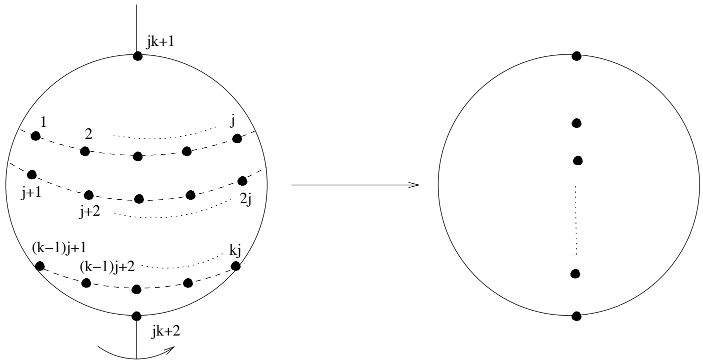


Figure 3.16. A finite-order diffeomorphism in genus zero.

The north and south poles may or may not be marked points, but they are always the only branch points for φ. The permutation associated to a rotation φ is always of the form $c_1 \cdots c_k$, where the c_i are disjoint cycles of length j such that

$$\begin{cases} jk = n & \text{if the north and south poles are not marked} \\ jk = n-1 & \text{if one of the two poles is marked} \\ jk = n-2 & \text{if both poles are marked points.} \end{cases}$$

In the following theorem, we compute the points of special automorphism associated to a permutation $[\varphi]$ which is a product of k disjoint cycles of length j only in the case $jk = n-2$ (i.e. when the two fixed points of φ are marked points), since in the other two cases, the special locus in $\mathcal{M}_{0,n}$ is just the image of the one we compute here in $\mathcal{M}_{0,n+1}$ or $\mathcal{M}_{0,n+2}$, under the morphism given by erasing the extra marked points.

Theorem 3.5.1. *Let S be a sphere with n marked numbered points, and let φ be the rotation shown in figure 3.16, with $n = jk+2$ (i.e. the fixed points of φ are marked points of S), so that*

$$[\varphi] = \big(1 \cdots j\big)\big(j+1 \cdots 2j\big) \cdots \big(j(k-1)+1 \cdots jk\big).$$

Let $G_\varphi \subset S_n$ be the subgroup generated by the disjoint cycles $c_1, \ldots, c_k$ of $[\varphi]$. Let T be the orbifold quotient S/φ, which has k marked points with ramification index 1 and 2 marked points with ramification index j.

(i) The set of fixed points of $[\varphi]$ in the ordered moduli space $\mathcal{M}(S)$ consists of $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint connected components given by

$$\mathcal{C}_\zeta = \Big(1, \zeta, \ldots, \zeta^{j-1}, a_1, a_1\zeta, \ldots, a_1\zeta^{j-1}, \cdots, a_{k-1}, a_{k-1}\zeta, \cdots, a_{k-1}\zeta^{j-1}, 0, \infty\Big), \quad (3.5.1)$$

where ζ runs through the primitive j-th roots of unity. Each component is isomorphic to a copy of $(\mathbb{P}^1 - \{0, 1, \zeta, \ldots, \zeta^{j-1}, \infty\})^{k-1}$ minus the $j(k-1)$ lines $a_i = a_r\zeta^s$ for $r \neq i$, $0 \le s \le j-1$, and is thus defined over $\mathbb{Q}^{\mathrm{ab}}$.

(ii) The special locus of φ in the quotient space $\mathcal{M}(S)/G_\varphi = \mathcal{M}_{G_\varphi}(S)$ also consists of $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint connected components $\overline{\mathcal{C}}_\zeta$, the images of the $\mathcal{C}_\zeta$ in the quotient space. Each $\overline{\mathcal{C}}_\zeta$ is isomorphic to

$$(\mathbb{P}^1 - \{0,1,\infty\})^{k-1} - \Delta \simeq \mathcal{M}(T) \simeq \mathcal{M}_{0,k+2}, \qquad (4.2.2)$$

and is thus defined over $\mathbb{Q}$; however the embeddings $\mathcal{M}(T) \to \overline{\mathcal{C}}_\zeta \subset \mathcal{M}(S)/G_\varphi$ are defined over $\mathbb{Q}^{\mathrm{ab}}$.

(iii) In the unordered moduli space $\mathcal{M}([S])$, the special locus of φ consists of a single connected component $\mathcal{C}$. It is isomorphic to the unordered moduli space $\mathcal{M}_G(T)$ which is the quotient of $\mathcal{M}(T)$ by the group G of "admissible" permutations, i.e. permutations of marked points of the same ramification index, and the space $\mathcal{M}_G(T)$ and the embedding $\mathcal{M}_G(T) \to \mathcal{C} \subset \mathcal{M}([S])$ are defined over $\mathbb{Q}$.

Proof. (i) Let a point on $\mathcal{M}(S)$ be given by its unique representative as a sphere with n marked points $(x_1,\ldots,x_n)$ such that $x_1 = 1$, $x_{n-1} = 0$ and $x_n = \infty$. It is immediate that the components $\mathcal{C}_\zeta$ of (3.5.1) are all disjoint on $\mathcal{M}(S)$, since the points $0,1,\infty$ are marked.

To see that each point of each component is fixed by the action of the permutation $[\varphi]$, it suffices to notice that the permutation is realized by multiplication by ζ^{-1}. To show that these points are the only points fixed by $[\varphi]$ is a straightforward computation. Finally, a point of $\mathcal{C}_\zeta$ uniquely determines a point $(a_1,\ldots,a_{k-1}) \in (\mathbb{P}^1)^{k-1}$, but conversely, a tuple $(a_1,\ldots,a_{k-1})$ gives rise to a point of $\mathcal{M}(S)$ via (3.5.1) if and only if the components of (3.5.1) are all distinct, i.e. if and the a_i fulfill all the inequality conditions of (i).

(ii) The fact that the $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint components remain disjoint in the quotient space $\mathcal{M}(S)/G_\varphi$ follows immediately from the simple remark that each component $\mathcal{C}_\zeta$ is stabilized (not pointwise fixed) by the whole of the group G_φ, so that the components $\overline{\mathcal{C}}_\zeta = \mathcal{C}_\zeta/G_\varphi$ remain disjoint. The action of G_φ on $\mathcal{C}_\zeta$ can be translated into an action on to a copy of $(\mathbb{P}^1 - \{0,1,\zeta,\ldots,\zeta^{j-1},\infty\})^{k-1}$ minus the $j(k-2)$ lines $a_i = a_r\zeta^s$ for $r \neq i$, $0 \le s \le j-1$; for $i \ge 1$, the $(i+1)$-th generator (i.e. the $(i+1)$-st disjoint cycle of $[\varphi]$) acts via $a_i \mapsto a_i\zeta$ for $1 \le i \le k-1$. The quotient is thus isomorphic to $(\mathbb{P}^1 - \{0,1,\infty\})^{k-1}$, parametrized by $(b_1,\ldots,b_{k-1})$ with $b_i = a_i^j$ for each i, minus the diagonals $b_i = b_j$; this space is isomorphic to $\mathcal{M}_{0,k+2} \simeq \mathcal{M}(T)$. However, each of the $|(\mathbb{Z}/j\mathbb{Z})^*|$ embeddings $f_\zeta : \mathcal{M}_{0,k+2} \to \overline{\mathcal{C}}_\zeta$ are defined over $\mathbb{Q}^{\mathrm{ab}}$ since ζ must be specified.

(iii) Finally, all these components (as well as all those corresponding to other rotations having the same cycle type as φ) become identified in the unordered moduli space $\mathcal{M}([S])$, since for any $i \in (\mathbb{Z}/j\mathbb{Z})^*$, the locus

$$\mathcal{C}_{\zeta^i} = \Big(1,\zeta^i,\ldots,\zeta^{i(j-1)},a_1,a_1\zeta^i,\ldots,a_1\zeta^{i(j-1)},\cdots,a_{k-1},a_{k-1}\zeta^i,\cdots,a_{k-1}\zeta^{i(j-1)},0,\infty\Big)$$

differs from

$$\mathcal{C}_\zeta = \Big(1, \zeta, \ldots, \zeta^{j-1}, a_1, a_1\zeta, \ldots, a_1\zeta^{j-1}, \cdots, a_{k-1}, a_{k-1}\zeta, \cdots, a_{k-1}\zeta^{j-1}, 0, \infty\Big)$$

only by a permutation.

The special locus $\mathcal{C}$ in the unordered moduli space $\mathcal{M}([S])$ is isomorphic to $\mathcal{C}_\zeta$ modulo its stabilizer in $S_n = S_{jk+2}$. Therefore, to determine it, we need to determine its stabilizer.

We begin by determining the order of the stabilizer, which is the quotient of $|S_n|$ by the number of different loci corresponding to groups $\langle\alpha\rangle$ where $\alpha \in S_n$ is a permutation of the same cycle type as $[\varphi]$, i.e. k cycles of length j. So let us enumerate the groups $\langle\alpha\rangle$, where α is a product of k j-cycles.

To begin with, there are $\binom{n}{2}$ ways of choosing two indices fixed by such a permutation α. Having fixed such a choice, let us count the ways of dividing the permutations of the remaining jk components into groups $\langle\alpha\rangle$. There are jk possible orderings of the remaining indices; packaging them into j-cycles gives a redundant version of the set of possible permutations α of the right form. But each of the k j-cycles can be written in j different ways giving the same permutation (by cyclically permuting them, i.e. (12345) is the same permutation as (23451)), and furthermore any permutation of the k disjoint cycles amongst each other again gives the same permutation. Therefore there are $jk/(k!j^k)$ possible permutations α of the right form, fixing two given indices. We actually want to count the groups $\langle\alpha\rangle$, so we must divide by the number of powers of α which are also products of k j-cycles; this is $\phi(j) = |(\mathbb{Z}/j\mathbb{Z})^*|$. Finally, we have found $\binom{n}{2} \cdot jk/(k!j^k\phi(j))$ different groups $\langle\alpha\rangle$. Since the locus of fixed points of each such cyclic group has $\phi(j)$ connected components, and $n = jk+2$, we find

$$\frac{(jk+2)!}{2 \cdot k! \cdot j^k}$$

connected components in the ordered moduli space $\mathcal{M}(S)$ for all the different α. The order of the stabilizer of one component is thus equal to $2j^k k!$, and the locus corresponding to the disjoint cycle-type of α is isomorphic to the quotient of $\mathcal{C}_\zeta$ of (3.5.1) by its stabilizer.

