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On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups

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Contents

§1. Introduction and main statements.

In this article, we introduce a certain group $\mathbb{I}\Gamma$ as a subgroup of the Grothendieck-Teichmüller group $\widehat{GT}$, by adding two newtype relations to the definition of $\widehat{GT}$. We show that the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is mapped into $\mathbb{I}\Gamma$ (in fact, injectively by virtue of Belyi's result [Be].) Although we still leave it open to settle (in-)equalities between consecutive terms of $G_{\mathbb{Q}} \subset \mathbb{I}\Gamma \subset \widehat{GT}$, we show that $\mathbb{I}\Gamma$ acts on all types of the profinite Teichmüller modular groups $\hat{\Gamma}^n_{g,m}$ in certain consistent ways respecting natural homomorphisms between them.

First, let us review briefly studies on the Grothendieck-Teichmüller group. Let B_n denote the Artin braid group on n strands, generated by standard generators $\tau_1, \dots, \tau_{n-1}$, subject to the relations $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ $(1 \le i < n))$ and $\tau_i\tau_j = \tau_j\tau_i$ $(|i-j| \ge 2)$. There is a canonical surjection of B_n onto S_n, the symmetric group of degree n, obtained by looking merely at the permutations of strands. The kernel is the pure braid group P_n generated by the elements $x_{ij} = \tau_{j-1} \cdots \tau_{i+1}\tau_i\tau_i\tau_{i+1}^{-1} \cdots \tau_{j-1}^{-1}$ for $1 \le i < j \le n$. We set $x_{ji} = x_{ij}$ and $x_{ii} = 1$. By convention, we denote the profinite completion of a discrete group Γ by $\hat{\Gamma}$.

In [D], V.G.Drinfeld introduced the Grothendieck-Teichmüller group as follows. First, let $\hat{F}_2$ be the free profinite group of rank 2 with free generators x, y, and let $\underline{\widehat{GT}}$ be the set of pairs $F = (\lambda, f) \in \widehat{\mathbb{Z}}^\times \times [\hat{F}_2, \hat{F}_2]$ (where the latter bracket means the commutator subgroup) satisfying the following three relations:

(I) $$f(x,y)f(y,x)=1,$$

(II) $$f(x,y)x^{\mu}f(z,x)z^{\mu}f(y,z)y^{\mu}=1,\ \text{where } \mu=(\lambda-1)/2,\ z=(xy)^{-1},$$

(III) $$f(x_{12},x_{23}x_{24})f(x_{13}x_{23},x_{34})=f(x_{23},x_{34})f(x_{12}x_{13},x_{24}x_{34})f(x_{12},x_{23}).$$

Here, the relation (III) is understood to hold as a relation in $\hat{B}_4$, under the rule that, for $f\in\hat{F}_2$ and elements a,b of a profinite group G, $f(a,b)$ represents the image $\phi(f)$ by the homomorphism $\phi:\hat{F}_2\to G$ defined by $\phi(x)=a$, $\phi(y)=b$. An element $F\in\underline{\widehat{GT}}$ induces an endomorphism of $\hat{F}_2$ given by $F(x)=x^{\lambda}$ and $F(y)=f^{-1}y^{\lambda}f$, and the composition of these endomorphisms makes $\underline{\widehat{GT}}$ a monoid. The Grothendieck-Teichmüller group $\widehat{GT}$ is by definition the group of invertible elements of $\underline{\widehat{GT}}$, which can naturally be identified with a subgroup of $\mathrm{Aut}(\hat{F}_2)$.

In [I1], Y.Ihara pointed out that the above third relation (III) is equivalent to the following 5-cyclic relation in the profinite Teichmüller modular group $\hat{\Gamma}_0^5$:

(III) $$f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23})=1.$$

Before explaining the notation appearing above, let us introduce the Teichmüller modular groups in their most general form, in order to keep the consistency of notation with the following paragraphs. Let $\Sigma_{g,m}^n$ be a compact oriented topological surface of genus g with m boundary components and n marked points. Let $\Gamma_{g,m}^{[n]}$ denote the mapping class group of $\Sigma_{g,m}^n$, i.e. the group of isotopy classes of diffeomorphisms fixing boundary points and permuting the marked points, and write $\Gamma_{g,m}^n=\Gamma(\Sigma_{g,m}^n)$ for its "pure" subgroup consisting of the classes of diffeomorphisms not permuting the marked points. For shortness, we write $\Sigma_{g,m}=\Sigma_{g,m}^0$, $\Sigma_g^n=\Sigma_{g,0}^n$, $\Gamma_g^n=\Gamma_{g,0}^n$ etc. It is well known that there is a canonical surjection of B_n to $\Gamma_0^{[n]}$ through which one can define elements τ_i, x_{ij} of $\Gamma_0^{[n]}$ as the images of those of B_n. The generators of Γ_0^5 used by Ihara in the above latter form of (III) are the images of the corresponding generators of B_5.

In this article, we call the profinite completions of surface mapping class groups the *profinite Teichmüller modular groups.* The profinite group $\hat{\Gamma}_g^n$ can be naturally identified with the algebraic fundamental group of the moduli stack $M_{g,n}/\overline{\mathbb{Q}}$ of smooth projective curves of genus g with n ordered marked points (cf. Oda [O]). (Notation: Whenever dealing with a space X defined over $\mathbb{Q}$, we write $X/\overline{\mathbb{Q}}$ for the same space with scalars extended to $\overline{\mathbb{Q}}$, and $X/\mathbb{Q}$ if it is necessary to recall that we are considering it over $\mathbb{Q}$.) From this interpretation, we have a canonical outer $G_{\mathbb{Q}}$-action on $\hat{\Gamma}_g^n$. In the special case of $g=0,n=4$, the moduli space $M_{0,4}$ is isomorphic to $\mathbb{P}^1-\{0,1,\infty\}$, and by comparing the $\widehat{GT}$ action on $\hat{F}_2$ with the Belyi lifting of the canonical outer $G_{\mathbb{Q}}$-action on $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}}-\{0,1,\infty\})\cong\hat{\Gamma}_0^4$, one obtains an injection $G_{\mathbb{Q}}\hookrightarrow\widehat{GT}$ (cf. Belyi [B] for injectivity of $G_{\mathbb{Q}}\to\mathrm{Aut}\hat{F}_2$; cf. Ihara [I2], [N0, Appendix] for first rigorous proofs that the image satisfies (I),(II) and (III).)

The group $\widehat{GT}$ acts on $\hat{B}_n$ universally with respect to n; if $F=(\lambda,f)\in\widehat{GT}$ and $n\geq 3$, the transformation of the standard braid generators

(1.1) $$\begin{cases} F(\tau_1) &= \tau_1^{\lambda}, \\ F(\tau_i) &= f(\tau_i^2,y_i)\tau_i^{\lambda}f(y_i,\tau_i^2) \quad (1<i\leq n-1) \end{cases}$$

(where $y_i = \tau_{i-1} \cdots \tau_1 \tau_1 \cdots \tau_{i-1}$) extends to an automorphism of $\hat{B}_n$. The above beautiful formula (1.1) was discovered by Drinfeld [Dr] in the context of the pro-unipotent braid groups acting on tensored modules of quasi-Hopf algebras, and in the profinite context, the extendability to $\mathrm{Aut}\,\hat{B}_n$ was confirmed first by Ihara [IM, Appendix] and then by [S]-[LS] with independent methods. Passing to the quotient $\widehat{\Gamma}_0^{[n]}$ of $\hat{B}_n$, we obtain $\widehat{GT}$-actions on the genus zero tower of profinite Teichmüller modular groups (see also [HS]). The $\widehat{GT}$-action on $\widehat{B}_n$ and $\widehat{\Gamma}_0^{[n]}$ by formula (1.1) is called the *standard* $\widehat{GT}$*-action*.

One of the motivating clues to the present article was a result by the first named author that $G_{\mathbb{Q}}$ acts on the Lickorish twist generators of (higher genus) profinite Teichmüller modular groups $\widehat{\Gamma}_{g,1}$ in a similar fashion to the above standard action ([N1], cf. also §3(3.2) below). Moreover, by explicitly comparing Galois representations in $\hat{\Gamma}_0^5$ and $\hat{\Gamma}_1^2$ ([N2], Theorem 4.16), he encountered a mysterious newtype relation in $\hat{B}_3$:

$$\text{(IV)} \qquad f(\tau_1, \tau_2^4) = \tau_2^{8\rho_2(F)} f(\tau_1^2, \tau_2^2) \tau_1^{4\rho_2(F)} (\tau_1\tau_2)^{-6\rho_2(F)}$$

satisfied by the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. Here, ρ_2 represents a "Kummer 1-cocycle with respect to the roots of 2", which can be extended to a 1-cocycle map $\widehat{GT} \to \hat{\mathbb{Z}}$ (cf. §5 below). Then, our discussions (partly with Pierre Lochak) aiming to understand the relation (IV) in view of moves of pants decomposition of Riemann surfaces, produced a second newtype relation (III′):

Theorem 1.1. *For any element $F = (\lambda, f)$ of $\widehat{GT}$, let $g(x,y) \in \hat{F}_2$ denote the unique element satisfying $f(x,y) = g(y,x)^{-1} g(x,y)$ introduced in [LS2]. Then,*

$$\text{(III}'\text{)} \qquad f(\tau_1\tau_3, \tau_2^2) = g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45})$$

holds in $\hat{\Gamma}_0^{[5]}$ for the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$.

Indeed, the relation (III′) implies (III) easily; apply the involution induced from that of $\hat{B}_4$ interchanging $\tau_1 \leftrightarrow \tau_3$ and fixing τ_2 to (III′), and eliminate $f(\tau_1\tau_3, \tau_2^2)$ from the resulting formulae, then (III) follows at once. (Note here that $x_{45} = (\tau_1\tau_2)^3$, $x_{51} = (\tau_2\tau_3)^3$ in $\hat{\Gamma}_0^{[5]}$.) Observation of these two newtype relations (IV),(III′) playing certain roles in moves of pants decomposition leads us to introduce the following

Definition 1.1. We define a subset $\mathrm{I}\!\Gamma$ of $\widehat{GT}$ to be the collection of all $(\lambda, f) \in \widehat{GT}$ satisfying (III′) and (IV).

Our first task is now to establish

Theorem 1.2. *$\mathrm{I}\!\Gamma$ forms a subgroup of $\widehat{GT}$, which contains the absolute Galois group $G_{\mathbb{Q}}$.*

Notice that, from the above mentioned results, we already know the second statement, that $\mathrm{I}\!\Gamma$ is nontrivially big enough to contain the absolute Galois group $G_{\mathbb{Q}}$.

Next step of our program is to investigate close-compatibilities of $\mathrm{I}\!\Gamma$-actions on the profinite Teichmüller modular groups under moving pants decompositions of

The character $\mathrm{I}\!\Gamma$ may be typeset, say in Latex, by `$\mathrm{I}\!\Gamma$`.

Riemann surfaces. In [LNS], we stated results of Theorems 1.1 and 1.2 together with certain evidence for the above compatibilities in the special case of $\hat{\Gamma}_g^1$. After writing the note [LNS], the undergrounding philosophy of moves on complexes of curves was realized in [HLS]. In this article we generalize that philosophy and use this to extend the results of [LNS] to the general case, and to obtain the further theorems 1.3 and 1.4 below. One of the essential generalizations is the following. In [HLS], it is shown that imposing the following additional relation (R) to the elements of $\widehat{GT}$ with $\lambda = 1$ is crucial to define certain automorphisms of the tower of $\hat{\Gamma}_{g,m}^n$:

$$\text{(R)} \qquad f(e_3, a_1)f(a_2^2, a_3^2)f(e_2, e_3)f(e_1, e_2)f(a_1^2, a_2^2)f(a_3, e_1) = 1.$$

Here a_i, e_i $(i = 1, 2, 3)$ are certain elements of $\hat{\Gamma}_{1,2}$ (given as Dehn twists along certain circles on $\Sigma_{1,2}$). Moreover, the last named author found that the elements of $\mathbb{I}\Gamma$ with $\lambda = 1$, $\rho_2 = 0$ satisfy the above relation (R). In this article, we continue this investigation more to extend our program to the total $\mathbb{I}\Gamma$. In particular, we generalize (R) to the following refined form (see §8 for details):

$$\text{(R}'\text{)} \qquad \begin{gathered} f(e_3, a_1)a_3^{-8\rho_2}f(a_2^2, a_3^2)(a_3a_2a_3)^{2\mu}f(e_2, e_3)e_2^{2\mu}f(e_1, e_2)a_2^{-2\mu} \\ f(a_1^2, a_2^2)a_1^{8\rho_2}(a_1a_2a_1)^{2\mu}f(a_3, e_1)\epsilon_1^{-\mu}\epsilon_2^{-\mu} = 1. \end{gathered}$$

One of our outstanding features here is to introduce a notion of "quilt-decomposition" (or just called "quilt" for shortness) of a surface Σ which refines the notion of pants decomposition of Σ (see §7 for the precise definition). Roughly speaking, a quilt Q over a given pants decomposition P of Σ (written Q/P) is an isotopy class of the ways of dividing each pair of pants of P into two hexagonal patches. Starting from a quilt Q/P, we define an action of $\mathbb{I}\Gamma$ on all (infinitely many) Dehn twists in $\hat{\Gamma}(\Sigma)$ in well-defined manners (§8) by using the description of the simplicial complex of pants decompositions given in [HLS]. And then we show in §§9-10,

Theorem 1.3. *For any surface $\Sigma = \Sigma_{g,m}^n$ with a quilt-decomposition Q/P given, one can define a representation in the profinite Teichmüller modular group $\hat{\Gamma}(\Sigma)$:*

$$\rho_{Q/P}^{\Sigma} : \mathbb{I}\Gamma \longrightarrow \operatorname{Aut}\hat{\Gamma}(\Sigma)$$

in a certain systematic way.

