The Fay relations satisfied by the elliptic associator

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To Moshe Jarden on the occasion of his 80th birthday

Abstract. We recall the construction by B. Enriquez of the elliptic associator $\mathbf{A}_\tau$, a power series in two non-commutative variables a, b defined as an iterated integral of the Kronecker function, and turn our attention to a family of *Fay relations* satisfied by $\mathbf{A}_\tau$, derived from the original well-known Fay relation satisfied by the Kronecker function. The Fay relations of $\mathbf{A}_\tau$ were studied by Broedel, Matthes and Schlotterer, and determined up to non-explicit correction terms that arise from the necessity of regularizing the non-convergent integral. In this article, we work modulo $2\pi i$. Since $\mathbf{A}_\tau$ reduces to 1 mod $2\pi i$, we set $A_\tau = \mathbf{A}_\tau^{1/2\pi i}$, and study the reduction $\bar{A}_\tau$ of A_τ mod $2\pi i$. We recall a different construction of $\bar{A}_\tau$ in three steps, due to Matthes, Lochak and the author: first one defines the reduced *elliptic generating series* $\bar{E}_\tau$ which comes from the reduced Drinfel'd associator $\overline{\Phi}_{KZ}$ and whose coefficients generate the same ring $\bar{R}$ as those of $\bar{A}_\tau$; then one defines Ψ to be the automorphism of the free associative ring $\bar{R}\langle\langle a, b\rangle\rangle$ defined by $\Psi(a) = \bar{E}_\tau$ and $\Psi([a, b]) = [a, b]$; finally one shows that the reduced elliptic associator $\bar{A}_\tau$ is equal to $\Psi\big(\frac{ad(b)}{e^{ad(b)}-1}(e^a)\big)$. Using this construction and mould theory and working with Lie-like versions of the elliptic generating series and associator, we prove the following results: first, a mould satisfies the Fay relations if and only if a closely related mould satisfies the well-known "swap circ-neutrality" relations defining the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$; second, the reduced elliptic generating series $\bar{E}_\tau$ satisfies a family of Fay relations with extremely simple correction terms coming directly from the well-known constants needed to ensure that the Drinfel'd associator satisfies the double shuffle relations; third, the correction terms for the Fay relations satisfied by the reduced elliptic associator $\bar{A}_\tau$ can be deduced explicitly from these.

§0. Introduction

The present article investigates the family of *Fay relations* satisfied by the elliptic associator $\mathbf{A}_\tau$ defined by Enriquez in [E], arising from the definition of $\mathbf{A}_\tau$ as an iterated integral of the classical Kronecker function (cf. §2.1) which satisfies the well-known three-term Fay relation. The series $\mathbf{A}_\tau$ is a group-like power series in two non-commutative variables a and b, with coefficients in a subring R of $\mathcal{O}(\mathcal{H})$ whose explicit description was determined in [LMS] and whose definition is recalled below (cf. §2.2). The Fay relations for the elliptic associator were studied in [BMS]. They are expressed in the notation of Écalle's mould theory, which we now briefly introduce (see [Ec], especially the beginning, for an overall introduction, or alternatively the introductory article [S2] which contains complete proofs).

Fix a base ring R. In this article, the word *mould* will always refer to a rational-function mould, which is a family $A = (A_r)_{r\geq 0}$ such that each A_r lies in the ring of rational functions on an infinite number of commutative variables $u_1, u_2, \ldots$, satisfying the unique property that each A_r is a function of r variables; in particular, $A(\emptyset) \in R$. We generally drop the subscript r and write $A(u_1, \ldots, u_r)$ for A_r, which is called the *depth r part* of the mould A.

A *polynomial mould* is one for which each $A(u_1, \ldots, u_r)$ lies in the subring of polynomials $R[u_1, u_2, \ldots, u_r]$, and a *power series mould* is one for which $A(u_1, \ldots, u_r) \in R[[u_1, \ldots, u_r]]$. The set of moulds forms an R-module by adding moulds and multiplying them by scalars componentwise, i.e. in each depth.

Consider the subspace of power series in $R\langle\langle a, b\rangle\rangle$ that lie in the kernel of the derivation mapping $a \mapsto 1$ and $b \mapsto 0$ (which is the case for all Lie-like and group-like power series and in particular all those considered in this article). These are precisely the power series that can be written as power series in the variables $c_i = ad(a)^{i-1}(b)$ for $i \geq 1$. There is a canonical map ma from these power series to power-series-valued moulds given by linearly extending the map on monomials

$$ma : c_{a_1} \cdots c_{a_r} \mapsto u_1^{a_1-1} \cdots u_r^{a_r-1}.$$

For any mould A, let A' denote the mould defined by

$$A'(u_1,\dots,u_r) = \frac{1}{u_1\cdots u_r}A(u_1,\dots,u_r). \tag{0.1}$$

Define the *Fay operator* $\mathcal{F}$ on a mould B by the formula

$$\mathcal{F}(B)(u_1,\dots,u_r) = B(u_1,\dots,u_r) + B(u_2,\dots,u_r,-\overline{u}_r) + \sum_{i=1}^{r-1} B(u_2,\dots,u_i,-\overline{u}_i,\overline{u}_{i+1},u_{i+2},\dots,u_r)$$

where $\overline{u}_i = u_1+\cdots+u_i$.

Let $\mathbf{A}(\tau) = ma(\mathbf{A}_\tau)$ be the polynomial mould associated to the elliptic associator $\mathbf{A}_\tau \in R\langle\langle a,b\rangle\rangle$. The family of Fay relations satisfied by $\mathbf{A}_\tau$ and studied in [BMS] can be written in the following form: for each $r\geq 2$, we have an equality of the form

$$\mathcal{F}\big(A'(\tau)\big)(u_1,\dots,u_r) = \text{an undetermined correction term in lower depth,}$$

where "lower depth" indicates that the while the correction term is of course a mould in the same depth r as the left-hand side, it consists of a polynomial expression in lower-depth parts $A'(\tau)(u_1,\dots,u_s)$ with $s<r$ and linear terms in $u_1,\dots,u_r$.

Generally speaking, we will say that a mould B satisfies the *strict Fay relations* if $\mathcal{F}(B')=0$, or *corrected Fay relations* if there is a mould $C_{B'}$ such that $\mathcal{F}(B') = C_{B'}$ and in each depth r, $C_{B'}$ is a sum of linear terms in the u_i and polynomial expressions in the parts of B' of depth less than r. Thus the main result of [BMS] states that the elliptic associator $\mathbf{A}_\tau$ satisfies a corrected family of Fay relations, although the correction terms are not explicitly known.

In the present article we use mould theory to give a different interpretation of how the Fay relations arise, which allows the explicit determination of the correction terms. There is however one caveat, which is that we can only apply the main results of mould theory to this situation on the condition of working modulo $2\pi i$, in the following sense. The ring $R\subset\mathcal{O}(\mathcal{H})$ generated by the coefficients of the power series $\mathbf{A}_\tau$ contains the element $2\pi i$ (viewed as a constant function on $\mathcal{H}$). The quotient ring $R' = R/\langle 2\pi i\rangle$ is non-trivial; its structure was determined completely in [LMS] and is recalled in §2.2 below. Unfortunately, it so happens that reducing the coefficients of $\mathbf{A}_\tau$ from R to $\bar{R}$ gives zero, however this can be rectified by showing that the power series $A_\tau := \mathbf{A}_\tau^{1/2\pi i}$ also has coefficients in R, but its reduction $\bar{A}_\tau$ modulo $2\pi i$ is highly non-trivial.

The main focus of this article is to determine the explicit family of Fay relations satisfied by the *reduced elliptic associator* $\bar{A}_\tau$. On the way there, we deduce the explicit (and much simpler) Fay relations satisfied by the *reduced elliptic generating series* $\bar{E}_\tau$ (cf. Def. 2.5), and we prove the close relationship between the Fay relations in the linearized context and the defining relations of the elliptic Kashiwara-Vergne Lie algebra.

Outline of the article.

In §1, we recall the elements of mould theory necessary to define the strict and corrected Fay relations, and establish the following mould-theoretic statement (Corollary of Theorem 1.3):

Theorem A. *The space of all polynomial alternal push-invariant moulds A satisfying a family of corrected Fay relations of the form*

$$\mathcal{F}(A')(u_1,\dots,u_r) = k_r(u_2+\cdots+u_r)$$

for some set of constants $(k_r)_{r\geq 2}$ is isomorphic to the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$.

This theorem will be crucial in determining the Fay relations of the Lie-like elliptic generating series.

In §2, we recall the definition and construction of the reduced elliptic generating series and elliptic associator. Let $\mathcal{Z}$ denote the $\mathbb{Q}$-algebra of multiple zeta values, and $\overline{\mathcal{Z}}$ its quotient modulo the ideal generated by $\zeta(2)$. Let $\overline{\Phi}_{KZ}\in\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$ denote the Drinfel'd associator reduced mod $\zeta(2)$. This power series lies in the pro-unipotent Grothendieck-Teichmüller group $GRT(\overline{\mathcal{Z}})$. Enriquez in [En1] defined an elliptic version GRT_{ell} with a canonical surjection to GRT, and he also defined a section map

$$\Gamma : GRT \to GRT_{ell}.$$

In [En2], he defined a certain key automorphism g_τ of the power series ring $\mathcal{O}(\mathcal{H})\langle\langle a,b\rangle\rangle$. Let us briefly recall some of the main facts from [LMS]. Let $\mathcal{E}^{\mathrm{geom}}$ denote the subring of $\mathcal{O}(\mathcal{H})$ generated by the coefficients of g_τ (cf. §2.1 below for the precise definition of these coefficients). It was shown in [LMS] that one can form the tensor product

$$R := \mathcal{Z}[2\pi i] \otimes_{\mathbb{Q}} \mathcal{E}^{\mathrm{geom}} \subset \mathcal{O}(\mathcal{H})$$

(cf. Thm. 2.6 of [LMS]; here elements of $\mathcal{Z}[2\pi i]$ are considered as constant functions on $\mathcal{H}$), that the reduction $\bar{R}$ of R modulo the ideal generated by $2\pi i$ is isomorphic to $\overline{\mathcal{Z}} \otimes_{\mathbb{Q}} \mathcal{E}^{\mathrm{geom}}$, and that the coefficients of $\bar{A}_\tau$ (along with the element $2\pi i\tau \otimes 1$) generate $\bar{R}$ (cf. Thm. 3.6 of [LMS]). The automorphism g_τ restricts to an automorphism of $R\langle\langle a,b\rangle\rangle$ that passes to an automorphism of $\bar{R}\langle\langle a,b\rangle\rangle$.

The reduced elliptic generating series comes in three flavors. First we have the *reduced group-like elliptic generating series*, defined by

$$\bar{C}_\tau := g_\tau\Big(\Gamma(\overline{\Phi}_{KZ})\Big) \in \bar{R}\langle\langle a,b\rangle\rangle. \tag{0.2}$$

It was shown in [LMS] that there exists a unique automorphism Ψ_{KZ} of $\bar{R}\langle\langle a,b\rangle\rangle$ such that $\Psi_{KZ}(e^a) = \Gamma(\overline{\Phi}_{KZ})$ and $\Psi_{KZ}(e^{[a,b]}) = e^{[a,b]}$. Let $\Psi = g_\tau \circ \Psi_{KZ}$, so that $\bar{C}_\tau = \Psi(e^a)$. Both automorphisms Ψ_{KZ} and Ψ lie in the group $GRT_{ell}(\bar{R})$. We also define the *reduced elliptic generating series*

$$\bar{E}_\tau := \Psi(a) = \log \bar{C}_\tau.$$

Finally, we let the *reduced Lie-like elliptic generating series* be defined by $\bar{\mathbf{e}}_\tau = \psi(a)$ where ψ is the derivation $\log(\Psi)$ of $\bar{R}\langle\langle a,b\rangle\rangle$. The power series $\bar{\mathbf{e}}_\tau$ represents a *double linearization* of $\bar{C}_\tau$, first by passing to the log of the series, and then by passing to the log of the automorphism.

Let t_{01} be the power series defined by

$$t_{01} = \frac{ad(b)}{e^{ad(b)}-1}(a) \in \bar{R}\langle\langle a,b\rangle\rangle. \tag{0.3}$$

One of the main results of [LMS] is the following alternative construction of the reduced elliptic associator $\bar{A}_\tau$:

$$\bar{A}_\tau = \Psi(e^{t_{01}}).$$

We conclude §2 by defining the *reduced Lie-like elliptic associator* $\bar{\mathbf{a}}_\tau := \psi(t_{01})$, which is obtained from $\bar{A}_\tau$ by the same double linearization as above with $\bar{C}_\tau$, $\bar{E}_\tau$ and $\bar{\mathbf{e}}_\tau$, first passing from $e^{t_{01}}$ to t_{01}, i.e. taking $\log\bar{A}_\tau = \Psi(t_{01})$, and then setting $\bar{\mathbf{a}}_\tau = \psi(t_{01})$ where $\psi = \log(\Psi)$ as above.

Finally, §3 is devoted to the Fay relations. To determine the explicit Fay relations satisfied by $\bar{A}_\tau$, we proceed as follows. We first explicitly determine the Fay relations satisfied by the Lie-like series $\bar{\mathbf{e}}_\tau$, which are particularly simple, with correction terms closely related to the Drinfel'd associator. We then use this to explicitly determine the Fay relations satisfied by $\bar{\mathbf{a}}_\tau$. Finally, we show in §3.3 that the group-like Fay relations for $\bar{A}_\tau$ can be deduced from the Lie-like Fay relations for $\bar{\mathbf{a}}_\tau$ directly and explicitly, although naturally they appear much more complicated.

More precisely, the main results of §3.1 and §3.2 are stated in the following theorem, which regroups the statements of Theorems 3.1 and 3.4. Let $C = (c_r)_{r\geq 0}$ be the well-known constant correction mould associated to the Drinfel'd associator $\overline{\Phi}_{KZ}$ (by which the associator must be modified to ensure that it satisfies the double shuffle relations), namely

$$c_r = \begin{cases} \frac{\zeta(r)}{r} & \text{for } r \geq 3 \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{0.4}$$

and let $T = ma(t_{01} + a + \frac{1}{2}[a,b])$ where t_{01} is defined in (0.2).

Theorem B. *The reduced Lie-like elliptic generating series $\bar{\mathbf{e}}_\tau$ satisfies the corrected Fay relations*

$$\mathcal{F}(\bar{\mathbf{e}}'_\tau)(u_1,\ldots,u_r) = -rc_r(u_2+\cdots+u_r).$$

The reduced Lie-like elliptic associator $\bar{\mathbf{a}}_\tau$ satisfies the corrected Fay relations

$$\mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1,\dots,u_r) = -rc_r(u_2+\cdots+u_r) + \sum_{\substack{i=3\\ i \text{ odd}}}^{r-1} \zeta(i)\Big(T'(u_2,u_3,\dots,u_{r-i+1}) - T'(u_{i+1},u_{i+2},\dots,u_r)\Big).$$

In particular, we have

$$\mathcal{F}\big(\bar{\mathbf{e}}'(\tau)\big)(u_1,\dots,u_r) = \mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1,\dots,u_r) = 0 \quad \textit{for all even values of } r.$$

The first statement is proved in §3.1, using the identity $\bar{\mathbf{e}}_\tau = \psi(a)$, mould theory and the equivalence between the Fay relation and the elliptic Kashiwara-Vergne of Theorem A (Corollary of Theorem 1.3). The second statement is proved in §3.2, using the identity $\bar{\mathbf{a}}_\tau = \psi(t_{01})$ and mould theory. The third statement follows from the fact that T is zero in odd depth. The group-like situation is dealt with in §3.3, where we show how to use the results of Theorem B to compute the explicit form of the Fay relations satisfied by the the reduced elliptic generating series $\bar{E}_\tau$ and the reduced elliptic associator $\bar{A}_\tau$.

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§1. The Fay relations on moulds

§1.1. The Fay operator on moulds

For the purposes of this article, a *mould* is nothing other than a family $(A_r)_{r\geq 0}$ of rational functions in commutative variables over a given ring, such that for each r, A_r is a rational function of r variables $u_1,\dots,u_r$; in particular $A_0(\emptyset)$ is a constant. When there is no ambiguity, we drop the subscript and just write $A(u_1,\dots,u_r)$ instead of $A_r(u_1,\dots,u_r)$, called *depth r part of A*. All the results in this section are general and hold over $\mathbb{Q}$, which can be assumed to the base ring, but it can be replaced by any larger ring, and indeed it will be in the subsequent sections. Moulds can be multiplied by scalars and added componentwise (depth by depth). We write ARI for the vector space of moulds A having $A(\emptyset)=0$. We write ARI^{pol} for the vector subspace of polynomial-valued moulds.

Let $\mathbb{Q}\langle\langle a,b\rangle\rangle$ denote the free completed associative algebra on two non-commutative variables a, b, and consider the subspace $\mathrm{Ker}\partial$, where ∂ is the derivation mapping $a\mapsto 1$, $b\mapsto 0$. This subspace, which we denote by $\mathbb{Q}[C]$, is precisely the space of power series in the free variables $c_i = ad(a)^{i-1}(b)$. There is a standard linear map from $\mathbb{Q}[C]$ to ARI^{pol} given by linearly extending the map on monomials

$$ma : c_{a_1}\cdots c_{a_r} \mapsto u_1^{a_1-1}\cdots u_r^{a_r-1}.$$

A mould $A\in ARI$ is said to be *alternal* if

$$A\Big(sh\big(v_1,\dots,v_i\big),\big(v_{i+1},\dots,v_r\big)\Big) = 0,$$

for $1 \leq i \leq r-1$, where sh denotes the shuffle operator, considered additively in the sense that with our notation, we have for example

$$A\big(sh((u_1),(u_2))\big) = A(u_1,u_2) + A(u_2,u_1).$$

We write ARI_{al} for the subspace of alternal moulds in ARI. It is well-known that the restriction of ma to polynomial alternal moulds induces an isomorphism

$$ma : Lie[C] \xrightarrow{\sim} ARI_{al}^{pol},$$

where Lie$[C]$ denotes the free Lie algebra on the letters c_i, $i \geq 1$ viewed as a subspace of $\mathbb{Q}[C]$.

