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Automorphisms of curves and their role in Grothendieck–Teichmüller theory

Leila Schneps[*1]

[1] 175 rue du Chevaleret, 75013 Paris, France

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The goal of this article is to consider the role played by finite-order elements in the mapping class groups and special loci on moduli spaces, within the framework of Grothendieck–Teichmüller theory, and in particularly in the genus zero case. Quotienting topological surfaces by finite-order automorphisms induces certain morphisms between moduli spaces; we consider the corresponding special homomorphisms between mapping class groups. In genus zero, these morphisms are always defined over $\mathbb{Q}$, so that the canonical outer Galois action on profinite genus zero mapping class groups respects the induced homomorphisms. For simplicity, we consider only the subgroup $\widehat{GT}^1_{0,0}$ of elements $F = (\lambda, f) \in \widehat{GT}$ with $\lambda = 1$ and conditions on the Kummer characters $\rho_2(F) = \rho_3(F) = 0$. We define a subgroup $\widehat{GS}^1_{0,0} \subset \widehat{GT}^1_{0,0}$ by considering only elements of $\widehat{GT}^1_{0,0}$ respecting these homomorphisms on the first two levels in genus zero. Our main result states that the subgroup $\widehat{GS}^1_{0,0}$, which is thus defined using only properties occurring in genus zero, possesses many remarkable geometric Galois-type properties not visibly satisfied by $\widehat{GT}$ itself, the most striking of which is that it is also an automorphism group of the profinite mapping class groups in all genera.

1 Introduction

Let $S = S_{g,n}$ denote a topological surface of genus g with n ordered marked points, and let $\mathcal{M}(S)$ denote the moduli space of Riemann surfaces of topological type S (this moduli space is often denoted $\mathcal{M}_{g,n}$). The permutation group $\mathfrak{S}_n$ acts naturally on $\mathcal{M}(S)$ by permuting the marked points on the Riemann surfaces; for any subgroup $G \subset \mathfrak{S}_n$, we write $\mathcal{M}_G(S) = \mathcal{M}(S)/G$. Topologically, the spaces $\mathcal{M}_G(S)$ are *orbifolds* in general; in fact $\mathcal{M}_G(S)$ is a quotient of a simply connected space of complex dimension $3g - 3 + n$, the Teichmüller space $\mathcal{T}(S)$, by the action of a discrete group $\Gamma_G(S)$, the mapping class group, which acts properly discontinuously but not, in general, freely. The mapping class group is the group of orientation-preserving diffeomorphisms of $S_{g,n}$ which permute the marked points only according to the permutations contained in the group G, modulo those diffeomorphisms which are isotopic to the identity. By the above, they can also be considered as the orbifold fundamental groups of the moduli spaces. If $G = \{1\}$, we write $\mathcal{M}(S) = \mathcal{M}_{\{1\}}(S)$ and $\Gamma(S) = \Gamma_{\{1\}}(S)$.

When $S = S_{0,n}$ is a sphere equipped with n ordered marked points, the full mapping class groups $\Gamma_{\mathfrak{S}_n}(S)$ are particularly well understood, as they are quotients of the Artin braid groups B_n. The group B_n is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ satisfying $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and $\Gamma(S_{0,n})$ is the quotient of B_n by the center relation, $(\sigma_1 \ldots \sigma_{n-1})^n = 1$, and the sphere relation $\sigma_1 \ldots \sigma_{n-1} \cdot \sigma_{n-1} \ldots \sigma_1 = 1$. Set $x_{i,i+1} = \sigma_i^2$. We write $\widehat{\Gamma}_G(S)$ for the profinite completion of $\Gamma_G(S)$, so that $\widehat{\Gamma}_G(S)$ can be considered as the geometric orbifold fundamental group of $M_G(S)$. The moduli space $M(S_{0,4})$ is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$, so that its fundamental group $\Gamma(S_{0,4})$ is isomorphic to the free group on two generators F_2.

The Grothendieck–Teichmüller group $\widehat{GT}$ was first defined by Drinfel'd in [2]. Let us recall the definition here, with the modified version of relation (III) given by Ihara–Matsumoto [11].

[*] e-mail: leila@math.jussieu.fr, Phone: +00 33 1 4427 7516, Fax: +00 33 1 4427 8551

Definition 1.1 Let $\widehat{GT}$ be the group defined as follows:

$$\widehat{GT} = \Big\{(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2' \mid (\lambda, f) \text{ defines an automorphism of } \widehat{F}_2 \text{ via } x \mapsto x^\lambda,\ y \mapsto f^{-1}y^\lambda f \quad (1.1)$$
$$\text{and } (\lambda, f) \text{ satisfies the three following relations :}$$

$$\text{(I)}\quad f(y,x)f(x,y) = 1\,,$$
$$\text{(II)}\quad f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1 \text{ where } m = (\lambda-1)/2 \text{ and } xyz = 1\,, \quad (1.2)$$
$$\text{(III)}\quad f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23}) = 1 \text{ in } \widehat{\Gamma}(S_{0,5})\Big\}\,.$$

The multiplication law is given by composing the corresponding automorphisms of $\widehat{F}_2$.

This group has been much studied. Drinfel'd indicated in [2] that it contained the absolute Galois group $G_{\mathbb{Q}}$; a complete and detailed proof was given by Ihara in [9]. Drinfel'd also showed that $\widehat{GT}$ acts on the genus zero profinite groups $\widehat{\Gamma}(S_{0,n})$ via the formula

$$(\lambda, f)(\sigma_i) = f\big(\sigma_i^2, y_i\big)\sigma_i^\lambda f\big(y_i, \sigma_i^2\big) \quad (1.3)$$

where $y_i = \sigma_{i-1}\dots\sigma_1\cdot\sigma_1\dots\sigma_{i-1}$. Nakamura ([13, Appendix]) (for $n = 5$) and Ihara and Matsumoto ([11]) in general showed that this action extends the $G_{\mathbb{Q}}$-action on these groups which occurs naturally by considering them as fundamental groups of moduli spaces and using a tangential base point to lift the canonical outer automorphisms to actual automorphisms.

Over the last ten years, various refined versions of $\widehat{GT}$ have been defined. These groups are subgroups of $\widehat{GT}$ (which are never actually known to be strict subgroups). They contain $G_{\mathbb{Q}}$, or at least some "typical large subgroup" of $G_{\mathbb{Q}}$ (see below), and they are defined on purpose in order to ensure the possession of various geometric properties which are known for $G_{\mathbb{Q}}$, but not for $\widehat{GT}$. Such geometric properties, for example, are that like those of $G_{\mathbb{Q}}$, the elements of the group should act as automorphisms on the profinite mapping class groups in all genera, where they should preserve conjugacy classes of Dehn twists, and they should also respect various homomorphisms between these groups which come from natural $\mathbb{Q}$-morphisms between the moduli spaces.

These "new versions of $\widehat{GT}$" are defined by adding relations to $\widehat{GT}$ of types similar to (I), (II), (III) above. In this article, we define (yet) another new subgroup of $\widehat{GT}$, called $\widehat{GS}{}^1_{0,0}$, with two new relations coming from the action of finite-order automorphisms of genus zero surfaces.

Technical restriction. For simplicity, we choose to work in this article over the extension K of $\mathbb{Q}$ generated by all roots of unity and all roots of 2 and 3. In this way, the past and new relations added to $\widehat{GT}$ that we recall below will have much simpler forms. Let us explain exactly what is meant by this, before stating the main theorems of this article.

We begin by recalling Ihara's definition of Kummer characters (cf. [16])

$$\rho_n : \widehat{GT} \longrightarrow \widehat{\mathbb{Z}}^*\,.$$

The quotient group $\widehat{F}_2'/\widehat{F}_2''$ is acted on by the ring $\widehat{\mathbb{Z}}\big[\big[\widehat{\mathbb{Z}}^2(1)\big]\big]$-module via conjugation, and in fact it is a free $\widehat{\mathbb{Z}}\big[\big[\widehat{\mathbb{Z}}^2(1)\big]\big]$-module of rank 1. Thus for $F = (\lambda, f) \in \widehat{GT}$, since $f \in \widehat{F}_2'$, there exists a unique element of $\widehat{\mathbb{Z}}\big[\big[\widehat{\mathbb{Z}}^2(1)\big]\big]$ such that

$$f \equiv A_F(x,y)\cdot[x,y] \mod \widehat{F}_2''\,.$$

For every integer $n \geq 2$, we set

$$\rho_n(F) = \frac{1}{n}\sum_{i=0}^{n-1}(\zeta_n^i - 1)A_F(\zeta_n^i, 1)$$

where ζ_n is a primitive n-th root of unity. Then it is easy to see that restricted to elements $(\lambda_\sigma, f_\sigma) \in \widehat{GT}$ corresponding to $\sigma \in G_{\mathbb{Q}}$, these $\rho_n(\sigma)$ satisfy

$$\sigma\big(\sqrt[r]{n}\big) = \zeta_n^{\rho_n(\sigma)}\sqrt[r]{n}\,.$$

In this article, we consider only the subgroup $\widehat{GT}^{\,1}_{0,0}$ of $\widehat{GT}$ consisting of elements $F = (\lambda, f) \in \widehat{GT}$ having $\lambda = 1$ and $\rho_2(F) = \rho_3(F) = 0$, so that rather than $G_{\mathbb{Q}}$, $\widehat{GT}^{\,1}_{0,0}$ contains the "large subgroup" which is the absolute Galois group G_K. This restriction is purely technical and serves to simplify all the formulae.