We can now easily compute the stabilizer of $\mathcal{C}_\zeta$, one of the loci associated to

$$[\varphi] = (1 \cdots j)(j+1 \cdots 2j) \cdots ((k-1)j \cdots jk).$$

It is generated by three natural subgroups; the first, of order $k!$, corresponding to permuting the k disjoint cycles of $[\varphi]$, the second, of order j^k (which is actually just G_φ), is generated by the j cycles themselves, and the third, of order 2, is generated by the permutation

$$\tau = \begin{cases} (1, jk)(2, jk-1)\cdots((jk/2), (jk+2)/2)\cdot(jk+1, jk+2) & \text{if } jk \text{ is even} \\ (1, jk)(2, jk-1)\cdots((jk-1)/2, (jk+1)/2)\cdot(jk+1, jk+2) & \text{if } jk \text{ is odd.} \end{cases}$$

This last permutation is easily seen to stabilize $\mathcal{C}_\zeta$ since applying it to the point corresponding to $(a_1,\ldots,a_{k-1})$ gives

$$\big(a_{k-1}\zeta^{j-1},\ldots,a_{k-1},a_{k-2}\zeta^{j-1},\ldots,a_{k-2},\ldots,a_1\zeta^{j-1},\ldots,a_1,\zeta^{j-1},\ldots,1,\infty,0\big),$$

and then applying the mapping $z\mapsto a_{k-1}\zeta^{j-1}/z$ gives

$$\big(1,\zeta,\ldots,\zeta^{j-1},c_1,\ldots,c_1\zeta^{j-1},\ldots,c_{k-2},\ldots,c_{k-2}\zeta^{j-1},c_{k-1},\ldots,c_{k-1}\zeta^{j-1}\big)$$

with $c_m=a_{k-1}/a_{k-m-1}$ (writing $a_0=1$), which also lies on $\mathcal{C}_\zeta$.

It remains only to compute the quotient of $\mathcal{C}_\zeta$ by this stabilizer, by determining the action of the generators of the stabilizer on the a_i. We already saw that

$$\mathcal{C}_\zeta/G_\varphi=\overline{\mathcal{C}}_\zeta\simeq(\mathbb{P}^1-\{0,1,\infty\})^{k-1}-\Delta,$$

parametrized by $b_1,\ldots,b_{k-1}$. To quotient by the rest of the stabilizer, we identify $(\mathbb{P}^1-\{0,1,\infty\})^{k-1}-\Delta$ with $\mathcal{M}(T)=\mathcal{M}_{0,k+2}$, by identifying the point $(b_1,\ldots,b_{k-1})$ with the orbifold sphere of topological type T, having marked points $(1,b_1,\ldots,b_{k-1},0,\infty)$ where 0 and ∞ have associated ramification index j and $1,b_1,\ldots,b_{k-1}$ have associated ramification index 1. Then the permutations of the k disjoint cycles of $[\varphi]$ act like permutations of $b_0=1,b_1,\ldots,b_{k-1}$, i.e. we quotient by all possible permutations of the marked points of T of ramification index equal to 1, and the permutation τ acts like

$$(b_0,b_1,\ldots,b_{k-1},0,\infty)\xrightarrow{\tau}(b_{k-1},b_{k-2},\ldots,b_0,\infty,0)=(1,b_{k-1}/b_{k-2},\ldots,b_{k-1}/b_0,0,\infty),$$

the last two tuples representing the same point in the moduli space $\mathcal{M}(T)$ via the transformation $z\to b_{k-1}/z$. Finally, then, we find that the special locus $\mathcal{C}\subset\mathcal{M}([S])$ is isomorphic to $\mathcal{M}(T)/S_k\times(\mathbb{Z}/2\mathbb{Z})$, which is naturally the unordered moduli space $\mathcal{M}([T])$ of the orbifold T, in which marked points of different ramification orders cannot be permuted. This moduli space is defined over $\mathbb{Q}$, as is the isomorphism with $\mathcal{C}$. $\diamondsuit$

Corollary. *The isomorphisms of the special loci with moduli spaces in parts (ii) and (iii) of theorem 3.5.1 hold for any finite-order element $\varphi\in\Gamma([S])$.*

Proof. Any finite-order φ is the conjugate of a φ having a permutation of the type $[\varphi]$ of theorem 3.5.1, which was fixed only for convenience. Obviously, for any other product of k j-cycles, one needs to modify the expression (3.5.1); everything else in the proof passes to the general case. $\diamondsuit$

4. Special loci and special homomorphisms

When dealing with special loci in moduli space, the most natural question to ask is *What varieties can appear?* The brief answer is that *the special loci look very much like moduli spaces themselves.* In fact, the special locus of a finite-order element φ in moduli space of a topological surface S of type (g, n) is closely related to the moduli space of the quotient $T = S/\varphi$. Throughout this section, *we consider only the case where all branch points of the cover $S \to T$ are marked points of T, images of marked points on S*, and we investigate the nature of the special loci in the moduli space of S.

Recall from §2.1 that a "good" orbifold $\mathcal{M}$ is the quotient of a simply connected topological space $\mathcal{T}$ by a discrete group Γ acting properly discontinuously but not necessarily freely, and that all the moduli spaces of Riemann surfaces are such orbifolds. The *locus of special orbifold points* on the orbifold $\mathcal{M}$ is the set of points in $\mathcal{M}$ which are the images of points in $\mathcal{T}$ having non-trivial isotropy in Γ. A *special locus* of points in $\mathcal{M}$ is any subset of the locus of special orbifold points.

Definition/Notation. Let S be a topological surface of type (g, n), and let $G \subset S_n$ be a subgroup. Let $\mathcal{M}(S)$ denote the ordered moduli space $\mathcal{M}_{g,n}$ (whose points are isomorphism classes of Riemann surfaces of topological type S), and let $\mathcal{M}_G(S)$ denote the quotient of the ordered moduli space $\mathcal{M}(S)$ by the permutation group G. Let $\Gamma(S)$ denote the pure mapping class group $\Gamma_{g,n}$ (isomorphic to $\mathrm{Diff}^+(S)/\mathrm{Diff}^0(S)$, cf. §2.1) and let $\Gamma([S])$ denote the full mapping class group $\Gamma_{g,[n]}$. We also write $\Gamma_G(S)$ for the preimage of G under the canonical surjection $\Gamma([S]) \to S_n$.

If φ is a finite-order element of the full mapping class group $\Gamma([S])$, we let $\mathcal{M}_G(S, \varphi)$ denote the image in $\mathcal{M}_G(S)$ of the set of points in the Teichmüller space $\mathcal{T}(S) = \mathcal{T}_{g,n}$ which are fixed by φ under the canonical action of $\Gamma([S])$ on $\mathcal{T}(S)$ (cf. §2.1). This locus is called the *special locus* of φ in $\mathcal{M}_G(S)$ if and only if $\varphi \in \Gamma_G(S)$, ensuring that the points of the locus are special orbifold points of φ. Otherwise, as we saw in §3, we simply call the points of $\mathcal{M}_G(S, \varphi)$ "points with special automorphism group". Note that the special locus $\mathcal{M}_G(S, \varphi)$ depends only on $[\langle\varphi\rangle]$, the conjugacy class in $\Gamma_G(S)$ of the group generated by φ.

Every finite-order element φ of $\Gamma([S])$ can be realized as an automorphism of some Riemann surface X of topological type S. This result, long known as the *Nielsen realization problem*, was finally proved by S. Kerckhoff ([K1], [K2]). It implies that the special locus of φ consists of isomorphism classes of Riemann surfaces admitting automorphisms of topological type φ. The relation of the special locus $\mathcal{M}_G(S, \varphi)$ to the moduli space of the quotient $T = S/\varphi$ arises from the natural map from the special locus to the moduli space of T given by associating to each point X of the special locus the quotient Riemann surface

X/φ. This morphism makes the special locus into a finite cover of the moduli space of T, in which the orbifold structure coming from the presence of the automorphism φ at each point of the special locus may or may not be "trivial" in a certain sense (cf. §4.1). When the cover is 1-to-1 and the orbifold structure is trivial, there is a bijection between the special locus and the moduli space of T. The goals of this section are to justify these statements (§4.1), to determine these conditions explicitly in terms of fundamental groups in the case where $G = G_\varphi$ is the group generated by the disjoint cycles of φ and $\mathcal{M}(T)$ is the ordered moduli space of T (§4.2), to show that in genus zero, the conditions are fulfilled and therefore the bijection exists for all φ (§4.3), and to give examples (§§4.4, 4.5).

We consistently use the word *special* to refer to the finite-order diffeomorphisms or automorphisms of Riemann surfaces, as in the terms *curve with special automorphisms* and *special locus associated to a finite-order diffeomorphism*. Therefore, we use the term *special homomorphism between mapping class groups* to denote the homomorphisms between mapping class groups we construct in §4.2, which exist whenever there is a bijection between the special locus and the moduli space of T.

§4.1. Special loci in moduli space

In theorem 3.5.1 (ii), we saw that each component of the special locus of a finite-order element $\varphi \in \Gamma([S])$ in the genus zero moduli space $\mathcal{M}_\varphi(S)$ is itself isomorphic to the ordered moduli space of $T = S/\varphi$. In this section, we give the general picture underlying this fact. This general picture is weaker than the genus zero case, in that in arbitrary genus, a special locus of $\varphi \in \Gamma([S])$ is *related* to the moduli space $\mathcal{M}(T)$, but not necessarily isomorphic to it.

Let us begin by citing some important results of González-Díez and Harvey ([GH]), applied to the case where the finite group $G \subset \Gamma([S])$ they consider is the cyclic group $\langle\varphi\rangle$.

Theorem 4.1.1. ([GH], [HM]) *Let S be a topological surface of type (g, n), and let φ be a finite-order element of $\Gamma([S])$. Assume that the quotient $T = S/\varphi$ is of genus g' with n' marked points, including all the branch points.*

(i) Denote by $\mathcal{T}(S,\varphi)$ the subset of points of the Teichmüller space $\mathcal{T}(S) = \mathcal{T}_{g,n}$ fixed by φ. Then $\mathcal{T}(S,\varphi)$ is isomorphic to $\mathcal{T}_{g',n'} = \mathcal{T}(T)$.

(ii) The set of elements of $\Gamma([S])$ globally preserving the subset $\mathcal{T}(S,\varphi)$ in $\mathcal{T}(S)$ is exactly the subgroup $\mathrm{Norm}_{\Gamma([S])}(\varphi)$.

(iii) For every $G \subset S_n$ containing the permutation $[\varphi]$ associated to φ, the quotient $\widetilde{\mathcal{M}}_G(S,\varphi) = \mathcal{T}(S,\varphi)/\mathrm{Norm}_{\Gamma_G(S)}(\varphi)$ is isomorphic to the normalisation of the special locus $\mathcal{M}_G(S,\varphi) \subset \mathcal{M}_G(S)$.