The usefulness of moulds is that they easily allow us to work with denominators and not only polynomials, thus generalizing the situation of power series in the c_i, which is necessary, as we will see, to express the Fay relations and many other important properties that cannot be expressed within the power series ring. Our first use of denominators will appear via the mould operator dar and its inverse dar^{-1} defined as follows: fixed $A(\emptyset)$, and for $r > 0$ we set

$$darA(u_1,\dots,u_r) = u_1\cdots u_r\, A(u_1,\dots,u_r) \quad\text{and}\quad dar^{-1}A(u_1,\dots,u_r) = \frac{A(u_1,\dots,u_r)}{u_1\cdots u_r}. \tag{1.1.1}$$

Throughout this article we will use the notation $A' := dar^{-1}A$. As studied in [BMS], the Fay operator associated to the elliptic associator $\mathbf{A}_\tau$ actually acts on A'_τ, which is the motivation for the following definition.

Definition 1.1. The *Fay operator* on moulds, denoted $\mathcal{F}$, is defined by

$$\mathcal{F}(A)(u_1,\dots,u_r) := A(u_1,\dots,u_r) + A(u_2,\dots,u_r,-\overline{u}_r) + \sum_{i=1}^{r-1} A(u_2,\dots,u_i,-\overline{u}_i,\overline{u}_{i+1},u_{i+2},\dots,u_r)$$

where $\overline{u}_i = u_1 + \cdots + u_i$. A mould $A \in ARI$ is said to *satisfy the strict Fay relations* if

$$\mathcal{F}(A')(u_1,\dots,u_r) = 0 \tag{1.1.2}$$

for $r \geq 2$. The mould A is said to satisfy *a family of corrected Fay relations* if there exists a "correction" mould $C_{A'}$ whose depth r part is a polynomial expression in the parts of A' of depth $< r$ (possibly plus a linear expression in the u_i), such that

$$\mathcal{F}(A')(u_1,\dots,u_r) = C_{A'}(u_1,\dots,u_r). \tag{1.1.3}$$

§1.2. The Fay operator and swap circ-neutrality

We begin this subsection by introducing several more simple but important mould operators, each of which preserves the value $A(\emptyset)$. Firstly, for $r > 0$, we have

$$durA(u_1,\dots,u_r) = (u_1+\cdots+u_r)A(u_1,\dots,u_r)$$

$$\Delta A(u_1,\dots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)A(u_1,\dots,u_r)$$

$$push_u A(u_1,\dots,u_r) = A(-u_1-\cdots-u_r,u_1,\dots,u_{r-1}).$$

We also have the swap operator

$$swapA(v_1,\dots,v_r) = A(v_r, v_{r-1}-v_r,\dots,v_1-v_2),$$

which sends moulds $A \in ARI$ to moulds $swapA \in \overline{ARI}$, where $\overline{ARI}$ is identical to ARI except that the moulds are rational functions of commutative variables $v_1, v_2, \dots$ (The use of differently named variables for $\overline{ARI}$ is purely a convenience to distinguish between moulds and their swaps.)

We now introduce three operators defined on $\overline{ARI}$. The first is the inverse of the *swap* (also denoted *swap*), defined by

$$swapB(u_1,\dots,u_r) = B(u_1+\cdots+u_r, u_1+\cdots+u_{r-1},\dots,u_1+u_2,u_1)$$

for $B\in\overline{ARI}$. The second is the push-operator $push_v$ defined by

$$push_v B(v_1,\dots,v_r) = B(-v_r, v_1-v_r,\dots,v_{r-1}-v_r).$$

Finally, we introduce the *circ*-operator on $\overline{ARI}$, defined by

$$circB(v_1,v_2,\dots,v_r) = B(v_2,\dots,v_r,v_1).$$

Observe that the operators $push_u$ and $push_v$ are both of order $r+1$ in depth r, and *circ* is of order r. Let ARI^Δ denote the subspace of moulds $A\in ARI$ such that $\Delta A\in ARI^{pol}$; in other words, ARI^Δ is the subspace of rational moulds whose denominator in depth r is "at worst" $u_1\cdots u_r(u_1+\cdots+u_r)$.

Lemma 1.1. *We have the following relations between mould operators:*

(i) $\Delta = dar\cdot dur = dur\cdot dar$

(ii) $swap\cdot push_u^{-1} = push_v\cdot swap$

(iii) A mould $A\in ARI$ *is* $push_u$*-invariant if and only if* $swapA$ *is* $push_v$*-invariant.*

Proof. The first statement is obvious. For the second, we use the definitions to compute

$$\begin{aligned}(push_v\cdot swapA)(v_1,\dots,v_r) &= swapA(-v_r, v_1-v_r,\dots,v_{r-1}-v_r)\\ &= A(v_{r-1}-v_r, v_{r-2}-v_{r-1},\dots,v_1-v_2,-v_1)\\ &= push_u^{-1}A(v_r, v_{r-1}-v_r,\dots,v_1-v_2)\\ &= (swap\cdot push_u^{-1}A)(v_1,\dots,v_r).\end{aligned}$$

For the third, we note that by (ii), $push_v\cdot swapA$ is equal to $swap\cdot push_u^{-1}A$ which is itself equal to $swapA$ thanks to the $push_u$-invariance of A. ♢

Theorem 1.2. *Let* $M\in ARI^\Delta$ *be alternal and push-invariant. Let* $A=\Delta M$ *and* $A' = dar^{-1}A = durM$*. Then the following are equivalent:*

(i) A satisfies the strict Fay relations

$$\mathcal{F}(A')(u_1,\dots,u_r) = 0; \tag{1.2.1}$$

(ii) swapM satisfies the first alternality relation

$$swapM\big(sh\big((v_1),(v_2,\dots,v_r)\big)\big) = 0; \tag{1.2.2}$$

(iii) swapM satisfies the **circ-neutrality relation**

$$swapM(v_1,\dots,v_r) + swapM(v_2,\dots,v_r,v_1) + \cdots + swapM(v_r,v_1,\dots,v_{r-1}) = 0. \tag{1.2.3}$$

Proof. Let us show that (i) and (ii) are equivalent. We do this by swapping the Fay relation (1.2.1):

$$\begin{aligned}swap\Big(\mathcal{F}(A')(u_1,\dots,u_r)\Big) &= swap\Big(A'(u_1,\dots,u_r) + \sum_{i=1}^r A'(u_2,\dots,u_i,-\overline{u}_i,\overline{u}_{i+1},u_{i+2},\dots,u_r)\Big)\\ &= A'(v_r, v_{r-1}-v_r,\dots,v_2-v_3,v_1-v_2)\\ &\quad \sum_{i=1}^r A'(v_{r-1}-v_r,\dots,v_{r-i+1}-v_{r-i+1},-v_{r-i+1},v_{r-i},v_{r-i-1}-v_{r-i},\dots,v_1-v_2)\\ &= swap(A')(v_1,\dots,v_r) + \sum_{i=1}^r swap(A')(v_1-v_r,\dots,v_{r-i}-v_r,-v_r,v_{r-i+1}-v_r,\dots,v_{r-1}-v_r)\\ &= swap(A')(v_1,\dots,v_r) + swap(A')\Big(sh\big((-v_r),(v_1-v_r,\dots,v_{r-1}-v_r)\big)\Big)\\ &= swap(A')(v_1,\dots,v_r) + \big(push_v\cdot swap(A')\big)\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big).\end{aligned} \tag{1.2.4}$$

Thus the condition (1.2.1) of (i) is equivalent to

$$swap(A')(v_1,\dots,v_r) + \big(push_v \cdot swap(A')\big)\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = 0. \tag{1.2.5}$$

We now rewrite (1.2.5) as the equivalent condition (1.2.6) obtained by applying $push_v^{-1}$ to (1.2.5):

$$push_v^{-1} \cdot swap(A')(v_1,\dots,v_r) + swap(A')\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = 0. \tag{1.2.6}$$

Since we have $A' = durM$ and

$$swap\, durM(v_1,\dots,v_r) = v_1\, swapM(v_1,\dots,v_r),$$

we can use this to rewrite (1.2.6) in the equivalent form (1.2.7) as follows:

$$\begin{aligned}
&\big(push_v^{-1} \cdot swap\, durM\big)(v_1,\dots,v_r) + swap\, durM\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big)\\
&\quad = push_v^{-1}\big(v_1\, swapM(v_1,\dots,v_r)\big) + v_1\, swapM(v_1,\dots,v_r) + v_2\, swapM\Big(v_2, sh\big((v_1),(v_3,\dots,v_r)\big)\Big)\\
&\quad = (v_2-v_1)push_v^{-1}swapM(v_1,\dots,v_r) + v_1\, swapM(v_1,\dots,v_r) + v_2\, swapM\Big(v_2, sh\big((v_1),(v_3,\dots,v_r)\big)\Big)\\
&\quad = (v_2-v_1)swapM(v_1,\dots,v_r) + v_1\, swapM(v_1,\dots,v_r) + v_2\, swapM\Big(v_2, sh\big((v_1),(v_3,\dots,v_r)\big)\Big)\\
&\quad = v_2\, swapM\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = 0,
\end{aligned} \tag{1.2.7}$$

where the third inequality comes from the $push_v$-invariance of $swapM$ which follows from the $push_u$-invariance of M and Lemma 1 (iii). But clearly (1.2.7) is equivalent to the first alternality relation (1.2.2). This proves the equivalence of (i) and (ii).

We now prove the equivalence of (ii) and (iii). Since $swapM$ is $push_v$-invariant, the circ-neutrality expression

$$swapM(v_1,\dots,v_r) + circ\, swapM(v_1,\dots,v_r) + \dots + circ^{r-1}swapM(v_1,\dots,v_r) \tag{1.2.8}$$

can be written

$$\begin{aligned}
&push_v^r \cdot swapM(v_1,\dots,v_r) + circ\, push_v swapM(v_1,\dots,v_r)\\
&\qquad + circ^2 push_v^2 swapM(v_1,\dots,v_r) + \dots + circ^{r-1}push_v^{r-1}\, swapM(v_1,\dots,v_r).
\end{aligned} \tag{1.2.9}$$

Let us compute each term of (1.2.9). For the first term, we have

$$\big(push_v^r \cdot swapM\big)(v_1,\dots,v_r) = swapM(v_2-v_1,\dots,v_r-v_1,-v_1). \tag{1.2.10}$$

For the second term, we have

$$(circ\, push_v swapM)(v_1,\dots,v_r) = push_v swapM(v_2,\dots,v_r,v_1) = swapM(-v_1, v_2-v_1,\dots,v_{r-1}-v_1). \tag{1.2.11}$$

Using the general formula for $push_v^i$ for $i > 2$

$$push_v^i A(v_1,\dots,v_r) = A(v_{r-i+2}-v_{r-i+1}, v_{r-i+3}-v_{r-i+1},\dots,v_r-v_{r-i+1}, -v_{r-i+1}, v_1-v_{r-i+1},\dots,v_{r-i}-v_{r-i+1}),$$

adding i to every index (modulo r) we have

$$push_v^i A(v_{i+1},\dots,v_r,v_1,\dots,v_i) = A(v_2-v_1, v_3-v_1,\dots,v_i-v_1,-v_1,v_{i+1}-v_1,\dots,v_r-v_1), \tag{1.2.12}$$

which is thus equal to

$$\big(circ^i push_v^i swapM\big)(v_1,\dots,v_r)$$

for $i = 2,\dots,r-1$. But the sum of the terms (1.2.10), (1.2.11) and (1.2.12) for $i = 2,\dots,r-1$ is exactly equal to

$$swapM\big(sh\big((-v_1),(v_2-v_1,v_3-v_1,\dots,v_r-v_1)\big)\big),$$

yet at the same time to the circ-neutrality expression (1.2.8). Therefore $swapM$ satisfies the first alternality relation if and only if it is satisfies the circ-neutrality relation (1.2.3). This concludes the proof of Theorem 1.2. ♢

§1.3. Correction terms

In this subsection we generalize the statement of Theorem 1.2 to moulds whose swaps only satisfy the first alternality relation and/or the circ-neutrality relation up to addition of a constant mould (i.e. a mould whose value is a constant in each depth).

Theorem 1.3. *Let $M \in ARI^{\Delta}$ be alternal and push-invariant. Let $A = \Delta M$ and $A' = durM = dar^{-1}(A)$. Then the following are equivalent for the same constant mould $C = (c_r)_{r\geq 0}$:*

(i) $swapM + C$ satisfies the first alternality relation;

(ii) $swapM + C$ satisfies the circ-neutrality relation;

(iii) $\mathcal{F}(A')(u_1, \dots, u_r) = -rc_r(u_2 + \cdots + u_r)$. (1.3.1)

Proof. We saw in the proof of Theorem 1.2 that

$$swapM(v_1, \dots, v_r) + circ(swapM)(v_1, \dots, v_r) + \cdots + circ^{r-1}(swapM)(v_1, \dots, v_r)$$
$$= swapM\big(sh\big((-v_1), (v_2 - v_1, v_3 - v_1, \dots, v_r - v_1)\big)\big).$$

If $swapM + C$ is circ-neutral, this means that

$$swapM(v_1, \dots, v_r) + circ\, swapM(v_1, \dots, v_r) + \cdots + circ^{r-1} swapM(v_1, \dots, v_r) = -rc_r$$

for $r \geq 2$; similarly, if $swapM + C$ satisfies the first alternality relation, then we must have

$$= swapM\big(sh\big((v_1), (v_2, v_3, \dots, v_r)\big)\big) = -rc_r,$$

but this must then also hold for any variable change in the v_i, so we must also have

$$= swapM\big(sh\big((-v_1), (v_2 - v_1, v_3 - v_1, \dots, v_r - v_1)\big)\big) = -rc_r.$$

Therefore circ-neutrality and the first alternality relation are still equivalent in the presence of a constant correction.

Now assume that $swapM + C$ satisfies the first alternality relation. We saw in (1.2.7) that

$$v_2\, swapM\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) = \big(push_v^{-1} swap\, durM\big)(v_1, \dots, v_r) + swap\, durM\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big).$$

Thus if

$$swapM\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) = -rc_r,$$

we have

$$\big(push_v^{-1} \cdot swap\, durM\big)(v_1, \dots, v_r) + swap\, durM\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) = -rc_r v_2.$$

Applying $push_v$ to both sides gives

$$swap\, durM(v_1, \dots, v_r) + push_v \cdot swap\, durM\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) = -r(v_1 - v_r)c_r$$

or writing $A' = durM$,

$$swap(A')(v_1, \dots, v_r) + push_v \cdot swap(A')\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) = -r(v_1 - v_r)c_r.$$

But by (1.2.4), this is then equivalent to

$$\big(swap\, \mathcal{F}(A')\big)(v_1, \dots, v_r) = -r(v_1 - v_r)c_r.$$

Therefore, swapping both sides, we end up with

$$\mathcal{F}(A')(u_1, \dots, u_r) = -rc_r(u_2 + u_3 + \cdots + u_r).$$

§1.4. The elliptic Kashiwara-Vergne Lie algebra

In [En1], Enriquez defined an elliptic version $\mathfrak{grt}_{ell}$ of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$, contained in the vector space $\mathrm{Der}^0\mathrm{Lie}[a,b]$ of derivations of the free Lie algebra $\mathrm{Lie}[a,b]$ that annihilate the bracket $[a,b]$; Enriquez explicitly constructed a section map $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$. In [AT], a simple proof of the Lie algebra inclusion $\iota : \mathfrak{grt} \hookrightarrow \mathfrak{krv}$ was found. The article [RS] is devoted to the construction of an elliptic version $\mathfrak{krv}_{ell}$ of the Kashiwara-Vergne Lie algebra, also as a subspace of $\mathrm{Der}^0\mathrm{Lie}[a,b]$, that has various good properties including a section map γ from the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ extending Enriquez's section, making the diagram

$$\begin{tikzcd}
\mathfrak{grt} \arrow[rr, "\iota"] \arrow[d, "\gamma"'] & & \mathfrak{krv} \arrow[d, "\gamma"] \\
\mathfrak{grt}_{ell} \arrow[dr] & & \mathfrak{krv}_{ell} \arrow[dl] \\
& \mathrm{Der}^0\mathrm{Lie}[a,b] &
\end{tikzcd}$$

commute. The following definition of the elliptic Kashiwara-Vergne Lie algebra given in [RS] shows how closely the Fay relations are related to $\mathfrak{krv}_{ell}$.

Definition 1.2. The underlying vector space of the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$ is the space of power series $f \in \mathbb{Q}\langle\langle a,b\rangle\rangle$ whose associated mould $F \in ARI^{pol}$ satisfies the following properties: $M := \Delta^{-1}(F)$ is alternal, push-invariant and there exists a constant mould C such that $swapM + C$ is circ-neutral.

In view of Theorem 1.3, however, we now have another equivalent definition using the Fay relations, as follows.