We now proceed to describe the main contents of the article.

Definition 1.2 Let $\widehat{GS}^{\,1}_{0,0}$ denote the subset of elements $f \in \widehat{GT}^{\,1}_{0,0}$ satisfying the following two relations, of which the first takes place in the group $\widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$ and the second in the group $\widehat{\Gamma}_{\mathfrak{S}_6}(S_{0,6})$:

$$\begin{aligned} &f\big(\sigma_3^{-1}\sigma_2\sigma_3, \sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_3^2\sigma_2\sigma_3\big) f\big(\sigma_1^2\sigma_3^2, \sigma_3^{-1}\sigma_2\sigma_3\big) \\ &\quad = f\big(x_{51}, \sigma_4 x_{51}\sigma_4^{-1}\big) f(x_{34}, x_{23}) f(x_{12}, x_{51})\,, \end{aligned} \tag{$*$}$$

$$\begin{aligned} &f\big(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}, \sigma_1^2\sigma_3^2\sigma_5^2\big) f\big(\sigma_2^2\sigma_4^2\sigma_{61}^2, \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\big) \\ &\quad = f(x_{12,3}, x_{34}) f(x_{23}, x_{12}) f(x_{23,4}, x_{1,23}) f(x_{234,5}, x_{1,234}) f(x_{45}, x_{23,4})\,, \end{aligned} \tag{$**$}$$

where the element $\sigma_{61} \in \Gamma_{\mathfrak{S}_6}(S_{0,6})$ is according to the notation

$$\sigma_{n\,1} \;=\; \sigma_1 \ldots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \ldots \sigma_1^{-1} \in \Gamma_{\mathfrak{S}_n}(S_{0,n})\,,$$

and the braid x_{AB} for two adjacent packets of strands A and B is the flat double crossing of the two packets.

Remarks 1.3 (1) The first main result of this article (see below and Theorem 3.1) explains that the relations $(*)$ and $(**)$ are just explicit expressions of the commutation of two commutative diagrams which arise from natural morphisms between moduli spaces. In some ways, this more natural definition would be better as a direct definition of $\widehat{GS}^{\,1}_{0,0}$. However, we prefer not to use it as a definition because it takes large parts of Section 2 and part of Section 3 to define the morphisms in the diagrams, whereas the relations can be given directly.

(2) An equivalent form of relation $(*)$ is the conjugation form

$$\sigma_2^{f(\sigma_1^2\sigma_3^2, \sigma_2)} \;=\; \sigma_2^{f(x_{34}, x_{23}) f(x_{12}, x_{51})} \;=\; \sigma_2^{f(x_{12}, x_{23}) f(x_{34}, x_{45})}\,, \tag{1.4}$$

where $x^\alpha = \alpha^{-1}x\alpha$; here, the first equality is obvious after substituting

$$f\big(\sigma_3^2, \sigma_2^{-1}\sigma_3^2\sigma_2\big) \;=\; f\big(\sigma_2^2, \sigma_2^{-1}\sigma_3^2\sigma_2\big) f\big(\sigma_3^2, \sigma_2^2\big)$$

for the middle term on the right-hand side of $(*)$ by relation (II), and the second equality follows from the pentagon relation (III).

(3) Relation $(*)$ was originally written in the more complicated form

$$f\big(\sigma_2, \sigma_2^{-1}\sigma_1^2\sigma_3^2\sigma_2\big) f\big(\sigma_1^2\sigma_3^2, \sigma_2\big) \;=\; f\big(\sigma_3^2\sigma_2^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big) f\big(\sigma_3^2, \sigma_2^{-1}\sigma_3^2\sigma_2\big) f\big(\sigma_1^2, \sigma_2\sigma_3^2\sigma_2\big)\,. \tag{1.5}$$

The discovery that this relation (conjugated by σ_3) is equivalent to the better form given above is due to H. Tsunogai [20] (see details in the proof of Theorem 3.1, §3.1). He also observed that $(*)$ is equivalent to the relation (3.12) which plays a key role in the proof of Theorem 3.4 (see §3.2) and was originally introduced as an independent relation. He furthermore generalized the expressions of these relations to the fully general situation over $\mathbb{Q}$ rather than K ([21]).

In this article, we prove two results, whose precise statements are given as Theorems 3.1 and 3.4 of §3. Necessary definitions and background are provided in §2. Essentially, the first result is as follows.

• *The two defining relations $(*)$ and $(**)$ of $\widehat{GS}^{\,1}_{0,0}$ come from requiring elements of $\widehat{GT}^{\,1}_{0,0}$ to respect the two first non-trivial special homomorphisms between genus zero moduli spaces, i.e., homomorphisms coming from automorphisms of genus zero curves. These two homomorphisms are given explicitly at the end of* §2. *As a corollary, this characterization of relations $(*)$ and $(**)$ shows that $\widehat{GS}^{\,1}_{0,0}$ is a group.*

In order to state the second result, we need to have a closer look at some of the relations which have been added, at different times, to the three original defining relations of $\widehat{GT}$ in order to define subgroups of $\widehat{GT}$ having specific "geometric Galois properties". We continue to restrict to $\widehat{GT}^{\,1}_{0,0}$ in order to simplify the expressions of these relations. Note that this list is not exhaustive; there are variants of these relations, and entire families of other relations which have been added to $\widehat{GT}$ (see in particular the definition of $\widehat{GTA}$ due to Ihara [10]) which are perhaps also consequences of genus zero geometry. We consider the five relations in the list below. For the first relation (R), we recall that $\Gamma(S_{1,2})$ is isomorphic to the quotient of $B_4 = \langle\sigma_1, \sigma_2, \sigma_3\rangle$ modulo its center, which is generated by $(\sigma_1\sigma_2\sigma_3)^4$.

List of relations added to $\widehat{GT}$ in the various papers [8], [14], and [15]:

(R) $f(e_3, \sigma_1)f(\sigma_2^2, \sigma_3^2)f(e_2, e_3)f(e_1, e_2)f(\sigma_1^2, \sigma_2^2)f(\sigma_3, e_1)$ in $\widehat{\Gamma}(S_{1,2})$, with $e_1 = (\sigma_1\sigma_2)^6$. $e_3 = (\sigma_2\sigma_3)^6$, $e_2 = \sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}$.

(III′) $f(\sigma_1\sigma_3, \sigma_2^2) = f(\sigma_4^2, \sigma_4\sigma_{51}\sigma_4)f(x_{12}, x_{23})f(x_{34}, x_{45})$ in $\widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$,

(IV) $f(\sigma_1^4, \sigma_2) = f(\sigma_1^2, \sigma_2^2)$ in $\widehat{\Gamma}_{\mathfrak{S}_4}(S_{0,4})$,

(NT1) $f(\sigma_1^2, \sigma_2^2) = f(\sigma_1\sigma_2\sigma_1, \sigma_2^2)f(\sigma_1^2, \sigma_1\sigma_2\sigma_1)$ in $\widehat{B}_3$,

(NT2) $f(\sigma_1^2, \sigma_2^2) = f(\sigma_2\sigma_1, \sigma_2^2)f(\sigma_1^2, \sigma_2\sigma_1)$, also in $\widehat{B}_3$.

Geometric properties associated to these relations. The first of these relations was added to $\widehat{GT}^{\,1}_{0,0}$ in [8] in order to pick out the elements of $\widehat{GT}^{\,1}_{0,0}$ which extend to automorphisms of the profinite mapping class groups in all genera respecting point-erasing and subsurface-inclusion homomorphisms. Relations (III′) and (IV) were introduced in [14] (which actually gives their general forms, corresponding to the full group $\widehat{GT}$, as well as the general form of (R)); in particular, relation (IV) actually implies (R). We show in §3 that relations (III′) and (IV) can be subsumed into just one relation, coming from respecting a single genus zero special homomorphism. Finally, relations (NT1) and (NT2) (which are also introduced in their general forms in [15]) were introduced in order to make sure that for all subgroups $G \subset \mathfrak{S}_4$, $\widehat{GT}$ respects the two homomorphisms

$$\widehat{\Gamma}(S_{0,4}) \longrightarrow \widehat{\Gamma}_G(S_{0,4})$$

coming from the two natural morphisms

$$M(S_{0,4}) \longrightarrow M_G(S_{0,4}),$$

namely the quotient and the inclusion.