Let G_φ be the subgroup of S_n generated by the disjoint cycles of the permutation $[\varphi]$.

Let us write $\mathcal{M}_\varphi(S)$ for the quotient space $\mathcal{M}_{G_\varphi}(S) = \mathcal{M}(S)/G_\varphi$, and $\mathcal{M}_\varphi(S,\varphi)$ for the whole of the special locus of φ in $\mathcal{M}_\varphi(S)$. We also write $\Gamma_\varphi(S)$ for the group $\Gamma_{G_\varphi}(S)$, the preimage in $\Gamma([S])$ of $\langle[\varphi]\rangle$ under the surjection $\Gamma([S]) \to S_n$.

There is a criterion (cf. [GH, Section 2]) to determine when the special locus $\mathcal{M}_\varphi(S,\varphi)$ is normal.

(N) We have $\widetilde{\mathcal{M}}_\varphi(S,\varphi) \neq \mathcal{M}_\varphi(S,\varphi)$ if and only if there exists a Riemann surface X of topological type S whose automorphism group G contains φ and another element φ' of $\Gamma_\varphi(S)$, such that φ' is a conjugate of φ inside $\Gamma_\varphi(S)$ but not inside G.

We now introduce a proposition whose proof is the goal of §4.2 (cf. proposition 4.2.7).

Proposition 4.1.2. *There is a canonical injective homomorphism*

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \hookrightarrow \Gamma(T), \tag{4.1.1}$$

whose image is of finite index.

This leads to the main result of this section.

Proposition 4.1.3. *The morphism*

$$\mathcal{M}_\varphi(S,\varphi) \to \mathcal{M}(T) \tag{4.1.2}$$

defined by associating to a point of $\mathcal{M}_\varphi(S,\varphi)$ (which is the isomorphism class of a Riemann surface X admitting φ as an automorphism) the quotient $X/\varphi \in \mathcal{M}(T)$ is a covering map of finite degree.

Proof. Let X be a point where $\mathcal{M}_\varphi(S,\varphi)$ is not normal, and suppose that $X \mapsto Y \in \mathcal{M}(T)$ under (4.1.2). We lift the morphism (4.1.2) to $\widetilde{\mathcal{M}}_\varphi(S,\varphi)$ by sending all the points lying above X to Y, and we show that the morphism

$$\widetilde{\mathcal{M}}_\varphi(S,\varphi) \to \mathcal{M}(T) \tag{4.1.3}$$

is a covering map of finite degree. For this, we use theorem 4.1.1 and proposition 4.1.2. Firstly, since $\langle\varphi\rangle$ fixes every point of $\mathcal{T}(S,\varphi)$, the action of $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$ factors through the quotient group $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$, and there is a canonical one-to-one correspondence

$$\mathcal{T}(S,\varphi)/\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \leftrightarrow \mathcal{T}(S,\varphi)/\big(\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle\big) \tag{4.1.4}$$

(the difference between the two spaces is hidden in the orbifold structure due to the action of $\langle\varphi\rangle$ fixing each point).

Now, using theorem 4.1.1, (4.1.4) and proposition 4.1.2, we have a natural sequence of morphisms

$$\mathcal{T}(S,\varphi) \to \mathcal{T}(S,\varphi)/\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \widetilde{\mathcal{M}}(S,\varphi) \leftrightarrow \mathcal{T}(S,\varphi)/\big(\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle\big)$$
$$\simeq \mathcal{T}(T)/\big(\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle\big) \to \mathcal{T}(T)/\Gamma(T) \simeq \mathcal{M}(T). \tag{4.1.5}$$

This shows that $\mathcal{M}_\varphi(S,\varphi)$ is a cover of $\mathcal{M}(T)$, and the fact that it is of finite degree follows from the fact that $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$ is of finite index in $\Gamma(T)$ (proposition 4.1.2). $\diamondsuit$

The goal of the next section, apart from proving proposition 4.1.2, is to give two conditions on the fundamental groups which are equivalent to two conditions on the covering map (4.1.3), which taken together mean that (4.1.3) is "as close as possible to an isomorphism", given the difference in orbifold structure due to the group $\langle\varphi\rangle$.

§4.2. Special homomorphisms between mapping class groups

In this section, we study the *special homomorphisms* between mapping class groups coming from the special morphisms

$$\mathcal{M}_\varphi(S,\varphi) \to \mathcal{M}(T),$$

or rather, from their liftings

$$\widetilde{\mathcal{M}}_\varphi(S,\varphi) \to \mathcal{M}(T)$$

introduced in the previous section.

By theorem 4.1.1, that the orbifold fundamental group of $\widetilde{\mathcal{M}}_\varphi(S,\varphi)$ is $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$, so the morphism (4.1.3) gives rise to a homomorphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \to \Gamma(T).$$

The goal of this section is to study this morphism (using the identification of the mapping class groups with outer automorphism groups of fundamental groups), in order to prove that not only does it factor through $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$, but that the induced morphism

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T)$$

is an injective homomorphism with image of finite index. This is the final result of this section (proposition 4.2.7) given as proposition 4.1.2 above. In order to prove it, we characterize the normalizer of φ in $\Gamma_\varphi(S)$ as the set of inertia-preserving outer automorphisms of $\pi_1(S)$ which extend to $\pi_1(T)$ (cf. lemma 4.2.3), and work in that context.

Let S be a topological surface of type (g,n), and write $\Gamma([S]) = \Gamma_{g,[n]}$ for the mapping class group of S permuting points and $\Gamma(S) = \Gamma_{g,n}$ for the pure mapping class group. Let

φ be a finite-order element of $\Gamma([S])$, and set $T = S/\varphi$; *we assume that all branch points of this cover (and their preimages) are marked points.* Let g' denote the genus of T and n' the number of marked points; the fundamental group of T is given by generators and relations as

$$\pi_1(T) = \Big\langle a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_{n'} \,\Big|\, \prod_{i=1}^{g'} (a_i, b_i) c_1 \cdots c_{n'} = 1 \Big\rangle. \tag{4.2.1}$$

Recall that the group of inertia-preserving automorphisms of $\pi_1(T)$, $\mathrm{Aut}^*\big([\pi_1(T)]\big)$, is defined to be the group of automorphisms

$$\mathrm{Aut}^*\big([\pi_1(T)]\big) = \big\{\psi \in \mathrm{Aut}\big(\pi_1(T)\big) \mid \exists \sigma \in S_n \text{ such that } \psi(c_i) \sim c_{\sigma(i)} \text{ for } 1 \le i \le n'\big\},$$

where $\sim$ means "is conjugate to". We saw (cf. Theorem 2.2.1) that

$$\Gamma([T]) = \mathrm{Out}^*([\pi_1(T)]), \tag{4.2.2}$$

the quotient of $\mathrm{Aut}^*([\pi_1(T)])$ by $\mathrm{Inn}(\pi_1(T))$. As in §2.2, we have a natural surjective homomorphism

$$\begin{aligned} \mathrm{Aut}^*\big([\pi_1(T)]\big) &\to S_n \\ \psi &\mapsto \sigma, \end{aligned} \tag{4.2.3}$$

which passes to $\mathrm{Out}^*([\pi_1(T)])$.

Because $T = S/\varphi$, we have the exact sequence

$$1 \to \pi_1(S) \to \pi_1(T) \to \langle\varphi\rangle \to 1. \tag{4.2.4}$$

The quotient $\langle\varphi\rangle$ is an outer automorphism group of $\pi_1(S)$, which is naturally identified with the subgroup $\langle\varphi\rangle$ of $\Gamma([S]) = \mathrm{Out}^*([\pi_1(S)])$, cf. the following lemma.

Lemma 4.2.1. *The homomorphism*

$$\begin{aligned} \pi_1(T) &\hookrightarrow \mathrm{Aut}([\pi_1(S)]) \\ t &\mapsto \mathrm{inn}(t)|_{\pi_1(S)} \end{aligned} \tag{4.2.5}$$

is injective, and its image lies in $\mathrm{Aut}^*([\pi_1(S)])$.

Proof. Let $\tilde\varphi$ be a lifting of φ to $\pi_1(T)$ in (4.2.4). If r is the order of φ, then $\tilde\varphi^r \in \pi_1(S)$, so every element of $\pi_1(T)$ can be written $s\tilde\varphi^m$ with $s \in \pi_1(S)$ and $0 \le m \le r-1$. Suppose that $s\tilde\varphi^m$ maps to the identity in (4.2.5). If $m = 0$, then $s = 1$ because the restriction of (4.2.5) to $\pi_1(S)$ is injective, since $\pi_1(S)$ is center-free. If $1 \le m \le r-1$, then the image of $s\tilde\varphi^m$ under the induced homomorphism

$$\pi_1(T) \to \mathrm{Aut}([\pi_1(S)])/\mathrm{Inn}(\pi_1(S))$$

is equal to φ^m, which is non-trivial since $1 \leq m \leq r-1$, so $s\tilde{\varphi}^m$ cannot lie in the kernel of (4.2.5). To see that the image of (4.2.5) lies in $\mathrm{Aut}^*([\pi_1(S)])$, note that $\pi_1(T)$ is generated by $\pi_1(S)$ and $\tilde{\varphi}$, and the image of $\pi_1(S)$ certainly lies in $\mathrm{Aut}^*([\pi_1(S)])$. But so does the image of $\tilde{\varphi}$, since its reduction modulo $\pi_1(S)$ (identified with $\mathrm{Inn}(\pi_1(S))$) is equal to φ, which is in $\Gamma([S]) = \mathrm{Out}^*([\pi_1(S)])$. $\diamondsuit$

Lemma 4.2.2. *Let φ be an element of finite order in $\Gamma([S])$ and $T = S/\varphi$ as usual, so $\pi_1(S) \subset \pi_1(T)$ as in (4.2.4). If an automorphism of $\pi_1(S)$ extends to an automorphism of $\pi_1(T)$, then it extends uniquely.*

Proof. Let r denote the order of φ, and let $\tilde{\psi}$ be an automorphism of $\pi_1(S)$ extending to $\pi_1(T)$. Suppose that Ψ and Φ are two extensions of $\tilde{\psi}$ to automorphisms of $\pi_1(T)$, and set $\chi = \Psi \circ \Phi^{-1}$, so that χ is an automorphism of $\pi_1(T)$ which restricts to the trivial automorphism on the subgroup $\pi_1(S)$. Let $\tilde{\varphi}$ be a lifting of φ to $\pi_1(T)$, so $\pi_1(T)$ is generated by $\pi_1(S)$ and $\tilde{\varphi}$. Then $\chi(\tilde{\varphi}) = s\tilde{\varphi}^m$ for some $s \in \pi_1(S)$ and some m with $0 \leq m \leq r-1$. For all $s' \in \pi_1(S)$, we have $\tilde{\varphi}^{-1}s'\tilde{\varphi} \in \pi_1(S)$ since $\pi_1(S)$ is normal in $\pi_1(T)$, but by assumption, χ fixes elements of $\pi_1(S)$, so

$$\tilde{\varphi}^{-1}s'\tilde{\varphi} = \chi(\tilde{\varphi}^{-1}s'\tilde{\varphi}) = \chi(\tilde{\varphi}^{-1})s'\chi(\tilde{\varphi}) = \tilde{\varphi}^{-m}s^{-1}s's\tilde{\varphi}^m.$$