Corollary of Theorem 1.3. *The underlying vector space of the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$ is isomorphic to the vector space of power series $f \in \mathbb{Q}\langle\langle a,b\rangle\rangle$ whose associated mould $F \in ARI^{pol}$ satisfies the following properties: F is alternal and push-invariant and there exists a constant mould $C = (c_r)_{r\geq 2}$ such that for $r \geq 2$, we have*

$$\mathcal{F}(A')(u_1,\ldots,u_r) = -rc_r(u_2+\cdots+u_r).$$

Remark. It is proved in [RS] that the image of the inclusion map of $\mathfrak{krv}_{ell}$ into $\mathrm{Der}^0\mathrm{Lie}[a,b]$ is closed under the natural Lie bracket on $\mathrm{Der}^0\mathrm{Lie}[a,b]$, making $\mathfrak{krv}_{ell}$ into a Lie algebra.

§2. The elliptic associator $\mathbf{A}_\tau$ and its reduced, Lie and mould versions

§2.1. Enriquez' elliptic associator $\mathbf{A}_\tau$

Let $\mathcal{Z}$ denote the $\mathbb{Q}$-algebra of multizeta values, and let $\overline{\mathcal{Z}}$ be the quotient of this algebra by the ideal generated by $\zeta(2)$. Let $\mathbb{Q}\langle\langle x,y\rangle\rangle$ denote the free completed polynomial algebra on two non-commutative variables x and y*. Let $\Phi_{KZ}(x,y) \in \mathcal{Z}\langle\langle x,y\rangle\rangle$ denote the Drinfel'd associator, and let us define three Bernoulli power series inside $\mathbb{Q}\langle\langle x,y\rangle\rangle$ by

$$t_{01} := Ber_x(-y), \qquad t_{02} := Ber_{-x}(y), \qquad t_{12} := [y,x], \tag{2.1.1}$$

* In this introductory section containing definitions, we use the variables x and y so as to conform with the usual notation in articles on the subject. In the next section we will identify these variables with a and b via $x = b$, $y = a$.

where

$$Ber_x(y) := \frac{ad(x)}{e^{ad(x)} - 1}(y).$$

Let ϵ_{2i}, $i \geq 0$, denote the well-known derivations defined on $\mathbb{Q}\langle\langle x, y\rangle\rangle$ by

$$\epsilon_{2i}(x) = ad(x)^{2i}y, \quad \epsilon_{2i}([x, y]) = 0.$$

Note that the second condition determines the value of $\epsilon_{2i}(y)$ uniquely. In particular, $\epsilon_0(x) = y$, $\epsilon_0(y) = 0$. We write $\mathfrak{u}$ for the Lie subalgebra of $\operatorname{Der}\operatorname{Lie}[x, y]$ generated by the ϵ_{2i}, and $\mathcal{U}\mathfrak{u}$ for its enveloping algebra.

Definition 2.1. Let g_τ be the automorphism of $\mathbb{Q}\langle\langle x, y\rangle\rangle$ defined in [En2, Prop. 5.1] (see also [LMS, §2.3]). It is a solution of the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau}g_\tau = -\Big(\epsilon_0 + \sum_{k\geq 1}\frac{2}{(2k-2)!}G_{2k}(\tau)\epsilon_{2k}\Big)g_\tau,$$

where

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1}\sigma_{2k-1}(n)q^n$$

is the Hecke-normalized Eisenstein series, with $q = e^{2\pi i\tau}$. Enriquez singles out the solution g_τ by specifying its asymptotic behavior as $\tau \to i\infty$.

An explicit expression for g_τ can be given in the form

$$g_\tau = \mathrm{id} + \sum_{\underline{k}=(2k_1,\ldots,2k_r)} \mathcal{G}_{\underline{k}}\epsilon_{\underline{k}}, \tag{2.1.2}$$

where the sum runs over all tuples $(2k_1, \ldots, 2k_r)$ of even integers ≥ 0, with $r \geq 1$, and $\mathcal{G}_{\underline{k}}$ is defined as a certain regularized iterated integral of $G_{2k_1}, \ldots, G_{2k_n}$ [LMS, §2.2]. The operators $\epsilon_{\underline{k}}$ are not linearly independent. Choose a basis for the $\mathbb{Q}$-vector space they generate, write g_τ in that basis, and let $\mathcal{E}^{\mathrm{geom}} \subset \mathcal{O}(\mathcal{H})$ denote the $\mathbb{Q}$-vector space spanned by the coefficients in g_τ of the basis elements. These coefficients are all linear combinations of the $\mathcal{G}_{\underline{k}}$, so $\mathcal{E}^{\mathrm{geom}}$ is a subspace of the $\mathbb{Q}$-vector space spanned by the $\mathcal{G}_{\underline{k}}$, and it is independent of the choice of basis. It was shown in [LMS] that $\mathcal{E}^{\mathrm{geom}}$ is not merely a vector space but actually forms a subring of the ring of functions $\mathcal{O}(\mathcal{H})$ on the Poincaré upper half-plane (this is simply a direct consequence of the definition of $g(\tau)$ which shows it to be a group-like power series). We can also view the multizeta value algebra $\mathcal{Z}[2\pi i]$ as a subring of constant functions of $\mathcal{O}(\mathcal{H})$. It is shown in [LMS] (Cor. 2.10) that the subring of $\mathcal{O}(\mathcal{H})$ generated by $\mathcal{E}^{\mathrm{geom}}$ and by $\mathcal{Z}[2\pi i]$ is isomorphic to the tensor product $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$.

Definition 2.2. Let R denote the subring $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i] \subset \mathcal{O}(\mathcal{H})$. Set

$$\mathbf{A} = \Phi_{KZ}(t_{01}, t_{12})e^{2\pi i t_{01}}\Phi_{KZ}(t_{01}, t_{12})^{-1} \in \mathcal{Z}[2\pi i]\langle\langle x, y\rangle\rangle. \tag{2.1.3}$$

The *elliptic associator* is the group-like power series

$$\mathbf{A}_\tau = g_\tau(\mathbf{A}) \in R\langle\langle x, y\rangle\rangle. \tag{2.1.4}$$

This elliptic associator was introduced by Enriquez in [En1]. The formula (2.1.4) is not the original definition; Enriquez views it rather as a property of the elliptic associator, which can be defined in various more intrinsic ways, in particular via iterated integrals of the Kronecker function as in (2.1.5) and (2.1.6) below ([En2, §6.2]). However, (2.1.4) emerges as the most apt definition for our purposes. The definition by iterated integrals, on the other hand, which we now describe, is that studied in [BMS] which gave rise to the discovery of the Fay relations satisfied by $\mathbf{A}_\tau$.

Definition 2.3. Let $F\begin{pmatrix} u \\ v \end{pmatrix}$ denote the Kronecker function

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\theta(u+v)}{\theta(u)\theta(v)}$$

where $\theta(u)$ is the only Jacobi theta-function that is odd in z,

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2} e^{(2n+1)iz}$$

with $q = e^{i\pi\tau}$. This function satisfies

$$\theta(z+\tau) = qe^{-2i\pi z}\theta(z), \quad \theta(z+1) = -\theta(z), \quad \frac{d\theta}{dz}(0) = 1,$$

and the only zeros of θ are the points of the lattice $\mathbb{Z} + \mathbb{Z}\tau$.

The Kronecker function satisfies the famous elementary *Fay relation*

$$F\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} F\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = F\begin{pmatrix} u_1+u_2 \\ v_1 \end{pmatrix} F\begin{pmatrix} u_2 \\ v_2-v_1 \end{pmatrix} + F\begin{pmatrix} u_1+u_2 \\ v_2 \end{pmatrix} F\begin{pmatrix} u_1 \\ v_1-v_2 \end{pmatrix}. \tag{2.1.5}$$

Consider the iterated-integral mould defined as $\mathcal{I}(\tau)(\emptyset) = 1$ and for $r \geq 1$ by

$$\mathcal{I}(\tau)(u_1,\ldots,u_r) = \int_{0<v_r<\cdots<v_1<1} F\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \cdots F\begin{pmatrix} u_r \\ v_r \end{pmatrix} dv_r \cdots dv_1,$$

where the integral is regularized as usual; this is the original mould used by Enriquez to construct his "elliptic analogues of multizeta values" in [En2] (see def. 2.6 in which he uses x_i instead of u_i). Enriquez shows that $\mathbf{I}(\tau)$ is a rational-function-valued mould with denominator $u_1 \ldots u_r$ in depth r. Thus the mould $dar\big(\mathbf{I}(\tau)\big)$ is polynomial-valued, where dar is defined in (1.1.1). Let $\mathbf{I}_\tau(x,y)$ denote the power series in x,y such that $ma(\mathbf{I}_\tau) = \mathbf{I}(\tau)$, where ma is the map defined just before (0.1). The power series $\mathbf{I}_\tau$ differs from the elliptic associator $\mathbf{A}_\tau$ only by some corrective factors; the two are related explicitly by the formula

$$\mathbf{A}_\tau = e^{\frac{i\pi}{2}[x,y]}\ (2\pi)^{-[x,y]}\ \mathbf{I}_\tau(x/2\pi i, 2\pi i y)\ (2\pi)^{[x,y]}\ e^{\frac{i\pi}{2}[x,y]}. \tag{2.1.6}$$

However, the expression of $\mathbf{I}(\tau)$ as an iterated integral allows certain properties to become visible that are not easy to see directly on the power series $\mathbf{A}_\tau$, especially when it is defined by (2.1.4).

The elementary Fay relation satisfied by the Kronecker function "propagates", in a manner described in [BMS]. The result shown there is that, writing $\mathbf{A}(\tau) := ma(\mathbf{A}_\tau)$ for the polynomial-valued mould associated to the power series $\mathbf{A}_\tau$, and $\mathbf{A}'(\tau) = dar^{-1}\mathbf{A}(\tau)$, the mould $\mathbf{A}(\tau)$ satisfies a family of Fay relations of the form (1.1.3), in which the correction mould $C_{\mathbf{A}'_\tau}$ arises from the need to regularize the iterated integrals defining $\mathbf{A}_\tau$. The correction term was not explicitly determined in [BMS].

§2.2. Reducing the elliptic associator mod $2\pi i$

The goal of this article is to show that, at least modulo $2\pi i$, the mould construction of the elliptic associator based on the Drinfel'd associator reveals a different explanation for the Fay relations, and furthermore allows us to explicitly determine the correction term, which arises directly from that of the Drinfel'd associator.

From now until the end of this article, we make the variable change $a := y$, $b := x$, and consider t_{01}, t_{02}, t_{12}, A, and $\mathbf{A}_\tau$ from (2.1.1), (2.1.3) and (2.1.4) as power series in the variables a, b. The reason for this is that it is a more convenient notation for the application of mould theory starting in §2.3 below.

Let $R = \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$ as in Definition 2.2, and let $\bar{R}$ denote the quotient of this ring by the ideal generated by $1 \otimes 2\pi i$; thus

$$\bar{R} \simeq \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$$

where $\overline{\mathcal{Z}} = \mathcal{Z}/\big(\zeta(2)\big)$.

Since $\mathbf{A}$ is a conjugate of $e^{2\pi i t_{01}}$, the same holds for $\mathbf{A}_\tau = g_\tau(\mathbf{A})$, and therefore every coefficient in the power series $\mathbf{A}_\tau$ vanishes in $\bar{R}$.

Lemma 2.1. *The power series $A_\tau := \mathbf{A}_\tau^{1/2\pi i}$ lies in $R\langle\langle a,b\rangle\rangle$,*

Proof. We first show that the power series $A := \mathbf{A}^{1/2\pi i}$ lies in $\mathcal{Z}\langle\langle a,b\rangle\rangle$. Indeed, this follows directly from (2.1.3), since

$$A = \Phi_{KZ}(t_{01},t_{12})e^{t_{01}}\Phi_{KZ}(t_{01},t_{12})^{-1}, \tag{2.2.1}$$

and all coefficients of the Drinfel'd associator lie in $\mathcal{Z}$. In order to show that $A_\tau \in R\langle\langle a,b\rangle\rangle$, we note that (2.1.4) is still valid, i.e. we have

$$A_\tau = g_\tau(A).$$

Since the coefficients of g_τ generate $\mathcal{E}^{\mathrm{geom}}$, all the coefficients of $g_\tau(A)$ lie in the ring generated by $\mathcal{E}^{\mathrm{geom}}$ and $\mathcal{Z}$, which is a subring of R. ♢

Definition 2.4. Let the *reduced elliptic associator*, denoted $\bar{A}_\tau$, be the power series obtained by reducing the coefficients of $A_\tau = \mathbf{A}_\tau^{1/2\pi i}$ from R to $\bar{R}$.

It is easy to see that $\bar{A}_\tau$ is highly non-trivial, since the reduction $\bar{A}$ of A mod $\zeta(2)$ is already non-trivial. Indeed, it is shown in [LMS] (Thm. 3.6) that the coefficients of $\bar{A}_\tau$ together with the single element $2\pi i\tau$ generate the entire ring $\bar{R}$.

§2.3. Automorphism construction of the reduced group-like and Lie-like elliptic associators

The main result that we will use for our analysis of the power series $\bar{A}_\tau$ (and also the reduction $\bar{A}$ of A mod $2\pi i$) is the construction of these series as the images of $e^{t_{01}}$ by specific automorphisms of the ring $\bar{R}\langle\langle a,b\rangle\rangle$. Let

$$\Gamma : GRT \to GRT_{ell}$$

denote the section map defined by Enriquez in [En1] from the prounipotent Grothendieck-Teichmüller group to its elliptic version. Enriquez also defined a graded Lie algebra version $\mathfrak{grt}_{ell}$ of GRT_{ell} equipped with a natural surjection $\mathfrak{grt}_{ell} \to \mathfrak{grt}$ and a Lie algebra section

$$\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$$

(cf. §1.4). The reduced Drinfel'd associator $\overline{\Phi}_{KZ}$ lies in $GRT(\overline{\mathcal{Z}})$, and its image $\Gamma(\overline{\Phi}_{KZ})$ lies in the group $GRT_{ell}(\overline{\mathcal{Z}})$, which Enriquez considers as a subgroup of automorphisms of the ring $\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$ fixing the bracket $[a,b]$. Let Ψ_{KZ} denote the automorphism of $\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$ given by $\Gamma(\overline{\Phi}_{KZ}) \in GRT_{ell}(\overline{\mathcal{Z}})$. Like Ψ_{KZ}, the automorphism g_τ of defined in §2.1 can also be viewed as an automorphism of the ring $\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$ fixing $[a,b]$, since each derivation ϵ_{2i} annihilates $[a,b]$ and g_τ is a group-like power series in the ϵ_{2i}.

Let r_τ and ψ_{KZ} denote the derivations of $\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$

$$r_\tau = log(g_\tau), \qquad \psi_{KZ} = log(\Psi_{KZ}),$$

and set

$$\Psi := g_\tau \circ \Psi_{KZ} = e^{r_\tau} \circ e^{\psi_{KZ}} = e^{\psi} \in \mathrm{Aut}(\bar{R}\langle\langle a,b\rangle\rangle), \tag{2.3.1}$$

where letting CH denote the Campbell-Hausdorff law in the derivation Lie algebra $\mathrm{Der}\,\mathrm{Lie}[a,b]$, we have

$$\psi := CH(r_\tau, \psi_{KZ}). \tag{2.3.2}$$

Definition 2.5. The *reduced group-like elliptic generating series* is the power series

$$\bar{C}_\tau := \Psi(e^a) \in \bar{R}\langle\langle a,b\rangle\rangle.$$

The *reduced elliptic generating series* is the logarithm of $\bar{C}_\tau$, namely the power series $\bar{E}_\tau$ given by

$$\bar{E}_\tau := \Psi(a) \in \bar{R}\langle\langle a,b\rangle\rangle. \tag{2.3.3}$$

The *reduced Lie-like elliptic generating series* is defined by

$$\bar{\mathbf{e}}_\tau := \psi(a). \tag{2.3.4}$$

By the previous section, the *reduced elliptic associator* is given by

$$\bar{A}_\tau = \Psi(e^{t_{01}}). \tag{2.3.5}$$

Finally, the *reduced Lie-like elliptic associator* is defined by

$$\bar{\mathbf{a}}_\tau := \psi(t_{01}). \tag{2.3.6}$$

The power series $\bar{E}_\tau$ was introduced in [LMS]. Indeed, it was shown there that up to adjoining the single element $2\pi i\tau$, the coefficients of $\bar{E}_\tau$ generate the full ring $\bar{R}$ of reduced elliptic multiple zeta values, and the same statement holds for the coefficients of $\bar{A}_\tau$.