We can now state the main result of this article, given explicitly as Theorem 3.4.

• *Elements of $\widehat{GS}^{\,1}_{0,0}$ satisfy the above five relations.*

Putting the two results together, we perceive that requiring elements of $\widehat{GT}^{\,1}_{0,0}$ to satisfy the first two genus zero special homomorphisms implies that several other geometric Galois properties are automatically satisfied. In particular, the fact that by imposing conditions on elements of $\widehat{GT}^{\,1}_{0,0}$ coming purely from the geometry in genus zero we automatically obtain the passage to all genera is the most striking and unexpected result of this article.

We are very grateful to H. Tsunogai for the helpful remarks, computations and simplifications specified above, which much improved this article.

2 Special loci and special homomorphisms: definitions and genus zero case

It is well-known that for every subgroup $G \subset \mathfrak{S}_n$, the partially pure mapping class group group $\Gamma_G(S)$ acts properly discontinuously on the Teichmüller space $\mathcal{T}(S)$, but not always freely (although the pure mapping class groups $\Gamma(S)$ act freely when S is of genus zero). In general, however, some points of Teichmüller space have isotropy groups inside the mapping class groups. The following facts are well-known:

(1) the isotropy groups in the mapping class groups are always of finite order;

(2) every element of finite order in the mapping class group $\Gamma_G(S)$ has at least one fixed point in $\mathcal{T}(S)$;

(3) the isotropy group of a point in Teichmüller space is equal to the group of automorphisms of the corresponding Riemann surface.

The quotient of a simply connected space by a group acting in this way is called a *topological orbifold*, and the groups themselves are the *orbifold fundamental groups* of the quotient spaces (cf. [5] for an introduction to these groups). Thus the mapping class group $\Gamma_G(S)$ should be considered as the topological orbifold fundamental group of the moduli space $\mathcal{M}_G(S)$, which we simply denote by $\pi_1(\mathcal{M}_G(S))$.

Definition 2.1 The images in moduli space of the points with non-trivial isotropy in Teichmüller space are called *special orbifold points*. If φ is an element of finite order in $\Gamma_G(S)$, then we consider the set of points in $\mathcal{T}(S)$ fixed by φ; the image of this set in the quotient moduli space $\mathcal{M}_G(S)$ is called the *special locus of* φ and denoted $\mathcal{M}_G(S,\varphi)$. It depends only on the conjugacy class of the cyclic group generated by φ.

Much work has been devoted to studying and determining these special loci (cf. for example [3], [7] for a geometric approach, [1] for a complete classification of the maximal special loci in the case $g \geq 1$, $n = 0$, or [12] for more explicit descriptions).

Let us recall here the essential facts concerning special loci which we will use in this article; they come from [7] and [18], and concern essentially the group theoretic aspect of the special loci, i.e., the way in which inclusions of special loci into moduli spaces can be reflected in homomorphisms of the mapping class groups.

Theorem 2.2 ([7]) *Let φ be a finite element of $\Gamma_{\mathfrak{S}_n}(S)$. Then φ permutes the marked points of S. Let $[\varphi]$ denote the associated permutation, and let $G \subset \mathfrak{S}_n$ be the subgroup generated by the disjoint cycles of $[\varphi]$. Then in the moduli space $\mathcal{M}_G(S)$, the special locus of φ is closely related to the moduli space $\mathcal{M}(T)$ of the topological quotient $T = S/\varphi$. More explicitly, if $\mathcal{T}(S,\varphi)$ denotes the set of points in the Teichmüller space fixed by φ, then $\mathcal{T}(S,\varphi)/\mathrm{Norm}_{\Gamma_G(S)}(\varphi)$ is isomorphic to the normalization $\widetilde{\mathcal{M}}_G(S,\varphi)$ of the special locus $\mathcal{M}_G(S,\varphi)$, and associating to the Riemann surface corresponding to a point of $\mathcal{M}_G(S,\varphi)$ its quotient by the action of the automorphism φ yields a natural covering map of finite degree*

$$\widetilde{\mathcal{M}}_G(S,\varphi) \longrightarrow \mathcal{M}(T)\,. \tag{2.1}$$

Corollary 2.3 (Well-known, cf. [18]) *The morphism* (2.1) *corresponds to a group homomorphism*

$$\mathrm{Norm}_{\Gamma_G(S)}(\varphi) \longrightarrow \Gamma(T) \tag{2.2}$$

whose kernel is $\langle\varphi\rangle$. Indeed, these two groups are the orbifold fundamental groups of $\widetilde{\mathcal{M}}_G(S,\varphi)$ and of $\mathcal{M}(T)$ respectively. This group homomorphism can also be deduced directly from the fact that a diffeomorphism of S passes to T if and only if it normalizes φ.

Definition 2.4 If the homomorphism (2.2) is surjective, we say that φ *has the surjectivity property*. This corresponds to the restriction of the morphism (2.1) to each connected component being one-to-one, i.e., a degree 1 covering. Indeed, it is not difficult to see (cf. [18]) that in general the image of (2.2) will always be of finite index in $\Gamma(T)$, and therefore it determines a finite cover of $\mathcal{M}(T)$. If the homomorphism (2.2) is split, we say that φ *has the splitting property*, which corresponds to a certain triviality of the orbifold structure automatically possessed by $\widetilde{\mathcal{M}}_G(S,\varphi)$ because of the automorphism φ at each point. When φ has both the surjectivity and the splitting properties, we have non-canonical inverse homomorphisms

$$\Gamma(T) \longrightarrow \mathrm{Norm}_{\Gamma_G(S)}(\varphi) \subset \Gamma_{\mathfrak{S}_n}(S)\,, \tag{2.3}$$

called *special homomorphisms*.

Let us now turn to the special case of genus zero. As it turns out, the special homomorphisms exist for every φ in genus zero. Let us recall the three main genus zero results from [18], which are are necessary to compute the three fundamental examples at the end of this section, on which the definition in §3 of a new version of $\widehat{GT}$ is based.

Fact 1. *All finite-order elements of the genus zero mapping class groups $\Gamma_G(S)$ (for S a sphere with marked points) are rotations which can be explicitly described.*

Indeed, for $n \geq 5$, the fact that all finite-order elements are rotations follows from [7, Corollary p. 508]; this result states that in $\Gamma_{\mathfrak{S}_n}(S)$, (i) there are no elements of finite order strictly greater than n, (ii) there is exactly one conjugacy class of elements of order n (resp. $n-1$, resp. $n-2$), (iii) every finite-order element is conjugate to a power of an element of order n, $n-1$ or $n-2$. It is easy to then display a rotation of $\Gamma_{\mathfrak{S}_n}(S)$ of order n, (resp. $n-1$, resp. $n-2$) as the rotation given by placing n (resp. $n-1$, resp. $n-2$) points on the equator and making a $2\pi/n$, $2\pi/(n-1)$ or $2\pi/(n-2)$ rotation around a north-south axis as in the left-hand figure below (where 0, 1 or 2 of the points at the north and south poles may be marked, according to how many lie on the equator). By the above, all other finite-order elements are conjugates of powers of these, so that a permutation of the marked points associated a rotation is necessarily of the form $c_1 \ldots c_k$, where the c_i are disjoint cycles of length j with $jk = n$, $n-1$ or $n-2$; we represent such a rotation as in the right-hand figure below.

Fact 2. *Let φ be a finite-order element of $\Gamma_{\mathfrak{S}_n}(S)$ with S of genus 0 and n marked points, and let $G \subset \mathfrak{S}_n$ be the subgroup generated by the disjoint cycles of the permutation associated to φ. Then the special locus $M_G(S,\varphi)$ in the genus zero moduli space $\mathcal{M}_G(S)$ can be explicitly described and is defined over $\mathbb{Q}^{\mathrm{ab}}$.*

This is a consequence of the following theorem from [18], where a more complete version is given, explicitly describing special loci in genus zero in the ordered moduli space $M(S)$ and the unordered space $M_{\mathfrak{S}_n}(S)$ as well as the intermediate space $M_G(S)$ we consider here.