The content of Theorem 1.3 as proved in §§9-10 includes our explicit description of $\mathbb{I}\Gamma$-action (§8) on the Dehn twist generators of the Teichmüller modular group in terms of the main parameter $(\lambda, f) \in \mathbb{I}\Gamma$ and the auxiliary parameter ρ_2 introduced in §5. Moreover, it will be shown that, if another quilt Q'/P' on the surface Σ is chosen, then the difference of the two lifted representations $\rho_{Q/P}^{\Sigma}$ and $\rho_{Q'/P'}^{\Sigma}$ can be computed to be an explicitly given inner automorphism of $\hat{\Gamma}(\Sigma)$. In particular, we obtain a *canonical* exterior representation

$$\rho^{\Sigma} : \mathbb{I}\Gamma \to \operatorname{Out}\hat{\Gamma}(\Sigma)$$

which is independent of choices of quilt-decompositions of Σ. Moreover, by construction, we have the following compatibility theorem for this type of representations:

Theorem 1.4. *Let Q/P be a quilt-decomposition of a surface Σ and let $\Sigma' \subset \Sigma$ be a connected subsurface of Σ consisting of (closures of) pairs of pants from P. Then $\rho^{\Sigma}_{Q/P}(\mathbb{I}\Gamma)$ preserves the image of the natural homomorphism $\hat{\Gamma}(\Sigma') \to \hat{\Gamma}(\Sigma)$, and if Q'/P' denotes the quilt on Σ' induced from Q/P by restriction, then the two actions $\rho^{\Sigma}_{Q/P}$, $\rho^{\Sigma'}_{Q'/P'}$ fit in the commutative diagram:*

$$\begin{array}{ccc} \hat{\Gamma}(\Sigma') & \longrightarrow & \hat{\Gamma}(\Sigma) \\ {\scriptstyle \rho^{\Sigma'}_{Q'/P'}(F)} \downarrow & & \downarrow {\scriptstyle \rho^{\Sigma}_{Q/P}(F)} \\ \hat{\Gamma}(\Sigma') & \longrightarrow & \hat{\Gamma}(\Sigma) \end{array}$$

for all $F \in \mathbb{I}\Gamma$.

The definition of $\rho^{\Sigma}_{Q/P}$ encodes our Galois-theoretic knowledge concerning the effects of change of tangential basepoints on Galois representations in $\pi_1(M_{g,n})$. In [IN], we defined tangential basepoints on $M_{g,n}$ by deformation of maximally degenerate stable marked curves endowed with combinatorial data — so called "tangential structures" on dual graphs. Our notion of quilts has been abstracted from certain detailed study of behaviors of such tangential basepoints on specific types of moduli spaces $M_{g,n}$ (cf. [IM] for $M_{0,n}$, [Ma], [N1] for $M_{g,1}$, [N2] for $M_{1,2}$). Roughly speaking, moves of quilted pants decompositions correspond to moves of tangential basepoints along 1-dimensional strata of the stable compactifications of $M_{g,n}$ in the sense of Deligne-Mumford-Knudsen. The essence of this philosophy was indicated in "Esquisse d'un Programme" [Gr] by A.Grothendieck. Our use of terminology on quilts has been inspired from an interesting paper by Conway-Hsu [CH] on Moonshine, although the objects they define as quilts are not the same as those defined here. After our completing the main part of this work, we learned of appearences of related topological work by Bakalov-Kirillov [BK], Funar-Gelca [FG] concerning Teichmüller groupoids and Moore-Seiberg's questions [MS]. We expect future investigations which will clarify and develop relations between their formulations and ours.

In §11, we finally compute, for a standard quilt on $\Sigma^n_{g,m}$, the $\mathbb{I}\Gamma$-action on a finite number of twist generators of $\hat{\Gamma}^n_{g,m}$ of Lickorish-Humphries type, and give explicit formulae of transformations of those generators in terms of $(\lambda, f) \in \mathbb{I}\Gamma$. *Indeed, this action properly extends the* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-action on* $\pi_1(M_{g,n}/\overline{\mathbb{Q}})$ *given by a standard* ($\mathbb{Q}$*-rational*) *tangential base point on* $M_{g,n}$ (cf. [N1] §6). Still, in this paper, we do not entirely present the whole dictionary between our topological manipulations of quilts here and its algebro-geometric correspondents. We hope to discuss some of them in a future publication.

Besides our group $\mathbb{I}\Gamma \subset \widehat{GT}$ discussed in this paper, there is another subgroup "GTA"$\subset \widehat{GT}$ introduced by Y.Ihara [I3] from an independent arithmetic motivation of (hyper-)adelic beta and gamma functions. Both $\mathbb{I}\Gamma$ and GTA contain the Galois group $G_{\mathbb{Q}}$, but at the time of writing this paper, neither $\mathbb{I}\Gamma$ nor GTA is known to be equal to $G_{\mathbb{Q}}$ or strictly smaller than $\widehat{GT}$. Moreover, relations between $\mathbb{I}\Gamma$ and GTA are not fully understood yet. But some techniques of our treating $\mathbb{I}\Gamma$ in the present paper are influenced from the "profinite free differential calculus" which has been introduced in Ihara's work to play a crucial role there. Still, we would expect more intrinsic relationships between GTA and $\mathbb{I}\Gamma$ to be inspected in future studies.

More recently, H.Tsunogai [T] investigated geometry of $M_{0,5}$ from a motivation to understand our relation (III$'$) in a more direct way. T.Ichikawa's study [Ich] seems to indicate some positive evidence for understanding our moving process (§§8,9) in view of Mumford's uniformization of degenerate curves. We hope that their interesting related studies will appear in the near future.

Before proceeding to the main text of this article, we give one technical lemma on $\widehat{GT}$ which we will use several times below.

Lemma 1.5. *Suppose that three elements x, y, z in a profinite group G satisfy the conditions that the product $\omega := xyz$ commutes with each of x, y, z; Then, for $F = (\lambda, f) \in \widehat{GT}$, we have*

$$f(x,y)x^{\mu}f(z,x)z^{\mu}f(y,z)y^{\mu} = \omega^{\mu} \tag{1.5.1}$$

$$f(x,y)x^{-1-\mu}f(z,x)z^{-1-\mu}f(y,z)y^{-1-\mu} = \omega^{-1-\mu}, \tag{1.5.2}$$

where $\mu = (\lambda - 1)/2$.

Proof. In the case where $G = \hat{F}_2$ with free generators x, y and $z = (xy)^{-1}$, (1.5.1) is the same as the relation (II) satisfied by (λ, f). For (1.5.2) in this case, we note that the element $(\lambda, f)(-1, 1) = (-\lambda, f)$ also belongs to $\widehat{GT}$, so that the relation (II) for this element is given by $f(x,y)x^{\mu'}f(z,x)z^{\mu'}f(y,z)y^{\mu'} = 1$ where $\mu' = (-\lambda - 1)/2$, i.e. $\mu' = -\mu - 1$.

For the general case, let $x, y, z \in G$ be as in the assumption, and let X, Y and $Z = (XY)^{-1}$ now denote the generators of $\hat{F}_2$. Then, we have a homomorphism $\hat{F}_2 \to G$ given by $X \mapsto x$, $Y \mapsto y$ and $Z \mapsto z' = z\omega^{-1}$, which brings the relation (II) for $\hat{F}_2$ to

$$f(x,y)x^{\mu}f(z',x)(z')^{\mu}f(y,z')y^{\mu} = f(x,y)x^{\mu}f(z\omega^{-1},x)z^{\mu}f(y,z\omega^{-1})y^{\mu}\omega^{-\mu} = 1.$$

To conclude, we note that for any elements γ, a, b in G such that γ commutes with a and b, we have $f(\gamma a, b) = f(a, \gamma b) = f(a,b)$ since $f \in \hat{F}_2'$. Thus $f(z\omega^{-1}, x) = f(z,x)$ and $f(y, z\omega^{-1}) = f(y,z)$, which proves (1.5.1) for G. The proof of (1.5.2) for G follows identically from the validity of (1.5.1) for $\hat{F}_2$. □

§2. The 1-cocycle $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$.

Let $\hat{\mathbb{Z}}(1)$ denote the Tate twist of $\hat{\mathbb{Z}}$, i.e. $\hat{\mathbb{Z}}(1)$ is equal to $\hat{\mathbb{Z}}$ as a set, but it is equipped with the $G_{\mathbb{Q}}$ action given by $\sigma(x) = \chi(\sigma) \cdot x$, where χ is the cyclotomic character. Define the Kummer 1-cocycle $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ of (positive) roots of 2 by

$$\sigma(\sqrt[n]{2}) = \zeta_n^{\rho_2(\sigma)}\sqrt[n]{2} \qquad (n \geq 1)$$

for $\sigma \in G_{\mathbb{Q}}$, where $\zeta_n = \exp(2\pi i/n)$. If $\tau, \sigma \in G_{\mathbb{Q}}$, we have

$$\tau\sigma(\sqrt[n]{2}) = \tau\big(\zeta_n^{\rho_2(\sigma)}\sqrt[n]{2}\big) = \zeta_n^{\chi(\tau)\rho_2(\sigma)+\rho_2(\tau)}\sqrt[n]{2},$$

so that $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ is a crossed homomorphism, i.e. one which satisfies

$$\rho_2(\sigma\tau) = \tau\big(\rho_2(\sigma)\big) + \rho_2(\tau) = \chi(\tau)\rho_2(\sigma) + \rho(\tau).$$

As shown in [N1-2], this 1-cocycle ρ_2 plays certain crucial roles in descriptions of Galois representations in profinite Teichmüller modular groups. In this section, we summarize several aspects of the behavior of ρ_2 on the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$, which will be compared later again in §5 when we extend ρ_2 to the whole of $\widehat{GT}$.

Let us first review geometric interpretation of the image $(\lambda_\sigma, f_\sigma) \in \widehat{GT}$ of a Galois element $\sigma \in G_{\mathbb{Q}}$ given by Ihara ([I1,2]). Let $\mathbf{P}^1_t$ be the projective t-line with standard coordinate t, and consider the fundamental groupoid of $X = \mathbf{P}^1_t - \{0, 1, \infty\}$ with tangential basepoints $\overrightarrow{01}, \overrightarrow{10}$. Here, $\overrightarrow{01}$ is defined by the geometric point $\mathrm{Spec}\,\overline{\mathbb{Q}}\{\{t\}\} \to X$ valued in the Puiseux field $\overline{\mathbb{Q}}\{\{t\}\} = \bigcup_{n=1}^{\infty} \overline{\mathbb{Q}}((t^{1/n}))$, and $\overrightarrow{10}$ is defined by $\mathrm{Spec}\,\overline{\mathbb{Q}}\{\{1-t\}\} \to X$. These tangential basepoints are illustrated as in Figure 2.1, and we introduce standard loops x, y based at $\overrightarrow{01}$ and a path γ from $\overrightarrow{01}$ to $\overrightarrow{10}$ as in Figure 2.1.

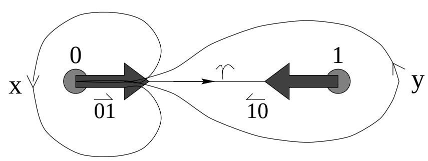


Figure 2.1

We have a canonical Galois action on the chains of this groupoid, and $(\lambda_\sigma, f_\sigma) \in \widehat{GT}$ is defined by

$$\sigma(x) = x^{\lambda}, \quad \sigma(\gamma) = f_\sigma(x, y)^{-1}\gamma. \tag{2.1}$$

This gives $\sigma(y) = f_\sigma(x,y)^{-1} y^{\lambda} f_\sigma(x,y)$ since $y = \gamma\theta(x)\gamma^{-1}$, where θ denotes the automorphism of $\mathbf{P}^1_t$ given by $\theta(t) = 1 - t$. Here we employ a systematically fixed convention of path composition introduced in [N2] §2, where we compose paths from left to right under the rule that each path draws fibre-objects backward. (Our $f_\sigma(x,y)$ here is $\mathfrak{f}_\sigma(x,y)$ of loc.cit. and is $f_\sigma(x^{-1}, y^{-1})$ of Ihara [I1,2], but the difference is not theoretically essential except for small alterations of indices in formulae). It is known that $\sigma \mapsto \lambda_\sigma$ is the cyclotomic character on $G_{\mathbb{Q}}$ and that f_σ is contained in the commutator subgroup of $\pi_1(X_{\overline{\mathbb{Q}}}, \overrightarrow{01}) \cong \hat{F}_2$. We often regard x, y as free non-commutative generators of $\hat{F}_2$ and $f_\sigma(x,y)$ as a "pro-word" in variables x, y.