Lemma 2.2. *The power series $\psi_{KZ}(a)$ lies in the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$. More precisely, setting $\bar{\mathbf{e}} := ma\big(\psi_{KZ}(a)\big)$, the mould $\Delta^{-1}\bar{\mathbf{e}}$ is alternal, push-invariant and $swap(\bar{\mathbf{e}}) + C$ is circ-neutral, where C is the mould in (0.4).*

Proof. Recall that $\overline{\Phi}_{KZ}$ is associated with an element of $GRT(\overline{\mathcal{Z}})$ in the sense that there is an automorphism of $\overline{\mathcal{Z}}\langle\langle a, b\rangle\rangle$ in $GRT(\overline{\mathcal{Z}})$ that maps $a \mapsto \overline{\Phi}_{KZ}$ and fixed the Lie bracket $[a, b]$. Let φ_{KZ} denote the *Lie-like Drinfel'd associator*, which is defined to be the power series in the completed free Lie algebra on a, b which is the value on a of the derivation which is the log of that automorphism. Since $\log\big(GRT(\overline{\mathcal{Z}})\big)$ is the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, we have $\varphi_{KZ} \in \mathfrak{grt}$. Let $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$ denote the Enriquez section recalled in §1.4. Then $\psi_{KZ}(a) = \gamma(\varphi_{KZ})(a)$. Let $\nabla := \Delta^{-1} \circ \gamma$, and let $\bar{\mathbf{e}} = ma(\varphi_{KZ})$. Then $\bar{\mathbf{e}}$ is alternal since elements of $\mathfrak{grt}$ (or rather, their values on a) lie in $\mathrm{Lie}[a, b]$. Taking $F = \bar{\mathbf{e}}$ in [S1, Theorem 1.3.2] (in which the morphism ∇ is called $Ad_{ari}(invpal)$ following Écalle's notation), we see that $swap(\nabla(\bar{\mathbf{e}})) + C$ is alternal for the C in (0.4), which is the usual constant correction mould associated to φ_{KZ}. Thus $\bar{\mathbf{e}}$ is bialternal, i.e. alternal and with alternal swap up to adding the constant mould C. But all bialternal moulds that are even in depth 1 (which is the case for $\bar{\mathbf{e}}$) are push-invariant by [S2], Lemma 2.5.5. Therefore $\bar{\mathbf{e}}$ is alternal and push-invariant and $swap(\bar{\mathbf{e}})$ is alternal, so in particular it satisfies the first alternality relation, and therefore by Theorem 1.3, $swap(\bar{\mathbf{e}}) + C$ satisfies the circ-neutrality relation. This concludes the proof.$\diamondsuit$

§2.4. Mould versions of the reduced Lie-like elliptic generating series and associator

To express the action of derivations in mould terms, we need to introduce a few more basic mould theory notions. To start with, the standard formula for mould multiplication is given by

$$mu(A, B)(u_1, \ldots, u_r) = \sum_{i=0}^{r} A(u_1, \ldots, u_i)B(u_{i+1}, \ldots, u_r).$$

It is easy to see that the mu multiplication extends the usual multiplication of power series in that

$$ma(fg) = mu\big(ma(f), ma(g)\big)$$

for $f, g \in \mathbb{Q}[C]$. Recall that ARI denotes the vector space of (rational) moulds satisfying $A(\emptyset) = 0$. One can make ARI into a Lie algebra by equipping it with the standard Lie bracket

$$lu(A, B) = mu(A, B) - mu(B, A).$$

The vector space ARI equipped with the lu bracket is a Lie algebra denoted ARI_{lu}. We can add a certain very important yet "trivial" mould, called a, to ARI_{lu}; this mould takes the value a on the empty set and 0 in all depths ≥ 1. We write ARI^a for the vector space generated by ARI and the mould a, and ARI^a_{lu} for the Lie algebra obtained by extending the bracket to ARI^a via the formula

$$lu(Q, a) = durQ.$$

We note that this formula merely extends to all moulds in ARI the known identity for a power series f:

$$ma\big([f,a]\big) = lu(ma(f),a) = dur\big(ma(f)\big)$$

([R], App. A, Prop. 5.1, see also [S2], Lemma 3.3.1). The advantage of adding the mould a to ARI is that extending ma to the letter a by mapping it to the mould also denoted by a makes ma extend to isomorphisms

$$ma : \mathbb{Q}\langle\langle a,b\rangle\rangle \to ARI^{a,pol}$$

and restrict to an isomorphism

$$ma : \mathrm{Lie}[a,b] \to (ARI^a)^{pol}_{al}.$$

Thus, using a bracket notation for moulds associated to power series, we may set

$$\begin{cases} \bar{E}(\tau) := ma(\bar{E}_\tau) \\ \bar{A}(\tau) := ma(\bar{A}_\tau) \\ g(\tau) := ma\big(g_\tau(a)\big) \\ \bar{\mathbf{e}}(\tau) := ma(\bar{\mathbf{e}}_\tau) \\ \bar{\mathbf{a}}(\tau) := ma(\bar{\mathbf{a}}_\tau). \end{cases}$$

Set $U_1 = ma([a,b])$, so that $U_1(u_1) = -u_1$. For every mould $P \in ARI$, there exists an associated derivation $Darit(P)$ of ARI^a_{lu} (defined explicitly in Appendix A) having the two following properties:

i) $Darit(P)\cdot a = P$;

ii) $Darit(P)\cdot U_1 = 0$. (2.4.1)

Thanks to these two properties, we see that knowing the value of the derivation $Darit(P)$ on a yields the mould P uniquely. In particular, therefore, the derivation $Darit\big(\bar{\mathbf{e}}(\tau)\big)$ restricted to $(ARI^a)^{pol}_{al}$ is equal to the derivation ψ defined in (2.3.2), in the sense that for every $f \in \mathrm{Lie}[a,b]$, we have

$$ma\big(\psi(f)\big) = Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot ma(f). \tag{2.4.2}$$

Since by Definition 2.5 we have $\bar{\mathbf{e}}_\tau = \psi(a)$ and $\bar{\mathbf{a}}_\tau = \psi(t_{01})$, we can translate these into the mould equalities

$$\bar{\mathbf{e}}(\tau) = ma\big(\bar{\mathbf{e}}_\tau\big) = ma\big(\psi(a)\big) = Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot a. \tag{2.4.3}$$

$$\bar{\mathbf{a}}(\tau) = ma\big(\bar{\mathbf{a}}_\tau\big) = ma\big(\psi(t_{01})\big) = Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot ma(t_{01}). \tag{2.4.4}$$

Finally, since $\psi = log(\Psi)$ with $\Psi(a) = \bar{E}_\tau$ and $\bar{A}_\tau = \Psi(e^{t_{01}})$, we also have the mould versions

$$\bar{E}(\tau) = \exp\Big(Darit\big(\bar{\mathbf{e}}(\tau)\big)\Big)\cdot a = \sum_{n\geq 0}\frac{1}{n!}Darit\big(\bar{\mathbf{e}}(\tau)\big)^n\cdot a \tag{2.4.5}$$

and

$$\bar{A}(\tau) = \exp\Big(Darit\big(\bar{\mathbf{e}}(\tau)\big)\Big)\cdot ma(e^{t_{01}}) = \sum_{n\geq 0}\frac{1}{n!}Darit\big(\bar{\mathbf{e}}(\tau)\big)^n\cdot ma(e^{t_{01}}). \tag{2.4.6}$$

In the next section §3, we will use the results of §1 and the expressions (2.4.3) and (2.4.4) to compute the Fay correction of the Lie-like elliptic generating series and the Lie-like elliptic associators, and then based on the expressions (2.4.5) and (2.4.6), we will show how to deduce the Fay corrections for the group-like versions.

§3. The Fay relations of the elliptic generating series and elliptic associator

The results of the previous sections make it quite natural and easy to determine the Fay relations satisfied by the elliptic generating series and the elliptic associator, particularly in their Lie-like forms (and mod $2\pi i$).

§3.1. The Fay relations for the Lie-like elliptic generating series

Theorem 3.1. *The Lie-like elliptic generating series* $\bar{\mathbf{e}}(\tau)$ *satisfies the Fay relations*

$$\mathcal{F}\big(\bar{\mathbf{e}}'(\tau)\big) = \begin{cases} -\zeta(r)(u_2 + \cdots + u_r) & r \geq 2 \text{ odd} \\ 0 & r \geq 2 \text{ even.} \end{cases} \tag{3.1.1}$$

Proof. Recall that $\bar{\mathbf{e}}_\tau = \psi(a)$ where $\psi = CH(r_\tau, \psi_{KZ})$ from (2.3.2). Let $\varphi_{KZ} \in \mathfrak{grt}$ be the Lie-like version of the Drinfel'd associator, which lies in the Grothendieck-Teichmüller Lie algebra. Then we have $\psi_{KZ} = \gamma(\varphi_{KZ})$, where $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$ is the Enriquez section on the level of the Lie algebras. As recalled in §1.4, it was shown in [AT] that there is an inclusion $\iota : \mathfrak{grt} \to \mathfrak{krv}$, and in [RS] that the section map $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$ extends to a section map $\gamma : \mathfrak{krv} \to \mathfrak{krv}_{ell}$. Therefore $\gamma(\varphi_{KZ}) \in \mathfrak{krv}_{ell}$, i.e. $\psi_{KZ} \in \mathfrak{krv}_{ell}$. Furthermore, it is shown in [RS] that all the derivations ϵ_{2i} defined in §2.1 lie in $\mathfrak{krv}_{ell}$. Let $\mathfrak{u}$ denote the Lie subalgebra of $\mathfrak{krv}_{ell}$ generated by the ϵ_{2i}. It is shown in [LMS] (just following Lemma 3.1, based on a result in [HM]) that any Lie bracket of an element of $\mathfrak{u}$ with $\gamma(\varphi_{KZ})$ lies in $\mathfrak{u}$. In particular, any Lie bracket of the derivations r_τ and $\gamma(\varphi_{KZ}) = \psi_{KZ}$ lies in $\mathfrak{u}$. Therefore the Campbell-Hausdorff product $\psi = CH(r_\tau, \psi_{KZ})$ can be written $\psi = \psi_{KZ} + D$ with $D \in \mathfrak{u}$. Since $\psi_{KZ} \in \mathfrak{krv}_{ell}$ and the ϵ_{2i}, $i \geq 0$ are in $\mathfrak{krv}_{ell}$, this shows that $\psi \in \mathfrak{krv}_{ell}$, so in terms of moulds we have

$$ma\big(\psi(a)\big) = \bar{\mathbf{e}}(\tau) \in ma(\mathfrak{krv}_{ell}).$$

Thus by Definition 1.2, $\Delta^{-1}\bar{\mathbf{e}}(\tau)$ is alternal, push-invariant and its swap is circ-neutral up to addition of a constant mould that we now determine.

Let $\bar{\mathbf{e}} := \psi_{KZ}(a)$ as in Lemma 2.2, and let $\mathcal{D} := ma\big(D(a)\big)$. Since $\psi = \psi_{KZ} + D$, and $\bar{\mathbf{e}}(\tau) = ma\big(\psi(a)\big)$ and $\bar{\mathbf{e}} = ma\big(\psi_{KZ}(a)\big)$, we have

$$swap\big(\Delta^{-1}\bar{\mathbf{e}}(\tau)\big) = swap\big(\Delta^{-1}\bar{\mathbf{e}}\big) + swap\big(\Delta^{-1}\mathcal{D}\big).$$

To determine the constant correction mould for $swap\big(\Delta^{-1}\bar{\mathbf{e}}(\tau)\big)$, we determine the constant corrections of $swap\big(\Delta^{-1}\mathcal{D}\big)$ and $swap\big(\Delta^{-1}\bar{\mathbf{e}}\big)$ separately.

We first show that $swap\big(\Delta^{-1}\mathcal{D}\big)$ is strictly circ-neutral. To see this, let $U_{2i} = ma\big(\epsilon_{2i}(a)\big)$ be the mould associated to each derivation ϵ_{2i}; then $U_{2i}(u_1) = u_1^{2i}$ and U_{2i} is zero in every depth different from 1. We have $\Delta^{-1}U_{2i} = U_{2i-2}$. In particular, we see that the moulds $swap(U_{2i-2})$ are all automatically circ-neutral simply because circ-neutrality is a property that only concerns the parts of a mould in depths ≥ 2. The moulds U_{2i-2} are also trivially alternal for the same reason, and they are also push-invariant since they are even in depth 1. It is shown in [RS] that the vector space of alternal, push-invariant moulds with strictly circ-neutral swap forms a Lie algebra; in particular, if δ denotes the image under ma of any bracket of the derivations ϵ_{2i}, then $\Delta^{-1}\delta$ has strictly circ-neutral swap. This holds in particular for $\delta = D$, which is a sum of Lie brackets of the ϵ_{2i}. Thus $\Delta^{-1}\mathcal{D})$ has strictly circ-neutral swap.

We now need to determine the constant correction mould of $swap(\Delta^{-1}\bar{\mathbf{e}})$; however, this was done in Lemma 2.2, where we showed that $swap(\Delta^{-1}\bar{\mathbf{e}}) + C$ is circ-neutral for the constant mould C defined in (0.4). Therefore $swap\big(\bar{\mathbf{e}}(\tau)\big) + C$ is circ-neutral for the same mould C. Thus, by Theorem 1.3, for each $r \geq 2$ we have the Fay relation

$$\mathcal{F}\big(\bar{\mathbf{e}}'(\tau)\big)(u_1, \ldots, u_r) = -rc_r(u_2 + \cdots + u_r),$$

which is exactly (3.1.1) for this particular constant mould C This concludes the proof. ♢

§3.2. The Fay relations for the Lie-like elliptic associator

In this section, we use the results of §2 to explicitly deduce the Fay relations satisfied by the *reduced Lie-like elliptic associator* $\bar{\mathbf{a}}(\tau)$ from the simple family satisfied by the Lie-like elliptic generating series $\bar{\mathbf{e}}(\tau)$ given in Theorem 3.1 and the characteristics of the mould $T_{01} := ma(t_{01})$.

We begin by separating t_{01} into the sum $t_0 + \hat{t}_{01}$ where $t_0 = -a - \frac{1}{2}[a,b]$ and $\hat{t}_{01}$ is of minimal degree 2 in b. Write the associated moulds $T_{01} = T_0 + \hat{T}_{01}$ where $T_0 = ma(t_0) = -a - \frac{1}{2}U_1$ and $\hat{T}_{01} = ma(\hat{t}_{01})$, which is given explicitly by

$$\hat{T}_{01}(u_1, \dots, u_r) = \frac{B_r}{r!} \sum_{i=1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \tag{3.2.1}$$

where B_r is the r-th Bernoulli number. In particular, $\hat{T}_{01}(u_1, \dots, u_r) = 0$ for all even r, and also for $r = 1$. Let $\hat{T}'_{01} = dar^{-1}\hat{T}_{01}$.

We saw in (2.4.4) that

$$\bar{\mathbf{a}}(\tau) = Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot T_{01}.$$

Since $T_{01} = T_0 + \hat{T}_{01}$, we have

$$\bar{\mathbf{a}}(\tau) = Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot T_0 + Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}. \tag{3.2.2}$$

But $T_0 = -a - (1/2)U_1$, and by the properties of the *Darit*-derivation given in (2.4.1), we thus have

$$Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot T_0 = -\bar{\mathbf{e}}(\tau),$$

so by (3.2.2), we have

$$\bar{\mathbf{a}}(\tau) = -\bar{\mathbf{e}}(\tau) + Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}. \tag{3.2.3}$$

Thus the Fay relations satisfied by $\bar{\mathbf{a}}(\tau)$ come from the Fay relations satisfied by $\bar{\mathbf{e}}(\tau)$ on the one hand, and those coming from $Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}$ on the other. The Fay relations satisfied by $\bar{\mathbf{e}}(\tau)$ are given in Theorem 3.1, and Theorem 3.2 below gives a formula for the Fay relations of a mould of the type $P = Darit(N) \cdot R$ under certain hypothesis on N and R that do hold when $N = \bar{\mathbf{e}}(\tau)$ and $R = \hat{T}_{01}$. In Lemma 3.3, we show that the Fay correction mould for $\hat{T}_{01}$ is zero. Based on these results, we can compute the explicit Fay correction for $\bar{\mathbf{a}}(\tau)$; it is given in Theorem 3.4.

Theorem 3.2. *Let M be an alternal and push-invariant mould such that $swapM + C$ is circ-neutral for some constant-valued mould $C = (c_r)_{r\geq 2}$. Set $N = \Delta M$. Let $R \in ARI$ be even in depth 1 and satisfy a family of Fay relations with correction mould $C_{R'}$, i.e.*

$$\mathcal{F}(R')(u_1, \dots, u_r) = C_{R'}(u_1, \dots, u_r).$$

Then the mould $P = Darit(N) \cdot R$ satisfies the Fay relations

$$\mathcal{F}(P')(u_1, \dots, u_r) = C_{P'}(u_1, \dots, u_r)$$

with correction mould $C_{P'}$ given for $r \geq 2$ by

$$\begin{aligned} C_{P'}(u_1, \dots, u_r) = &\sum_{1<i<j\leq r} C_{R'}(u_1, \dots, u_{i-1}, u_i + \dots + u_j, u_{j+1}, \dots, u_r) M(u_{i+1}, \dots, u_j) \\ &- \sum_{1<i<j\leq r} C_{R'}(u_1, \dots, u_i, u_{i+1} + \dots + u_{j+1}, u_{j+2}, \dots, u_r) M(u_{i+1}, \dots, u_j) \\ &- \sum_{i=2}^{r-1} i c_i R'(u_{i+1}, \cdots, u_r) + \sum_{i=2}^{r-1} i c_i R'(u_2, \dots, u_{r-i+1}). \end{aligned} \tag{3.2.4}$$

In particular, in the case where the mould R satisfies the strict Fay relations, i.e. $C_{R'} = 0$, the mould $P = Darit(N) \cdot R$ satisfies the Fay relations

$$C_{P'}(u_1, \dots, u_r) = \sum_{i=2}^{r-1} i c_i \Big(R'(u_2, u_3, \dots, u_{r-i+1}) - R'(u_{i+1}, u_{i+2}, \dots, u_r) \Big). \tag{3.2.5}$$

Thus if R satisfies the strict Fay relations and furthermore swap$M$ is strictly circ-neutral, then $P = Darit(N) \cdot R$ satisfies the strict relations:

$$\mathcal{F}(P')(u_1, \ldots, u_r) = 0. \tag{3.2.6}$$

Proof. The proof, which is long and technical, can be found in Appendix A. ♢

Let $N = \bar{\mathbf{e}}(\tau)$ and $M = \Delta^{-1}\bar{\mathbf{e}}(\tau)$. It was shown in the proof of Theorem 3.1 that this mould is alternal, push-invariant and $swap(M) + C$ is circ-neutral for the mould C of(0,4). Thus $M = \Delta^{-1}\bar{\mathbf{e}}(\tau)$ satisfies the properties required for M in Theorem 3.2. In order to apply the result of Theorem 3.2 to compute the Fay correction for $Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}$, we also need to know the Fay correction for the mould $R = \hat{T}_{01}$. The following lemma shows that in fact $\hat{T}_{01}$ satisfies the strict Fay relations. Its proof, which is annoyingly technical, is banished to Appendix B.