Thus, $s\tilde{\varphi}^{m-1}$ commutes with s' for all $s' \in \pi_1(S)$, so $s\tilde{\varphi}^{m-1}$ lies in the kernel of the homomorphism (4.2.5). Thus by lemma 4.2.1, $s\tilde{\varphi}^{m-1} = 1$, so $s = 1$ and $m - 1 = 0$. This means that $\chi(\tilde{\varphi}) = \tilde{\varphi}$, i.e. χ is the identity on all of $\pi_1(T)$. $\diamondsuit$

Lemma 4.2.3. *Let φ be a finite-order element of $\Gamma([S])$. Then the group $\mathrm{Norm}_{\Gamma([S])}(\varphi)$ is exactly the subgroup of elements ψ of $\Gamma([S]) = \mathrm{Out}^*([\pi_1(s)])$ whose liftings to automorphisms $\tilde{\psi} \in \mathrm{Aut}^*([\pi_1(S)])$ extend to automorphisms of $\pi_1(T)$.*

Proof. Let r be the order of φ in $\Gamma([S])$, and let $\tilde{\varphi}$ be a lifting of φ to $\pi_1(T)$ in (4.2.4); we consider $\tilde{\varphi}$ in $\mathrm{Aut}^*([\pi_1(S)])$ via the inclusion

$$\pi_1(T) \xrightarrow{\sim} \mathrm{Inn}(\pi_1(T)) \subset \mathrm{Aut}^*([\pi_1(S)])$$

of lemma 4.2.1. Let $\psi \in \mathrm{Norm}_{\Gamma([S])}(\varphi)$, so that $\psi\varphi\psi^{-1} = \varphi^m$ for some m. Let $\tilde{\psi}$ be an arbitrary lifting of ψ to $\mathrm{Aut}^*([\pi_1(S)])$; then there exists $s \in \pi_1(S)$ such that

$$\tilde{\psi}\tilde{\varphi}\tilde{\psi}^{-1} = s\tilde{\varphi}^m, \tag{4.2.6}$$

where s is identified with $\mathrm{inn}(s)$ in $\mathrm{Aut}^*([\pi_1(S)])$.

The action of the automorphism $\tilde{\psi}$ on $\pi_1(S)$ is recovered by conjugating the copy of $\pi_1(S)$ inside $\mathrm{Aut}^*([\pi_1(S)])$, identified with $\mathrm{Inn}(\pi_1(S))$, by $\tilde{\psi}$ inside $\mathrm{Aut}^*([\pi_1(S)])$. Indeed, for every $s, s' \in \pi_1(S)$, we have

$$\begin{aligned}(\tilde{\psi}\,\mathrm{inn}(s)\tilde{\psi}^{-1})(s') &= (\tilde{\psi}\,\mathrm{inn}(s))(\tilde{\psi}^{-1}(s')) = \tilde{\psi}\big(s\tilde{\psi}^{-1}(s')s^{-1}\big)\\ &= \tilde{\psi}(s)s'\tilde{\psi}(s)^{-1} = \mathrm{inn}(\tilde{\psi}(s))(s').\end{aligned}$$

Furthermore, (4.2.6) shows that the automorphism $\tilde{\psi}$ of $\pi_1(S)$ (given by conjugation by $\tilde{\psi}$ inside $\mathrm{Aut}^*([\pi_1(S)])$) preserves $\pi_1(T)$ since $\pi_1(T)$ is generated by $\pi_1(S)$ and $\tilde{\varphi}$, so that it gives an automorphism of $\pi_1(T)$. This proves one direction of the lemma.

Now, we take $\psi \in \Gamma([S]) = \mathrm{Out}^*([\pi_1(S)])$ such that any lifting $\tilde{\psi}$ of ψ to an automorphism of $\pi_1(S)$ extends to an automorphism of $\pi_1(T)$ (note that if one lifting extends then they all do since they differ by elements of $\mathrm{Inn}(\pi_1(S))$). Then since $\pi_1(T)$ is generated by $\pi_1(S)$ and a lifting $\tilde{\varphi}$ of φ to $\pi_1(T)$, we must have $\tilde{\psi}(\tilde{\varphi}) = \tilde{\psi}\tilde{\varphi}\tilde{\psi}^{-1} = s\tilde{\varphi}^m$ for some m and some $s \in \pi_1(S)$. Thus $\psi\varphi\psi^{-1} = \varphi^m$ in $\Gamma([S])$. Since conjugation by $\tilde{\psi}$ is an automorphism, we find that m must be relatively prime to r, so ψ normalizes φ in $\Gamma([S])$. $\diamondsuit$

Definition. Let $\mathrm{Aut}^*([S/T])$ denote the subgroup of $\mathrm{Aut}^*([\pi_1(T)])$ consisting of elements which preserve the subgroup $\pi_1(S) \subset \pi_1(T)$. This subgroup is of finite index in $\mathrm{Aut}^*(\pi_1(T))$, since $\pi_1(S)$ is of finite index in the finitely generated group $\pi_1(T)$. Restriction from $\pi_1(T)$ to $\pi_1(S)$ gives a homomorphism

$$\mathrm{Aut}^*([S/T]) \to \mathrm{Aut}([\pi_1(S)]).$$

Indeed, we easily see that this restriction actually gives a homomorphism

$$\mathrm{Aut}^*([S/T]) \to \mathrm{Aut}^*([\pi_1(S)]), \tag{4.2.7}$$

since because S is a topological cover of T, any loop surrounding a marked point on S must be a conjugate of a power of a loop surrounding the corresponding marked point of T, so the restriction of an element of $\mathrm{Aut}^*([S/T])$ must then send it to a conjugate of another such loop.

Lemma 4.2.4. *The homomorphism*

$$\mathrm{Aut}^*([S/T]) \to \mathrm{Aut}^*([\pi_1(S)])$$

of (4.2.7) is injective.

Proof. If $\psi \in \mathrm{Aut}^*([S/T])$ lies in the kernel, then it is an element of $\mathrm{Aut}^*([\pi_1(T)])$ which acts like the identity on the subgroup $\pi_1(S)$. Lemma 4.2.2 then shows that ψ is the identity on $\pi_1(T)$. $\diamondsuit$

Lemma 4.2.5. $\mathrm{Aut}^*([S/T])$ *can be identified with the subgroup of* $\mathrm{Aut}^*([\pi_1(S)])$ *of automorphisms of* $\pi_1(S)$ *which extend to* $\pi_1(T)$.

Proof. The previous lemma shows that we can consider $\mathrm{Aut}^*([S/T])$ as a subgroup of $\mathrm{Aut}^*([\pi_1(S)])$, and clearly every element of this subgroup is an automorphism of $\pi_1(S)$ which extends to $\pi_1(T)$. Therefore, we only need to show the converse, i.e. that if

$\psi \in \mathrm{Aut}^*([\pi_1(S)])$ is an automorphism of $\pi_1(S)$ which extends to an automorphism of $\pi_1(T)$, then it lies in $\mathrm{Aut}^*([S/T])$. This means that we must show that if an element $\psi \in \mathrm{Aut}^*([\pi_1(S)])$ extends to $\pi_1(T)$ at all, then it extends to an inertia-preserving automorphism of $\pi_1(T)$, i.e. it gives an element of $\mathrm{Aut}^*([\pi_1(T)])$.

By lemma 4.2.2, if ψ extends to an automorphism of $\pi_1(T)$, then it does so uniquely. Let Ψ be the unique extension of ψ to $\pi_1(T)$; we need to show that it lies in $\mathrm{Aut}^*([\pi_1(T)])$.

Recall that we assume that all branch points of the cover $S \to T$ are marked points on T, images of marked points of S. Let the n marked points of S be labeled $(x_1, \ldots, x_n)$ and the n' marked points of T be labeled $(y_1, \ldots, y_{n'})$. Let $c_1, \ldots, c_{n'}$ be loops around $y_1, \ldots, y_{n'}$ forming part of a generating system of $\pi_1(T)$ as in (4.2.1), and let $e_1, \ldots, e_n$ be analogous loops around $x_1, \ldots, x_n$ on S. We need to show that for every $k \in \{1, \ldots, n'\}$, $\Psi(c_k)$ is a conjugate of c_l for some $l \in \{1, \ldots, n'\}$. Choose k, and choose an $i \in \{1, \ldots, n\}$ such that the marked point $x_i \in S$ lies over y_k. The restriction of Ψ to the subgroup $\pi_1(S)$ is equal to $\psi \in \mathrm{Aut}^*([\pi_1(S)])$, so we have $\Psi(e_i) = ae_ja^{-1}$ for some $j \in \{1, \ldots, n\}$. Let y_l be the image in T of the marked point $x_j \in S$. Then the loop $e_i \in \pi_1(S)$, seen as an element of $\pi_1(T)$ via $\pi_1(S) \subset \pi_1(T)$, is a conjugate of a power of c_k, say $e_i = bc_k^r b^{-1}$, where r is the ramification index of y_k. Similarly, the loop e_j is a conjugate of a power of c_l, say $e_j = dc_l^s d^{-1}$ where s is the ramification index of y_l. Thus

$$\Psi(e_i) = ae_ja^{-1}$$

can be written as

$$\Psi(bc_k^r b^{-1}) = \Psi(b)\Psi(c_k)^r\Psi(b)^{-1} = adc_l^s d^{-1}a^{-1},$$

so

$$\Psi(c_k)^r = \Psi(b)^{-1}adc_l^s d^{-1}a^{-1}\Psi(b).$$

Now, $\pi_1(T)$ is a free group on finitely many generators, with no torsion, and this means that if x and y are such that $x^r = y^s$ in this group, then there exists an element z such that $x = z^a$ and $y = z^b$ with $ra = sb$. Taking $x = \Psi(c_k)$ and $y = \Psi(b)^{-1}adc_l d^{-1}a^{-1}\Psi(b)$, this shows that there exists $z \in \pi_1(T)$ and integers a, b with $ra = sb$, such that

$$\Psi(c_k) = z^a \quad \text{and} \quad \Psi(b)^{-1}adc_l d^{-1}a^{-1}\Psi(b) = z^b.$$

The right hand element is a simple loop around the point y_l of T, and as (the conjugate of) a fundamental generator of $\pi_1(T)$, it cannot be a non-trivial power of any z, so $b = 1$. Thus

$$\Psi(c_k) \sim c_l^a.$$

Let Φ be the inverse of Ψ, and set $w = \Phi(c_l)$. Then $\Phi(c_l^a) = w^a \sim c_k$, which means that c_k is the a-th power of a conjugate of w, so since again c_k is a fundamental generator of $\pi_1(T)$, we must have $a = 1$ and $\Psi(c_k) \sim c_l$. This shows that $\Psi \in \mathrm{Aut}^*([\pi_1(T)])$. $\diamondsuit$

Lemma 4.2.6. *Let $T = S/\varphi$ for a finite-order element $\varphi \in \Gamma([S])$ as usual. Set*

$$\begin{cases} A = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(S)) \subset \Gamma([S]) \\ B = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(T)) \subset \Gamma([T]). \end{cases}$$

Then $\varphi \in A$ and $\langle\varphi\rangle$ is the kernel of the natural surjection

$$g : A \to B. \tag{4.2.8}$$

Remark. Note that by lemmas 4.2.3 and 4.2.5, $A = \mathrm{Norm}_{\Gamma([S])}(\varphi)$.