The appearance of the Kummer 1-cocycle ρ_2 is typically observed in the following

Theorem 2.1. ([N2] Theorem 4.16) *Let $B_3 = \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle$ be the Artin braid group of 3 strands. Then, for each $\sigma \in G_{\mathbb{Q}}$, the relation*

$$\text{(IV)} \qquad f_\sigma(\tau_1, \tau_2^4) = \tau_2^{8\rho_2(\sigma)} f_\sigma(\tau_1^2, \tau_2^2) \tau_1^{4\rho_2(\sigma)} (\tau_1\tau_2)^{-6\rho_2(\sigma)} \qquad (\sigma \in G_{\mathbb{Q}})$$

holds in $\hat{B}_3$.

The above relation was found (and proved) by comparing Galois representations in $\hat{\Gamma}^5_0$ and $\hat{\Gamma}^2_1$ at certain explicitly constructed tangential base points ([N2] §4). Here we shall give an alternative proof using only double covers of the projective line minus three points. Indeed, Theorem 2.1 is now a corollary of the following more general

Theorem 2.2. *Notations being as in Theorem 2.1, we have the following relation in $\hat{B}_3$.*

(IV′) $$f_\sigma(\tau_1,\tau_2^2) = \tau_2^{4\rho_2(\sigma)} f_\sigma(\tau_1^2,\tau_2^2)\tau_1^{2\rho_2(\sigma)}(\tau_1\tau_2^2)^{-2\rho_2(\sigma)}$$
$$= \tau_2^{-4\rho_2(\sigma)} f_\sigma(\tau_1,\tau_2^4)\tau_1^{-2\rho_2(\sigma)}(\tau_1\tau_2^2)^{2\rho_2(\sigma)} \quad (\sigma \in G_{\mathbb{Q}}).$$

Proof. Let $X = X_t$ be the projective t-line $\mathbf{P}^1_t$ (over $\mathbb{Q}$) minus the three points $t = 0, 1, \infty$, and let Y_i ($i = 1, 2$) be the projective u_i-line minus the four points $u_i = 0, \pm 1, \infty$ respectively realized as a double cover over X by

$$t = 1 - \frac{(1-u_1)^2}{(1+u_1)^2} = \frac{(1-u_2)^2}{(1+u_2)^2}.$$

If $p_i : \mathbf{P}^1_{u_i} \to \mathbf{P}^1_t$ denotes the natural projections for $i = 1, 2$, then p_1 maps $0, \infty, 1, -1$ to $0, 0, 1, \infty$ respectively, and p_2 maps $0, \infty, 1, -1$ to $1, 1, 0, \infty$ respectively. (Ramifications occur at $t = 1, \infty$ for Y_1 and at $t = 0, \infty$ for Y_2.) For each of $i = 1, 2$, let Y_i^* be the Y_i plus one point $u_i = -1$, which is $\mathbf{P}^1_{u_i} - \{0, 1, \infty\}$, and take chains x_i, y_i, γ_i (analogous to the x, y, γ on X cf. Figure 2.1) from $\overrightarrow{01}_{/Y_i}$ to $\overrightarrow{10}_{/Y_i}$ which have

$$\sigma(\gamma_i) = f_\sigma(x_i, y_i)^{-1}\gamma_i \quad (\sigma \in G_{\mathbb{Q}},\ i = 1, 2).$$

(Here, $\overrightarrow{01}_{/Y_i}$, $\overrightarrow{10}_{/Y_i}$ denote the tangential base points valued in $\overline{\mathbb{Q}}\{\{u_i\}\}$, $\overline{\mathbb{Q}}\{\{1-u_i\}\}$ respectively.) Essentially we have $x_1 = x$, $y_1 = y^2$, $x_2 = y$, $y_2 = x^2$, but these equalities are not precise because, say, $p_1(\overrightarrow{01}_{/Y_1})$ has a different scale than $\overrightarrow{01}$ due to the principal coefficient of t expanded in u_1 being not 1. Taylor expansions show the primary approximations $t \sim 4u_1 \sim \frac{1}{4}(1-u_2)$ near $t = 0$ and $1 - t \sim \frac{1}{4}(1-u_1) \sim 4u_2$ near $t = 1$, and these measurements should be symbolically expressed as $\overrightarrow{01} = 4p_1(\overrightarrow{01}_{/Y_1}) = \frac{1}{4}p_2(\overrightarrow{10}_{/Y_2})$, $\overrightarrow{10} = 4p_2(\overrightarrow{01}_{/Y_2}) = \frac{1}{4}p_1(\overrightarrow{10}_{/Y_1})$. More precisely, one can interpret these estimates in terms of Galois actions on standard chains between the adjacent tangential base points; for example, if $\epsilon : \overrightarrow{01} \to \frac{1}{4}\overrightarrow{01}$ be the path defined by the field isomorphism of Puiseux fields $\overline{\mathbb{Q}}\{\{t\}\} \overset{\sim}{\leftarrow} \overline{\mathbb{Q}}\{\{t/4\}\}$ ($t^{1/n}/\sqrt[n]{4} \leftarrow (t/4)^{1/n}$), then $\sigma \in G_{\mathbb{Q}}$ acts on ϵ by $\sigma(\epsilon) = x^{2\rho_2(\sigma)}\epsilon$. Summing up the piece-by-piece actions of $\sigma \in G_{\mathbb{Q}}$ on the decompositions $\gamma = (\overrightarrow{01} \overset{\epsilon}{\to} \frac{1}{4}\overrightarrow{01} \to 4\overrightarrow{10} \to \overrightarrow{10}) = (\overrightarrow{01} \to 4\overrightarrow{01} \to \frac{1}{4}\overrightarrow{10} \to \overrightarrow{10})$, we obtain

$$\sigma(\gamma) = x^{2\rho_2(\sigma)} f_\sigma(x, y^2)^{-1} y^{2\rho_2(\sigma)}\gamma = x^{-2\rho_2(\sigma)} f_\sigma(x^2, y)^{-1} y^{-2\rho_2(\sigma)}\gamma$$

in $\pi_1(X, e_\infty|2, \overrightarrow{01}, \overrightarrow{10})$, where '$e_\infty|2$' means that this π_1 classifies only covers with ramification indices over $t = \infty$ dividing 2. Comparing this with the equality $\sigma(\gamma) = f_\sigma(x, y)^{-1}\gamma$ from (2.1), we obtain

(2.2) $$f_\sigma(x, y) = y^{-2\rho_2(\sigma)} f_\sigma(x, y^2) x^{-2\rho_2(\sigma)} = y^{2\rho_2(\sigma)} f_\sigma(x^2, y) x^{2\rho_2(\sigma)}$$

in $\pi_1(X, e_\infty|2, \overrightarrow{01})$. Let A_3 be the subgroup of B_3 generated by $\{\tau_1, \tau_2^2\}$; these generators have only a single relation $[\tau_2^2, \tau_1\tau_2^2\tau_1] = 1$. (We write $[a, b]$ to designate the commutator $aba^{-1}b^{-1}$.) Then, there exists a homomorphism ϕ of $\hat{A}_3$ to $\pi_1(X, e_\infty|2, \overrightarrow{01})$ (which is the quotient of $\hat{F}_2$ modulo the normal closure of

$z^2 = (xy)^{-2}$) by sending $\tau_1 \mapsto y$, $\tau_2^2 \mapsto x$. Since the kernel of ϕ is a cyclic group generated by the central element $(\tau_1\tau_2^2)^2$, the relations (2.2) in $\pi_1(X, e_\infty|2, \overrightarrow{01})$ lift to relations in $\hat{A}_3 \subset \hat{B}_3$ of the form

(2.3)

$$f_\sigma(\tau_2^2, \tau_1) = \tau_1^{-2\rho_2(\sigma)} f_\sigma(\tau_2^2, \tau_1^2)\tau_2^{-4\rho_2(\sigma)}(\tau_1\tau_2^2)^a = \tau_1^{2\rho_2(\sigma)} f_\sigma(\tau_2^4, \tau_1)\tau_2^{4\rho_2(\sigma)}(\tau_1\tau_2^2)^b$$

for some a and b. To determine a and b, we reduce these equalities the normal closure of $\langle \tau_2^2 \rangle$ in $\hat{A}_3$ (i.e., by pulling out the third strand of braids). Because f_σ lies in the derived subgroup of the free group and τ_2^2 maps to 1 when the third strand is pulled out, the images of the f_σ terms above are trivial, and in the quotient we obtain

$$1 = \tau_1^{-2\rho_2(\sigma)}\tau_2^{-4\rho_2(\sigma)}(\tau_1\tau_2^2)^a = \tau_1^{2\rho_2(\sigma)}\tau_2^{4\rho_2(\sigma)}(\tau_1\tau_2^2)^b,$$

so that $a = 2\rho_2(\sigma)$ and $b = -2\rho_2(\sigma)$. Thus, we obtain the theorem. □

Remark. Based on the idea used in the above proof, one can investigate similarly what happens in the cover given by the S_3-quotient of $\mathbf{P}^1 - \{0, 1, \infty\}$. From this context, a few more equations satisfied by the image of $G_\mathbb{Q} \hookrightarrow \widehat{GT}$ can be found, in some of which ρ_3, the Kummer 1-cocycle of roots of 3 also appears to play roles. See [NT].

Let us now give another interpretation of the Kummer 1-cocycle ρ_2 in terms of the profinite Blanchfield-Lyndon calculus developed by Ihara ([I3]). Let G' (resp. G'') denote the commutator (resp. double-commutator) subgroup of a profinite group G, and let G^{ab} denote the abelianization G/G' of G. Then, a special case of Ihara's profinite Blanchfield-Lyndon theorem asserts that for $G = \hat{F}_2$ (the free profinite group on two generators x, y) the quotient module $\hat{F}_2'/\hat{F}_2''$ under the conjugate action by the profinite group algebra $\hat{\mathbb{Z}}[[\hat{F}_2^{ab}]]$ is a free module of rank one generated by the class of $[x, y] = xyx^{-1}y^{-1}$ ([I3] Proposition 1.4.1). Applying this to our $f_\sigma(x, y) \in \hat{F}_2'$, we obtain a unique element $A_\sigma(\bar{x}, \bar{y}) \in \hat{\mathbb{Z}}[[\hat{F}_2^{ab}]]$ such that

$$f_\sigma(x, y) \equiv A_\sigma(\bar{x}, \bar{y}) * [x, y] \bmod \hat{F}_2''. \tag{2.4}$$

Here, $*$ means the conjugate (left) action of $\hat{\mathbb{Z}}[[\hat{F}_2^{ab}]]$ on $\hat{F}_2'/\hat{F}_2''$, and $\bar{x}$, $\bar{y}$ represent the images of x, y (topologically) generating $\hat{F}_2^{ab} \cong \hat{\mathbb{Z}}^{\oplus 2}$. The profinite group homomorphism $\hat{F}_2^{ab} \to \hat{\mathbb{Z}}$ defined by $\bar{x} \mapsto -1$, $\bar{y} \mapsto 1$ can be continuously extended to a unique ring homomorphism $\hat{\mathbb{Z}}[[\hat{F}_2^{ab}]] \to \hat{\mathbb{Z}}$. Denote by $A_\sigma(-1, 1)$ the image of $A_\sigma(\bar{x}, \bar{y})$ in $\hat{\mathbb{Z}}$ by this map. Then, we have the following result characterizing ρ_2.

Lemma 2.3. $\rho_2(\sigma) = -A_\sigma(-1, 1) \quad (\sigma \in G_\mathbb{Q})$. □

Proof. As before, let τ_1, τ_2 be the standard generators of B_3 as in Theorem 2.1, and write $\omega_3 = (\tau_1\tau_2^2)^2$ which (topologically) generates the center of $\hat{B}_3$. Then, the pure subgroup $\hat{P}_3$ can be decomposed as a direct product $\langle \tau_1^2, \tau_2^2 \rangle \times \langle \omega_3 \rangle$. Considering (the inverse of) equation (2.3) in $\hat{P}_3$ (with $a = 2\rho_2(\sigma)$, $b = -2\rho_2(\sigma)$) and reducing it modulo $[\hat{P}_3, \hat{P}_3]\langle \omega_3 \rangle$, we obtain

$$f_\sigma(\tau_1, \tau_2^2) \equiv \tau_1^{2\rho_2(\sigma)}\tau_2^{4\rho_2(\sigma)} \equiv \tau_1^{-2\rho_2(\sigma)}\tau_2^{-4\rho_2(\sigma)} f_\sigma(\tau_1, \tau_2^4). \tag{2.5}$$

As above, we note that since f_σ lies in the commutator subgroup of $\hat{F}_2$, both $f_\sigma(\tau_1, \tau_2^2)$ and $f_\sigma(\tau_1, \tau_2^4)$ lie in $\hat{P}_3$.