Lemma 3.3. *The mould $\hat{T}_{01}$ satisfies the strict Fay relations*

$$\mathcal{F}\big(\hat{T}'_{01}\big)(u_1, \ldots, u_r) = 0, \quad r \geq 2.$$

We will now apply Theorems 3.1 and 3.2 and Lemma 3.3 directly to give the explicit Fay correction for $\bar{\mathbf{a}}(\tau)$.

Theorem 3.4. *The Lie-like elliptic associator $\bar{\mathbf{a}}(\tau)$ satisfies the Fay relations*

$$\mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big) = C_{\bar{\mathbf{a}}'(\tau)}(u_1, \ldots, u_r)$$

for $r \geq 2$, where the correction mould $C_{\bar{\mathbf{a}}'(\tau)}$ is given for even r by

$$C_{\bar{\mathbf{a}}'(\tau)}(u_1, \ldots, u_r) = 0$$

and for odd r by

$$C_{\bar{\mathbf{a}}'(\tau)}(u_1, \ldots, u_r) = \zeta(r)(u_2 + \cdots + u_r) + \sum_{\substack{i=3 \\ i \text{ odd}}}^{r-1} \zeta(i)\Big(\hat{T}'_{01}(u_2, u_3, \ldots, u_{r-i+1}) - \hat{T}'_{01}(u_{i+1}, u_{i+2}, \ldots, u_r)\Big). \tag{3.2.7}$$

Proof. By (3.2.3), the Fay correction satisfied by $\bar{\mathbf{a}}(\tau)$ is the sum of that of $-\bar{\mathbf{e}}(\tau)$ and that of $Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}$. For even r, the Fay correction for $-\bar{\mathbf{e}}(\tau)$ is zero by Theorem 3.1. Since $\hat{T}_{01}$ satisfies the strict Fay relations by Lemma 3.3, the Fay correction for $Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}$ is given by (3.2.5). But in the case where r is even, the terms in the sum in (3.2.5) where i is even are all zero because $c_i = 0$, and the terms where i is odd are also all zero, because the terms $R'(u_2, \ldots, u_{r-i+1})$ and $R'(u_{i+1}, \ldots, u_r)$ are both of odd depth $r - i$, and the mould $R = \hat{T}_{01}$ is zero in all odd depths because of the Bernoulli coefficient in its definition (3.2.1).

When r is odd, the Fay correction term coming from $-\bar{\mathbf{e}}(\tau)$ is given in Theorem 3.1, and the Fay correction term coming from $Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}$ comes directly from (3.2.5), giving the desired formula (3.2.7) for the total Fay correction. This concludes the proof. ♢

Up to depth 5, the Fay correction for $\bar{\mathbf{a}}(\tau)$ is given by

$$\begin{cases} \mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1, u_2) = 0 \\ \mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1, u_2, u_3) = \zeta(3)(u_2 + u_3) \\ \mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1, u_2, u_3, u_4) = 0 \\ \mathcal{F}\big(\bar{\mathbf{a}}'(\tau)\big)(u_1, u_2, u_3, u_4, u_5) = \zeta(5)(u_2 + u_3 + u_4 + u_5) + \zeta(3)\Big(\dfrac{u_2 - u_3}{12u_2u_3} - \dfrac{u_4 - u_5}{12u_4u_5}\Big). \end{cases} \tag{3.2.8}$$

§3.3. The group-like elliptic generating series and elliptic associator.

In this section we briefly recall how to "undo" the linearization of the mould $\bar{\mathbf{e}}(\tau)$ and the double linearization of the mould $\bar{\mathbf{a}}(\tau)$, so as to explicitly compute the Fay corrections of the *reduced group-like elliptic generating series* $\bar{E}(\tau)$ and the *reduced group-like elliptic associator* $\bar{A}(\tau)$. We underline the point that the natural place for simple closed formulas is in the Lie-like situation; the formulas in the group-like situation are deduced from these directly, for which reason we do not attempt to give a closed formula, but simply show the process.

In fact, we saw in (3.2.6) of Theorem 3.2 that if in fact $swap(\Delta^{-1}N)$ is strictly circ-neutral and R satisfies the strict Fay relations, then $P = Darit(N)\cdot R$ also satisfies the strict Fay relations. Thus also

$$\exp\big(Darit(N)\big)\cdot R = R + Darit(N)\cdot R + \frac{1}{2}Darit(N)^2\cdot R + \cdots$$

satisfies the strict Fay relations, so in the case of moulds satisfying the strict relations, the closed formula is immediate. It is only the correction moulds involved in the construction of $\bar{\mathbf{e}}(\tau)$ and $\bar{\mathbf{a}}(\tau)$ which make a step-by-step calculation of the correction moulds in the group-like case necessary.

Let us start with the group-like elliptic generating series. Recall from (2.3.3) that $\bar{E}_\tau = \Psi(a) = exp(\psi)(a)$. Thus in mould terms, since we saw in (2.4.2) that the derivation ψ corresponds to the derivation $Darit\big(\bar{\mathbf{e}}(\tau)\big)$, we have

$$\bar{E}(\tau) = exp\Big(Darit\big(\bar{\mathbf{e}}(\tau)\big)\Big)\cdot a = \sum_{n\geq 0}\frac{1}{n!}Darit^n\big(\bar{\mathbf{e}}(\tau)\big)\cdot a$$

which since $Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot a = \bar{\mathbf{e}}(\tau)$ by (2.4.1), we rewrite as

$$\bar{E}(\tau) = a + \bar{\mathbf{e}}(\tau) + \sum_{n\geq 2}\frac{1}{n!}Darit^{n-1}\big(\bar{\mathbf{e}}(\tau)\big)\cdot\bar{\mathbf{e}}(\tau). \tag{4.2.1}$$

The depth 0 term a does not contribute to the Fay correction mould for $\bar{E}(\tau)$. Theorem 3.1 gives the contribution of the term $\bar{\mathbf{e}}(\tau)$. Now set $N = R = \bar{\mathbf{e}}(\tau)$. Then the Fay correction mould $C_{R'}$ is given in Theorem 3.1, and the constant correction mould C such that $swap\big(\Delta^{-1}\bar{\mathbf{e}}(\tau)\big) + C$ is circ-neutral is the mould defined in (0.4). Therefore (3.2.4) from Theorem 3.2 gives us the Fay correction mould for the $n = 2$ term $Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot\bar{\mathbf{e}}(\tau)$ of (4.2.1). Next we set $N = \bar{\mathbf{e}}(\tau)$ and $R = Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot\bar{\mathbf{e}}(\tau)$; then using the Fay correction just computed for this mould R, Theorem 3.2 gives the Fay correction mould for the $n = 3$ term $Darit\big(\bar{\mathbf{e}}(\tau)\big)^2\cdot\bar{\mathbf{e}}(\tau)$. Proceeding step by step in this manner, we compute each depth of the Fay correction mould. Up to depth 4, it is given by

$$\begin{cases}\mathcal{F}\big(\bar{E}'(\tau)\big)(u_1,u_2) = 0\\ \mathcal{F}\big(\bar{E}'(\tau)\big)(u_1,u_2,u_3) = 0\\ \mathcal{F}\big(\bar{E}'(\tau)\big)(u_1,u_2,u_3,u_4) = 2\zeta(3)\sum_{r\geq 3\ \mathrm{odd}}\zeta(r)\big(u_2^r - u_4^r\big).\end{cases} \tag{4.2.1}$$

We now turn to the case of $\bar{A}(\tau)$. Recall from (2.3.5) that $\bar{A}_\tau = \Psi(e^{t_{01}})$. Since $\bar{\mathbf{a}}(\tau)$ is doubly linearized with respect to $\bar{A}(\tau)$, our first step is to undo one level of linearization, by computing the Fay correction for the log of the elliptic associator, given by $\Psi(t_{01})$, or in mould terms by

$$\log\bar{A}(\tau) = \exp\Big(Darit\big(\bar{\mathbf{e}}(\tau)\big)\Big)\cdot T_{01}.$$

Since by (3.2.3) we have

$$\bar{\mathbf{a}}(\tau) = -\bar{\mathbf{e}}(\tau) + Darit\big(\bar{\mathbf{e}}(\tau)\big)\cdot\hat{T}_{01},$$

the log elliptic associator is given by

$$\begin{aligned}
\log \bar{A}(\tau) &= exp\big(Darit(\bar{\mathbf{e}}(\tau)\big) \cdot T_{01} \\
&= \sum_{n\geq 0} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^n \cdot T_{01} \\
&= T_{01} + Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot T_{01} + \sum_{n\geq 2} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^n \cdot T_{01} \\
&= T_{01} + \bar{\mathbf{a}}(\tau) + \sum_{n\geq 2} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^{n-1} \cdot \bar{\mathbf{a}}(\tau) \\
&= T_{01} + \bar{\mathbf{a}}(\tau) + \sum_{n\geq 2} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^{n-1} \cdot \Big(-\bar{\mathbf{e}}(\tau) + Darit\big(\bar{\mathbf{e}}(\tau)\big) \cdot \hat{T}_{01}\Big) \\
&= T_{01} + \bar{\mathbf{a}}(\tau) - \sum_{n\geq 2} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^{n-1} \cdot \bar{\mathbf{e}}(\tau) + \sum_{n\geq 2} \frac{1}{n!} Darit\big(\bar{\mathbf{e}}(\tau)\big)^{n} \cdot \hat{T}_{01}.
\end{aligned}$$

We already know how to determine the Fay corrections of T_{01} (which is zero) and $\bar{\mathbf{a}}(\tau)$ (which is given in Theorem 3.4). The third term is exactly the term calculated in the preceding part for the elliptic generating series. The fourth term is computed in the exact same way, using Theorem 3.2 repeatedly.

Up to depth 4, we find that the log elliptic associator satisfies the Fay relations up to depth 4 given by (3.4) and (3.5):

$$\begin{cases}
\mathcal{F}\big(\log \bar{A}'(\tau)\big)(u_1, u_2) = 0 \\
\mathcal{F}\big(\log \bar{A}'(\tau)\big)(u_1, u_2, u_3) = \zeta(3)(u_2 + u_3) \\
\mathcal{F}\big(\log \bar{A}'(\tau)\big)(u_1, u_2, u_3, u_4) = -\zeta(3) \sum_{r\geq 3 \text{ odd}} \zeta(r)(u_2^r - u_4^r)
\end{cases}$$

where we note that since dar is an automorphism, we have

$$\big(\log \bar{A}(\tau)\big)' = \log \bar{A}'(\tau).$$

We now undo the second level of linearization by computing the Fay correction for $\bar{A}(\tau)$ as the exponential of $\log \bar{A}(\tau)$. We show how to compute the Fay relations for $\exp(\mathcal{P})$ where $\mathcal{P} \in ARI$ is any mould satisfying a family of Fay relations with correction mould $C_{\mathcal{P}'}$. Just as for the log, since dar is an automorphism we have

$$exp(P)' = dar^{-1} exp(\mathcal{P}) = exp(\mathcal{P}') = 1 + \mathcal{P}' + \frac{1}{2}(\mathcal{P}')^2 + \frac{1}{6}(\mathcal{P}')^3 + \cdots$$

and furthermore

$$dar^{-1}(\mathcal{P}^r) = (\mathcal{P}')^r.$$

Thus

$$C_{exp(\mathcal{P})'}(u_1, \ldots, u_r) = \mathcal{F}\big(exp(\mathcal{P}')\big)(u_1, \ldots, u_r) = C_{\mathcal{P}'}(u_1, \ldots, u_r) + \sum_{r\geq 2} \frac{1}{r!} \mathcal{F}\big((\mathcal{P}')^r\big)(u_1, \ldots, u_r).$$

Then we write $\mathcal{P}' = log\big(exp(\mathcal{P}')\big)$ to get the final answer in terms of $exp(\mathcal{P}')$. Since $\mathcal{P}'(u_1) = exp(\mathcal{P}')(u_1)$ in depth 1, we find that in depth 2,

$$\begin{aligned}
C_{exp(\mathcal{P}')}(u_1, u_2) = C_{\mathcal{P}'}(u_1, u_2) &+ \frac{1}{2} exp(\mathcal{P}')(u_1) exp(\mathcal{P}')(u_2) \\
&+ \frac{1}{2} exp(\mathcal{P}')(-u_1) exp(\mathcal{P}')(u_1 + u_2) + \frac{1}{2} exp(\mathcal{P}')(u_2) exp(\mathcal{P}')(-u_1 - u_2).
\end{aligned}$$

and indeed the general formula for the Fay correction of $exp(\mathcal{P})$ is given by

$$C_{\mathcal{P}'}(u_1, \ldots, u_r) = \mathcal{F}(dar\mathcal{P}'')(u_1, \ldots, u_r), \tag{3.6}$$

where the mould $\mathcal{P}''$ is defined by

$$\mathcal{P}''(w) = \sum_{n=2}^{r} \frac{1}{n!} \sum_{w=a_1 \ldots a_n} \prod_{j=1}^{n} \mathcal{P}'(a_j)$$

with $w = (u_1, \ldots, u_r)$.

Thus the Fay correction mould for $\bar{A}(\tau)$ can be computed directly from that of $\log \bar{A}(\tau)$ with a closed formula. In depth 2, for example, we obtain the expression:

$$\mathcal{F}(\bar{A}'_\tau)(u_1, u_2) = \frac{1}{2}\bar{A}'_\tau(u_1)\bar{A}'_\tau(u_2) + \frac{1}{2}\bar{A}'_\tau(-u_1)\bar{A}'_\tau(u_1 + u_2) + \frac{1}{2}\bar{A}'_\tau(u_2)\bar{A}'_\tau(-u_1 - u_2).$$

Appendix A: Proof of Theorem 3.2.

Let N, M, C, R and $C_{R'}$ be as in the statement of the proposition, so M is bialternal with constant correction C and $N = \Delta M$, while R satisfies the Fay relations $\mathcal{F}(R') = C_{R'}$. Set $Q = R' = dar^{-1}R$; since R is assumed even in depth 1, the mould Q is odd in depth 1. We will use the identity

$$P = Darit(N) \cdot R = dar \cdot arat(M) \cdot dar^{-1}R = dar \cdot arat(M) \cdot Q,$$

where for each $M \in ARI$, $arat(M)$ is a derivation* of ARI given explicitly by

$$\big(arat(M) \cdot Q\big)(u_1, \dots, u_r) = \sum_{\substack{w=abc \\ ac,b\neq\emptyset}} \Big(Q(a\rceil c)M(b) - Q(a\lceil c)M(b)\Big),$$

where $w = (u_1, \dots, u_r)$, and the "flexion" notation is defined as follows: in any decomposition of $w = (u_1, \dots, u_r)$ into words $a_1 \cdots a_m$ (for example $w = abc$) with $a_l = (u_i, \dots, u_j)$, $a_{l+1} = (u_{j+1}, \dots, u_k)$, we set

$$\begin{cases} a_l\rceil = (u_i, \dots, u_{j-1}, u_j + u_{j+1} + \cdots + u_l) \\ \lceil a_{l+1} = (u_i + \cdots + u_j + u_{j+1}, u_{j+2}, \dots, u_k). \end{cases}$$

Starting from (A.3) below, we will also use the flexion notation for moulds in $\overline{ARI}$, which is defined as follows: in any decomposition of $w = (v_1, \dots, v_r)$ into words $a_1 \cdots a_m$ with $a_l = (v_i, \dots, v_j)$, $a_{l+1} = (v_{j+1}, \dots, v_k)$, we set

$$\begin{cases} a_l\rfloor = (v_i - v_{j+1}, \dots, v_j - v_{j+1}) \\ \lfloor a_{l+1} = (v_{j+1} - v_j, \dots, v_k - v_j). \end{cases}$$

The goal of this proof is to compute the Fay correction mould $C_{P'}$ for the Fay relations

$$C_{P'} := \mathcal{F}(P') = \mathcal{F}\big(arat(M) \cdot Q\big).$$

To do this, we will resort to the swap version of the Fay relations given in (2.3):

$$swap(P')\big(sh\big((v_1), (v_2, \dots, v_r)\big)\big) + push^{-1} \cdot swap(P')(v_1, \dots, v_r) = push^{-1} \cdot swap(C_{P'})(v_1, \dots, v_r). \quad (A.1)$$

Let us simplify the terminology by setting

$$\begin{cases} \mathbf{M} = swap(M) \\ \mathbf{Q} = swap(Q) \\ \mathbf{P} = swap(P') = swap(arat(M) \cdot Q) \\ C_{\mathbf{P}} = push^{-1} \cdot swap(C_{P'}). \end{cases}$$

Then (A.1) translates to

$$C_{\mathbf{P}}(v_1, \dots, v_r) := \mathbf{P}\Big(sh\big((v_1), (v_2, \dots, v_r)\big)\Big) + push^{-1}\mathbf{P}(v_1, \dots, v_r). \quad (A.2)$$

Letting $w = (v_1, \dots, v_r)$ and using the known commutation relations of *swap* with *arit* and with *lu* (cf. §2.4 of [S2]) together with the fact that M is push-invariant, we find that

$$\mathbf{P}(w) = swap\Big(arat\big(M\big) \cdot Q\Big)(w) = \sum_{\substack{w=abc \\ a,b\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(\lfloor b) - \sum_{\substack{w=abc \\ ac,b\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor) + \sum_{\substack{w=ab \\ a,b\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(b). \quad (A.3)$$

We can now proceed to the computation of the Fay correction mould $C_{P'}$ via the computation of $C_{\mathbf{P}}$, for which we compute and sum the two terms from the right-hand side of (A.3).