Proof. Let $\tilde{\varphi}$ be a lifting of φ to $\pi_1(T)$. Then by lemmas 4.2.1 and 4.2.4, we have inclusions

$$\pi_1(T) \subset \mathrm{Aut}^*([S/T]) \subset \mathrm{Aut}^*([\pi_1(S)]),$$

and $\tilde{\varphi} \in \pi_1(T)$, so $\varphi \in \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(S)) = A \subset \Gamma([S])$. Now, let ψ be in the kernel of g, and let $\tilde{\psi}$ be a lifting of ψ to $\mathrm{Aut}^*([S/T])$. Then since ψ is in the kernel of g, we must have $\tilde{\psi} = \mathrm{inn}(t)$ for some $t \in \pi_1(T)$, and we can write $t = s\tilde{\varphi}^m$ for some $0 \le m \le r-1$ and some $s \in \pi_1(S)$, i.e.

$$\mathrm{inn}(t) = \tilde{\psi} = \mathrm{inn}(s\tilde{\varphi}^m) \quad \text{in} \quad \mathrm{Inn}(\pi_1(T)) \subset \mathrm{Aut}^*([S/T]) \subset \mathrm{Aut}^*([\pi_1(S)]). \tag{4.2.9}$$

Therefore,

$$\psi = \varphi^m \ \text{ in } \ A = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(S)) \subset \mathrm{Out}^*([\pi_1(S)]) = \Gamma([S]).$$

This shows that the kernel of g lies inside the group $\langle\varphi\rangle \subset A$. To show that it is equal to $\langle\varphi\rangle$, it suffices to check that φ itself lies in the kernel. But that is of course true, since $\tilde{\varphi} \in \pi_1(T) \subset \mathrm{Aut}^*([S/T])$, so we have

$$\begin{array}{ccccc} \mathrm{Aut}^*([S/T]) & \to & A = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(S)) & \to & B = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(T)) \\ \tilde{\varphi} & \mapsto & \varphi & \mapsto & 1. \end{array}$$

This proves the lemma. $\diamondsuit$

We introduce the following notation for the group B of lemma 4.2.6. Set

$$\Gamma_{[S/T]} = \mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(T)) \subset \mathrm{Out}^*([\pi_1(T)]) = \Gamma([T]), \tag{4.2.10}$$

and

$$\Gamma_{S/T} = \mathrm{Aut}^*(S/T)/\mathrm{Inn}(\pi_1(T)) = \Gamma_{[S/T]} \cap \Gamma(T) \subset \Gamma(T). \tag{4.2.11}$$

Proposition 4.2.7. *Let G_φ be the subgroup of S_n generated by the disjoint cycles of the permutation $[\varphi]$ associated to φ, and let $\Gamma_\varphi(S)$ be the subgroup of $\Gamma([S])$ which is the preimage of G_φ under the surjection $\Gamma([S]) \to S_n$. We have the isomorphisms*

$$\mathrm{Norm}_{\Gamma([S])}(\varphi)/\langle\varphi\rangle \xrightarrow{\sim} \Gamma_{[S/T]} \subset \mathrm{Out}^*([\pi_1(T)]) = \Gamma([T]). \tag{4.2.12}$$

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \xrightarrow{\sim} \Gamma_{S/T} \subset \mathrm{Out}^*(\pi_1(T)) = \Gamma(T). \tag{4.2.13}$$

Furthermore, the subgroups $\Gamma_{[S/T]}$ and $\Gamma_{S/T}$ are of finite index in $\Gamma([T])$ and $\Gamma(T)$, respectively.

Proof. By lemmas 4.2.3 and 4.2.5, the subgroup $A \subset \Gamma([S])$ is equal to $\mathrm{Norm}_{\Gamma([S])}(\varphi)$, and by lemma 4.2.6, since $\Gamma_{[S/T]}$ is the subgroup B, we obtain an injective homomorphism

$$\overline{g} : A/\langle\varphi\rangle \hookrightarrow B$$

which is exactly (4.2.12). For the second statement, we first observe that the restriction of the injection $\overline{g}$ to the subgroup $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$ produces an image which lies inside $\Gamma_{[S/T]}$, but also inside $\Gamma(T)$. Indeed, let $\psi \in \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$ and let $\tilde{\psi}$ be a lifting of ψ to $\mathrm{Aut}^*([\pi_1(S)])$. Then the permutation associated to ψ (by its action on the loops around marked points of S) lies in G_φ, which means that $\tilde{\psi}$ permutes only loops in $\pi_1(S)$ around points of S which map to a single point in T. Thus the extension of $\tilde{\psi}$ to an automorphism of $\pi_1(T)$ lies in $\mathrm{Aut}^*(\pi_1(T))$, so the image of $\tilde{\psi}$ in $\Gamma([T])$ lies in $\Gamma(T)$.

This shows that the image of $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$ under (4.2.13) lies inside $\Gamma_{S/T}$. But if ϕ is an element of $\Gamma_{S/T} = \mathrm{Aut}^*(S/T)/\mathrm{Inn}(\pi_1(T))$ and Φ is a lifting of ϕ to $\mathrm{Aut}^*(S/T)$, then Φ extends to $\pi_1(T)$, so the image of Φ in $\mathrm{Aut}^*(S/T)/\mathrm{Inn}(\pi_1(S))$ lies in $\mathrm{Norm}_{\Gamma([S])}(\varphi)$. Since its associated permutation as an inertia-preserving automorphism of T is trivial, its associated permutation as an inertia-preserving automorphism of $\pi_1(S)$ must lie in G_φ, so it in fact lies in $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$, which proves (4.2.13).

For the last statement, we recall that $\mathrm{Aut}^*([S/T])$ is of finite index in $\mathrm{Aut}^*([\pi_1(T)])$ since $\pi_1(S)$ is of finite index in $\pi_1(T)$ (cf. the definition of $\mathrm{Aut}^*([S/T])$), and $\mathrm{Aut}^*(S/T)$ is of finite index in $\mathrm{Aut}^*(\pi_1(T))$. Thus $\mathrm{Aut}^*([S/T])/\mathrm{Inn}(\pi_1(T)) = \Gamma_{[S/T]}$ is of finite index in $\mathrm{Aut}^*([\pi_1(T)])/\mathrm{Inn}(\pi_1(T)) = \Gamma([T])$ and $\mathrm{Aut}^*(S/T)/\mathrm{Inn}(\pi_1(T))$ is of finite index in $\mathrm{Aut}^*(\pi_1(T))/\mathrm{Inn}(\pi_1(T)) = \Gamma(T)$. This concludes the proof. ♢

Definition. (1) We say that a finite-order element $\varphi \in \Gamma([S])$ satisfies the *surjectivity condition* if $\Gamma_{S/T} = \Gamma(T)$, i.e. every element of $\mathrm{Aut}^*(\pi_1(T))$ preserves the subgroup $\pi_1(S)$, in other words the homomorphism (4.2.13)

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \to \Gamma(T)$$

is surjective. In geometric terms, this means that the covering map

$$\widetilde{\mathcal{M}}_\varphi(S,\varphi) \to \mathcal{M}(T)$$

of (4.1.3) is one-to-one, consisting only in forgetting the orbifold structure of $\widetilde{\mathcal{M}}_\varphi(S,\varphi)$ due to the action of φ.

(2) We say that φ satisfies the *splitting condition* if the surjection

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \to \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \simeq \Gamma_{S/T}$$

splits; in other words, if we have a semi-direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle\varphi\rangle \rtimes \Gamma_{S/T}. \tag{4.2.14}$$

Whenever we have $T = S/\varphi$, where $\varphi \in \Gamma([S])$ satisfies both the surjectivity and the splitting conditions, we have

$$\langle\varphi\rangle \rtimes \Gamma(T) \simeq \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \subset \Gamma([S]);$$

for each choice of splitting, we can thus define (non-canonical) *special homomorphisms*

$$\Gamma(T) \to \Gamma([S]). \tag{4.2.15}$$

In geometric terms, this means that $\widetilde{\mathcal{M}}_\varphi(S,\varphi)$ is as close to $\mathcal{M}(T)$ as possible, in the following sense. The morphism (4.1.3) is given by the "forgetting the action of $\langle\varphi\rangle$" map

$$\mathcal{T}(T)/\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \to \mathcal{T}(T)/\big(\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle\big)$$

of (4.1.5). A point of the left-hand space can be represented by a pair (t,h) where t is the image of a point in $\mathcal{T}(T)$ and $h \in \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$, whereas a point of the right-hand space can be represented as $(t,\bar{h})$ where $\bar{h} \in \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle$. With the splitting condition, the action of $\langle\varphi\rangle$ can be defined on $\mathcal{M}(T)$, which is the right-hand space, by $\varphi(t,\bar{h}) = (t,\varphi\cdot\bar{h})$, where $\varphi\cdot\bar{h} \in \langle\varphi\rangle \rtimes \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \simeq \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$.

Taken together, the splitting and surjectivity conditions mean that (4.1.3) is as close as possible to an isomorphism given the difference of orbifold structures. In §4.3 we will show that when S is of genus zero, every finite-order element $\varphi \in \Gamma([S])$ satisfies both the splitting and the surjectivity conditions.

§4.3. The case of genus zero

In the case of the genus zero moduli spaces, the results of the previous section become simpler and more explicit; not only are the splitting condition always satisfied (theorem 4.3.1) as well as the surjectivity condition (theorem 4.3.2), but the semi-direct product (4.2.14) is always a direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \simeq \langle\varphi\rangle \times \Gamma(T),$$

and an algorithm allows us to compute the splitting morphism $\Gamma(T) \to \Gamma([S])$ explicitly (proof of theorem 4.3.2), so that we actually obtain a method for computing normalizers of finite-order elements in genus zero mapping class groups.