Now let us make use of (2.4). By an easy computation, we see that $[\tau_1, \tau_2^2] \equiv (\tau_1^2\tau_2^4)^{-1}$, $[\tau_1, \tau_2^4] \equiv (\tau_1^4\tau_2^8)^{-1}$ modulo $\hat{P}_3'\langle\omega_3\rangle = [\hat{P}_3, \hat{P}_3]\langle\omega_3\rangle$. Since τ_2^2, τ_2^4 lie in $\hat{P}_3$, their actions by conjugation on $\hat{P}_3/\hat{P}_3'\langle\omega_3\rangle$ are trivial, while computation shows that the actions by conjugation of τ_1 on $[\tau_1, \tau_2^2]$, $[\tau_1, \tau_2^4]$ turn out to be 'inversion' modulo $\hat{P}_3'\langle\omega_3\rangle$. Noticing then that the double commutator subgroups $\langle\tau_1, \tau_2^2\rangle''$ and $\langle\tau_1, \tau_2^4\rangle''$ are contained in $\hat{P}_3'\langle\omega_3\rangle$, we obtain:

$$(2.6) \qquad \begin{cases} f_\sigma(\tau_1, \tau_2^2) \equiv (\tau_1^2\tau_2^4)^{-A_\sigma(-1,1)}, \text{ mod } \hat{P}_3'\langle\omega_3\rangle \\ f_\sigma(\tau_1, \tau_2^4) \equiv (\tau_1^4\tau_2^8)^{-A_\sigma(-1,1)} \text{ mod } \hat{P}_3'\langle\omega_3\rangle. \end{cases}$$

Comparing this with (2.5) proves the result. □

The Kummer 1-cocycle ρ_2 also appears in a rather different manner through a certain proword $g_\sigma(x, y) \in \hat{F}_2$ studied in [LS2]. We shall first recall the geometric definition of $g_\sigma(x, y)$ associated to $\sigma \in G_\mathbb{Q}$. The point is to introduce the basepoint 1/2 in addition to tangential ones, i.e., let r be the simple path from 0 to 1/2 along the real line, and regard it as an element of $\pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, \overrightarrow{01}, 1/2)$ on which $G_\mathbb{Q}$ acts canonically. Then we introduce and define $g_\sigma(x, y)$ by:

$$(2.7) \qquad \sigma(r) = g_\sigma(x, y)^{-1} r.$$

The element $g_\sigma(x, y)$, lying in $\pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, \overrightarrow{01})$, was first introduced in [LS2]. Let θ denote the automorphism of $\mathbf{P}^1 - \{0, 1, \infty\}$ interchanging $0 \leftrightarrow 1$ and fixing ∞, i.e. $\theta(t) = 1 - t$. Then obviously $\gamma = r\theta(r)^{-1}$, $\theta(x) = \gamma^{-1} y \gamma$ etc. Since θ is defined over $\mathbb{Q}$, we easily obtain

$$(2.8) \qquad f_\sigma(x, y) = g_\sigma(y, x)^{-1} g_\sigma(x, y).$$

While $f_\sigma(x, y)$ is known to lie in the commutator subgroup $[\hat{F}_2, \hat{F}_2]$, g_σ is in general not. In fact,

Proposition 2.4. *For any $\sigma \in G_\mathbb{Q}$, $g_\sigma(x, y) \equiv (xy)^{\rho_2(\sigma)}$ mod $[\hat{F}_2, \hat{F}_2]$.*

Proof. Let β (resp. β') denote the map

$$G_\mathbb{Q} \to \pi_1(\mathbf{P}^1_\mathbb{Q} - \{0, 1, \infty\}, p)$$

where $p = \overrightarrow{01}$ (resp. $p = 1/2$) obtained by splitting the short exact sequence

$$1 \to \pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, p) \to \pi_1(\mathbf{P}^1_\mathbb{Q} - \{0, 1, \infty\}, p) \to G_\mathbb{Q} \to 1.$$

The action of σ on a loop in $\pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, p)$ is given in $\pi_1(\mathbf{P}^1_\mathbb{Q} - \{0, 1, \infty\}, p)$ by conjugation by $\beta(\sigma)$ if $p = \overrightarrow{01}$, by $\beta'(\sigma)$ if $p = 1/2$. The Galois action on the path r from $\overrightarrow{01}$ to 1/2 is given by

$$\sigma(r) = \beta(\sigma) r \beta'(\sigma)^{-1} = g_\sigma(x, y)^{-1} r,$$

which is equivalent to

$$\beta'(\sigma) r^{-1} = r^{-1} g_\sigma(x, y) \beta(\sigma).$$

Each side represents a path from 1/2 to $\overrightarrow{01}$ which corresponds to a mapping of the value fields of tangential base points: $\Omega_{\overrightarrow{01}} = \overline{\mathbb{Q}}\{\{t\}\} \to \Omega_{1/2} = \overline{\mathbb{Q}}$ (where t denotes the canonical coordinate of $\mathbf{P}^1$). Applying the rule that the path r^{-1} maps both $t^{1/N}$ and $(1-t)^{1/N} = \sum_k \binom{1/N}{k}(-1)^k t^k$ in $\Omega_{\overrightarrow{01}}$ to $1/\sqrt[N]{2} \in \overline{\mathbb{Q}}$ respectively, we see that the above LHS carries both of them to $(\sqrt[N]{2}\zeta_N^{\rho_2(\sigma)})^{-1}$. On the other hand, if $g_\sigma(x,y) \equiv x^a y^b \bmod [\hat{F}_2, \hat{F}_2]$, then, since $t^{1/N}, (1-t)^{1/N}$ generates only abelian extensions over $\overline{\mathbb{Q}}(t)$, it turns out that the RHS of the above carries $t^{1/N}$ (resp. $(1-t)^{1/N}$) to $(\sqrt[N]{2}\zeta_N^a)^{-1}$ (resp. to $(\sqrt[N]{2}\zeta_N^b)^{-1}$). Thus we conclude $a = b = \rho_2(\sigma)$. □

Before closing this section, we quote the following result from [N2] as another type of example where $\rho_2(\sigma)$ appears curiously.

Theorem 2.5. ([N2] Corollary 4.13) *In* $\mathrm{GL}_2(\hat{\mathbb{Z}})$*, we have*

$$\begin{aligned} f_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right) &= \pm\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\lambda_\sigma&8\rho_2(\sigma)\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}\lambda_\sigma&8\rho_2(\sigma)\\0&1\end{smallmatrix}\right)^{-1} \\ &= \pm\begin{pmatrix}\lambda_\sigma^{-1} & -8\rho_2(\sigma)\lambda_\sigma^{-1} \\ -8\rho_2(\sigma)\lambda_\sigma^{-1} & \lambda_\sigma + 64\rho_2(\sigma)^2\lambda_\sigma^{-1}\end{pmatrix}\end{aligned}$$

for $\sigma \in G_\mathbb{Q}$*, where* $\pm$ *is according to* $\lambda_\sigma \equiv \pm 1$ *mod* 4. □

Combining Theorems 2.2 and 2.5, we also obtain

Corollary 2.6. *In* $\mathrm{GL}_2(\hat{\mathbb{Z}})$*, we have*

$$f_\sigma\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right) = \pm\begin{pmatrix}\lambda_\sigma^{-1} & -6\rho_2(\sigma)\lambda_\sigma^{-1} \\ -12\rho_2(\sigma)\lambda_\sigma^{-1} & \lambda_\sigma + 72\rho_2(\sigma)^2\lambda_\sigma^{-1}\end{pmatrix}$$

for $\sigma \in G_\mathbb{Q}$*, where* $\pm$ *is according to* $\lambda_\sigma \equiv \pm 1$ *mod* 4. □

Proof. The formula is a consequence of applying the composition of homomorphisms

$$\begin{aligned} &\hat{B}_3 \to \mathrm{GL}_2(\mathbb{Z})^\wedge \to \mathrm{GL}_2(\hat{\mathbb{Z}}), \\ &\tau_1, \tau_2 \mapsto \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\-1&0\end{smallmatrix}\right) \end{aligned}$$

to the equation (IV′) of Theorem 2.2. □

Remark 2.7. *The expressions of Theorem 2.5 and Corollary 2.6 have the following intriguingly simple forms:*

$$\begin{cases} f_\sigma\left(\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right) = \pm\left(\begin{smallmatrix}1&0\\-8\rho_2(\sigma)&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\lambda_\sigma^{-1}&0\\0&\lambda_\sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-8\rho_2(\sigma)\\0&1\end{smallmatrix}\right), \\ f_\sigma\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right)\right) = \pm\left(\begin{smallmatrix}1&0\\-12\rho_2(\sigma)&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\lambda_\sigma^{-1}&0\\0&\lambda_\sigma\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-6\rho_2(\sigma)\\0&1\end{smallmatrix}\right). \end{cases}$$

§3. $G_\mathbb{Q}$ satisfies (III′).

In this section, we shall prove Theorem 1.1 of Section 1, i.e., that the image $F_\sigma = (\lambda_\sigma, f_\sigma) \in \widehat{GT}$ of each Galois element $\sigma \in G_\mathbb{Q}$ satisfies:

$$\text{(III}'\text{)} \qquad f_\sigma(\tau_1\tau_3, \tau_2^2) = g_\sigma(x_{45}, x_{51}) f_\sigma(x_{12}, x_{23}) f_\sigma(x_{34}, x_{45})$$

in $\hat{\Gamma}_0^{[5]}$ (see §1 for notation used here). A sketch of the proof was given in [LNS]; here we will fill details of the (original) proof. (An alternative new proof was later found by H.Tsunogai. cf. [T].) We begin by

Lemma 3.1. $f_\sigma(\tau_1\tau_3, \tau_2^2) \equiv (x_{45}x_{51})^{\rho_2(\sigma)} \bmod [\hat{\Gamma}_0^5, \hat{\Gamma}_0^5]$.

Proof. We first show the following congruence:

$$f_\sigma(\tau_1\tau_3, \tau_2^2) \equiv (x_{14}x_{23}^{-1})^{A_\sigma(-1,1)} \bmod [\hat{\Gamma}_0^5, \hat{\Gamma}_0^5]. \tag{3.1}$$

The argument for verifying (3.1) goes exactly in a similar way to (2.6): since $\tau_1\tau_3$ generates an abelian subgroup ($\cong \mathbb{Z}/2\mathbb{Z}$) in the image of $\hat{\Gamma}_0^{[5]} \to S_5$, the double commutator group $\langle \tau_1\tau_3, \tau_2^2\rangle''$ is contained in $[\hat{\Gamma}_0^5, \hat{\Gamma}_0^5]$. By direct computation using only the (Artin) braid relations, one proves the congruence $[\tau_1\tau_3, \tau_2^2] \equiv x_{14}x_{23}^{-1} \bmod [\hat{\Gamma}_0^5, \hat{\Gamma}_0^5]$. Moreover, it is not difficult to see that the conjugate action of $\tau_1\tau_3$ (resp. τ_2^2) on $x_{14}x_{23}^{-1} \bmod [\hat{\Gamma}_0^5, \hat{\Gamma}_0^5]$ is (-1)-multiplication (resp. trivial). Putting these observations into (2.4), we obtain (3.1). Using the extra sphere braid relations $\prod_{i=1}^4 x_{j,j+i} = 1$ $(j = 1, \ldots, 5)$, we obtain $x_{23} \equiv x_{14}x_{15}x_{45}$ in the abelianization of Γ_0^5. Lemma 3.1 follows from this and Lemma 2.3. □

In [N2], we constructed a tangential base point $\vec{v}$ on $M_{g,1}$ by linearly patching g copies of the Tate elliptic curve Tate(q^2) of level 2. This tangential base point $\vec{v}$ defined a section $s_{\vec{v}} : G_{\mathbb{Q}} \to \pi_1(M_{g,1})$ such that the conjugate actions of $s_{\vec{v}}(\sigma)$ on the Dehn twists along circles $a_1, \ldots, a_{2g}, d_{\pm j}, e_j \in \hat{\Gamma}_g^1$ (see Figure 3.1) are given by:

$$\left\{\begin{array}{ll} D_{a_i} \mapsto w^{-4\rho_2(\sigma)} f_\sigma(D_{a_i}^2, w_i) D_{a_i}^{\lambda_\sigma} f_\sigma(w_i, D_{a_i}^2) w^{4\rho_2(\sigma)} & (1 \le i \le 2g), \\ D_{d_i} \mapsto D_{d_i}^{\lambda_\sigma},\ D_{d_{-i}} \mapsto D_{d_{-i}}^{\lambda_\sigma},\ D_{e_j} \mapsto D_{e_j}^{\lambda_\sigma} & (1 \le i \le g,\ 1 \le j \le g-1), \end{array}\right. \tag{3.2}$$

for $\sigma \in G_{\mathbb{Q}}$, where $w_1 = 1, w_i = (D_{a_1} \cdots D_{a_{i-1}})^i$ and $w = \prod_i w_{2i}$. (We write D_c for the Dehn twist along a circle c.)