* In the more familiar notation used by Écalle and in [S1], we have $arat(M) \cdot Q = -arit(M) \cdot Q + lu(M, Q)$.

We do this by adapting a method due to A. Salerno (cf. [SS, appendix]), consisting in separating the terms running the v_1 through a, b and c as follows. We write $w' = v_2 \cdots v_r$, so $w = v_1 w'$.

$$\mathbf{P}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(\lfloor b) + \sum_{\substack{w'=abc\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(\lfloor sh(v_1,b)\big)$$

$$+ \sum_{\substack{w'=abc\\ a,b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b) - \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(b\rfloor) - \sum_{\substack{w'=abc\\ ac\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(sh(v_1,b)\rfloor\big)$$

$$- \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(b\rfloor) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(sh(v_1,b)). \tag{A.4}$$

We split up the second and fifth sums of (A.4) as follows:

$$\sum_{\substack{w'=abc\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(\lfloor sh(v_1,b)\big) - \sum_{\substack{w'=abc\\ ac\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(sh(v_1,b)\rfloor\big) =$$

$$\sum_{\substack{w'=abc\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(\lfloor sh(v_1,b)\big) - \sum_{\substack{w'=abc\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}\big(sh(v_1,b)\rfloor\big) - \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} \mathbf{Q}(c)\mathbf{M}\big(sh(v_1,b)\rfloor\big) - \mathbf{Q}(w')\mathbf{M}(v_1-v_2).$$

Since $M+C$ is alternal, where $C=(c_i)_{i\geq 0}$ is the constant correction mould for M, we can write

$$\mathbf{M}\big(sh(v_1,b)\big) = \mathbf{M}\big(\lfloor sh(v_1,b)\big) = \mathbf{M}\big(sh(v_1,b)\rfloor\big) = -(|b|+1)c_{|b|+1}$$

for $|b|>1$, so we see that all the terms for $b\neq\emptyset$ in the first two sums cancel out, and it becomes equal to

$$\sum_{\substack{w'=ac\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(v_1-a_l) - \sum_{\substack{w'=ac\\ a\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(v_1-c_f) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} \mathbf{Q}(c)(|b|+1)c_{|b|+1} - \mathbf{Q}(w')\mathbf{M}(v_1-v_2).$$

Writing $a=(v_2,\dots,v_i)$ and $c=(v_{i+1},\dots,v_r)$, and setting $v_{r+1}=0$, this can be rewritten as

$$\sum_{i=2}^{r} \mathbf{Q}(v_2,\dots,v_r)\mathbf{M}(v_1-v_i) - \sum_{i=2}^{r} \mathbf{Q}(v_2,\dots,v_r)\mathbf{M}(v_1-v_{i+1}) - \mathbf{Q}(v_2,\dots,v_r)\mathbf{M}(v_1-v_2) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(c),$$

in which most terms cancel out, so that finally the second and fifth term in (A.4) sum to just

$$-\mathbf{Q}(v_2,\dots,v_r)\mathbf{M}(v_1) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(c).$$

Replacing this in (A.4), we obtain

$$\mathbf{P}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(\lfloor b) + \sum_{\substack{w'=abc\\ a,b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b)$$

$$- \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(b\rfloor) - \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(b\rfloor) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b)$$

$$+ \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(sh(v_1,b)) - \mathbf{Q}(v_2,\dots,v_r)\mathbf{M}(v_1) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(c). \tag{A.5}$$

Let us now add and subtract the term $a = \emptyset$ to the second sum of (A.5), in order to make the first four sums all over the same indices:

$$\sum_{\substack{w'=abc\\ a,b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b) = \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b) - \sum_{\substack{w'=bc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,c)\big)\mathbf{M}(b).$$

We will also modify the form of the sixth term in (A.5), noting that because of the $a\rfloor$ term, the shuffle term in M is not an independent factor: for example, if $r = 4$, in the decomposition $w' = ab = (v_2)(v_3, v_4)$, the sixth term is

$$\mathbf{Q}(v_2 - v_1)\mathbf{M}(v_1, v_3, v_4) + \mathbf{Q}(v_2 - v_3)\mathbf{M}(v_3, v_1, v_4) + \mathbf{Q}(v_2 - v_3)\mathbf{M}(v_3, v_4, v_1),$$

since the term $a\rfloor$ is with respect to v_1 in the first shuffle term (v_1, v_3, v_4), but with respect to v_3 for all the other shuffle terms of $sh\big((v_1), (v_3, v_4)\big)$. To take advantage of the alternality of $\mathbf{M}$, we rewrite the sixth term as

$$\sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(sh(v_1,b)) = \mathbf{Q}(v_2 - v_1, \ldots, v_r - v_1)\mathbf{M}(v_1) + \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(sh(v_1,b))$$

$$+ \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1 b) - \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1 b)$$

$$= \mathbf{Q}(v_2 - v_1, \ldots, v_r - v_1)\mathbf{M}(v_1) + \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1 b) - \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1 b) - \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)(|b|+1)c_{|b|+1}$$

Plugging these two modifications into (A.5), we obtain

$$\mathbf{P}\Big(sh\big((v_1), (v_2, \ldots, v_r)\big)\Big) = \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(\lfloor b) + \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b) - \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(b\rfloor)$$

$$- \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(b\rfloor) - \sum_{\substack{w'=bc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,c)\big)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b)$$

$$+\mathbf{Q}(v_2 - v_1, \ldots, v_r - v_1)\mathbf{M}(v_1) + \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1 b) - \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1 b)$$

$$- \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)(|b| + 1)c_{|b|+1} - \mathbf{Q}(v_2, \ldots, v_r)\mathbf{M}(v_1) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b| + 1)c_{|b|+1}\mathbf{Q}(c). \qquad (A.6)$$

The seventh term is the $b = \emptyset$ case of the eighth term and the eleventh term is the $b = \emptyset$ case of the ninth term, so we can simplify this as

$$\mathbf{P}\Big(sh\big((v_1), (v_2, \ldots, v_r)\big)\Big) = \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(\lfloor b) + \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(\lfloor b)$$

$$- \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,a)c\big)\mathbf{M}(b\rfloor) - \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(a\, sh(v_1,c)\big)\mathbf{M}(b\rfloor) - \sum_{\substack{w'=bc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,c)\big)\mathbf{M}(b)$$

$$+ \sum_{\substack{w'=ab\\ b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1 b) - \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1 b)$$

$$- \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)(|b| + 1)c_{|b|+1} + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b| + 1)c_{|b|+1}\mathbf{Q}(c). \qquad (A.7)$$

We will now recompute the sum of the first four terms in (A.7), with the decompositions

$$w' = abc = (v_1, \dots, v_i)(v_{i+1}, \dots, v_j)(v_{j+1}, \dots, v_r).$$

Set $a = a'v_i$ and $c = v_{j+1}c'$, and observe that in the first term $\mathbf{Q}\big(sh(v_1, a)c\big)\mathbf{M}(\lfloor b)$, the flexion $\lfloor b$ is given by

$$\lfloor b = \begin{cases} (v_{i+1} - v_i, \dots, v_j - v_i) & \text{for every term in the shuffle } sh(v_1, a')v_i \\ (v_{i+1} - v_1, \dots, v_j - v_1) & \text{for the term } av_1. \end{cases}$$

In the second term $\mathbf{Q}\big(a\, sh(v_1, c)\big)\mathbf{M}(\lfloor b)$, however, the left flexion is always $\lfloor b = (v_{i+1} - v_i, \dots, v_j - v_i)$. Similarly, in the third term $\mathbf{Q}\big(sh(v_1, a)c\big)\mathbf{M}(b\rfloor)$ the right flexion is always $b\rfloor = (v_{i+1} - v_{j+1}, \dots, v_j - v_{j+1})$, but in the fourth term $\mathbf{Q}\big(a\, sh(v_1, c)\big)\mathbf{M}(b\rfloor)$, it is given by

$$b\rfloor = \begin{cases} (v_{i+1} - v_{j+1}, \dots, v_j - v_{j+1}) & \text{for every term in } v_{j+1}sh(v_1, c') \\ (v_{i+1} - v_1, \dots, v_j - v_1) & \text{for } v_1c. \end{cases}$$

Observing that writing $a = a'v_i$ and $c = v_{j+1}c'$, we have

$$\mathbf{Q}\big(sh(v_1, a)c\big) = \mathbf{Q}\big(sh(v_1, a')v_ic\big) + \mathbf{Q}(av_1c).$$

and

$$\mathbf{Q}\big(a\, sh(v_1, c)\big) = \mathbf{Q}(av_1c) + \mathbf{Q}\big(av_{j+1}sh(v_1, c')\big),$$

we recombine the first four terms in (A.7) as

$$\begin{aligned}
\mathbf{P}\Big(sh\big((v_1), (v_2, \dots, v_r)\big)\Big) = &\sum_{\substack{w'=abc \\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1, ac)\big)\mathbf{M}(b - a_l) + \sum_{\substack{w'=abc \\ b\neq\emptyset}} \mathbf{Q}\big(av_1c\big)\mathbf{M}(b - v_1) \\
&- \sum_{\substack{w'=abc \\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1, ac)\big)\mathbf{M}(b - c_f) - \sum_{\substack{w'=abc \\ b\neq\emptyset}} \mathbf{Q}\big(av_1c\big)\mathbf{M}(b - v_1) \\
&- \sum_{\substack{w'=bc \\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1, c)\big)\mathbf{M}(b) + \sum_{\substack{w'=ab \\ b\neq\emptyset}} \mathbf{Q}(sh(v_1, a)\rfloor)\mathbf{M}(b) \\
&+ \sum_{\substack{w'=ab \\ a\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1b) - \sum_{\substack{w'=ab \\ a\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1b) \\
&- \sum_{\substack{w'=ab \\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)(|b| + 1)c_{|b|+1} + \sum_{\substack{w'=bc \\ b,c\neq\emptyset}} (|b| + 1)c_{|b|+1}\mathbf{Q}(c)
\end{aligned} \tag{A.8}$$

where we use the notation

$$w' = abc = (v_1, \dots, v_i)(v_{i+1}, \dots, v_j)(v_{j+1}, \dots, v_r)$$

and

$$a_l = \begin{cases} a_{last} = v_i & \text{if } a \neq \emptyset \\ a_l = 0 & \text{if } a = \emptyset \end{cases}, \qquad c_f = \begin{cases} c_{first} = v_{j+1} & \text{if } c \neq \emptyset \\ c_f = 0 & \text{if } c = \emptyset. \end{cases}$$

The second and fourth terms in (A.8) cancel out, leaving

$$\begin{aligned}
\mathbf{P}\Big(sh\big((v_1), (v_2, \dots, v_r)\big)\Big) = &\sum_{\substack{w'=abc \\ b,ac\neq\emptyset}} \mathbf{Q}\big(sh(v_1, ac)\big)\mathbf{M}(b - a_l) - \sum_{\substack{w'=abc \\ b,ac\neq\emptyset}} \mathbf{Q}\big(sh(v_1, ac)\big)\mathbf{M}(b - c_f) \\
&- \sum_{\substack{w'=bc \\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1, c)\big)\mathbf{M}(b) + \sum_{\substack{w'=ab \\ b\neq\emptyset}} \mathbf{Q}(sh(v_1, a)\rfloor)\mathbf{M}(b) \\
&+ \sum_{\substack{w'=ab \\ a\neq\emptyset}} \mathbf{Q}(a - v_1)M(v_1b) - \sum_{\substack{w'=ab \\ a\neq\emptyset}} \mathbf{Q}(a - b_1)\mathbf{M}(v_1b) \\
&- \sum_{\substack{w'=ab \\ a,b\neq\emptyset}} \mathbf{Q}(a - b_1)(|b| + 1)c_{|b|+1} + \sum_{\substack{w'=bc \\ b,c\neq\emptyset}} (|b| + 1)c_{|b|+1}\mathbf{Q}(c)
\end{aligned} \tag{A.9}$$

(where we note that the terms with $ac = \emptyset$ cancel out in the first two sums, so we can assume $ac \neq \emptyset$ in these sums).

We will now finish our calculation by using the swapped version of the Fay relation satisfied by $\mathbf{Q}$:

$$\mathbf{Q}(sh(a,b)) = -push^{-1}\mathbf{Q}(ab) + C_{\mathbf{Q}}(ab)$$

for any non-empty sequences a, b, to simplify the first four terms of (A.9).

$$\sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,ac)\big)\mathbf{M}(b-a_l) - \sum_{\substack{w'=abc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,ac)\big)\mathbf{M}(b-c_f) = \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-c_f)$$
$$- \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-a_l) + \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-a_l) - \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-c_f).$$

Furthermore, we use the swapped Fay to simplify the third term in (A.9) as

$$\begin{aligned}
-\sum_{\substack{w'=bc\\ b\neq\emptyset}} \mathbf{Q}\big(sh(v_1,c)\big)\mathbf{M}(b) &= -\sum_{\substack{w'=bc\\ b,c\neq\emptyset}} \mathbf{Q}\big(sh(v_1,c)\big)\mathbf{M}(b) - \mathbf{Q}(v_1)\mathbf{M}(v_2,\dots,v_r)\\
&= \sum_{\substack{w'=bc\\ b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1c)\mathbf{M}(b) - \sum_{\substack{w'=bc\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1c)\mathbf{M}(b) - \mathbf{Q}(v_1)\mathbf{M}(v_2,\dots,v_r),
\end{aligned}$$

and the fourth term in (A.9) as

$$\begin{aligned}
\sum_{\substack{w'=ab\\ b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b) &= \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(sh(v_1,a)\rfloor)\mathbf{M}(b) + \mathbf{Q}(v_1-v_2)\mathbf{M}(v_2,\dots,v_r)\\
&= -\sum_{\substack{w'=ab\\ a,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1a\rfloor)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1a\rfloor)\mathbf{M}(b) + \mathbf{Q}(v_1-v_2)\mathbf{M}(v_2,\dots,v_r).
\end{aligned}$$

Replacing the first four terms in (A.9) in this way, we obtain

$$\begin{aligned}
\mathbf{P}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) &= \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-c_f) - \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-a_l)\\
&+ \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} push^{-1}\mathbf{Q}(v_1c)\mathbf{M}(b) - \mathbf{Q}(v_1)\mathbf{M}(v_2,\dots,v_r)\\
&- \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1a\rfloor)\mathbf{M}(b) + \mathbf{Q}(v_1-v_2)\mathbf{M}(v_2,\dots,v_r)\\
&+ \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a-v_1)M(v_1b) - \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a-b_1)\mathbf{M}(v_1b)\\
&- \sum_{\substack{w'=bc\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1c)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1a\rfloor)\mathbf{M}(b)\\
&+ \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-a_l) - \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-c_f)\\
&- \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a-b_1)(|b|+1)c_{|b|+1} + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(c)
\end{aligned} \tag{A.10}$$

Let us write (A.10) as

$$\mathbf{P}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = \mathcal{A} + \mathcal{C},$$

where $\mathcal{C}$ denotes the main part given by the first eight terms and $\mathcal{C}$ denotes the correction part given in the last six terms. In order to prove (A.2), we will show in (A.11)-(A.17) below that

$$push(\mathcal{A})(v_1,\dots,v_r) = -P(v_1,\dots,v_r), \quad \text{i.e.} \quad (\mathcal{A})(v_1,\dots,v_r) = -push^{-1}P(v_1,\dots,v_r),$$

and thus $\mathcal{C}$ is the desired correction mould $C_{\mathbf{P}}$.