Theorem 2.5 ([18, Theorem 3.5.1 (ii)]) *Let S be a sphere with n numbered marked points, and let φ be a rotation as in Figure 1, with $n = jk + 2$ (i.e., we assume that the north and south poles, fixed points of the rotation, are marked points of S). Up to replacing φ by a conjugate of φ, which has the same special locus as φ, we may assume that the points of S are numbered so that the permutation associated to φ is given by*

$$[\varphi] \;=\; \big(1 \ldots j\big)\big((j+1) \ldots 2j\big) \ldots \big((j(k-1)+1) \ldots jk\big)\,.$$

Let $G \subset \mathfrak{S}_n$ be the subgroup generated by the disjoint cycles $c_1, \ldots, c_k$ of $[\varphi]$. Let T be the orbifold quotient S/φ, which has k marked points with ramification index 1 and 2 marked points with ramification index j. Then the special locus of φ in the quotient space $\mathcal{M}(S)/G = \mathcal{M}_G(S)$ consists of $|(\mathbb{Z}/j\mathbb{Z})^*|$ disjoint connected components C_ζ indexed by the primitive j-th roots of unity ζ. Each C_ζ is isomorphic to

$$\big(\mathbb{P}^1 - \{0,1,\infty\}\big)^{k-1} - \Delta \;\simeq\; \mathcal{M}(T)\,,$$

and is thus defined over $\mathbb{Q}$; however the embeddings $\mathcal{M}(T) \to C_\zeta \subset \mathcal{M}_G(S)$ are defined over $\mathbb{Q}(\zeta)$.

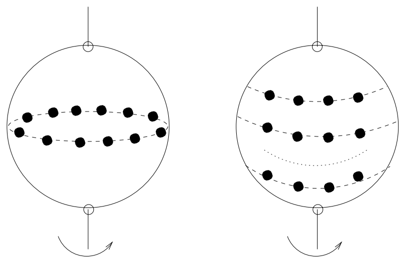

Fig. 1 Rotation of order n, $n-1$ or $n-2$ and power of such a rotation

Fact 3. *Every finite-order element of the genus zero mapping class groups gives rise to special homomorphisms, which can be explicitly computed.*

For this we use the results from [18] summarized in the following theorem.

Theorem 2.6 ([18, Theorems 4.3.1 and 4.3.2, and Corollary]) *When $g = 0$, i.e., when S is a sphere with marked points, every finite-order element φ of $\Gamma_{\mathfrak{S}_n}(S)$ possesses both the surjectivity and splitting properties, and therefore there exist special homomorphisms*

$$\Gamma(T) \;\longrightarrow\; \Gamma_{\mathfrak{S}_n}(S) \tag{2.4}$$

associated to φ as in Definition 2.4.

The homomorphisms between mapping class groups in (2.4) are simply the homomorphisms of fundamental groups associated to the morphisms of moduli spaces

$$\mathcal{M}(T) \;\longrightarrow\; C_\zeta \subset M_G(S)$$

of Theorem 2.5; the important point is that they can be computed explicitly. Details are given in [18, §5]; we restrict ourselves here to giving the two basic examples of special homomorphisms which will be used in the subsequent sections to define a new version of $\widehat{GT}$. For each of the examples below, we proceed by first determining a lifting of each generator of $\Gamma(T)$ to $\Gamma_{\mathfrak{S}_n}(S)$, and then checking that (i) these liftings define a homomorphism on $\Gamma(T)$, (ii) the liftings all normalise φ. An informal indication of the procedure is shown in the figures.

Example 2.7 ***S* of type $(0,5)$, *T* of type $(0,4)$.** The mapping class group $\Gamma_{\mathfrak{S}_5}(S)$ is generated by σ_1, σ_2, σ_3 and σ_4 with the usual braid relations and the relations

$$\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4 \;=\; (\sigma_1\sigma_2\sigma_3\sigma_4)^5 \;=\; 1\,.$$

The element φ' we consider in this example is

$$\varphi' \;=\; \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\,.$$

Its associated permutation is $(15)(24)$. Direct computation (manipulation with the braid relations) shows that ${\varphi'}^2 = (\sigma_4\sigma_3\sigma_2\sigma_1)^5 = 1$. In fact, φ' corresponds to the 180^o rotation around the axis shown in the figure below.

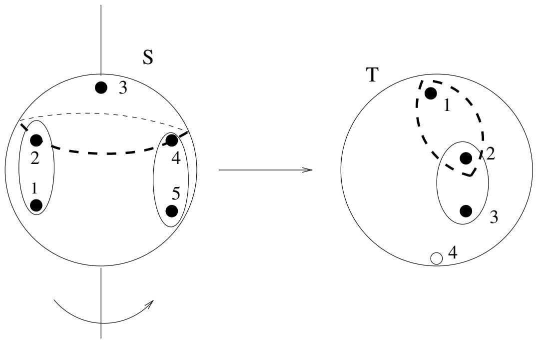

Fig. 2 The finite-order diffeomorphism φ'

The quotient T has 4 marked points coming from the 5 marked points of S and the orbifold point at the south pole.

Writing

$$\pi_1(S) \;=\; \langle x_1, x_2, x_3, x_4, x_5 \mid x_1x_2x_3x_4x_5 = 1\rangle$$

and

$$\pi_1(T) \;=\; \left\langle c_1, c_2, c_3, c_4 \mid c_1c_2c_3c_4 = 1, c_4^2 = 1\right\rangle,$$

the inclusion of $\pi_1(S)$ in $\pi_1(T)$ corresponding to the cover $S \to T = S/\varphi'$ is given by

$$\begin{aligned}
\pi(S) &\hookrightarrow \pi_1(T)\\
x_1 &\longmapsto c_1c_2c_3c_2^{-1}c_1^{-1}\\
x_2 &\longmapsto c_1c_2c_1^{-1}\\
x_3 &\longmapsto c_1^2\\
x_4 &\longmapsto c_2\\
x_5 &\longmapsto c_3\,.
\end{aligned}$$

Now, we know that $\Gamma(T)$ is isomorphic to the pure braid group on three strands, which in fact is a free group on two generators x and y, which can be taken to be the Dehn twists along the dotted and undotted loops shown in the right-hand part of Figure 2, respectively. We can take two splittings of the type (2.4), given by

$$\begin{aligned}
\widetilde{F}_{\varphi'} \;:\; \Gamma(T) &\longrightarrow \Gamma_{\mathfrak{S}_5}(S)\\
x &\longmapsto \sigma_2\sigma_3\sigma_2\,,\\
y &\longmapsto \sigma_1^2\sigma_4^2\,,
\end{aligned} \tag{2.5}$$

or

$$\begin{aligned} F_{\varphi'} \ :\ \Gamma(T) &\longrightarrow \Gamma_{\mathfrak{S}_5}(S) \\ x &\longmapsto \sigma_{51}\,, \\ y &\longmapsto \sigma_1^2\sigma_4^2\,. \end{aligned} \tag{2.6}$$

The latter becomes a little simpler if we replace φ' by $\varphi = (\sigma_1\sigma_2\sigma_3\sigma_4)^2\varphi(\sigma_1\sigma_2\sigma_3\sigma_4)^{-2}$. Then the splitting $F_\varphi = \mathrm{inn}(\sigma_1\sigma_2\sigma_3\sigma_4)^3 \circ F_{\varphi'}$ is given by

$$\begin{aligned} F_{\varphi} \ :\ \Gamma(T) &\longrightarrow \Gamma_{\mathfrak{S}_5}(S) \\ x &\longmapsto \sigma_{2}\,, \\ y &\longmapsto \sigma_1^2\sigma_3^2\,. \end{aligned} \tag{2.7}$$

Example 2.8 ***S* of type $(0,6)$, *T* of type $(0,4)$.** The element ψ we consider in this example is

$$\psi \ =\ \sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\,.$$

It is of order 3, and its associated permutation is $(135)(246)$. The diffeomorphism (up to isotopy) ψ is represented by the $2\pi/3$ rotation around the axis shown in the figure below.

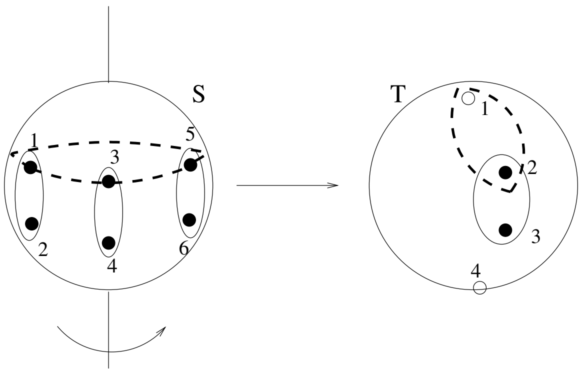

Fig. 3 The order three rotation ψ

The orbifold T now has two marked points and two branch points of ramification index 3. The group $\pi_1(T)$ is now given by

$$\pi_1(T) \ =\ \langle c_1, c_2, c_3, c_4 \mid c_1c_2c_3c_4 = c_1^3 = c_4^3 = 1\rangle\,,$$

$\Gamma(T)$ is the same as in the previous example with the same generators x and y, and we have the splitting homomorphism

$$\begin{aligned} F_{\psi} \ :\ \Gamma(T) &\longrightarrow \Gamma_{\mathfrak{S}_6}(S) \\ x &\longmapsto \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\,, \\ y &\longmapsto \sigma_1^2\sigma_3^2\sigma_5^2\,, \end{aligned} \tag{2.8}$$

indicated by the loops in the figure, whose image is easily checked to normalize ψ.