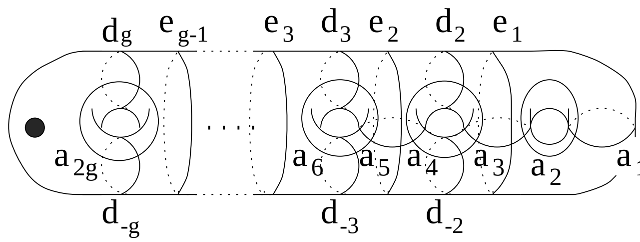


Figure 3.1

Now, let us write $\mathcal{M}_{g,n}$ to designate the compactification of $M_{g,n}$ obtained by adding the points of (marked) stable curves (Deligne-Mumford, Knudsen). We will consider the special case $(g, n) = (2, 1)$ here, denoting by D the union of all singular divisors on $\mathcal{M}_{2,1}$. There is a special irreducible component D_1 of D isomorphic to the product $\mathcal{M}_{1,1} \times \mathcal{M}_{1,2}$. Consider the formal completion $(\mathcal{M}_{2,1}/D_1)^\wedge$ of $\mathcal{M}_{2,1}$ along D_1. Then, by construction, the above $\vec{v}$ can be viewed as giving a base point for $\pi_1^D((\mathcal{M}_{2,1}/D_1)^\wedge)$, the fundamental group of the formal completion of $\mathcal{M}_{2,1}$ along D_1 admitting (tame) ramification along D in the sense of Grothendieck-Murre (cf. [GM]). Pushing down the basepoint by the canonical projection $\pi_1^D((\mathcal{M}_{2,1}/D_1)^\wedge) \to \pi_1(M_{1,2})$, we obtain a tangential base point $\vec{v}'$ on $M_{1,2}$ representing Tate$(q^2)/\mathbb{Q}((q))$ with two marked points "$1, q \bmod^\times q^2$". To see that the induced Galois action on $\hat{\Gamma}_1^2$ precisely inherits that on $\hat{\Gamma}_2^1$ by $\vec{v}$, we prove the following lemma:

Lemma 3.2. *Let $\hat{\phi} : \hat{\Gamma}_{1,1}^1 \to \hat{\Gamma}_2^1$ be the homomorphism induced from the surface embedding $\Sigma_{1,1}^1 \hookrightarrow \Sigma_2^1$ of Figure 3.2. Then $\hat{\phi}$ is injective.*

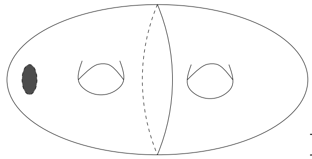


Figure 3.2

Proof. Introduce the forgetful homomorphisms $\hat{\Gamma}_{1,1}^1 \to \hat{\Gamma}_{1,1}$ and $\hat{\Gamma}_2^1 \to \hat{\Gamma}_2$ whose kernels are the profinite completions of $\pi_1(\Sigma_{1,1})$, $\pi_1(\Sigma_2)$ respectively, and consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\pi}_1(\Sigma_{1,1}) & \longrightarrow & \hat{\Gamma}_{1,1}^1 & \longrightarrow & \hat{\Gamma}_{1,1} & \longrightarrow & 1 \\ & & \downarrow{\hat{\phi}_1} & & \downarrow{\hat{\phi}} & & \downarrow{\hat{\phi}_2} & & \\ 1 & \longrightarrow & \hat{\pi}_1(\Sigma_{2,0}) & \longrightarrow & \hat{\Gamma}_2^1 & \longrightarrow & \hat{\Gamma}_2 & \longrightarrow & 1. \end{array}$$

We reduce the injectivity of $\hat{\phi}$ to those of $\hat{\phi}_1$ and $\hat{\phi}_2$. The injectivity of $\hat{\phi}_1$ follows from a result of L.Ribes ([R] Theorem 2.1) insuring that the $\hat{\Pi}_{2,0}$ is the amalgamated product (of profinite groups) of two copies of $\hat{\Pi}_{1,1}$ over $\hat{\mathbb{Z}}$. Let us consider that of $\hat{\phi}_2$. First we have a natural identification of $\iota : \hat{B}_3 \cong \hat{\Gamma}_{1,1}$. But Birman-Hilden [BH] tells us that there is a natural surjection $p^{BH} : \hat{\Gamma}_2 \to \hat{\Gamma}_0^{[6]}$. Since the composition $p^{BH} \circ \hat{\phi}_2 \circ \iota$ gives a familiar embedding of $\hat{B}_3$ into $\hat{\Gamma}_0^{[6]}$, the proof is completed. □

Using Lemma 3.2, we shall extract the $G_{\mathbb{Q}}$-action (3.2) on the part generated by d_2, a_4, d_{-2} of Figure 3.1 in a more economical surface supporting them. Thus, renaming $d_2 =: a_1$, $a_4 =: a_2$, $d_{-2} =: a_3$ respectively in formula (3.2) with $g = 2$, we see that $\vec{v}'$ induces a section $s_{\vec{v}'} : G_{\mathbb{Q}} \to \pi_1(M_{1,2})$ such that the conjugation by $s_{\vec{v}'}(\sigma)$ ($\sigma \in G_{\mathbb{Q}}$) acts on $\hat{\Gamma}_1^2$ as follows:

(3.3)

$$\begin{cases} D_{a_i} & \mapsto D_{a_i}^{\lambda_\sigma} \ (i = 1, 3), \\ D_{a_2} & \mapsto (D_{a_1} D_{a_3})^{-4\rho_2(\sigma)} f_\sigma(D_{a_2}^2, D_{a_1} D_{a_3}) D_{a_2}^{\lambda_\sigma} f_\sigma(D_{a_1} D_{a_3}, D_{a_2}^2)(D_{a_1} D_{a_3})^{4\rho_2(\sigma)}. \end{cases}$$

Note that the injectivity of Lemma 3.2 guarantees that the induced $G_{\mathbb{Q}}$-action on $\pi_1(M_{1,2})$ has to coincide with that given inside $\pi_1(M_{2,1})$ in (3.2).

On the other hand, in [N2] §4, we constructed another tangential base point $\vec{e}$ on $M_{1,2}$ lying in the fibre Tate(q^2) over '$\frac{1}{16}\overrightarrow{01}$' on $M_{1,1}^{\text{level } 2} \approx \mathbb{P}^1 - \{0, 1, \infty\}$ which gave a section $s_{\vec{e}} : G_{\mathbb{Q}} \to \pi_1(M_{1,2})$ such that the conjugation by $s_{\vec{e}}(\sigma)$ ($\sigma \in G_{\mathbb{Q}}$) acts on $\pi_1(M_{1,2})$ in the following way:

(3.4)

$$\begin{cases} D_{a_i} \mapsto D_{a_i}^{\lambda_\sigma} \ (i = 1, 3), \\ D_{a_2} \mapsto f_\sigma(x_{45}^2, D_{a_3}) D_{a_1}^{-8\rho_2(\sigma)} f_\sigma(D_{a_2}^2, D_{a_1}^2) D_{a_2}^{\lambda_\sigma} f_\sigma(D_{a_1}^2, D_{a_2}^2) D_{a_1}^{8\rho_2(\sigma)} f_\sigma(D_{a_3}, x_{45}^2). \end{cases}$$

From the fact that $\vec{v}'$ and $\vec{e}$ concentrate on the same cusp of $\mathcal{M}_{1,2}$ and have the same image under the canonical projection $M_{1,2} \to M_{1,1}$, the difference between $s_{\vec{v}'}(\sigma)$ and $s_{\vec{e}}(\sigma)$ is of the form $(D_{a_1} D_{a_3}^{-1})^{c_\sigma}$ for some constant $c_\sigma \in \hat{\mathbb{Z}}$. This connects two Galois actions (3.3) and (3.4). Now we shall carry our situation to the profinite Artin braid group $\hat{B}_4$ with standard generators τ_1, τ_2, τ_3 (see §1), where $\omega_4 = (\tau_1\tau_2\tau_3)^4$ generates the center of $\hat{B}_4$. Let us identify $\hat{\Gamma}_1^2 \cong \hat{B}_4/\langle\omega_4\rangle \hookrightarrow \hat{\Gamma}_0^{[5]}$ by mapping $D_{a_i} \mapsto \tau_i$ ($i = 1, 2, 3$) and compare the image of D_{a_2} under the actions of $\sigma \in G_{\mathbb{Q}}$ of (3.3) and (3.4), after replacing $f_\sigma(\tau_3, x_{45}^2)$ by $\tau_4^{8\rho_2(\sigma)} f_\sigma(\tau_3^2, \tau_4^2)\tau_3^{4\rho_2(\sigma)}\tau_1^{-4\rho_2(\sigma)}$ by relation (IV). (Note here that $(\tau_3\tau_4)^3 = \tau_1^2$). Then, noticing also that the centralizer of τ_2 in $\hat{B}_4/\langle\omega_4\rangle$ is $\langle\tau_2\rangle \times \langle x_{45}, x_{51}\rangle$ (cf. [N0]), we obtain

$$(3.5) \qquad f_\sigma(\tau_1\tau_3, \tau_2^2)(\tau_1\tau_3^{-1})^{c_\sigma} f_\sigma(x_{45}, x_{34}) f_\sigma(x_{23}, x_{12}) = \tau_2^\nu h_\sigma(x_{45}, x_{51})$$

for some $\nu \in \hat{\mathbb{Z}}$, $h_\sigma \in \hat{F}_2$. Applying Lemma 3.1 to the left-hand term, it becomes

$$(x_{45}x_{51})^{\rho_2(\sigma)}\tau_1^{c_\sigma}\tau_3^{-c_\sigma} \equiv \tau_2^\nu x_{45}^a x_{51}^b,$$

for certain elements $a,\ b \in \hat{\mathbb{Z}}$. The abelianization of $\hat{\Gamma}_0^5$ is free abelian on the generators $x_{i,i+1}$, so we must have $a = b = \rho_2(\sigma)$ and $c_\sigma = \nu = 0$. Then, applying the involution on $\hat{B}_4$ given by $\tau_1 \leftrightarrow \tau_3$, $\tau_2 \mapsto \tau_2$ and comparing the resulting equations with relation (III), we obtain

$$h_\sigma(x_{51}, x_{45})^{-1} h_\sigma(x_{45}, x_{51}) = f_\sigma(x_{45}, x_{51}).$$

Since g_σ is the unique pro-word satisfying this property ([LS2]), it follows that $h_\sigma = g_\sigma$. This completes the proof of Theorem 1.1.

§4. A subgroup $\mathbb{I}\Gamma' \subset \widehat{GT}$ defined by (III′).

Definition 4.1. We define $\mathbb{I}\Gamma'$ to be the subset of $\widehat{GT}$ consisting of all pairs (λ, f) with f satisfying

$$(\mathrm{III}') \qquad f(\tau_1\tau_3, \tau_2^2) = g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45})$$

in $\hat{\Gamma}_0^{[5]}$, where g is the unique proword in $\hat{F}_2$ such that $g(y,x)^{-1}g(x,y) = f(x,y)$.

The existence and uniqueness of $g(x,y)$ with $g(y,x)^{-1}g(x,y) = f(x,y)$ was shown in [LS2]. Note that we already know that $\mathbb{I}\Gamma'$ contains $G_{\mathbb{Q}}$ according to Theorem 1.1 settled in the previous section.

Proposition 4.1. *$\mathbb{I}\Gamma'$ forms a subgroup of $\widehat{GT}$.*

In order to prove this proposition, we have to check that the condition (III′) is closed under multiplication and inversion of $\widehat{GT}$

Throughout this section, we let each element $F = (\lambda, f) \in \widehat{GT}$ act on $\hat{\Gamma}_0^{[5]}$ in the following 'standard' way:

$$(4.1) \qquad \begin{cases} F(\tau_1) &= \tau_1^\lambda \\ F(\tau_2) &= f(x_{23}, x_{12})\tau_2^\lambda f(x_{12}, x_{23}) \\ F(\tau_3) &= f(x_{34}, x_{45})\tau_3^\lambda f(x_{45}, x_{34}) \\ F(\tau_4) &= \tau_4^\lambda. \end{cases}$$

and put $H_F := \mathrm{Inn}(f(x_{45}, x_{34})) \circ F \in \mathrm{Aut}\hat{\Gamma}_0^{[5]}$ ($\mathrm{Inn}(f)$ means the inner automorphism $* \mapsto f(*)f^{-1}$). The following lemma is useful for our purpose.