To apply the push-operator to $\mathcal{A}$, we first break the first two terms of $\mathcal{A}$ into two sums.

$$\begin{aligned}
\mathcal{A}(v_1,\dots,v_r) = &\sum_{\substack{w'=abc\\ b,c\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-c_f) + \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1a)\mathbf{M}(b)\\
&- \sum_{\substack{w'=abc\\ a,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1ac)\mathbf{M}(b-a_l) - \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} push^{-1}\mathbf{Q}(v_1c)\mathbf{M}(b)\\
&+ \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} push^{-1}\mathbf{Q}(v_1c)\mathbf{M}(b) - \mathbf{Q}(v_1)\mathbf{M}(v_2,\dots,v_r)\\
&- \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} push^{-1}\mathbf{Q}(v_1a\rfloor)\mathbf{M}(b) + \mathbf{Q}(v_1-v_2)\mathbf{M}(v_2,\dots,v_r)\\
&+ \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a-v_1)M(v_1b) - \sum_{\substack{w'=ab\\ a\neq\emptyset}} \mathbf{Q}(a-b_1)\mathbf{M}(v_1b).
\end{aligned} \tag{A.11}$$

The fourth and fifth terms cancel. Writing this out in indices with $abc = (v_2,\dots,v_i)(v_{i+1},\dots,v_j)(v_{j+1},\dots,v_r)$, we find

$$\begin{aligned}
\mathcal{A}(v_1,\dots,v_r) = &\sum_{1\le i<j<r} push^{-1}\mathbf{Q}(v_1,v_2,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_{j+1},\dots,v_j-v_{j+1})\\
&+ \sum_{1<i<r} push^{-1}\mathbf{Q}(v_1,v_2,\dots,v_i)\mathbf{M}(v_{i+1},\dots,v_r)\\
&- \sum_{1<i<j\le r} push^{-1}\mathbf{Q}(v_1,v_2,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_i,\dots,v_j-v_i)\\
&- \sum_{1<i<r} push^{-1}\mathbf{Q}(v_1-v_{i+1},\dots,v_i-v_{i+1})\mathbf{M}(v_{i+1},\dots,v_r)\\
&+ \sum_{1<i\le r} \mathbf{Q}(v_2-v_1,\dots,v_i-v_1)\mathbf{M}(v_1,v_{i+1},\dots,v_r)\\
&- \sum_{1<i\le r} \mathbf{Q}(v_2-v_{i+1},\dots,v_i-v_{i+1})\mathbf{M}(v_1,v_{i+1},\dots,v_r)\\
&- \mathbf{Q}(v_1)\mathbf{M}(v_2,\dots,v_r) + \mathbf{Q}(v_1-v_2)\mathbf{M}(v_2,\dots,v_r).
\end{aligned} \tag{A.12}$$

Now we compute $push(\mathcal{A})$ by applying the $push$ operator to (A.12): it maps $v_1 \mapsto -v_r$ and $v_k \mapsto v_{k-1} - v_r$

for $2 \le k \le r$, so we obtain

$$
\begin{aligned}
push(\mathcal{A})(v_1,\dots,v_r) = &\sum_{1\le i<j<r} push^{-1}\mathbf{Q}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r, v_j-v_r,\dots,v_{r-1}-v_r)\mathbf{M}(v_i-v_j,\dots,v_{j-1}-v_j)\\
&+\sum_{1<i<r} push^{-1}\mathbf{Q}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\sum_{1<i<j\le r} push^{-1}\mathbf{Q}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r, v_j-v_r,\dots,v_{r-1}-v_r)\mathbf{M}(v_i-v_{i-1},\dots,v_{j-1}-v_{i-1})\\
&-\sum_{1<i<r} push^{-1}\mathbf{Q}(-v_i, v_1-v_i,\dots,v_{i-1}-v_i)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&+\sum_{1<i\le r} \mathbf{Q}(v_1,\dots,v_{i-1})\mathbf{M}(-v_r, v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\sum_{1<i\le r} \mathbf{Q}(v_1-v_i,\dots,v_{i-1}-v_i)\mathbf{M}(-v_r, v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\mathbf{Q}(-v_r)\mathbf{M}(v_1-v_r,\dots,v_{r-1}-v_r)+\mathbf{Q}(-v_1)\mathbf{M}(v_1-v_r,\dots,v_{r-1}-v_r).
\end{aligned}
\tag{A.13}
$$

Now we unpush the $\mathbf{Q}$s (recall that Q is odd in depth 1), and also some of the $\mathbf{M}$'s (given that $\mathbf{M}$ being bialternal is push-invariant):

$$
\begin{aligned}
push(\mathcal{A})(v_1,\dots,v_r) = &\sum_{1\le i<j<r} \mathbf{Q}(v_1,\dots,v_{i-1},v_j,\dots,v_r)\mathbf{M}(v_i-v_j,\dots,v_{j-1}-v_j)\\
&+\sum_{1<i<r} \mathbf{Q}(v_1,\dots,v_{i-1},v_r)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\sum_{1<i<j\le r} \mathbf{Q}(v_1,\dots,v_{i-1},v_j,\dots,v_r)\mathbf{M}(v_i-v_{i-1},\dots,v_{j-1}-v_{i-1})\\
&-\sum_{1<i<r} \mathbf{Q}(v_1,\dots,v_i)\mathbf{M}(v_{i+1}-v_i,\dots,v_r-v_i)\\
&+\sum_{1<i\le r} \mathbf{Q}(v_1,\dots,v_{i-1})\mathbf{M}(v_i,\dots,v_r)\\
&-\sum_{1<i\le r} \mathbf{Q}(v_1-v_i,\dots,v_{i-1}-v_i)\mathbf{M}(v_i,\dots,v_r)\\
&+\mathbf{Q}(v_r)\mathbf{M}(v_1-v_r,\dots,v_{r-1}-v_r)-\mathbf{Q}(v_1)\mathbf{M}(v_2-v_1,\dots,v_r-v_1).
\end{aligned}
\tag{A.14}
$$

Now we rewrite this in terms of $w = abc = (v_1,\dots,v_{i-1})(v_i,\dots,v_{j-1})(v_j,\dots,v_r)$:

$$
\begin{aligned}
push(\mathcal{A})(v_1,\dots,v_r) = &\sum_{\substack{w=abc\\ b\neq\emptyset,|c|\ge 2}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor)+\sum_{\substack{w=abc\\ a,b\neq\emptyset,|c|=1}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor)-\sum_{\substack{w=abc\\ a,b,c\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(\lfloor b)\\
&-\sum_{\substack{w=ab\\ |a|\ge 2,b\neq\emptyset}} \mathbf{Q}(a)M(\lfloor b)+\sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a)\mathbf{M}(b)-\sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(b)\\
&+\mathbf{Q}(v_r)\mathbf{M}(v_1-v_r,\dots,v_{r-1}-v_r)-\mathbf{Q}(v_1)\mathbf{M}(v_2-v_1,\dots,v_r-v_1).
\end{aligned}
\tag{A.15}
$$

Term 7 is the $a=\emptyset$ case of term 2, and term 8 is the $|a|=1$ case of term 4:

$$
\begin{aligned}
push(\mathcal{A})(v_1,\dots,v_r) = &\sum_{\substack{w=abc\\ b\neq\emptyset,|c|\ge 2}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor)+\sum_{\substack{w=abc\\ b\neq\emptyset,|c|=1}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor)-\sum_{\substack{w=abc\\ a,b,c\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(\lfloor b)\\
&-\sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a)M(\lfloor b)+\sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a)\mathbf{M}(b)-\sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(b).
\end{aligned}
\tag{A.16}
$$

Finally, the first, second and fifth terms combine, and the third and fourth combine, to form

$$push(\mathcal{A})(v_1,\dots,v_r) = \sum_{\substack{w=abc\\ ac,b\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(b\rfloor) - \sum_{\substack{w=abc\\ a,b\neq\emptyset}} \mathbf{Q}(ac)\mathbf{M}(\lfloor b) - \sum_{\substack{w=ab\\ a,b\neq\emptyset}} \mathbf{Q}(a\rfloor)\mathbf{M}(b). \tag{A.17}$$

Comparing this with (A.3), we see that $push(\mathcal{A})(v_1,\dots,v_r)$ is equal to $-\mathbf{P}(v_1,\dots,v_r)$. Thus since we rephrased (A.10) as

$$\mathbf{P}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) = \mathcal{A}(v_1,\dots,v_r) + \mathcal{C}(v_1,\dots,v_r)$$

and (A.17) shows that

$$\mathcal{A}(v_1,\dots,v_r) = -push^{-1}\mathbf{P}(v_1,\dots,v_r),$$

comparison with (A.2) shows that $C_{\mathbf{P}}$ is given by $\mathcal{C}$, i.e. we have

$$\begin{aligned} C_{\mathbf{P}}(v_1,\dots,v_r) = &- \sum_{\substack{w'=bc\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1c)\mathbf{M}(b) + \sum_{\substack{w'=ab\\ b\neq\emptyset}} C_{\mathbf{Q}}(v_1a\rfloor)\mathbf{M}(b) + \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-a_l)\\ &- \sum_{\substack{w'=abc\\ ac,b\neq\emptyset}} C_{\mathbf{Q}}(v_1ac)\mathbf{M}(b-c_f) - \sum_{\substack{w'=ab\\ a,b\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(a-b_1) + \sum_{\substack{w'=bc\\ b,c\neq\emptyset}} (|b|+1)c_{|b|+1}\mathbf{Q}(c). \end{aligned} \tag{A.18}$$

To compute the desired correction $C_{P'}$, recall that

$$C_{\mathbf{P}} = push^{-1}\cdot swap(C_{P'}), \quad i.e.\ \ C_{P'} = swap\cdot push(C_{\mathbf{P}}).$$

To compute $C_{P'}$ explicitly, we first write (A.18) with indices, then push it, and finally swap the result.

$$\begin{aligned} C_{\mathbf{P}}(v_1,\dots,v_r) = &- \sum_{1<i\leq r} C_{\mathbf{Q}}(v_1,v_{i+1},\dots,v_r)\mathbf{M}(v_2,\dots,v_i)\\ &+ \sum_{1\leq i<r} C_{\mathbf{Q}}(v_1-v_{i+1},v_2-v_{i+1},\dots,v_i-v_{i+1})\mathbf{M}(v_{i+1},\dots,v_r)\\ &+ \sum_{1<i<j\leq r} C_{\mathbf{Q}}(v_1,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_i,\dots,v_j-v_i) + \sum_{1<i<r} C_{\mathbf{Q}}(v_1,v_{i+1},\dots,v_r)\mathbf{M}(v_2,\dots,v_i)\\ &- \sum_{1<i<j<r} C_{\mathbf{Q}}(v_1,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_{j+1},\dots,v_j-v_{j+1}) - \sum_{1<i<r} C_{\mathbf{Q}}(v_1,\dots,v_i)\mathbf{M}(v_{i+1},\dots,v_r)\\ &- \sum_{1<i<r} (r-i+1)c_{r-i+1}\mathbf{Q}(v_2-v_{i+1},\dots,v_i-v_{i+1}) + \sum_{1<i<r} iC_i\mathbf{Q}(v_{i+1},\dots,v_r). \end{aligned} \tag{A.19}$$

The first and fourth term combine to form

$$-\sum_{1<i\leq r} C_{\mathbf{Q}}(v_1,v_{i+1},\dots,v_r)\mathbf{M}(v_2,\dots,v_i) + \sum_{1<i<r} C_{\mathbf{Q}}(v_1,v_{i+1},\dots,v_r)\mathbf{M}(v_2,\dots,v_i) = -C_{\mathbf{Q}}(v_1)\mathbf{M}(v_2,\dots,v_r)$$

which then combines with the sixth sum, becoming

$$\begin{aligned} C_{\mathbf{P}}(v_1,\dots,v_r) = &\sum_{1\leq i<r} C_{\mathbf{Q}}(v_1-v_{i+1},v_2-v_{i+1},\dots,v_i-v_{i+1})\mathbf{M}(v_{i+1},\dots,v_r)\\ &+ \sum_{1<i<j\leq r} C_{\mathbf{Q}}(v_1,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_i,\dots,v_j-v_i)\\ &- \sum_{1<i<j<r} C_{\mathbf{Q}}(v_1,\dots,v_i,v_{j+1},\dots,v_r)\mathbf{M}(v_{i+1}-v_{j+1},\dots,v_j-v_{j+1})\\ &- \sum_{1\leq i<r} C_{\mathbf{Q}}(v_1,\dots,v_i)\mathbf{M}(v_{i+1},\dots,v_r)\\ &- \sum_{1<i<r} (r-i+1)c_{r-i+1}\mathbf{Q}(v_2-v_{i+1},\dots,v_i-v_{i+1}) + \sum_{1<i<r} iC_i\mathbf{Q}(v_{i+1},\dots,v_r). \end{aligned} \tag{A.20}$$

So we have

$$
\begin{aligned}
push(C_{\mathbf{P}})(v_1,\dots,v_r) = &\sum_{1\le i<r} C_{\mathbf{Q}}(-v_i, v_1-v_i,\dots,v_{i-1}-v_i)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&+\sum_{1<i<j\le r} C_{\mathbf{Q}}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r, v_j-v_r,\dots,v_{r-1}-v_r)\mathbf{M}(v_i-v_{i-1},\dots,v_{j-1}-v_{i-1})\\
&-\sum_{1<i<j<r} C_{\mathbf{Q}}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r, v_j-v_r,\dots,v_{r-1}-v_r)\mathbf{M}(v_i-v_j,\dots,v_{j-1}-v_j)\\
&-\sum_{1\le i<r} C_{\mathbf{Q}}(-v_r, v_1-v_r,\dots,v_{i-1}-v_r)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\sum_{1<i<r} (r-i+1)c_{r-i+1}\mathbf{Q}(v_1-v_i,\dots,v_{i-1}-v_i) + \sum_{1<i<r} iC_i\mathbf{Q}(v_i-v_r,\dots,v_{r-1}-v_r).
\end{aligned}
\tag{A.21}
$$

Rewriting this in terms of $push(C_{\mathbf{Q}})$ gives

$$
\begin{aligned}
push(C_{\mathbf{P}})(v_1,\dots,v_r) = &\sum_{1\le i<r} pushC_{\mathbf{Q}}(v_1,\dots,v_{i-1},v_i)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&+\sum_{1<i<j\le r} pushC_{\mathbf{Q}}(v_1,\dots,v_{i-1},v_j,\dots,v_{r-1},v_r)\mathbf{M}(v_i-v_{i-1},\dots,v_{j-1}-v_{i-1})\\
&-\sum_{1<i<j<r} pushC_{\mathbf{Q}}(v_1,\dots,v_{i-1},v_j,\dots,v_{r-1},v_r)\mathbf{M}(v_i-v_j,\dots,v_{j-1}-v_j)\\
&-\sum_{1\le i<r} pushC_{\mathbf{Q}}(v_1,\dots,v_{i-1},v_r)\mathbf{M}(v_i-v_r,\dots,v_{r-1}-v_r)\\
&-\sum_{1<i<r} (r-i+1)c_{r-i+1}\mathbf{Q}(v_1-v_i,\dots,v_{i-1}-v_i) + \sum_{1<i<r} iC_i\mathbf{Q}(v_i-v_r,\dots,v_{r-1}-v_r).
\end{aligned}
\tag{A.22}
$$

Now we compute $C_{P'} = swap\cdot push(C_{\mathbf{P}})$. We use the notation $u_i+\cdots+u_j = u_{i,j}$ for $i\le j$.

$$
\begin{aligned}
C_{P'}(u_1,\dots,u_r) &= swap\cdot push(C_{\mathbf{P}})(u_1,\dots,u_r)\\
&= \sum_{1\le i<r} pushC_{\mathbf{Q}}(u_{1,r},\dots,u_{1,r-i+2},u_{1,r-i+1})\mathbf{M}(u_{2,r-i+1},u_{2,r-i},\dots,u_2)\\
&+\sum_{1<i<j\le r} pushC_{\mathbf{Q}}(u_{1,r},\dots,u_{1,r-i+2},u_{1,r-j+1},\dots,u_{1,2},u_1)\mathbf{M}(-u_{r-i+2},\dots,-u_{r-j+3,r-i+2})\\
&-\sum_{1<i<j<r} pushC_{\mathbf{Q}}(u_{1,r},\dots,u_{1,r-i+2},u_{1,r-j+1},\dots,u_{1,2},u_1)\mathbf{M}(u_{r-j+2,r-i+1},\dots,u_{r-j+2})\\
&-\sum_{1\le i<r} pushC_{\mathbf{Q}}(u_{1,r},\dots,u_{1,r-i+2},u_1)\mathbf{M}(u_{2,r-i+1},\dots,u_2)\\
&-\sum_{1<i<r} (r-i+1)c_{r-i+1}\mathbf{Q}(u_{r-i+2,r},\dots,u_{r-i+2}) + \sum_{1<i<r} iC_i\mathbf{Q}(u_{2,r-i+1},\dots,u_2).
\end{aligned}
\tag{A.23}
$$

Now we use the fact that

$$swap(C)(u_1,\dots,u_r) = C(u_{1,r},u_{1,r-1},\dots,u_{1,2},u_1).$$

$$
\begin{aligned}
C_{P'}(u_1,\dots,u_r) &= swap\cdot push(C_{\mathbf{P}})(u_1,\dots,u_r)\\
&= \sum_{1\le i<r} swap\cdot push C_{\mathbf{Q}}(u_{1,r-i+1},u_{r-i+2},\dots,u_r)swap\mathbf{M}(u_2,\dots,u_{r-i+1})\\
&+ \sum_{1<i<j\le r} swap\cdot push C_{\mathbf{Q}}(u_1,\dots,u_{r-j+1},u_{r-j+2,r-i+2},u_{r-i+3},\dots,u_r)swap\mathbf{M}(u_{r-j+3},\dots,u_{r-i+2})\\
&- \sum_{1<i<j<r} swap\cdot push C_{\mathbf{Q}}(u_1,\dots,u_{r-j+1},u_{r-j+2,r-i+2},u_{r-i+3},\dots,u_r)swap\mathbf{M}(u_{r-j+2},\dots,u_{r-i+1})\\
&- \sum_{1\le i<r} swap\cdot push C_{\mathbf{Q}}(u_1,u_{2,r-i+2},u_{r-i+3},\dots,u_r)swap\mathbf{M}(u_2,\dots,u_{r-i+1})\\
&- \sum_{1<i<r}(r-i+1)c_{r-i+1}swap\mathbf{Q}(u_{r-i+2},\cdots,u_r) + \sum_{1<i<r} iC_i swap\mathbf{Q}(u_2,\dots,u_{r-i+1})
\end{aligned}
\tag{A.24}
$$

where the $\mathbf{M}$-factor in the second term is computed as

$$
\mathbf{M}(-u_{r-i+2},\dots,-u_{r-j+3,r-i+2}) = swap(M)(-u_{r-j+3,r-i+2},u_{r-j+3},\dots,u_{r-i+1}),
$$

which is equal to $swap(M)(u_{r-j+3},\dots,u_{r-i+2}$ thanks to the push-invariance of $swap(M)$. Now, recall from (1.2.4) that

$$
swap\big(\mathcal{F}(Q)\big)(v_1,\dots,v_r) = swap(Q)(v_1,\dots,v_r) + swap(Q)\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big),
$$

so since $\mathcal{F}(Q) = C_Q$ and

$$
C_{\mathbf{Q}}(v_1,\dots,v_r) := \mathbf{Q}\Big(sh\big((v_1),(v_2,\dots,v_r)\big)\Big) + push^{-1}\mathbf{Q}(v_1,\dots,v_r),
$$

we have $C_Q = swap\cdot push(C_{\mathbf{Q}})$. Replacing this in (A.24) as well as $Q = swap\mathbf{Q}$, $M = swap\mathbf{M}$, we obtain

$$
\begin{aligned}
C_{P'}(u_1,\dots,u_r) &= \sum_{1\le i<r} C_Q(u_{1,r-i+1},u_{r-i+2},\dots,u_r)M(u_2,\dots,u_{r-i+1})\\
&+ \sum_{1<i<j\le r} C_Q(u_1,\dots,u_{r-j+1},u_{r-j+2,r-i+2},u_{r-i+3},\dots,u_r)M(u_{r-j+3},\dots,u_{r-i+2})\\
&- \sum_{1<i<j<r} C_Q(u_1,\dots,u_{r-j+1},u_{r-j+2,r-i+2},u_{r-i+3},\dots,u_r)M(u_{r-j+2},\dots,u_{r-i+1})\\
&- \sum_{1\le i<r} C_Q(u_1,u_{2,r-i+2},u_{r-i+3},\dots,u_r)M(u_2,\dots,u_{r-i+1})\\
&- \sum_{i=2}^{r-1}(r-i+1)c_{r-i+1}Q(u_{r-i+2},\cdots,u_r) + \sum_{i=2}^{r-1} iC_i Q(u_2,\dots,u_{r-i+1}).
\end{aligned}
\tag{A.25}
$$