3 Properties of the group $\widehat{GS}^{\,1}_{0,0}$

3.1 Precise statement of the first main theorem

In this section, we explain in what sense the two defining relations $(*)$ and $(**)$ of $\widehat{GS}^{\,1}_{0,0}$ (cf. §1) come from special loci in the moduli spaces on the first two levels and their associated homomorphisms.

Letting T denote the 4-punctured sphere, S the 5-punctured sphere and S' the 6-punctured sphere, they come from considering the finite-order elements $\varphi \in \Gamma_{\mathfrak{S}_5}(S)$ and $\psi \in \Gamma_{\mathfrak{S}_6}(S')$ introduced in Examples 2.7 and 2.8 at the end of §2, namely

$$\begin{cases} \varphi \;=\; (\sigma_1\sigma_2\sigma_3\sigma_4)^2\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_3\sigma_4)^{-2} & \text{with} \quad G_\varphi \;=\; \langle(23),(14)\rangle\,, \\ \psi \;=\; \sigma_1^2\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4 & \text{with} \quad G_\psi \;=\; \langle(135),(246)\rangle\,, \end{cases}$$

and the corresponding morphisms

$$\eta_\varphi \;:\; \mathcal{M}(T) \;\longrightarrow\; \mathcal{M}_{G_\varphi}(S) \quad \text{and} \quad \eta_\psi \;:\; \mathcal{M}(T) \;\longrightarrow\; \mathcal{M}_{G_\psi}(S')\,,$$

with their associated homomorphisms

$$F_\varphi \;:\; \Gamma(T) \;\longrightarrow\; \Gamma_{G_\varphi}(S) \subset \Gamma_{\mathfrak{S}_5}(S) \quad \text{and} \quad F_\psi \;:\; \Gamma(T) \;\longrightarrow\; \Gamma_{G_\psi}(S) \subset \Gamma_{\mathfrak{S}_6}(S')$$

given by (2.7) and (2.8).

Let A denote the standard tangential base point on $\mathcal{M}(T)$ described by the left-hand figure below, and let B_φ and B_ψ denote the tangential base points $\eta_\varphi(A)$ on $\mathcal{M}_{G_\varphi}(S)$ and $\eta_\psi(A)$ on $\mathcal{M}_{G_\psi}(S')$ respectively. For later convenience, we will also consider two other tangential base points on $\mathcal{M}_{G_\psi}(S')$; the standard tangential base point A' and the tangential base point B'_φ, which reduces to B_φ if the 3th point is erased (and 4, 5, 6 are renumbered to 3, 4, 5). The point B'_φ is not tangential to a point of maximal degeneration, since it has a vertex of valency 4; this means that it is a symmetry point of a 1-dimensional stratum at the infinite divisor.

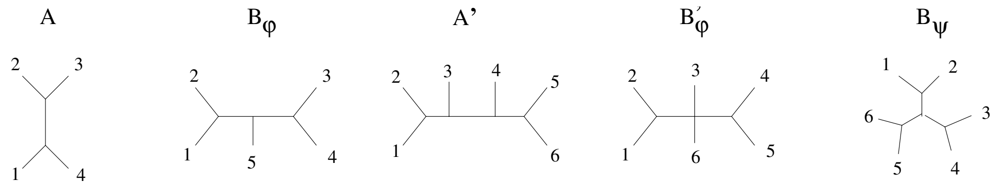

Fig. 4 Some useful tangential base points

We recall (cf. [19, §6 of] for details) that the diagrams in the figure above can be read as tangential base points on the moduli space $\mathcal{M}(T)$ and $\mathcal{M}(S)$ by considering the trees to be inscribed in a circle which represents the real line on $\mathbb{P}^1$, with their endpoints giving the marked points on $\mathbb{P}^1$ which are tangentially close. Explicitly, a topological description of these base points can be given by considering the regions on the corresponding moduli spaces described by the spheres with marked points given as follows, as ϵ varies in a small real positive segment $(0, \epsilon_0)$:

$$\begin{cases} (0,\epsilon,1,\infty) & \text{for} \quad A\,, \\ (0,\epsilon,1-\epsilon,1,\infty) & \text{for} \quad B_\varphi\,, \\ (-\epsilon-\epsilon^2,-\epsilon+\epsilon^2,\epsilon,1-\epsilon,1+\epsilon-\epsilon^2,1+\epsilon+\epsilon^2) & \text{for} \quad A'\,, \\ (-\epsilon,\epsilon,1,1/\epsilon,-1/\epsilon,-1) & \text{for} \quad B'_\varphi\,, \\ (-\epsilon,\epsilon,1-\epsilon,1+\epsilon,1/\epsilon,-1/\epsilon) & \text{for} \quad B_\psi\,. \end{cases}$$

Computing the action of an element $(\lambda = 1, f)$ of $\widehat{GT}$ as an automorphism of the associated profinite mapping class group with respect to each of the base points in Figure 4 is standard practice. We give these automorphism actions explicitly below, denoting by f_C the automorphism corresponding to the base point C. For the definition of $f_{B'_\varphi}$, recall that that g denotes the unique element in $\widehat{F}_2$ such that $f(x,y) = g(y,x)^{-1}g(x,y)$.

$$f_A \ : \ \Gamma(T) \ \longrightarrow \ \Gamma(T) \qquad \begin{cases} x \ \longmapsto \ f(x,y)xf(y,x)\,, \\ y \ \longmapsto \ y\,; \end{cases} \tag{3.1}$$

$$f_{B_\varphi} \ : \ \Gamma_{\mathfrak{S}_5}(S) \ \longrightarrow \ \Gamma_{\mathfrak{S}_5}(S) \qquad \begin{cases} \sigma_1 \ \longmapsto \ \sigma_1\,, \\ \sigma_2 \ \longmapsto \ f(x_{45},x_{34})f(x_{23},x_{12})\sigma_2 f(x_{12},x_{23})f(x_{34},x_{45})\,, \\ \sigma_3 \ \longmapsto \ \sigma_3\,, \\ \sigma_4 \ \longmapsto \ f(x_{45},x_{34})\sigma_4 f(x_{34},x_{45})\,; \end{cases} \tag{3.2}$$

$$f_{A'} \ : \ \Gamma_{\mathfrak{S}_6}(S') \ \longrightarrow \ \Gamma_{\mathfrak{S}_6}(S') \qquad \begin{cases} \sigma_1 \ \longmapsto \ \sigma_1\,, \\ \sigma_2 \ \longmapsto \ f(x_{23},x_{12})\sigma_2 f(x_{12},x_{23})\,, \\ \sigma_3 \ \longmapsto \ f(x_{34},x_{12,3})\sigma_3 f(x_{12,3},x_{34})\,, \\ \sigma_4 \ \longmapsto \ f(x_{45},x_{123,4})\sigma_4 f(x_{123,4},x_{45})\,, \\ \sigma_5 \ \longmapsto \ \sigma_5\,; \end{cases} \tag{3.3}$$

$$f_{B'_\varphi} \ : \ \Gamma_{\mathfrak{S}_6}(S') \longrightarrow \Gamma_{\mathfrak{S}_6}(S') \text{ is given by } f_{B'_\varphi} = \mathrm{inn}(f(x_{45},x_{123,4})g(x_{12,3},x_{3,45})) \circ f_{A'}\,, \tag{3.4}$$

$$f_{B_\psi} \ : \ \Gamma_{\mathfrak{S}_6}(S') \longrightarrow \Gamma_{\mathfrak{S}_6}(S') \text{ is given by } f_{B_\psi} = \mathrm{inn}(f(x_{34},x_{12,3})) \circ f_{A'}\,, \tag{3.5}$$

where $\mathrm{inn}(\alpha)(x) = \alpha^{-1}x\alpha$.

Saying that $(*)$ and $(**)$ come from the morphisms η_φ and η_ψ explicitly means the following.