Lemma 4.2. *Let $F \in \widehat{GT}$. Then, F satisfies* (III′) *if and only if $H_F(\tau_2) = f(\tau_2^2, \tau_1\tau_3)\tau_2^{\lambda} f(\tau_1\tau_3, \tau_2^2)$ holds.*

Proof. The "only if" part is immediate: if F satisfies (III′), then

$$\begin{aligned} H_F(\tau_2) &= f(x_{45}, x_{34}) f(x_{23}, x_{12}) \tau_2^{\lambda} f(x_{12}, x_{23}) f(x_{34}, x_{45}) \\ &= f(\tau_2^2, \tau_1\tau_3) g(x_{51}, x_{45}) \tau_2^{\lambda} g(x_{45}, x_{51})^{-1} f(\tau_1\tau_3, \tau_2^2) \\ &= f(\tau_2^2, \tau_1\tau_3) \tau_2^{\lambda} f(\tau_1\tau_3, \tau_2^2) \end{aligned}$$

since x_{45} and x_{51} commute with τ_2.

For the "if" part, suppose that $H_F(\tau_2) = f(\tau_1^2, \tau_1\tau_3)\tau_2^{\lambda} f(\tau_1\tau_3, \tau_2^2)$, i.e. that

$$f(x_{45}, x_{34}) f(x_{23}, x_{12}) \tau_2^{\lambda} f(x_{12}, x_{23}) f(x_{34}, x_{45}) = f(\tau_2^2, \tau_1\tau_3) \tau_2^{\lambda} f(\tau_1\tau_3, \tau_2^2).$$

Let $\alpha : \hat{B}_4 \to \Gamma_0^{[5]}$ be the natural map sending standard generators τ_1, τ_2, τ_3 to those denoted by the same symbols, and observe that both sides of this equality lie in $\mathrm{Im}(\alpha)$ (since $x_{34} = \tau_3^2$, $x_{45} = (\tau_1\tau_2)^3$). Since the centralizer of τ_2 in $\mathrm{Im}(\alpha)$ is $\langle \tau_2 \rangle \times \langle x_{45}, x_{51} \rangle$ (as before, cf. [N0]), there exists $\nu \in \hat{\mathbb{Z}}$ and $h \in \hat{F}_2$ such that

$$f(\tau_1\tau_3, \tau_2^2) f(x_{45}, x_{34}) f(x_{23}, x_{12}) = \tau_2^{\nu} h(x_{45}, x_{51}).$$

Then, by the same argument as given just after §3 (3.5) (passing to the abelianization), we show that $\nu = 0$ and then that $h(x, y) = g(x, y)$, obtaining relation (III′). □

Proof of Proposition 4.1. Let $F = (\lambda, f)$ and $F' = (\lambda', f')$ be two elements of $\widehat{GT}$ such that f and f' satisfy (III′), and let $\tilde{F} = (\tilde{\lambda}, \tilde{f})$ be the product $(\lambda, f) \cdot (\lambda', f')$ in $\widehat{GT}$, so that $\tilde{\lambda} = \lambda\lambda'$ and $\tilde{f}(x, y) = f(x, y) f'(x^{\lambda}, f(y, x) y^{\lambda} f(x, y)) = fF(f')$. Define H_F, $H_{F'}$ and $H_{\tilde{F}}$ as just after (4.1), and for simplicity of the notation, set $H = H_F$, $H' = H_{F'}$ and $\tilde{H} = H_{\tilde{F}}$. Then, we have

$$\begin{aligned} \tilde{f}(\tau_1\tau_3, \tau_2^2) &= f(\tau_1\tau_3, \tau_2^2) f'\big((\tau_1\tau_3)^{\lambda}, f(\tau_2^2, \tau_1\tau_3) \tau_2^{2\lambda} f(\tau_1\tau_3, \tau_2^2)\big) \\ &= f(\tau_1\tau_3, \tau_2^2) H\big(f'(\tau_1\tau_3, \tau_2^2)\big). \end{aligned} \tag{4.2}$$

To apply Lemma 4.2 to $\tilde{F}$, let us compute

$$\tilde{H}(\tau_2) = \tilde{f}(x_{45}, x_{34}) \tilde{f}(x_{23}, x_{12}) \tau_2^{\tilde{\lambda}} \tilde{f}(x_{12}, x_{23}) \tilde{f}(x_{34}, x_{45}). \tag{4.3}$$

Since

$$\begin{cases} \tilde{f}(x_{12}, x_{23}) &= f(x_{12}, x_{23}) f'\big(x_{12}^{\lambda}, f(x_{23}, x_{12}) x_{23}^{\lambda} f(x_{12}, x_{23})\big) \\ \tilde{f}(x_{34}, x_{45}) &= f(x_{34}, x_{45}) f'\big(x_{34}^{\lambda}, f(x_{45}, x_{34}) x_{45}^{\lambda} f(x_{34}, x_{45})\big) \end{cases}$$

we find that

$$\begin{aligned} &\tilde{f}(x_{12}, x_{23}) \tilde{f}(x_{34}, x_{45}) \\ &= f(x_{12}, x_{23}) f(x_{34}, x_{45}) f'\big(x_{12}^{\lambda}, f(x_{45}, x_{34}) f(x_{23}, x_{12}) x_{23}^{\lambda} f(x_{12}, x_{23}) f(x_{34}, x_{45})\big) \\ &\quad \cdot f'\big(x_{34}^{\lambda}, f(x_{34}, x_{45}) x_{45}^{\lambda} f(x_{34}, x_{45})\big) \\ &= f(x_{12}, x_{23}) f(x_{34}, x_{45}) H\big(f'(x_{12}, x_{23}) f'(x_{34}, x_{45})\big) \\ &= g(x_{45}, x_{51})^{-1} f(\tau_1\tau_3, \tau_2^2) H\big(g'(x_{45}, x_{51})\big)^{-1} H\big(f'(\tau_1\tau_3, \tau_2^2)\big), \end{aligned}$$

where we used (III′) twice satisfied by f and f'. Substituting this into (4.3) and noticing that x_{45}, x_{51} commute with τ_2, we see that $\tilde{H}(\tau_2)$ is equal to

$$\begin{aligned}
&H\big(f'(\tau_2^2, \tau_1\tau_3)g'(x_{45}, x_{51})\big)H(\tau_2^{\lambda'})H\big(g'(x_{45}, x_{51})^{-1}f'(\tau_1\tau_3, \tau_2^2)\big)\\
&= H\big(f'(\tau_2^2, \tau_1\tau_3)g'(x_{45}, x_{51})\tau_2^{\lambda'}g'(x_{45}, x_{51})^{-1}f'(\tau_1\tau_3, \tau_2^2)\big)\\
&= H\big(f'(\tau_2^2, \tau_1\tau_3)\big)f(\tau_2^2, \tau_1\tau_3)\,\tau_2^{\tilde{\lambda}}\,f(\tau_1\tau_3, \tau_2^2)H\big(f'(\tau_1\tau_3, \tau_2^2)\big).
\end{aligned}$$

Applying (4.2) to the above, we conclude

$$\tilde{H}(\tau_2) = \tilde{f}(\tau_2^2, \tau_1\tau_3)\tau_2^{\tilde{\lambda}}\tilde{f}(\tau_1\tau_3, \tau_2^2).$$

This settles our claim by Lemma 4.2.

Next, we shall prove that (III′) is also preserved under taking inverses in $\widehat{GT}$. Let $F = (\lambda, f) \in \widehat{GT}$ be such that f satisfies (III′) and let $F' = (\lambda^{-1}, f')$ denote the inverse of F in $\widehat{GT}$. Then,

$$fF(f') = f(x, y)f'\big(x^{\lambda}, f(y, x)y^{\lambda}f(x, y)\big) = 1.$$

Now, we compute:

$$\begin{aligned}
&F\big(f'(x_{34}, x_{45})f'(\tau_2^2, \tau_1\tau_3)\,\tau_2^{\lambda^{-1}}\,f'(\tau_1\tau_3, \tau_2^2)f'(x_{45}, x_{34})\big)\\
&= f'\big(f(x_{34}, x_{45})x_{34}^{\lambda}f(x_{45}, x_{34}), x_{45}^{\lambda}\big)\\
&\quad f'\big(f(x_{23}, x_{12})\tau_2^{2\lambda}f(x_{12}, x_{23}), \tau_1^{\lambda}f(x_{34}, x_{45})\tau_3^{\lambda}f(x_{45}, x_{34})\big)\\
&\quad f(x_{23}, x_{12})\,\tau_2 f(x_{12}, x_{23})\\
&\quad f'\big(\tau_1^{\lambda}f(x_{34}, x_{45})\tau_3^{\lambda}f(x_{45}, x_{34}), f(x_{23}, x_{12})\tau_2^{2\lambda}f(x_{12}, x_{23})\big)\\
&\quad f'\big(x_{45}^{\lambda}, f(x_{34}, x_{45})x_{34}^{\lambda}f(x_{45}, x_{34})\big)\\
&= f'\big(f(x_{34}, x_{45})x_{34}^{\lambda}f(x_{45}, x_{34}), x_{45}^{\lambda}\big)f(x_{34}, x_{45})\\
&\quad f'\big(f(x_{45}, x_{34})f(x_{23}, x_{12})\tau_2^{2\lambda}f(x_{12}, x_{23})f(x_{34}, x_{45}), \tau_1^{\lambda}\tau_3^{\lambda}\big)\\
&\quad f(x_{45}, x_{34})f(x_{23}, x_{12})\,\tau_2 f(x_{12}, x_{23})f(x_{34}, x_{45})\\
&\quad f'\big(\tau_1^{\lambda}\tau_3^{\lambda}, f(x_{45}, x_{34})f(x_{23}, x_{12})\tau_2^{2\lambda}f(x_{12}, x_{23})f(x_{34}, x_{45})\big)\\
&\quad f(x_{45}, x_{34})f'\big(x_{45}^{\lambda}, f(x_{34}, x_{45})x_{34}^{\lambda}f(x_{45}, x_{34})\big)\\
&= f'\big(f(\tau_2^2, \tau_1\tau_3)\tau_2^{2\lambda}f(\tau_1\tau_3, \tau_2^2), (\tau_1\tau_3)^{\lambda}\big)\\
&\quad f(\tau_2^2, \tau_1\tau_3)\tau_2 f(\tau_1\tau_3, \tau_2^2)\\
&\quad f'\big((\tau_1\tau_3)^{\lambda}, f(\tau_2^2, \tau_1\tau_3)\tau_2^{2\lambda}f(\tau_1\tau_3, \tau_2^2)\big)\\
&= \tau_2.
\end{aligned}$$

Since $F' \circ F = id$ in $\mathrm{Aut}\,\hat{\Gamma}_0^{[5]}$, the above computation implies that $H'(\sigma) = \mathrm{Inn}(f'(x_{45}, x_{34}))F'(\tau_2)$ is equal to $f'(\tau_2^2, \tau_1\tau_3)\,\tau_2^{\lambda^{-1}}\,f'(\tau_1\tau_3, \tau_2^2)$. Thus, Lemma 4.2 concludes our claim. □

§5. Extension of ρ_2 to $\widehat{GT}$.

In §2, we introduced several aspects of the Kummer 1-cocycle $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}(1)$ appearing from the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. We may then extend ρ_2 in several ways to a 1-cocycle on $\widehat{GT}$. To fix ideas, in this paper, we shall employ a way based on Proposition 2.4. For each $F = (\lambda, f) \in \widehat{GT}$, let $g_F(x,y) \in \hat{F}_2$ be the unique pro-word determined by

$$g_F(y,x)^{-1} g_F(x,y) = f(x,y);$$

the existence and uniqueness of g_F was proved in [LS2]; we recalled existence and uniqueness in the Galois case in (2.7). Writing $g_F(x,y) \equiv x^a y^b \bmod \hat{F}_2'$ for some $a, b \in \hat{\mathbb{Z}}$, the above formula implies $x^{a-b} y^{b-a} \equiv 0 \bmod \hat{F}_2'$, hence that $a = b$. Thus, we are allowed to make the following

Definition 5.1. We define the mapping $\rho_2 : \widehat{GT} \to \hat{\mathbb{Z}}$ by

(5.1) $$g_F(x,y) \equiv (xy)^{\rho_2(F)} \qquad \bmod \hat{F}_2'.$$

Note that, by Proposition 2.4, whenever $F = (\lambda_\sigma, f_\sigma)$ for some $\sigma \in G_{\mathbb{Q}}$, we have $\rho_2(F) = \rho_2(\sigma)$.