To begin with, we recall from §3.5 that if S is a sphere with n marked points, then a finite-order element of the mapping class group $\Gamma([S])$ is the class of a rotation around an axis:

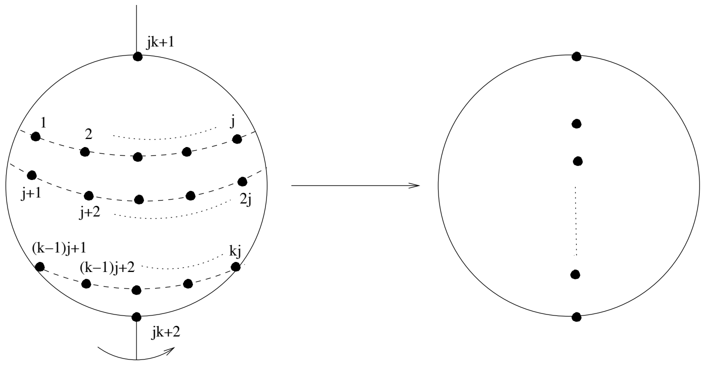


Figure 4.1. A finite-order diffeomorphism in genus zero.

In general, the north and south poles (fixed points of the rotation) may or may not be marked points, but in this article we restrict ourselves to the case where all branch points of T are marked, so we only consider the case where they are both marked. Thus, the permutation associated to φ is always of the form $c_1 \cdots c_k$, where the c_i are disjoint cycles of length j such that $jk = n - 2$.

The *splitting condition* introduced in the previous section has the following strong form in genus zero.

Theorem 4.3.1. *Let S be a sphere with n marked points, and φ a finite-order diffeomorphism of S whose fixed points are marked. Let $T = S/\varphi$, let $[\varphi]$ denote the permutation associated to φ, and let $G_\varphi \subset S_n$ be the group generated by the disjoint cycles of $[\varphi]$. Let $H_\varphi \subset G_\varphi$ be generated by any choice of all but one of the disjoint cycles of $[\varphi]$ ($H_\varphi = \{\mathrm{id}\}$ if $[\varphi]$ is a single cycle). Then we have the direct product isomorphism*

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) = \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi) \times \langle\varphi\rangle.$$

Proof. Take a subgroup $H_\varphi \subset G_\varphi$ as in the statement of the theorem, and consider the two groups

$$\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi) \subset \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi).$$

We will show that

- the first group does not contain any non-trivial power of φ;
- the second group is generated by the first group and the element φ;
- the first group is normal in the second.

These three facts taken together imply that we have a semi-direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) = \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi) \rtimes \langle\varphi\rangle. \tag{4.3.1}$$

But $\langle\varphi\rangle$ is also normal inside $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$, naturally! Now, let a lie in the normal subgroup $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$; then $\varphi^{-1}a\varphi = b \in \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$. But we also have $a\varphi a^{-1} = \varphi^m$, since a normalises φ. Thus

$$\varphi^{-1}a\varphi a^{-1} = ba^{-1} = \varphi^{-1}\varphi^m,$$

so this element lies in the intersection of the two groups in (4.3.1), so it is equal to 1, and we have $a = b$ and $m = -1$. Thus a commutes with φ, so all of $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$ commutes with φ; in fact this normaliser is a centraliser, and the semi-direct product (4.3.1) is a direct product

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) = \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi) \times \langle\varphi\rangle \tag{4.3.2}$$

as in the statement of the theorem. Note that this implies that in fact

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) = \mathrm{Cent}_{\Gamma_\varphi(S)}(\varphi). \tag{4.3.3}$$

It remains only to prove the three points above. To see that $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$ does not contain any power of φ, it suffices to consider the permutations associated to elements of this group; by definition they all lie in H_φ. Now, write $[\varphi] = c_1 \cdots c_k$ as a product of k disjoint cycles c_i each of length j, and assume that H_φ is generated by $c_1, \ldots, c_{k-1}$. Let $\{m_1, \ldots, m_{jk}\} \subset \{1, \ldots, n\}$ be the set of numbers not fixed by the permutation $[\varphi] \in S_n$, i.e. those occurring in the cycles $c_1, \ldots, c_k$. Then none of the m_i are left fixed by any non-trivial power of $[\varphi]$, since the lengths of all the cycles c_i are equal; on the other hand every element of H_φ fixes the numbers m_i occurring in the cycle c_k. This shows that no non-trivial power of $[\varphi]$ lies in H_φ.

For the second point, let $g \in \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$. If its associated permutation $[g]$ lies in H_φ, then by definition $g \in \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$. If $[g]$ is not in H_φ, then since it is in

G_φ, there exists an integer m such that $[g][\varphi]^m \in H_\varphi$, and since $g\varphi^m$ normalises φ, we see that $g\varphi^m \in \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$. This proves the second point. For the third point, it suffices to check that conjugation by φ preserves the subgroup $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$. Let $h \in \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$; we need to show that $g = \varphi h \varphi^{-1} \in \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$. But g certainly normalises φ, and its permutation is equal to the permutation of h since $[\varphi]$ commutes with $[h]$ (the group G_φ is abelian). This completes the proof of theorem 4.3.1. ♡

Next, we show that the surjectivity condition always holds in genus zero.

Theorem 4.3.2. *Let S be a topological sphere with n marked points, and φ a finite-order diffeomorphism of S whose fixed points are marked. Let $T = S/\varphi$. Then the homomorphism*

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \hookrightarrow \Gamma(T),$$

which is injective by proposition 4.2.7, is also surjective.

Proof. We present a rather charming method for giving an explicit lift of each of the generators of $\Gamma(T)$ to $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) \subset \Gamma([S])$. These lifts do not necessarily correspond to a splitting homomorphism, i.e. the lifts do not necessarily satisfy the same relations as the generators of $\Gamma(T)$, but we are only proving surjectivity here, so it is enough to show that each generator has a lift.

We saw (figure 4.1) that φ must be a rotation of permutation $c_1 \cdots c_k$, where each c_i is a cycle of length j, and $jk = n-2$. Then T has $k+2$ marked points, so $\Gamma(T)$ is generated by all but one of the x_{ij} such that $1 \le i < j \le k+1$. It is even more convenient to use the following system of generators. Set $y_{1j} = \sigma_{j-1}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{j-1}$; then via the relation

$$y_{1j} = x_{1j}x_{2j}\cdots x_{j-1,j},$$

we see that all but one of the elements y_{1j} for $1 \le j \le k+1$, together with the x_{ij} for $2 \le i < j \le k+1$, also form a system of generators for $\Gamma(T)$.

Now, S is a sphere with ordered numbered points, so $[\varphi]$ is a fixed permutation in S_n. Since we are working up to conjugation, we may assume that

$$[\varphi] = \big(1, k+1, \cdots, (j-1)k+1\big)\big(2, k+2, \cdots, (j-1)k+2\big)\cdots\big(k, 2k, \cdots jk\big),$$

which is more convenient than the numbering of figure 4.1 for giving a visual representation of the braids. We number the marked points on T by taking the image of the fixed points of φ to be 1 and $k+2$, and the image of the marked point numbered i on S to be numbered $i+1$ on T. Then the generator x_{ij} of $\Gamma(T)$ for $2 \le i < j \le k+1$ is represented by a braid, and by a motion of the points, as follows.

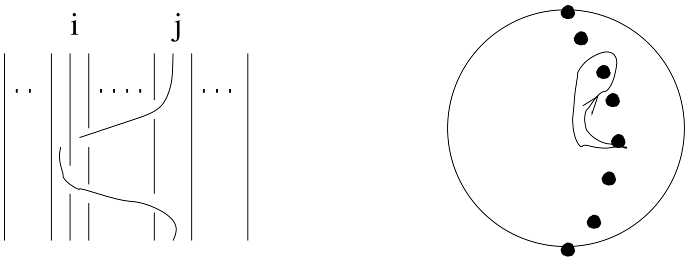


Figure 4.2. The generator x_{ij} of $\Gamma(T)$.

Each of these generators is lifted to a motion of points on S simply by j copies of x_{ij}, as follows,

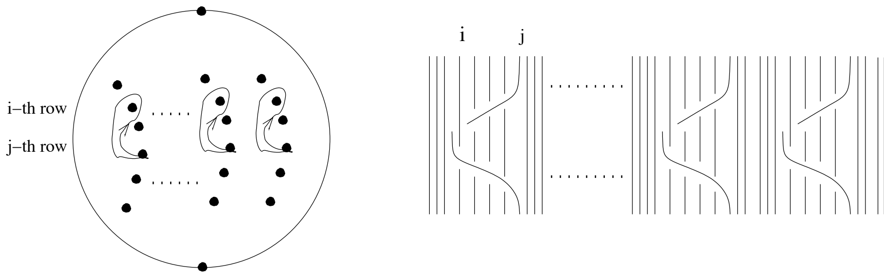


Figure 4.3. Lifting the generator x_{ij} to $\Gamma([S])$.

The element $y_{1j} \in \Gamma(T)$ is shown by the motion of points in the right-hand part of figure 4.4, and lifts to that of the left-hand side.

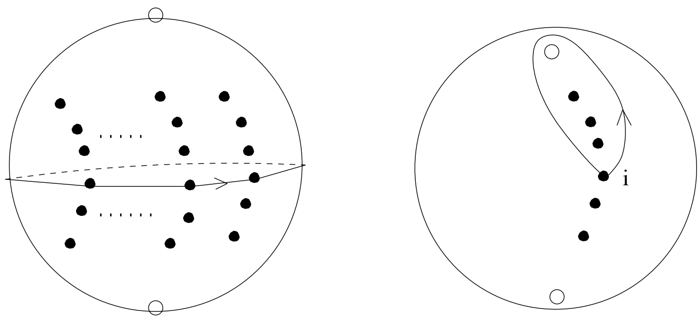


Figure 4.4. Lifting the generator y_{1j} to $\Gamma([S])$.

Having separately lifted each of the y_{1j} and the x_{ij}, we now make a final remark. In themselves, these liftings do not generate a copy of $\Gamma(T)$ inside $\Gamma([S])$. However, we know that it is enough to take all but one of these generators to generate $\Gamma(T)$. We claim that if we throw out any one of the y_{1j} with $2 \leq j \leq k+1$, then the lifts of the remaining generators as above do generate a copy of $\Gamma(T)$ inside of $\Gamma([S])$, and the k different copies obtained in this way are distinct. Indeed, it is easy to check directly that the lifting obtained by throwing out y_{1j} is exactly the subgroup $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$ where H_φ is generated by all of the cycles c_i except for c_j. This proof thus provides an explicit set of generators for the group $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$, namely the generators constructed above for any $\mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi)$, together with φ itself. $\diamondsuit$

The following corollary is an immediate consequence of theorems 4.3.1 and 4.3.2.