Theorem 3.1 *Let $f \in \widehat{GT}^{\,1}_{0,0}$. Then f satisfies relation $(*)$ (resp. relation $(**)$) of the definition of $\widehat{GS}^{\,1}_{0,0}$ if and only if the left-hand (resp. right-hand) diagram below commutes:*

$$\begin{array}{ccc} \widehat{\Gamma}(T) & \xrightarrow{F_\varphi} & \widehat{\Gamma}_{\mathfrak{S}_5}(S) \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_{B_\varphi}} \\ \widehat{\Gamma}(T) & \xrightarrow{F_\varphi} & \widehat{\Gamma}_{\mathfrak{S}_5}(S), \end{array} \qquad \begin{array}{ccc} \widehat{\Gamma}(T) & \xrightarrow{F_\psi} & \widehat{\Gamma}_{\mathfrak{S}_6}(S') \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_{B_\psi}} \\ \widehat{\Gamma}(T) & \xrightarrow{F_\psi} & \widehat{\Gamma}_{\mathfrak{S}_6}(S'). \end{array}$$

P r o o f. The proof of this theorem is a direct computation. The very first step is to prove that $(*)$ is equivalent to the form (1.5) given in §1. This observation is due to H. Tsunogai ([20]). One begins by substituting the equality $g(x_{51},x_{45}) = f(\sigma_{51}^2, \sigma_4\sigma_{51}\sigma_4)$ into the right-hand side of $(*)$. Then, conjugating $(*)$ by σ_3, we obtain

$$\begin{aligned} f\big(\sigma_2, \sigma_2^{-1}\sigma_1^2\sigma_3^2\sigma_2\big) f\big(\sigma_1^2\sigma_3^2, \sigma_2\big) \ &= \ f\big(\sigma_{51}^2, \sigma_3\sigma_4\sigma_{51}^2\sigma_4^{-1}\sigma_3^{-1}\big) f\big(\sigma_3^2, \sigma_3\sigma_2^2\sigma_3^{-1}\big) f(\sigma_1^2, \sigma_{51}^2) \\ &= \ f\big(\sigma_{51}^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big) f\big(\sigma_3^2, \sigma_3\sigma_2^2\sigma_3^{-1}\big) f(\sigma_1^2, \sigma_{51}^2) \\ &= \ f\big(\sigma_3^2\sigma_2^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big) f\big(\sigma_3^2, \sigma_2^{-1}\sigma_3^2\sigma_2\big) f\big(\sigma_1^2, \sigma_2\sigma_3^2\sigma_2\big)\,. \end{aligned} \tag{3.6}$$

Here, the second equality is obtained using the identity $x_{51} = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\sigma_4$ for the equality of the first terms, and the third uses the identities

$$f\big(\sigma_3^2\sigma_2^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big) \ = \ \sigma_2^{-1} f\big(\sigma_2\sigma_3^2\sigma_2, \sigma_1^2\big)\sigma_2 \ = \ \sigma_2^{-1} f\big(\sigma_{51}^2, \sigma_1^2\big)\sigma_2 \ = \ f\big(\sigma_{51}^2, \sigma_2^{-1}\sigma_1^2\sigma_{51}\big)$$

for the equalities of the first and last terms.

With (1.5), the proof of the theorem is a direct computation using the two diagrams of the statement, the definitions $F_\varphi(x) = \sigma_2$, $F_\varphi(y) = x_{12}x_{34}$ (cf. (2.7)) and $F_\psi(x) = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}$, $F_\psi(y) = x_{12}x_{34}x_{56}$ (cf.

(2.8)), and (3.1)–(3.5) above. We simply check directly that an element f of $\widehat{GT}^{\,1}_{0,0}$ belongs to $\widehat{GS}^{\,1}_{0,0}$ if and only if the two diagrams commute, i.e., if and only if on the two generators x and y of $\widehat{\Gamma}(T)$, we have

$$f_{B_\varphi}\circ F_\varphi \;=\; F_\varphi\circ f_A \quad\text{and}\quad f_{B_\psi}\circ F_\psi \;=\; F_\psi\circ f_A\,.$$

These two equalities are immediate on the generator y. Let us consider x. If f satisfies $(*)$, or equivalently, (1.5), we have

$$\begin{aligned} f_{B_\varphi}\circ F_\varphi(x) &= f_{B_\varphi}(\sigma_2)\\ &= f(x_{45},x_{34})f(x_{23},x_{12})\sigma_2 f(x_{12},x_{23})f(x_{34},x_{45})\\ &= f(x_{51},x_{12})f(x_{23},x_{34})\sigma_2 f(x_{34},x_{23})f(x_{12},x_{51}) \quad \text{by (III)}\\ &= \sigma_2 f\big(x_{51},\sigma_2^{-1}x_{12}\sigma_2\big)f\big(x_{23},\sigma_2^{-1}x_{34}\sigma_2\big)f(x_{34},x_{23})f(x_{12},x_{51})\\ &= \sigma_2 f\big(x_{51},\sigma_2^{-1}x_{12}\sigma_2\big)f\big(x_{34},\sigma_2^{-1}x_{34}\sigma_2\big)f(x_{12},x_{51}) \quad \text{by (II)}\\ &= \sigma_2 f\big(\sigma_2,\sigma_2^{-1}\sigma_1^2\sigma_3^2\sigma_2\big)f\big(\sigma_1^2\sigma_3^2,\sigma_2\big) \quad \text{by (1.5)}\\ &= f\big(\sigma_2,\sigma_1^2\sigma_3^2\big)\sigma_2 f\big(\sigma_1^2\sigma_3^2,\sigma_2\big)\\ &= F_\varphi\big(f(x,y)xf(y,x)\big)\\ &= F_\varphi\circ f_A(x)\,. \end{aligned} \tag{3.7}$$

This proves that relation $(*)$ implies the first equality, i.e., the commutation of the first diagram. Conversely, if we assume that the first diagram commutes, then the same sequence of equalities shows that f satisfies $(*)$.

Now assume that f satisfies $(**)$ and consider the second diagram. We use a standard formula

$$f_{A'}(\sigma_1\sigma_2\sigma_3\sigma_4) \;=\; f(x_{23},x_{12})f(x_{23,4},x_{1,23})f(x_{234,5},x_{1,234})\sigma_1\sigma_2\sigma_3\sigma_4\,.$$

Thus, since $f_{B_\psi} = \mathrm{inn} f(x_{34},x_{12,3})\circ f_{A'}$ by (3.5), we have

$$\begin{aligned} f_{B_\psi}(\sigma_1\sigma_2\sigma_3\sigma_4) &= f(x_{12,3},x_{34})f(x_{23},x_{12})f(x_{23,4},x_{1,23})f(x_{234,5},x_{1,234})\sigma_1\sigma_2\sigma_3\sigma_4 f(x_{34},x_{12,3})\\ &= f(x_{12,3},x_{34})f(x_{23},x_{12})f(x_{23,4},x_{1,23})f(x_{234,5},x_{1,234})f(x_{45},x_{23,4})\sigma_1\sigma_2\sigma_3\sigma_4\,. \end{aligned} \tag{3.8}$$

Since $f_{B_\psi}(\sigma_1)=\sigma_1$ and $f(B_\psi)(\sigma_3)=\sigma_3$, this yields

$$\begin{aligned} f_{B_\psi}(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}) \;=\; & f(x_{12,3},x_{34})f(x_{23},x_{12})f(x_{23,4},x_{1,23})f(x_{234,5},x_{1,234})\\ &\cdot f(x_{45},x_{23,4})\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\,. \end{aligned} \tag{3.9}$$

So we obtain

$$\begin{aligned} f_{B_\psi}\circ F_\psi(x) &= f_{B_\psi}\big(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\big)\\ &= f(x_{12,3},x_{34})f(x_{23},x_{12})f(x_{23,4},x_{1,23})f(x_{234,5},x_{1,234})\\ &\quad\cdot f(x_{45},x_{23,4})\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1} \quad \text{by (3.9)}\\ &= f\big(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1},\sigma_1^2\sigma_3^2\sigma_5^2\big)f\big(\sigma_2^2\sigma_4^2\sigma_{61}^2,\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\big)\\ &\quad\cdot \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1} \quad \text{by } (**)\\ &= f\big(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1},\sigma_1^2\sigma_3^2\sigma_5^2\big)\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}f\big(\sigma_1^2\sigma_3^2\sigma_5^2,\sigma_1\sigma_2\sigma_3\sigma_4\sigma_3^{-1}\sigma_1^{-1}\big)\\ &= F_\psi\big(f(x,y)xf(y,x)\big)\\ &= F_\psi\circ f_A(x)\,. \end{aligned}$$

As above, the same sequence of equalities shows that if f is assumed to make the second diagram commute, then f necessarily satisfies $(**)$, which concludes the proof of Theorem 3.1. □

Corollary 3.2 *$\widehat{GS}^{\,1}_{0,0}$ is a group.*

P r o o f. By Theorem 3.1, $\widehat{GS}{}^1_{0,0}$ is the set of elements $f \in \widehat{GT}{}^1_{0,0} \subset \widehat{GT}$ which make the two diagrams of Theorem 3.1 commute. For $i = 1, 2$, consider the set

$$E_i \;=\; \Big\{ f \in \widehat{GT}{}^1_{0,0} \mid f \text{ makes the } i\text{-th diagram commute} \Big\} .$$