The first sum corresponds to the term $C_Q(a\rceil c)M(b)$ for

$$
abc = (u_1)(u_2,\dots,u_{r-i+1})(u_{r-i+2},\dots,u_r),
$$

where the condition $i<r$ means that $b\neq\emptyset$, and the second sum corresponds to the same term $C_Q(a\rceil c)M(b)$ for

$$
abc = (u_1,\dots,u_{r-j+2})(u_{r-j+3},\dots,u_{r-i+2})(u_{r-i+3},\dots,u_r)
$$

with $j<r$, i.e. $|a|\ge 2$ and $b\neq\emptyset$; thus these two terms combine as the sum over $w=abc$ with $a,b\neq\emptyset$ of the terms $C_Q(a\rceil c)M(b)$. Similarly, the fourth sum corresponds to the term $C_Q(a\lceil c)M(b)$ for

$$
abc = (u_1)(u_2,\dots,u_{r-i+1})(u_{r-i+2},\dots,u_r)
$$

so $|a| = 1$ and again the condition $i < r$ means that $b \neq \emptyset$, while the third sum corresponds to the sum of the terms $C_Q(a\lceil c)M(b)$ over the decompositions

$$abc = (u_1, \dots, u_{r-j+1})(u_{r-j+2}, \dots, u_{r-i+1})(u_{r-i+2}, \dots, u_r),$$

in which the condition $j < r$ implies that $|a| \geq 2$ and the condition $i < j$ means that $b \neq \emptyset$, so again these two terms combine to form the sum of $C_Q(a\lceil c)M(b)$ over the decompositions $w = abc$ with $a, b \neq \emptyset$. In terms of indices, this means that using the standard form

$$abc = (u_1, \dots, u_i)(u_{i+1}, \dots, u_j)(u_{j+1}, \dots, u_r),$$

the terms of (A.25) can be reindexed as

$$\begin{aligned} C_{P'}(u_1, \dots, u_r) = & \sum_{1<i<j\leq r} C_Q(u_1, \dots, u_{i-1}, u_i + \dots + u_j, u_{j+1}, \dots, u_r) M(u_{i+1}, \dots, u_j) \\ & - \sum_{1<i<j\leq r} C_Q(u_1, \dots, u_i, u_{i+1} + \dots + u_{j+1}, u_{j+2}, \dots, u_r) M(u_{i+1}, \dots, u_j) \\ & - \sum_{i=2}^{r-1} iC_i Q(u_{i+1}, \cdots, u_r) + \sum_{i=2}^{r-1} iC_i Q(u_2, \dots, u_{r-i+1}). \end{aligned} \tag{A.26}$$

This concludes the proof of Theorem 3.2.

Appendix B: Proof of Lemma 3.3.

Since $-a + \frac{1}{2}ad(b)(a)$ is the start of the power series expansion of t_{01} in a, b, $\hat{t}_{01}$ is simply the depth $r \geq 2$ part of t_{01}, whose associated mould is extremely simple; indeed, for each even depth $r \geq 2$, we have

$$\hat{T}_{01}(u_1, \dots, u_r) = \frac{B_r}{r!} T_r(u_1, \dots, u_r),$$

where T_r is the mould concentrated in depth r defined by

$$T_r(u_1, \dots, u_r) := \sum_{i=1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i. \tag{B.1}$$

It suffices to check that each T_r satisfies the strict Fay relations for even r, i.e. that for all even $r \geq 2$,

$$T'_r(u_1, \dots, u_r) + \sum_{j=1}^{r-1} T'_r(u_2, \dots, u_j, -\overline{u}_j, \overline{u}_{j+1}, u_{j+2}, \dots, u_r) + T'_r(u_2, \dots, u_r, -u_{1,r}) = 0 \tag{B.2}$$

which we write out explicitly as

$$\begin{aligned} \frac{\sum_{i=1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i}{u_1 \cdots u_r} &= \frac{\sum_{i=1}^{r-1} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} - (-1)^{r-1} u_{1,r}}{u_2 \cdots u_r u_{1,r}} \\ + \sum_{j=1}^{r-1} & \frac{\sum_{i=1}^{j-1} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} - (-1)^{j-1} \binom{r-1}{j-1} u_{1,j} + (-1)^j \binom{r-1}{j} u_{1,j+1} + \sum_{i=j+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i}{u_2 \cdots u_j u_{1,j} u_{1,j+1} u_{j+2} \cdots u_r}. \end{aligned} \tag{B.3}$$

Let us put this over the common denominator

$$D = u_1 \cdots u_r u_{1,2} \cdots u_{1,r};$$

the equality we need to prove is then given by

$$u_{1,2} \cdots u_{1,r} \sum_{i=1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i$$

$$= \sum_{j=1}^{r-1} D_j \left(\sum_{i=1}^{j-1} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} - (-1)^{j-1} \binom{r-1}{j-1} u_{1,j} + (-1)^j \binom{r-1}{j} u_{1,j+1} + \sum_{i=j+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right). \tag{B.4}$$

where

$$D_j = \begin{cases} u_1 u_{j+1} u_{1,2} \cdots u_{1,j-1} u_{1,j+2} \cdots u_{1,r} & 1 \le j \le r-1 \\ u_1 u_{1,2} \cdots u_{1,r-1} & j = r. \end{cases}$$

To prove (B.4), we first show that the degree r polynomial on the right-hand side is divisible by $u_{1,k}$ for $k = 2, \ldots, r$. The factor $u_{1,k}$ appears in D_j for all $j \neq k-1, k$, so if we set $u_{1,k} = 0$, i.e. $u_k = -u_{k+1} - \cdots - u_r$ in the right-hand side of (B.4), only the two terms

$$D_{k-1} \left(\sum_{i=1}^{k-2} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} - (-1)^{k-2} \binom{r-1}{k-2} u_{1,k-1} + (-1)^{k-1} \binom{r-1}{k-1} u_{1,k} + \sum_{i=k+1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right)$$

$$+ D_k \left(\sum_{i=1}^{k-1} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} - (-1)^{k-1} \binom{r-1}{k-1} u_{1,k} + (-1)^k \binom{r-1}{k} u_{1,k+1} + \sum_{i=k+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right). \tag{B.5}$$

We write

$$D_{k-1} = D u_k u_{1,k+1} \quad \text{and} \quad D_k = D u_{k+1} u_{1,k-1}$$

with

$$D = u_1 u_{1,2} \cdots u_{1,k-2} u_{1,k+2} \cdots u_{1,r}.$$

Observe that if $u_{1,k} = 0$ then $u_{1,k-1} = -u_k$ and $u_{1,k+1} = u_{k+1}$. We use this to rewrite (B.5) as

$$D u_k u_{k+1} \left(\sum_{i=1}^{k-2} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} + (-1)^{k-2} \binom{r-1}{k-2} u_k + \sum_{i=k+1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right)$$

$$- D u_k u_{k+1} \left(\sum_{i=1}^{k-1} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} + (-1)^k \binom{r-1}{k} u_{k+1} + \sum_{i=k+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right),$$

or better

$$D u_k u_{k+1} \left(\sum_{i=1}^{k-2} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} + (-1)^{k-2} \binom{r-1}{k-2} u_k + (-1)^k \binom{r-1}{k} u_{k+1} + \sum_{i=k+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right)$$

$$- D u_k u_{k+1} \left(\sum_{i=1}^{k-2} (-1)^{i-1} \binom{r-1}{i-1} u_{i+1} + (-1)^{k-2} \binom{r-1}{k-2} u_k + (-1)^k \binom{r-1}{k} u_{k+1} + \sum_{i=k+2}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i \right)$$

which is identically equal to zero. This shows that the right-hand side of (B.4) is divisible by $u_{1,k}$ for $k = 2, \ldots, r$. To conclude the proof of the equality (B.4), we need to show that the right-hand side also vanishes when we set

$$\sum_{i=1}^{r} (-1)^{i-1} \binom{r-1}{i-1} u_i = 0,$$

an equality which can be written equivalently as

$$\sum_{i=j+2}^{r}(-1)^{i-1}\binom{r-1}{i-1}u_i=\sum_{i=1}^{j+1}(-1)^{i}\binom{r-1}{i-1}u_i$$

for $1\le j\le r$. Let us make this substitution in the right-hand side of (B.4) (and reindex the first sum from 2 to j instead of 1 to $j-1$):

$$=\sum_{j=1}^{r}D_j\left(\sum_{i=2}^{j}(-1)^{i-2}\binom{r-1}{i-2}u_i-(-1)^{j-1}\binom{r-1}{j-1}u_{1,j}+(-1)^{j}\binom{r-1}{j}u_{1,j+1}+\sum_{i=1}^{j+1}(-1)^{i}\binom{r-1}{i-1}u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left((-1)^{j}\binom{r-1}{j-1}u_{1,j}+(-1)^{j}\binom{r-1}{j}u_{1,j+1}-u_1-(-1)^{j}\binom{r-1}{j}u_{j+1}+\sum_{i=2}^{j}(-1)^{i}\left(\binom{r-1}{i-2}+\binom{r-1}{i-1}\right)u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+(-1)^{j}\binom{r-1}{j-1}u_{1,j}+(-1)^{j}\binom{r-1}{j}u_{1,j}+(-1)^{i}\sum_{i=2}^{j}\left(\binom{r-1}{i-2}+\binom{r-1}{i-1}\right)u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+(-1)^{j}\left(\binom{r-1}{j-1}+\binom{r-1}{j}\right)u_{1,j}+(-1)^{i}\sum_{i=2}^{j}\left(\binom{r-1}{i-2}+\binom{r-1}{i-1}\right)u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+(-1)^{j}\left(\frac{(r-1)!}{(j-1)!(r-j)!}+\frac{(r-1!)}{j!(r-j-1)!}\right)u_{1,j}+(-1)^{i}\sum_{i=2}^{j}\left(\frac{(r-1)!}{(i-2)!(r-i+1)!}+\left(\frac{(r-1)!}{(i-1)!(r-i)!}\right)u_i\right)\right.$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+(-1)^{j}\left(j\times\frac{(r-1)!}{j!(r-j)!}+(r-j)\times\frac{(r-1!)}{j!(r-j)!}\right)u_{1,j}\right.$$

$$\left.+\sum_{i=2}^{j}(-1)^{i}\left((i-1)\times\frac{(r-1)!}{(i-1)!(r-i+1)!}+(r-i+1)\times\left(\frac{(r-1)!}{(i-1)!(r-i+1)!}\right)u_i\right)\right.$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+(-1)^{j}\binom{r}{j}u_{1,j}+\sum_{i=2}^{j}(-1)^{i}\binom{r}{i-1}u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left(-u_1+\sum_{i=1}^{j}(-1)^{j}\binom{r}{j}u_i+\sum_{i=2}^{j}(-1)^{i}\binom{r}{i-1}u_i\right)$$

$$=\sum_{j=1}^{r}D_j\left(\sum_{i=1}^{j}\left((-1)^{j}\binom{r}{j}+(-1)^{i}\binom{r}{i-1}\right)u_i\right)$$

$$=\sum_{j=1}^{r}u_1u_{j+1}u_{1,2}\cdots u_{1,j-1}u_{1,j+2}\cdots u_{1,r}\left(\sum_{i=1}^{j}\left((-1)^{j}\binom{r}{j}+(-1)^{i}\binom{r}{i-1}\right)u_i\right). \qquad (B.6)$$

Let us call this polynomial P.

Claim. $u_{1,k}$ is a factor of P for $k=1,\dots,r$.

For $k=1$ this is obvious since u_1 appears as a factor in every term. Fix a k with $1<k<r$; then there are only two terms in P where $u_{1,k}$ does not appear, namely the term for $j=k-1$ and the term for $j=k$:

$$u_1u_ku_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^{i}\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{k+1}u_{1,2}\cdots u_{1,k-1}u_{1,k+2}\cdots u_{1,r}\left(\sum_{i=1}^{k}\left((-1)^k\binom{r}{k}+(-1)^i\binom{r}{i-1}\right)u_i\right).$$

Add and subtract a term between these two lines as follows:

$$u_1u_ku_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{1,k-1}u_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$-u_1u_{1,k-1}u_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{k+1}u_{1,2}\cdots u_{1,k-1}u_{1,k+2}\cdots u_{1,r}\left(\sum_{i=1}^{k}\left((-1)^k\binom{r}{k}+(-1)^i\binom{r}{i-1}\right)u_i\right).$$

$$=u_1u_{1,k}u_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$-u_1u_{1,2}\cdots u_{1,k-2}u_{1,k-1}(u_{1,k}+u_{k+1})\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{k+1}u_{1,2}\cdots u_{1,k-1}u_{1,k+2}\cdots u_{1,r}\left(\sum_{i=1}^{k}\left((-1)^k\binom{r}{k}+(-1)^i\binom{r}{i-1}\right)u_i\right).$$

$$=u_1u_{1,k}u_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$-u_1u_{1,2}\cdots u_{1,k-2}u_{1,k-1}u_{1,k}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$-u_1u_{k+1}u_{1,2}\cdots u_{1,k-2}u_{1,k-1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{k+1}u_{1,2}\cdots u_{1,k-1}u_{1,k+2}\cdots u_{1,r}\left(\sum_{i=1}^{k}\left((-1)^k\binom{r}{k}+(-1)^i\binom{r}{i-1}\right)u_i\right).$$

$$=u_1u_{1,k}u_{1,2}\cdots u_{1,k-2}u_{1,k+1}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$-u_1u_{1,2}\cdots u_{1,k-2}u_{1,k-1}u_{1,k}\cdots u_{1,r}\left(\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right)$$

$$+u_1u_{k+1}u_{1,2}\cdots u_{1,k-1}u_{1,k+2}\cdots u_{1,r}\left(\sum_{i=1}^{k}\left((-1)^k\binom{r}{k}+(-1)^i\binom{r}{i-1}\right)u_i-\sum_{i=1}^{k-1}\left((-1)^{k-1}\binom{r}{k-1}+(-1)^i\binom{r}{i-1}\right)u_i\right).$$

The factor $u_{1,k}$ appears in the first two lines of this, but in the third line, we will show that it appears in the large bracketed factor. For this, we compute the coefficient of u_i in the large bracketed factor. For $i > k$, u_i does not occur. The coefficient of u_k is

$$\begin{aligned}(-1)^k\binom{r}{k} + (-1)^k\binom{r}{k-1} &= (-1)^k\left(\frac{r!}{k!(r-k)!} + \frac{r!}{(k-1)!(r-k+1)!}\right)\\ &= (-1)^k\left((r-k+1)\times\frac{r!}{k!(r-k+1)!} + k\times\frac{r!}{k!(r-k+1)!}\right)\\ &= (-1)^k\left(\frac{(r+1)!}{k!(r-k+1)!}\right)\\ &= (-1)^k\binom{r+1}{k}.\end{aligned}$$

For $1 \le i < k$, the coefficient is

$$(-1)^k\binom{r}{k} + (-1)^i\binom{r}{i-1} - (-1)^{k-1}\binom{r}{k-1} - (-1)^i\binom{r}{i-1}$$

$$(-1)^k\binom{r+1}{k} + (-1)^i\binom{r}{i-1} - (-1)^i\binom{r}{i-1}$$

$$(-1)^k\binom{r+1}{k}.$$

Thus the large bracketed factor in the last line is nothing other than

$$(-1)^k\binom{r+1}{k}u_{1,k}$$

and thus every term contains the factor $u_{1,k}$; thus $u_{1,k}$ is a factor of P, proving the claim.

Since P is of degree r and $u_{1,k}$ is a factor of P for $1 \le k \le r$, we have $P = cu_{1,1}u_{1,2}\cdots u_{1,r}$. Since $u_{1,j} = u_1 + \cdots + u_j$, when we expand P in the variables $u_1, \ldots, u_r$, the monomial u_1^r appears with coefficient c. In order to compute the monomial cu_1^r, we consider the expression of P given in (B.6) and set $u_2 = \cdots = u_r = 0$. Because u_{j+1} appears in every term of P except for $j = r$, using the fact that $u_{1,j}$ becomes equal to u_1 when $u_2 = \cdots = u_r = 0$, we only need to consider the term $j = r$ and $i = 1$: we find

$$P(u_1, 0, \ldots, 0) = u_1^{r-1} \times \Big((-1)^r + (-1)\Big)u_1.$$

Given that r is even, this is equal to 0 and thus $P = 0$, which concludes the unfortunately complicated proof of Lemma 3.3. ♢

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