Corollary. *Let S be a sphere with n marked points, and φ a finite-order diffeomorphism of S, and set $T = S/\varphi$. Then*

$$\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)/\langle\varphi\rangle \simeq \mathrm{Norm}_{\Gamma_{H_\varphi}(S)}(\varphi) \simeq \Gamma(T),$$

and for each of the k possible choices of $H_\varphi \subset G_\varphi$, we obtain a special homomorphism

$$\Gamma(T) \to \Gamma([S]).$$

§4.4. An example in genus zero

S of type $(0,6)$, T of type $(0,4)$.

The mapping class group $\Gamma([S]) \simeq \Gamma_{0,[6]}$ is generated by σ_1, σ_2, σ_3, σ_4 and σ_5 with the usual braid relations and the relations

$$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5 = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^6 = 1. \tag{4.4.1}$$

The element φ we consider in this example is

$$\varphi = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1.$$

Its associated permutation is (15)(24). Direct computation (manipulation with the braid relations) shows that $\varphi^2 = (\sigma_4\sigma_3\sigma_2\sigma_1)^5$, and this element is equal to 1 in $\Gamma_{0,[6]}$, so φ is of order 2. In fact, φ is the class of the 180^o rotation Φ around the axis shown in the figure below.

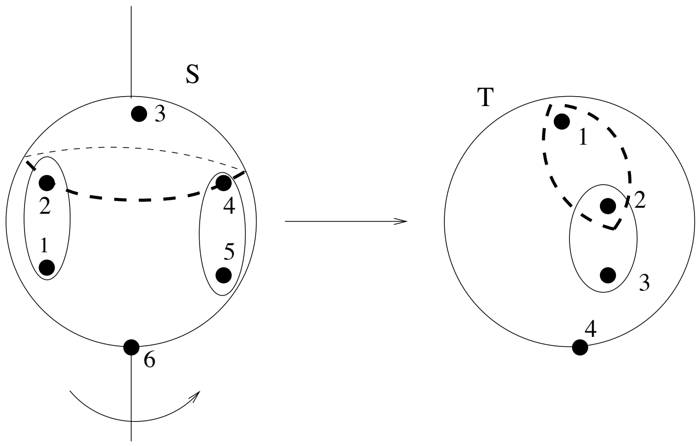


Figure 4.5. The finite-order diffeomorphism Φ lifting φ

The quotient T of S by Φ has 4 marked points coming from the 6 marked points of S; the north and south poles are fixed points of the rotation Φ, so they have ramification index equal to 2.

Let us compute the groups $\mathrm{Aut}^*([\pi_1(T)])$, $\mathrm{Aut}^*([S/T])$, $\Gamma_{[S/T]}$, and their pure versions.

Write

$$\pi_1(T) = \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_3c_4 = 1\rangle,$$

where c_i is a loop around the i-th marked point of T. The group $\mathrm{Aut}^*([\pi_(T)])$ of T is generated by $\mathrm{Inn}(\pi_1(T))$ and by elements τ_1, τ_2 and τ_3, acting via

$$\tau_i(c_j) = \begin{cases} c_{i+1} & \text{if } j = i \\ c_{i+1}^{-1}c_ic_{i+1} & \text{if } j = i+1 \\ c_j & \text{otherwise.} \end{cases} \tag{4.4.2}$$

Now, let us realize $\pi_1(S)$ as a subgroup of $\pi_1(T)$, and compute $\mathrm{Aut}^*([S/T])$. Writing

$$\pi_1(S) = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1x_2x_3x_4x_5 = 1\rangle,$$

then $\pi_1(S)$ can be included in $\pi_1(T)$ via

$$\begin{cases} x_1 = c_1c_2c_3c_2^{-1}c_1^{-1} \\ x_2 = c_1c_2c_1^{-1} \\ x_3 = c_1^2 \\ x_4 = c_2 \\ x_5 = c_3. \end{cases}$$

(To realize the cover $\Phi : S \to T$ geometrically, one can take for example the Riemann surface $\mathbb{P}^1\mathbb{C}$ with ordered marked points $2, 1, 0, -1, -2, \infty$; the loops x_1, x_2, x_3, x_4 and x_5 go around the first five points in order, based at the base point i. Then Φ corresponds to the quotient of this Riemann surface by $z \mapsto z^2$, and the quotient is the projective line $\mathbb{P}^1\mathbb{C}$ with marked points 4, 1, 0 and ∞, surrounded by loops c_1, c_2, c_3, c_4 based at -1.)

Now, in order to have a chance of restricting to $\pi_1(S)$, i.e. of belonging to $\mathrm{Aut}^*([S/T])$, an element of $\mathrm{Aut}^*([\pi_1(T)])$ cannot permute marked points of T having different ramification indices under φ. Therefore, the primary candidate for the subgroup $\mathrm{Aut}^*([S/T])$ is the subgroup of all elements of $\mathrm{Aut}^*([\pi_1(T)])$ whose permutations lie in the subgroup $\langle(14),(23)\rangle \subset S_4$. This group is generated by a lifting of each of the two transpositions, together with the whole of the pure subgroup $\mathrm{Aut}^*(\pi_1(T))$. A lift of the transposition (23) is given by τ_2, and a lift of (14) is given by the element $\tau_{14} = \tau_3\tau_2\tau_1\tau_2^{-1}\tau_3^{-1}$, whereas $\mathrm{Aut}^*(\pi_1(T))$ is generated by $\mathrm{Inn}(\pi_1(T))$ and by the six elements τ_1^2, $\tau_2\tau_1^2\tau_2^{-1}$,

$\tau_3\tau_2\tau_1^2\tau_2^{-1}\tau_3^{-1}$, τ_2^2, $\tau_3\tau_2^2\tau_3^{-1}$, τ_3^2. Therefore, the preimage of $\langle(14),(23)\rangle$ is generated in $\mathrm{Aut}^*([\pi_1(T)])$ by

$$\tau_1^2,\quad \tau_2\tau_1^2\tau_2^{-1},\quad \tau_3\tau_2\tau_1\tau_2^{-1}\tau_3^{-1},\quad \tau_2,\quad \tau_3\tau_2^2\tau_3^{-1},\quad \tau_3^2.$$

We can drop $\tau_2\tau_1^2\tau_2^{-1}$ from this set of generators, and since by the braid relation, we have $\tau_3\tau_2^2\tau_3^{-1} = \tau_2^{-1}\tau_3^2\tau_2$, we can also drop this element. Furthermore, $\tau_1^2 = \tau_3^2$ in $\mathrm{Aut}^*(\pi_1(T))$, so we are left with three generators

$$\tau_1^2,\quad \tau_2,\quad \tau_3\tau_2\tau_1\tau_2^{-1}\tau_3^{-1}.$$

Explicit computation shows that each of these three generators restricts to $\pi_1(S)$ as an element of $\mathrm{Aut}^*([\pi_1(S)])$. Indeed, we have

$$\tau_1^2 : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_4 \\ x_3 \mapsto x_4^{-1}x_3x_4 \\ x_4 \mapsto x_4^{-1}x_3^{-1}x_2x_3x_4 \\ x_5 \mapsto x_5 \\ x_6 \mapsto x_6, \end{cases} \qquad \tau_2 : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto x_2^{-1}x_1x_2 \\ x_3 \mapsto x_3 \\ x_4 \mapsto x_5 \\ x_5 \mapsto x_5^{-1}x_4x_5 \\ x_6 \mapsto x_6, \end{cases}$$

$$\tau_3\tau_2\tau_1\tau_2^{-1}\tau_3^{-1} : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_6 \\ x_4 \mapsto (x_1x_2x_3)x_4(x_1x_2x_3)^{-1} \\ x_5 \mapsto (x_1x_2x_3)x_5(x_1x_2x_3)^{-1} \\ x_5 \mapsto (x_1x_2x_3)x_3(x_1x_2x_3)^{-1}. \end{cases}$$

This shows that $\Gamma_{[S/T]} = \langle\tau_1^2, \tau_2, \tau_3\tau_2\tau_1\tau_2^{-1}\tau_3^{-1}\langle$. Since furthermore we have $\Gamma(T) = \langle\tau_1^2, \tau_2^2\rangle$, we see that $\Gamma_{S/T} = \Gamma(T)$, so φ satisfies the surjectivity condition of §4.2; this condition was of course guaranteed anyway in genus zero by theorem 4.3.2.

Let $\sigma_1, \ldots, \sigma_5$ be generators of $\mathrm{Aut}^*([\pi_1(S)])$, acting on the x_j exactly as τ_i acts on c_j in (4.4.2). Then it is easy to see that τ_1^2 acts on $\pi_1(S)$ like $\sigma_2\sigma_3\sigma_2$, and τ_2 acts like $\sigma_1\sigma_4$, so τ_2^2 acts like $\sigma_1^2\sigma_4^2$.

Furthermore, since $\Gamma(T)$ is a free group on the generators τ_1^2 and τ_2^2, we obtain a *special homomorphism*

$$\begin{aligned} \Gamma(T) &\to \Gamma([S]) \\ \tau_1^2 &\mapsto \sigma_2\sigma_3\sigma_2 \\ \tau_2^2 &\mapsto \sigma_1^2\sigma_4^2. \end{aligned} \tag{4.4.3}$$

This is an explicit version of the *splitting condition* (2) of §4.2, guaranteed in genus zero by theorem 4.3.1.

Let us now explain the meaning of these computations in geometric terms, i.e. by determining the points of special automorphism corresponding to φ in $\mathcal{M}(S)$, and the special locus of φ in the quotient moduli space $\mathcal{M}_\varphi(S)$, and relating the image of the splitting map (4.4.3) to the normalizer $\mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi)$.

The permutation $[\varphi] = (15)(24)$, and to find the set of points of special automorphism corresponding to φ, it is enough to parametrize the set of points fixed by (15)(24). Let us use the unique representative of the form $(1, a, 0, b, c, \infty)$ for each point of $\mathcal{M}(S) = \mathcal{M}_{0,6}$. This is odd-looking but ensures that 0 and ∞ correspond to the third and sixth marked points, i.e. the north and south poles of S which are the fixed points under φ (see figure 4.5). Applying (15)(24) gives $(c, b, 0, a, 1, \infty)$ which is equivalent to $(1, b/c, 0, a/c, 1/c, \infty)$ via $z \mapsto z/c$. The point $(1, a, 0, b, c, \infty)$ is a point of special automorphism corresponding to φ if and only if it is fixed under the action of (15)(24), i.e. if $c = -1$ and $a = -b$, so the set of special points is given by all points of the form

$$(1, a, 0, -a, -1, \infty). \tag{4.4.4}$$

This special locus has no orbifold structure since $\mathcal{M}(S)$ does not, and it is normal. It is easily seen to be a copy of $\mathbb{P}^1$ with four points removed, namely those corresponding to the values $a = 0, \pm 1, \infty$.