Then $\widehat{GS}{}^1_{0,0}$ is just the intersection $E_1 \cap E_2$. This intersection is a group simply because each E_i is a group. Indeed, suppose that $f, g \in \widehat{GT}{}^1_{0,0}$ lie in E_i. Then, taking (F, A, B) to be $(F_\varphi, A, B_\varphi)$ if $i = 1$ and (F_ψ, A, B_ψ) if $i = 2$, we have

$$F \circ ((fg)_A) \;=\; F \circ f_A \circ g_A \;=\; f_B \circ F \circ g_A \;=\; f_B \circ g_B \circ F \;=\; ((fg)_B) \circ F ,$$

so $fg \in E_i$, and

$$f_A \circ F \;=\; F \circ f_B \;\Longrightarrow\; F \circ f_B^{-1} \;=\; f_A^{-1} \circ F ,$$

so the inverse of f in $\widehat{GT}{}^1_{0,0}$ lies in E_i if f does. This shows that E_1 and E_2 are groups, so their intersection $\widehat{GS}{}^1_{0,0}$ is a group. □

Corollary 3.3 *Let K be the fixed field of the subgroup of $\sigma \in G_{\mathbb{Q}}$ such that $\chi(\sigma) = 1$, $\rho_2(\sigma) = \rho_3(\sigma) = 0$. Then $G_K \subset \widehat{GS}{}^1_{0,0}$.*

P r o o f. We know that $G_{\mathbb{Q}} \subset \widehat{GT}$, so the biggest subgroup of $G_{\mathbb{Q}}$ which could lie in $\widehat{GS}{}^1_{0,0}$ is exactly G_K, but this whole subgroup does indeed lie in $\widehat{GS}{}^1_{0,0}$ since all Galois elements make the two diagrams in Theorem 3.1 commute. □

3.2 The second main theorem

As explained in the introduction, the second main result of this article states that assuming the two conditions $(*)$ and $(**)$ yield many relations on elements of $\widehat{GT}$ previously introduced in the literature.

Theorem 3.4 *Elements of $\widehat{GS}{}^1_{0,0}$ satisfy the five relations* (R), (III′), (IV), (NT1) *and* (NT2) *introduced in* §1.

P r o o f. Let T denote the 4-punctured sphere, S the 5-punctured sphere and S' the 6-punctured sphere as before. Let us begin with the relations (NT1) and (NT2). We show that $(**)$ implies (NT2) (which in turn then implies relation (II) of $\widehat{GT}{}^1$), and then that $(*)$ and (I) imply (NT1).

Derivation of (NT2) *by pulling out the* 2nd *and* 4th *strands from* $(*)$. The relation $(**)$ is pure in the 2nd and 4th strands, so we can pull them out to obtain a relation in $\widehat{\Gamma}_{\mathfrak{S}_4}(T)$; renumbering 3 to 2, 5 to 3 and 6 to 4, and using $x_{34} = x_{12}$ and $x_{14} = x_{23}$ in $\widehat{\Gamma}(T)$, we obtain

$$f\big(\sigma_1\sigma_2, \sigma_1^2\big) f\big(\sigma_2^2, \sigma_1\sigma_2\big) \;=\; f(x_{23}, x_{12}) \;=\; f\big(\sigma_2^2, \sigma_1^2\big) . \tag{3.10}$$

Conjugating this relation by $\sigma_1\sigma_2\sigma_1$, which exchanges σ_1 and σ_2, we obtain

$$f\big(\sigma_2\sigma_1, \sigma_2^2\big) f\big(\sigma_1^2, \sigma_2\sigma_1\big) \;=\; f\big(\sigma_1^2, \sigma_2^2\big) , \tag{3.11}$$

which is (NT2). Let us recall how (NT2) implies relation (II) of $\widehat{GT}{}^1$. Let ω denote the order 3 inner automorphism of $\Gamma_{\mathfrak{S}_4}(T)$ given by

$$\omega(X) \;=\; (\sigma_2\sigma_1)^{-1} X (\sigma_2\sigma_1) .$$

If we let X be the left-hand side of (3.11) and compute $\omega^2(X)\omega(X)X$, we trivially obtain the identity. However, if we let X be the right-hand side $f\big(\sigma_1^2, \sigma_2^2\big)$ of (3.11) and compute $\omega^2(X)\omega(X)X$, we obtain

$$f\big(\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_1^2\big) f\big(\sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}\big) f\big(\sigma_1^2, \sigma_2^2\big) .$$

Thus this expression is equal to 1. This is exactly relation (II) of $\widehat{GT}^1$, since inside $\widehat{\Gamma}_{\mathfrak{S}_4}(T)$, setting $x = \sigma_1^2$ and $y = \sigma_2^2$, we have $z = (xy)^{-1} = \sigma_1\sigma_2^2\sigma_1^{-1}$.

Derivation of (NT1) *from* $(*)$ *with* (I), (II) *and* (III). We begin by showing that $(*)$ implies the relation

$$\begin{aligned} &f\big(\sigma_2\sigma_3\sigma_2, \sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_4^2\sigma_2\sigma_3\sigma_2\big) f\big(\sigma_1^2\sigma_4^2, \sigma_2\sigma_3\sigma_2\big) \\ &\quad = f\big(\sigma_2^{-1}\sigma_1^2\sigma_2, \sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\sigma_2\big) f\big(\sigma_4^2, \sigma_2^{-1}\sigma_3^{-1}\sigma_4^2\sigma_3\sigma_2\big) f\big(\sigma_1^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big)\,. \end{aligned} \tag{3.12}$$

This relation actually comes directly from the commutation of the diagram

$$\begin{array}{ccc} \widehat{\Gamma}(T) & \xrightarrow{\tilde{F}_{\varphi'}} & \widehat{\Gamma}_{\mathfrak{S}_5}(S) \\ {\scriptstyle f_A}\downarrow & & \downarrow{\scriptstyle f_{B_\varphi}} \\ \widehat{\Gamma}(T) & \xrightarrow{\tilde{F}_{\varphi'}} & \widehat{\Gamma}_{\mathfrak{S}_5}(S) \end{array}$$

analogous to the first diagram of Theorem 3.1, except with F_φ replaced by the homomorphism $\widetilde{F}_{\varphi'} : \widehat{\Gamma}(T) \to \widehat{\Gamma}_{\mathfrak{S}_5}(S)$ given in (2.5) of Example 2.7 in §2. It is not obvious that the commutation of this diagram actually follows from the other one, i.e., that $(*)$ implies (3.12). This key computation is due to Tsunogai ([20]). To prove it, we rewrite (3.12) as follows, making use several times of the following standard trick: since $f(x, y)$ lies in the derived subgroup $\widehat{F}_2'$ of $\widehat{F}_2$, for any γ commuting with both α and β in a profinite group G, we have

$$f(\alpha, \beta) = f(\alpha, \beta\gamma)\,. \tag{3.13}$$

For the left-hand side, we use the fact that $(\sigma_2\sigma_3\sigma_2)^{-1}\sigma_{51}$ commutes with both $\sigma_2\sigma_3\sigma_2$ and with $\sigma_1^2\sigma_4^2$, and for the right-hand side the identity $\sigma_{51} = \sigma_4\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1} = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_4$ which leads to $\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\sigma_2 = \sigma_{51}^{-1}\sigma_4^2\sigma_{51}$, so that (3.12) becomes

$$f\big(\sigma_{51}, \sigma_{51}^{-1}\sigma_1^2\sigma_4^2\sigma_{51}\big) f\big(\sigma_1^2\sigma_4^2, \sigma_{51}\big) = f\big(\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_{51}^{-1}\sigma_4^2\sigma_{51}\big) f\big(\sigma_4^2, \sigma_1\sigma_{51}^2\sigma_1^{-1}\big) f\big(\sigma_1^2, \sigma_1\sigma_2^2\sigma_1^{-1}\big)\,.$$

Conjugating this by $(\sigma_1\sigma_2\sigma_3\sigma_4)^2$, which advances all indices by 2 modulo 5, this gives

$$\begin{aligned} &f\big(\sigma_2, \sigma_2^{-1}\sigma_3^2\sigma_1^2\sigma_5\big) f\big(\sigma_1^2\sigma_3^2, \sigma_2\big) \\ &\quad = f\big(\sigma_3\sigma_4^2\sigma_3^{-1}, \sigma_2^{-1}\sigma_1^2\sigma_2\big) f\big(\sigma_1^2, \sigma_3\sigma_2^2\sigma_3^{-1}\big) f\big(\sigma_3^2, \sigma_3\sigma_4^2\sigma_3^{-1}\big)\,. \end{aligned} \tag{3.14}$$

We saw at the beginning of the proof of Theorem 3.1 that conjugating $(*)$ by σ_3 yields the equivalent relation

$$f\big(\sigma_2, \sigma_2^{-1}\sigma_1^2\sigma_3^2\sigma_2\big) f\big(\sigma_1^2\sigma_3^2, \sigma_2\big) = f\big(x_{51}, \sigma_2^{-1}\sigma_1^2\sigma_2\big) f\big(\sigma_3^2, \sigma_3\sigma_2^2\sigma_3^{-1}\big) f(x_{12}, x_{51})\,. \tag{3.15}$$

So in order to prove that $(*)$ implies (3.12), it suffices to show that the right-hand sides of (3.15) and (3.14) are equal. Conjugating them both by σ_3 and using the equality $\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3 = \sigma_4 x_{51}\sigma_4^{-1}$, this means we must show the equality

$$f\big(x_{51}, \sigma_4 x_{51}\sigma_4^{-1}\big) f\big(\sigma_3^2, \sigma_2^2\big) f(x_{12}, x_{51}\big) = f\big(\sigma_4^2, \sigma_4 x_{51}\sigma_4^{-1}\big) f\big(\sigma_1^2, \sigma_2^2\big) f\big(\sigma_3^2, \sigma_4^2\big)\,.$$

Rewriting this as

$$f\big(\sigma_4 x_{51}\sigma_4^{-1}, x_{45}\big) f\big(x_{51}, \sigma_4 x_{51}\sigma_4^{-1}\big) = f(x_{12}, x_{23}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34})\,,$$

we see by a simple application of relation (II) that the left-hand side is equal to $f(x_{51}, x_{45})$, and therefore that the entire equality holds by (III). We have thus shown that $(*)$ implies (3.12).