First we shall see that this is a 1-cocycle with respect to the action of $\widehat{GT}$ on $\hat{\mathbb{Z}}$ by multiplication by λ. We begin by

Lemma 5.1. *For $F = (\lambda, f), F' = (\lambda', f') \in \widehat{GT}$, we have*

$$g_{FF'}(x,y) = g_F(x,y) g_{F'}(x^\lambda, f(x,y)^{-1} y^\lambda f(x,y)).$$

Proof. This follows easily from the definitions: Compute

$$g_{F'}(y^\lambda, f(y,x)^{-1} x^\lambda f(y,x))^{-1} g_F(y,x)^{-1} g_F(x,y) g_{F'}(x^\lambda, f(x,y)^{-1} y^\lambda f(x,y))$$

and see that this is equal to $f'(f(x,y) x^\lambda f(x,y)^{-1}, y^\lambda) f(x,y)$. □

From this we immediately see the following

Corollary 5.2. *The above ρ_2 enjoys the 1-cocycle property:*

$$\rho_2(FF') = \rho_2(F) + \lambda \rho_2(F'),$$

where $F = (\lambda, f), F' = (\lambda', f') \in \widehat{GT}$. □

On the other hand, in the similar way to §2 (2.4), applying the argument of the profinite Blanchfield-Lyndon theorem, we may define $A_F(\bar{x}, \bar{y}) \in \hat{\mathbb{Z}}[[\hat{F}_2^{ab}]]$ for any $F = (\lambda, f) \in \widehat{GT}$ by

(5.2) $$f(x,y) \equiv A_F(\bar{x}, \bar{y}) * [x,y] \bmod \hat{F}_2''.$$

Then, just tracing our previous arguments given for A_σ ($\sigma \in G_{\mathbb{Q}}$), we obtain similar congruences to (2.6), (3.1) for A_F ($F \in \widehat{GT}$):

(5.3) $$f(\tau_1, \tau_2^2) \equiv (\tau_1^2 \tau_2^4)^{-A_F(-1,1)} \bmod \hat{P}_3' \langle \omega_3 \rangle$$

(5.4) $$f(\tau_1, \tau_2^4) \equiv (\tau_1^4 \tau_2^8)^{-A_F(-1,1)} \bmod \hat{P}_3' \langle \omega_3 \rangle$$

(5.5) $$f(\tau_1\tau_3, \tau_2^2) \equiv (x_{14} x_{23}^{-1})^{A_F(-1,1)} \bmod [\hat{\Gamma}_0^5, \hat{\Gamma}_0^5],$$

where the notation for $\hat{B}_3$, $\hat{\Gamma}_0^{[5]}$ are as in §§2,3.

Proposition 5.3. $\rho_2(F) = -A_F(-1,1)$ *for* $F \in \mathbb{\Gamma}'$. $\square$

Proof. The result follows by comparing (5.1), (5.2) and using the relation $x_{23} = x_{45}x_{14}x_{51}$ and relation (III′). $\square$

This will be used in the next section. Sometimes, it is useful to rewrite the relation (III′) in $\hat{\Gamma}_0^{[5]}$ in an equivalent form in $\hat{B}_4$:

Proposition 5.4. *Let* B_4 *be the 4-strand braid group with standard generators* τ_1, τ_2, τ_3, *and put* x_{ij} $(1 \le i, j \le 4)$ *be as in* §1. *We also define* $\mathbf{x}_{45} = (\tau_1\tau_2\tau_1)^2$, $\mathbf{x}_{51} = (\tau_3\tau_2\tau_3)^2$. *Then the relation* (III′) *in* $\hat{\Gamma}_0^{[5]}$ *is equivalent to*

$$(\mathrm{III}'_{bis}) \qquad f(\tau_1\tau_3, \tau_2^2)\omega^{\rho_2(F)} = g(\mathbf{x}_{45}, \mathbf{x}_{51})f(x_{12}, x_{23})f(x_{34}, \mathbf{x}_{45})$$

in $\hat{B}_4$, *where* $\omega = (\tau_1\tau_2\tau_3)^4 = x_{12}x_{13}x_{23}x_{14}x_{24}x_{34}$.

Proof. We have a natural homomorphism $j : \hat{B}_4 \to \hat{\Gamma}_0^{[5]}$ mapping $\tau_i \mapsto \tau_i$ for $i = 1, 2, 3$. Note in particular that $j(\mathbf{x}_{45}) = x_{45}$ and $j(\mathbf{x}_{51}) = x_{51}$, by the identities $x_{45} = (\tau_1\tau_2)^3$, $x_{51} = (\tau_2\tau_3)^3$; accordingly, $j(\omega) = 1$ since $\omega = (\tau_1\tau_2\tau_3)^4$ is killed in $\hat{\Gamma}_0^{[5]}$. So $(\mathrm{III}'_{bis}) \Rightarrow (\mathrm{III}')$ is obvious. Suppose (III′) holds in $\Gamma_0^{[5]}$ and let us consider the pull-back of the relation in $\hat{B}_4$ by j. Then since the kernel of j is the pro-cyclic group generated by ω, we have a relation of type (III'_{bis}) modulo some power of ω. To determine the exact exponent of ω, we shall reduce the equation modulo $[\hat{P}_4, \hat{P}_4]$. First, by a similar computation to (3.1)(or (5.5)), we get

$$f(\tau_1\tau_3, \tau_2^2) \equiv (x_{14}x_{23}^{-1})^{-\rho_2(F)} \bmod [\hat{P}_4, \hat{P}_4].$$

On the other hand, by Definition 5.1,

$$g(\mathbf{x}_{45}, \mathbf{x}_{51}) \equiv (\mathbf{x}_{45}\mathbf{x}_{51})^{\rho_2(F)} \equiv \{(x_{12}x_{13}x_{23})(x_{23}x_{24}x_{34})\}^{\rho_2(F)} \bmod \ [\hat{P}_4, \hat{P}_4].$$

Thus, combining the above two formulae in the abelianization of $\hat{P}_4$ (which is free abelian on the images of x_{ij} $(1 \le i < j \le 4)$), we obtain the congruence:

$$f(\tau_2^2, \tau_1\tau_3)g(\mathbf{x}_{45}, \mathbf{x}_{51}) \equiv (x_{12}x_{13}x_{23}x_{14}x_{24}x_{34})^{\rho_2(F)} (= \omega^{\rho_2(F)}) \bmod \ [\hat{P}_4, \hat{P}_4].$$

Meanwhile, since $x_{12}, x_{23}, x_{34}, \mathbf{x}_{45}$ belong to $\hat{P}_4$ themselves, the other two f terms $f(x_{12}, x_{23})$, $f(x_{34}, \mathbf{x}_{45})$ are killed in the abelianizaion of $\hat{P}_4$. Combining all these together leads us then to the congruence:

$$f(\tau_2^2, \tau_1\tau_3)g(\mathbf{x}_{45}, \mathbf{x}_{51})f(x_{12}, x_{23})f(x_{34}, \mathbf{x}_{45}) = \omega^{\rho_2(F)} \bmod \ [\hat{P}_4, \hat{P}_4]$$

which determines the desired exponent of ω in our formula. $\square$

We could define an extension of the Galois cocycle $\rho_2(F)$ to $\widehat{GT}$ by using $-A_F(-1,1)$ instead of $g_F(x,y)$. In that case, the machinery of profinite free differential calculus (cf. [I3]) tells a more general 1-cocycle property for $A_F(\bar{x}, \bar{y})$ which reduces to that for $A_F(-1,1)$. Using this machinery, Ihara [I3] defined many 1-cocycles on $\widehat{GT}$ including $\Psi_n^{(0)}$ $(n \in \mathbf{N})$ which extends the Kummer 1-cocycle on the roots of n. H.Tsunogai communicated to one of the authors that our $-A_F(-1,1)$ coincides with Ihara's $\Psi_2^{(0)}$ (cf. [I3] §2.6 (6) and Proposition 2.2.3).

§6. $\mathbb{I}\Gamma$ forms a subgroup of $\mathbb{I}\Gamma'$.

In §1, we defined $\mathbb{I}\Gamma$ to be the subset of $\widehat{GT}$ consisting of all pairs (λ, f) satisfying (III′) and (IV). In §4, we introduced $\mathbb{I}\Gamma' \subset \widehat{GT}$ only by using (III′), and showed that $\mathbb{I}\Gamma'$ forms a subgroup of $\widehat{GT}$. Thus, to prove Theorem 1.2, it suffices to show that the elements of $\mathbb{I}\Gamma'$ satisfying (IV) are closed under multiplication and inversion of $\mathbb{I}\Gamma'$.

Our proof of Theorem 1.2 here goes in a parallel way to §4. In this section, we let each $F = (\lambda, f) \in \widehat{GT}$ act on $\hat{B}_3 = \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle$ in the standard way: $F(\tau_1) = \tau_1^\lambda$, $F(\tau_2) = f(\tau_2^2, \tau_1^2)\tau_2^\lambda f(\tau_1^2, \tau_2^2)$, and define the automorphism H_F of $\hat{B}_3$ by $H_F := \mathrm{Inn}(\tau_1^{-4\rho_2(F)}) \circ F$.

Lemma 6.1. *Let $F \in \mathbb{I}\Gamma'$. Then, $F \in \mathbb{I}\Gamma'$ satisfies (IV) if and only if $H_F(\tau_2) = f(\tau_2^4, \tau_1)\tau_2^\lambda f(\tau_1, \tau_2^4)$ holds.*

Proof. For the "only if" part, assume that $F \in \mathbb{I}\Gamma'$ satisfies (IV). Then since $x_{12} = \tau_1^2$, $x_{23} = \tau_2^2$, we have

$$\begin{aligned} H_F(\tau_2) &= \tau_1^{-4\rho_2(F)} f(x_{23}, x_{12})\tau_2^\lambda f(x_{12}, x_{23})\tau_1^{4\rho_2(F)} \\ &= (\tau_1\tau_2)^{-6\rho_2(F)} f(\tau_2^4, \tau_1)\tau_2^{8\rho_2(F)}\tau_2^\lambda\tau_2^{-8\rho_2(F)} f(\tau_1, \tau_2^4)(\tau_1\tau_2)^{6\rho_2(F)} \\ &= f(\tau_2^4, \tau_1)\tau_2^\lambda f(\tau_1, \tau_2^4), \end{aligned}$$

since $(\tau_1\tau_2)^3$ commutes with τ_1 and τ_2.

Let us now discuss the "if" part. Recalling that the centralizer of τ_2 in $\hat{B}_3$ is generated by τ_2 and $\omega_3 = (\tau_1\tau_2^2)^2$, we see that the latter condition of the proposition implies that there exists $a, b \in \hat{\mathbb{Z}}$ such that

$$f(\tau_1^2, \tau_2^2) = \omega_3^a\tau_2^b f(\tau_1, \tau_2^4)\tau_1^{-4\rho_2(F)}. \tag{6.1}$$

Then, looking at this equation modulo $\hat{P}_3'\langle\omega_3\rangle$, we obtain from (5.4) and Proposition 5.3

$$\tau_2^b(\tau_1^4\tau_2^8)^{\rho_2(F)}\tau_1^{-4\rho_2(F)} \equiv 0 \bmod \hat{P}_3'\langle\omega_3\rangle.$$

Hence $b = -8\rho_2(F)$. Then the projection $\langle\tau_1, \tau_2^2\rangle \to \hat{\mathbb{Z}}$ via $\tau_1 \mapsto 1$, $\tau_2^2 \mapsto 0$ reduces the equation (6.1) to $0 = 2a - 4\rho_2(F)$; hence $a = 2\rho_2(F)$. Returning these values of a, b to (5.1), we obtain the relation (IV). □

Proof of Theorem 1.2. Let us first show that, for any elements $F = (\lambda, f)$ and $F' = (\lambda', f') \in \mathbb{I}\Gamma$, their product $\tilde{F} = FF' = (\tilde{\lambda}, \tilde{f})$ also satisfies (IV). Note that $\tilde{\lambda} = \lambda\lambda'$ and

$$\tilde{f}(x, y) = f(x, y)f'\big(x^\lambda, f(y, x)y^\lambda f(x, y)\big).$$

To use Lemma 6.1, put $H = H_F$, $H' = H_{F'}$ and $\tilde{H} = H_{\tilde{F}}$. Then, by taking

Corollary 5.2 into accounts, we compute:

$$\begin{aligned}
\tilde{F}(\tau_2) &= (F \circ F')(\tau_2)\\
&= \Big(\mathrm{Inn}(\tau_1^{4\rho_2(F)})H \circ \mathrm{Inn}(\tau_1^{4\rho_2(F')})H'\Big)(\tau_2)\\
&= \Big(\mathrm{Inn}(\tau_1^{4\rho_2(F)}) \circ \mathrm{Inn}(\tau_1^{4\lambda\rho_2(F')})HH'\Big)(\tau_2)\\
&= \tau_1^{4\rho_2(F)}\tau_1^{4\lambda\rho_2(F')}H\big(f'(\tau_2^4,\tau_1)\tau_2^{\lambda'}f'(\tau_1,\tau_2^4)\big)\tau_1^{-4\lambda\rho_2(F')}\tau_1^{-4\rho_2(F)}\\
&= \mathrm{Inn}\left(\tau_1^{4\rho_2(\tilde{F})}f'\big(f(\tau_2^4,\tau_1)\tau_2^{4\lambda}f(\tau_1,\tau_2^4),\tau_1^{\lambda}\big)f(\tau_2^4,\tau_1)\right)(\tau_2^{\tilde{\lambda}})\\
&= \tau_1^{4\rho_2(\tilde{F})}\tilde{f}(\tau_2^4,\tau_1)\tau_2^{\tilde{\lambda}}f(\tau_1,\tau_2^4)\tilde{\tau}_1^{4\rho_2(\tilde{F})}.
\end{aligned}$$

From this we obtain $H_{\tilde{F}}(\tau_2) = f(\tau_2^4,\tau_1)\tau_2^{\tilde{\lambda}}f(\tau_1,\tau_2^4)$ as desired.