Let us now consider the special locus $\mathcal{M}_\varphi(S, \varphi)$ inside the quotient moduli space $\mathcal{M}_\varphi(S)$; it is the image of the locus (4.4.4) in the quotient space $\mathcal{M}(S)/G_\varphi$ where $G_\varphi = \langle (15), (24) \rangle$. The permutation (15)(24) fixes each point, whereas (15) and (24) both act via $a \mapsto -a$; thus the points $a = 0$ and $a = \infty$ are fixed under the action of G_φ and the points $a = 1$ and $a = -1$ are exchanged. There are no other fixed points under G_φ. Thus, the special locus we are interested in, $\mathcal{M}_\varphi(S, \varphi)$, is a $\mathbb{P}^1$ with three points removed and a trivial orbifold structure with group $\mathbb{Z}/2\mathbb{Z}$ at each point. This is the geometric structure which is reflected by the fundamental groups since

$$\pi_1(\mathcal{M}_\varphi(S, \varphi)) = \mathrm{Norm}_{\Gamma_\varphi(S)}(\varphi) = \langle \sigma_2\sigma_3\sigma_2, \sigma_1^2\sigma_4^2, \varphi \rangle \simeq \langle \varphi \rangle \times \langle \sigma_2\sigma_3\sigma_2, \sigma_1^2\sigma_4^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times F_2.$$

The $F_2 \simeq \Gamma(T)$ factor in this direct product is the image of the splitting homomorphism (4.4.3).

§4.5. An example in genus one

S of type $(1,2)$, T of type $(1,1)$.

Here we quotient a torus S with 2 marked points by the diffeomorphism φ which is the 180^o rotation around the axis passing vertically through the hole. The quotient T is a torus with one marked point and no branch points.

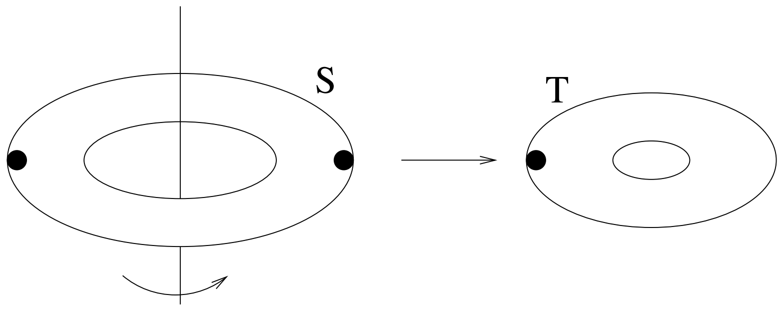


Figure 4.6. A diffeomorphism of type $(1, 2)$

The fundamental group $\pi_1(T)$ has the well-known presentation

$$\langle a, b, c \mid (a, b)c = 1 \rangle,$$

where (a, b) denotes the commutator $bab^{-1}a^{-1}$. The fundamental group $\pi_1(S)$ has the presentation

$$\langle a', b', c_1, c_2 \mid (a', b')c_1c_2 = 1 \rangle.$$

This group can be realized as a subgroup of $\pi_1(T)$ via the embedding

$$\begin{aligned} \pi_1(S) &\hookrightarrow \pi_1(T) \\ a' &\mapsto a \\ b' &\mapsto b^2 \\ c_1 &\mapsto c \\ c_2 &\mapsto bcb^{-1}. \end{aligned}$$

The group $\mathrm{Aut}^*(\pi_1(T))$ is generated by $\mathrm{Inn}(\pi_1(T))$ and by twists τ_1 and τ_2 along the loops α and β in figure 3.13. In the mapping class group $\Gamma(T)$, these twists satisfy the relations $\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_1 = \tilde{\tau}_2\tilde{\tau}_1\tilde{\tau}_2$ and $(\tilde{\tau}_1\tilde{\tau}_2)^3 = 1$, and in fact $\Gamma(T)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$.

Explicitly, the twists τ_1 and τ_2 act on $\pi_1(T)$ via

$$\tau_1(a) = a, \quad \tau_1(b) = ba^{-1}, \quad \tau_1(c) = c$$

and

$$\tau_2(a) = ab, \quad \tau_2(b) = b, \quad \tau_2(c) = c.$$

Since T has only one marked point, we have $\mathrm{Aut}^*([\pi_1(T)]) = \mathrm{Aut}^*(\pi_1(T))$ and $\mathrm{Aut}^*([S/T])$ $= \mathrm{Aut}^*(S/T)$. Let us show that the subgroup $\mathrm{Aut}^*(S/T)$ is given by $\langle \tau_1, \tau_2^2 \rangle$.

We first show that τ_1 lies in $\mathrm{Aut}^*(S/T)$, by computing its restriction to $\pi_1(S)$; we find that

$$\begin{cases} \tau_1(a') = \tau_1(a) = a = a' \\ \tau_1(b') = \tau_1(b^2) = ba^{-1}ba^{-1} = (a')^{-1}c_1b'(a')^{-1} \\ \tau_1(c_1) = \tau_1(c) = c = c_1 \\ \tau_1(c_2) = \tau_1(bcb^{-1}) = ba^{-1}cab^{-1} = b'(a')^{-1}(b')^{-1}c_1^{-1}a', \end{cases}$$

so τ_1 preserves the subgroup $\pi_1(S)$. We next show that τ_2^2 lies in $\mathrm{Aut}^*(S/T)$ by computing its restriction:

$$\begin{cases} \tau_2^2(a') = \tau_2^2(a) = ab^2 = a'b' \\ \tau_2^2(b') = \tau_2^2(b^2) = b^2 = b' \\ \tau_2^2(c_1) = \tau_2^2(c) = c = c_1 \\ \tau_2^2(c_2) = \tau_2^2(bcb^{-1}) = bcb^{-1} = c_2, \end{cases}$$

so again τ_2^2 preserves $\pi_1(S)$, so it lies in $\mathrm{Aut}^*(S/T)$. Now, the subgroup $\langle \tau_1, \tau_2^2 \rangle$ is of index 2 in $\Gamma(T) = \langle \tau_1, \tau_2 \rangle$, so if the subgroup $\mathrm{Aut}^*(S/T)$ was strictly greater than $\langle \tau_1, \tau_2^2 \rangle$, it would be all of $\Gamma(T)$. But this is not the case, since for instance $\tau_2 \notin \mathrm{Aut}^*(S/T)$; indeed, $\tau_2(a) = ab$ and $ab \notin \pi_1(S)$ (otherwise $b \in \pi_1(S)$ and $\pi_1(S) = \pi_1(T)$ which is absurd). Therefore we have shown that

$$\mathrm{Aut}^*(S/T) = \langle \tau_1, \tau_2^2 \rangle.$$

Thus, the surjectivity condition does not hold for this φ.

For the purposes of studying the surjectivity and splitting conditions, we should consider not the mapping class group $\mathrm{Diff}^+([S])/\mathrm{Diff}^0(S)$ but the orbifold fundamental group of the moduli space $\mathcal{M}([S])$. These two groups coincide in general, but in this particular case they differ, as we saw in §3.2. The fundamental group $\pi_1(\mathcal{M}([S])$ is generated by τ_1 and τ_2 subject to the braid relation and to the relation $(\tau_1\tau_2)^6 = 1$; it is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. The above arguments go through and the surjectivity condition naturally still does not hold, however the splitting condition holds. Indeed, τ_1 lifts to the element $\sigma_1\sigma_3$ of $\Gamma([S])$ and τ_2^2 lifts to σ_2. The only relation satisfied by τ_1 and τ_2^2 is $(\tau_1\tau_2^2\tau_1\tau_2^2) = (\tau_1\tau_2)^3 = 1$. So we need to check that $(\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2)^2 = 1$ in $\Gamma([S])$, but this is of course true since $\sigma_1\sigma_3\sigma_2$ is of order four in $\Gamma([S])$ (conjugate by σ_1 to $\sigma_3\sigma_2\sigma_1$).

Let us give the geometric description of the situation. As we saw in §3.4, each point of the moduli space $\mathcal{M}(S)$ of tori with two marked points can be represented uniquely by such a parallelogram, marked with two marked points whose sum is equal to the origin. In other words, a point of $\mathcal{M}(S)$ is determined by a pair (τ, z) of parameters, with τ in the fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ (cf. figure 3.12) defining the elliptic curve, and z a complex number $\neq 0$ in the fundamental parallelogram determined by 1 and τ; the marked points are taken to be z and $-z$ (both considered modulo the period lattice, i.e. in the fundamental parallelogram). The action of φ on the parallelogram is given by $z \mapsto z + \tau/2$. Thus the quotient of the torus by this map identifies the lower and upper half of the parallelogram as in the following figure.

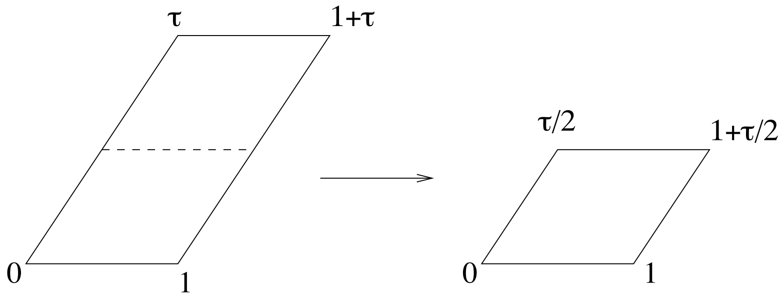


Figure 4.7. Action of φ on a parallelogram

The points of the moduli space $\mathcal{M}(S)$ which are preserved by the diffeomorphism φ, given by $z \mapsto z + \tau/2$, are those such that the two marked points are exchanged by this diffeomorphism, namely $z + \tau/2 = -z$ modulo the lattice $\langle 1, \tau \rangle$. This shows that the special locus of φ is not the whole moduli space, so it is the special locus of §3.4, which as we saw there is isomorphic to the quotient of $\mathbb{P}^1 - \{0, 1, \infty\}$ by $\frac{1}{2}(z^2 + \frac{1}{z^2})$. This corresponds to the isomorphism

$$\pi_1(\mathcal{M}(S, \varphi)) = \langle \tau_1, \tau_2^2 \rangle.$$

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