Now, relation (3.12) is pure in the 5th strand; pulling it out and using the identity $\sigma_1^2 = \sigma_3^2$ in $\widehat{\Gamma}(T)$ yields

$$f\big(\sigma_2\sigma_3\sigma_2, \sigma_2^2\big) f\big(\sigma_3^2, \sigma_2\sigma_3\sigma_2\big) = f\big(\sigma_2^{-1}\sigma_1^2\sigma_2, \sigma_2^2\big) f\big(\sigma_1^2, \sigma_2^{-1}\sigma_1^2\sigma_2\big) = f\big(\sigma_3^2, \sigma_2^2\big) \tag{3.16}$$

in $\widehat{\Gamma}(S_{0,4})$, the second equality being a simple consequence of relation (II) and $\sigma_1^2 = \sigma_3^2$.

Conjugating this relation by $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$ inside $\widehat{\Gamma}_{\mathfrak{S}_4}(T)$, which exchanges σ_1 and σ_3 and fixes σ_2, it becomes

$$f\big(\sigma_1^2,\sigma_2^2\big) \;=\; f\big(\sigma_1\sigma_2\sigma_1,\sigma_2^2\big)f\big(\sigma_1^2,\sigma_1\sigma_2\sigma_1\big)\,, \tag{3.17}$$

which is exactly (NT1). Note that since there is a unique g such that

$$f\big(\sigma_1^2,\sigma_2^2\big) \;=\; g\big(\sigma_2^2,\sigma_1^2\big)^{-1}g\big(\sigma_1^2,\sigma_2^2\big)\,,$$

we find that

$$g\big(\sigma_1^2,\sigma_2^2\big) \;=\; f\big(\sigma_1^2,\sigma_1\sigma_2\sigma_1\big)\,. \tag{3.18}$$

Derivation of a new relation from $(*)$ *using* (I), (II) *and* (III). For this, we use the conjugation form of $(*)$ given in (1.4), namely

$$\sigma_2^{f(\sigma_1^2\sigma_3^2,\sigma_2)} \;=\; \sigma_2^{f(x_{12},x_{23})f(x_{34},x_{45})}$$

in $\widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$. Thus, we have

$$f\big(\sigma_1^2\sigma_3^2,\sigma_2\big) \;=\; \alpha f(x_{12},x_{23})f(x_{34},x_{45}) \tag{3.19}$$

for some $\alpha \in \widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$ commuting with σ_2.

In order to compute α, we need to use the fact that f also satisfies relation (III) of $\widehat{GT}$. Using this relation, there is a standard trick which works as follows. Let η be the order 2 automorphism of $\widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$ given by $\eta(\sigma_1) = \sigma_3$, $\eta(\sigma_2) = \sigma_2$, $\eta(\sigma_3) = \sigma_1$, $\eta(\sigma_4) = \sigma_{51} = \sigma_4\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}$, and $\eta(\sigma_{51}) = \sigma_4$. Then (3.19) yields

$$\begin{aligned} 1 \;&=\; f\big(\sigma_2,\sigma_1^2\sigma_3^2\big)f\big(\sigma_1^2\sigma_3^2,\sigma_2\big)\\ &=\; \eta\big(f\big(\sigma_1^2\sigma_3^2,\sigma_2\big)\big)^{-1}f\big(\sigma_1^2\sigma_3^2,\sigma_2\big)\\ &=\; \eta(\alpha f(x_{12},x_{34})f(x_{34},x_{45}))^{-1}\cdot\alpha f(x_{12},x_{23})f(x_{34},x_{45})\\ &=\; f(x_{51},x_{12})f(x_{23},x_{34})\eta(\alpha)^{-1}\alpha f(x_{12},x_{23})f(x_{34},x_{45})\,. \end{aligned} \tag{3.20}$$

By the pentagon relation (III) satisfied by f, we find that

$$\eta(\alpha)^{-1}\alpha \;=\; f(x_{45},x_{51})\,.$$

Then we know that there is a unique solution for α, namely

$$\alpha \;=\; g(x_{45},x_{51}) \;=\; g\big(\sigma_4^2,\sigma_{51}^2\big) \;=\; f\big(\sigma_4^2,\sigma_4\sigma_{51}\sigma_4\big)$$

by (3.14) above. Therefore (3.19) can be rewritten precisely as

$$\begin{aligned} f\big(\sigma_1^2\sigma_3^2,\sigma_2\big) \;&=\; g(x_{45},x_{51})f(x_{12},x_{23})f(x_{34},x_{45})\\ &=\; f(x_{45},\sigma_4\sigma_{51}\sigma_4)f(x_{12},x_{23})f(x_{34},x_{45})\,. \end{aligned} \tag{3.21}$$

Let us show now that this striking relation implies both (IV) and (III′).

Derivation of (III′) *and* (IV) *from* $(*)$ *via* (3.17). We just saw that $(*)$ implies (3.21). Now, since (3.21) is pure in the 5th strand, we can pull that out; then, using $\sigma_1^2 = \sigma_3^2$, i.e., $x_{34} = x_{12}$ in $\widehat{\Gamma}_{\mathfrak{S}_4}(S_{0,4})$, we find

$$f\big(\sigma_1^4,\sigma_2\big) \;=\; f(x_{12},x_{23}) \;=\; f\big(\sigma_1^2,\sigma_2^2\big)\,,$$

which is relation (IV).

For relation (III′), we note that Equation (3.21) is similar but not equivalent to (III′), since the left-hand sides differ, (III′) being given by

$$f\big(\sigma_1\sigma_3, \sigma_2^2\big) \;=\; f(x_{45}, \sigma_4\sigma_{51}\sigma_4) f(x_{12}, x_{23}) f(x_{34}, x_{45})\,.$$

However, note that $\sigma_1^2 \mapsto \sigma_1\sigma_3$ and $\sigma_2 \mapsto \sigma_2$ gives a group homomorphism from the subgroup $\langle\sigma_1^2, \sigma_2\rangle$ of $\widehat{B}_4$ (in which (IV) actually takes place) to the subgroup $\langle\sigma_1\sigma_3, \sigma_2\rangle$ of $\widehat{\Gamma}_{\mathfrak{S}_5}(S_{0,5})$. Therefore, applying this homomorphism to (IV), we find that

$$f\big(\sigma_1^2\sigma_3^2, \sigma_2\big) \;=\; f\big(\sigma_1\sigma_3, \sigma_2^2\big)\,,$$

so that as (3.21) implies (IV), it also implies (III′).

Derivation of (R) *from* $(*)$. We just saw that (*) implies (IV), so to conclude the proof of Theorem 3.4, it suffices to recall that in fact (IV) (together with (I), (II) and (III)) implies (R). Indeed, we first rewrite (R) as

$$\begin{aligned} &f\big((\sigma_2\sigma_3)^6, \sigma_1\big) f(\sigma_2^2, \sigma_3^2) f\big(\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}, (\sigma_2\sigma_3)^6\big) \\ &\cdot f\big((\sigma_1\sigma_2)^6, \sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\big) f(\sigma_1^2, \sigma_2^2) f\big(\sigma_3, (\sigma_1\sigma_2)^6\big) \;=\; 1\,. \end{aligned}$$

Applying relation (IV) to the 1st, 3rd, 4th and 6th factors yields

$$f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{14}, x_{51}) f(x_{45}, x_{14}) f(x_{12}, x_{23}) f(x_{34}, x_{45})\,.$$

Applying (II) to the middle two terms yields

$$f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45})\,,$$

which is just the pentagon, so is equal to 1. This concludes the proof of Theorem 3.4. □

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