Next, let $F = (\lambda, f) \in \mathbb{I}\Gamma$ and consider its inverse $F' = (\lambda^{-1}, f')$ in $\mathbb{I}\Gamma'$ so that

$$f'(f(y,x)y^{\lambda}f(x,y),x^{\lambda}) = f(x,y).$$

Using this and $\rho_2(F) + \lambda\rho_2(F') = \rho_2(FF') = 0$ together with the relation (IV) for f, we compute:

$$\begin{aligned}
&F\big(\tau_1^{4\rho_2(F)}f'(\tau_2^4,\tau_1)\tau_2^{\lambda^{-1}}f'(\tau_1,\tau_2^4)\tau_1^{-4\rho_2(F)}\big)\\
&= \mathrm{Inn}\left(\tau_1^{4\lambda\rho_2(F')}f'\big(f(\tau_2^2,\tau_1^2)\tau_2^{4\lambda}f(\tau_1^2,\tau_2^2),\tau_1^{\lambda}\big)\right)\left(F(\tau_2^{\lambda^{-1}})\right)\\
&= \mathrm{Inn}\left(\tau_1^{4\lambda\rho_2(F')+4\rho_2(F)}f'\big(f(\tau_2^4,\tau_1)\tau_2^{4\lambda}f(\tau_1,\tau_2^4),\tau_1^{\lambda}\big)\tau_1^{-4\rho_2(F)}\cdot f(\tau_1^2,\tau_2^2)\right)(\tau_2)\\
&= \mathrm{Inn}\left(\tau_1^{4\lambda\rho_2(F')+4\rho_2(F)}f(\tau_1,\tau_2^4)\tau_1^{-4\rho_2(F)}f(\tau_1^2,\tau_2^2)\right)(\tau_2)\\
&= \tau_1^{4\lambda\rho_2(F')+4\rho_2(F)}\tau_2\tau_1^{-4\lambda\rho_2(F')-4\rho_2(F)} = \tau_2.
\end{aligned}$$

This implies $H_{F'}(\tau_2) = f'(\tau_2^4,\tau_1)\tau_2^{\lambda^{-1}}f'(\tau_1,\tau_2^4)$. By Lemma 6.1, we conclude that $F' \in \mathbb{I}\Gamma$, which proves Theorem 1.2. $\square$

Remark 6.2. In Theorem 2.5, we observed that $\rho_2(\sigma)$ for $\sigma \in G_{\mathbb{Q}}$ appears also in the ratio of the upper components of $f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))$. One can show that this holds true for $F = (\lambda, f) \in \widehat{GT}$ satisfying (IV) (especially for all elements of $\mathbb{I}\Gamma$). Indeed, considering (IV) by specializing $\tau_1 = (\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$, $\tau_2 = (\begin{smallmatrix}1&0\\-1&1\end{smallmatrix})$ in the finite adele group $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}}^{fin})$, we obtain

$$\begin{aligned}
&(\begin{smallmatrix}1&0\\0&2\end{smallmatrix})\, f((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))\,(\begin{smallmatrix}1&0\\0&2\end{smallmatrix})^{-1} = f((\begin{smallmatrix}1&1\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-4&1\end{smallmatrix}))\\
&= (\begin{smallmatrix}1&0\\-1&1\end{smallmatrix})^{8\rho_2} f((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})^{4\rho_2}(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix})^{-4\rho_2},
\end{aligned}$$

where $\rho_2 = \rho_2(F)$. Evaluating this after setting $f((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})) = (\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix})$, we obtain $\beta = -8\rho_2\alpha$ as desired.

§7. Basic moves and associated pro-words.

In the following few sections, we will mainly be concerned with a topological study of the subject. We fix a compact topological surface $\Sigma(\cong \Sigma_{g,r})$ of genus g with r boundary components, and assume $2-2g-r<0$. A pants decomposition P of Σ is by definition given by a finite collection of disjoint (non-oriented) simple closed curves (circles) on Σ such that each connected component of the complement of these curves is homeomorphic to the interior of $\Sigma_{0,3}$. We denote by $\mathcal{C}(P)$ (resp. $\Pi(P)$) the collection of those curves (resp. connected components) forming the pants decomposition P. We also set $\mathcal{C}^*(P)$ to be the union of $\mathcal{C}(P)$ and the circles parallel to boundary components. The circles of $\mathcal{C}^*(P)\backslash\mathcal{C}(P)$ will be called *boundary circles* for simplicity.

As easily seen, the cardinalities of $\mathcal{C}(P)$ and $\Pi(P)$ are $3g-3+r$, $2g-2+r$ respectively. We call each element of $\mathcal{C}(P)$ (resp. $\Pi(P)$) a circle (resp. a pair of pants) of the pants decomposition P.

We denote by $\mathbb{S}^*(\Sigma)$ the collection of isotopy classes of simple closed curves on Σ which are not homotopically trivial, and by $\mathbb{S}(\Sigma)$ the subset of $\mathbb{S}^*(\Sigma)$ consisting of those classes of curves not parallel to any boundary component. Any class $[c]\in\mathbb{S}^*(\Sigma)$ defines a Dehn twist $D_{[c]}$ of the mapping class group $\Gamma(\Sigma)$. For simplicity, we often identify a simple closed curve c on Σ with its isotopy class $[c]\in\mathbb{S}^*(\Sigma)$, and write for brevity $D_c=D_{[c]}$. Note then that for any pants decomposition P, the set $\mathcal{C}(P)$ will be regarded as a subset of $\mathbb{S}(\Sigma)$. Also, we will not distinguish two pants decomposition given by isotopic family of disjoint circles.

The geometric intersection form $i:\mathbb{S}(\Sigma)\times\mathbb{S}(\Sigma)\to\mathbb{Z}_{\geq 0}$ is defined by associating, to any two isotopy classes in $\mathbb{S}(\Sigma)$, the minimum number of intersection points of two curves representing them respectively. On the other hand, we have an algebraic intersection form $I: H_1(\Sigma,\partial\Sigma,\mathbb{Z})\times H_1(\Sigma,\mathbb{Z})\to H_0(\Sigma,\mathbb{Z})\cong\mathbb{Z}$ in standard topology theory. Since elements of $\mathbb{S}(\Sigma)$ give homology classes up to signs, for any pair $(c,c')\in\mathbb{S}(\Sigma)\times\mathbb{S}(\Sigma)$, the absolute value $|I(c,c')|$ makes sense.

Simple and associativity moves on pants decompositions

Now, we will introduce a graph structure on the set of all the (isotopy classes of) pants decompositions on a surface. The vertices are pants decompositions; to connect them by "edges", we specify certain types of pants decomposition pairs (S-moves and A-moves below), using the above terminology of geometric and algebraic intersection forms.

Definition 7.1.

(a) For any pants decomposition P on Σ and each circle $c\in\mathcal{C}(P)$, the *neighborhood* of c is defined to be the piece supporting the circle c when Σ is cut along all circles of $\mathcal{C}(P)\setminus\{c\}$. This neighborhood is either of type $(0,4)$ or of type $(1,1)$.

(b) Let P,P' be two pants decompositions of Σ, and suppose that they differ from each other only by one circle, i.e., $\mathcal{C}(P)\setminus\{c\}=\mathcal{C}(P')\setminus\{c'\}$ and $c\neq c'$ (as elements of $\mathbb{S}(\Sigma)$). Then, the pair (P,P') will be called a *simple move (or S-move)* if $i(c,c')=1$. An S-move can be performed if and only if the neighborhood of the circle $c\in\mathcal{C}(P)$ is of type $(1,1)$.

(c) The pair (P,P') is called an *associativity move (or A-move)* if $i(c,c')=2$ and $|I(c,c')|=0$; an A-move can be performed if and only if the neighborhood of c is of type $(0,4)$.

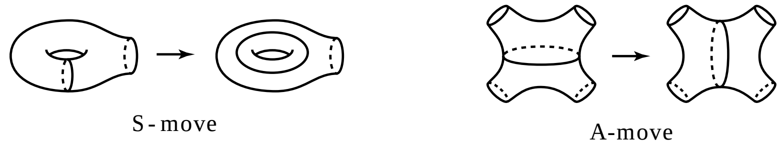


Figure 7.1

Quilt decompositions of Σ

Next, let us introduce a notion of "quilt-decompositions", also called "quilts" for shortness, which refines pants decompositions. Suppose that we are given a pants decomposition P of a surface Σ. We begin by defining a *quilt-decomposition* for each pair of pants $p \in \Pi(P)$.

Let us first consider the case when p is bounded by three distinct circles $c_i \in \mathcal{C}^*(P)$ ($i \in \mathbb{Z}/3\mathbb{Z}$). Then, it is easy to see that, any triple of disjoint lines l_i ($i \in \mathbb{Z}/3\mathbb{Z}$) such that l_i connects c_i and c_{i+1} cuts p into two hexagonal *patches*. We call this type of decomposition of p a quilt on p, and call l_1, l_2, l_3 *seams*. The endpoints of seams will be called *vertices*. Next, consider the case when the closure $\bar{p}$ of p in Σ is homeomorphic to $\Sigma_{1,1}$. This corresponds to the situation where two of the boundary components c_1, c_2, c_3 of p coincide in Σ. Suppose, say, that $c_1 \neq c_2 = c_3$. On such a p, we define a quilt by seams l_1, l_2, l_3 very much as above, but we additionally impose that the endpoints on c_2 $(= c_3)$ are exactly two points and that l_2 is not homotopic to part of c_2. See Figure 7.2 for typical examples of quilts on pairs of pants.

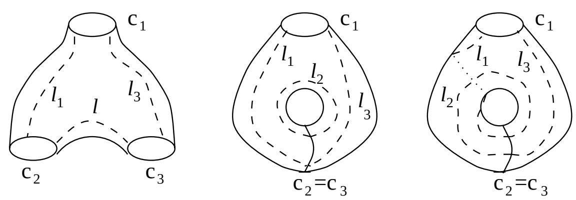


Figure 7.2

Finally, we define a quilt-decomposition of Σ over P to be a collection of quilt-decompositions of all pairs of pants of P such that each circle has exactly two vertices as meeting points of seams. In other words, if two pants p_1 and p_2 meet at a circle, and two seams of p_1 meet that circle at vertices v_1 and v_2, then the two seams of p_2 which meet the circle do so at the same vertices v_1 and v_2 (see Figure 7.3). We use the notation $\mathfrak{s}, \mathfrak{p}, v$ to denote seams, patches, vertices respectively, and, for any quilt Q over P, write $\mathfrak{s}(Q), \mathfrak{p}(Q), v(Q)$ to denote the sets of corresponding objects arising in Q.

Half-twist actions on quilts

If $\mathcal{Q}(P)$ denotes the set of (isotopy classes of) quilt-decompositions over a given pants decomposition P of Σ, then the Dehn twists D_c for $c \in \mathcal{C}(P)$ act naturally on $\mathcal{Q}(P)$. Now, let us also define a half-twist action "$D_c^{1/2}$" on $\mathcal{Q}(P)$; this action leaves the underlying pants decomposition invariant, but alters the seams in the neighborhood of the circle c as in Figure 7.3.

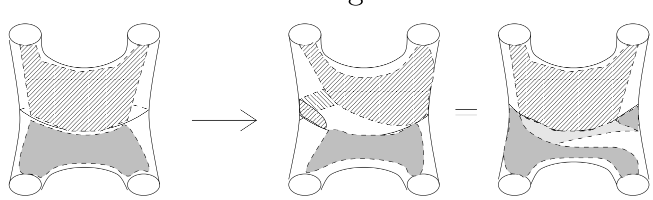


Figure 7.3

To define the orientation of our half twist operation rigorously, we shall give a topological characterization of the process as follows. Given a quilt Q/P on Σ

and a circle $c \in \mathcal{C}(P)$, cut off a small cylinder neighborhood B_c of c with two boundary circles parallel to c (as shown in the leftmost picture of Figure 7.4). Pick a boundary circle b of B_c. Deforming seams homotopically, we may assume that two seams meet b transversally. Let a_1, a_2 be those meeting points on b, and identify b with the unit circle $S^1 = \{\exp(2\pi i t) \mid 0 \leq t \leq 1\}$ so that a_1, a_2 correspond to the points $t = 1/8, 5/8$ respectively. Divid b into four (oriented) segments b_1, b_2, b_3, b_4 corresponding to $t \in [0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$ respectively and identify b_1 with b_3^{-1} and b_2 with b_4^{-1} to form a surface B'_c of type (1,1) (see the second picture of Figure 7.4) where the points a_1 and a_2 are also patched together. Let c_1, c_2 be circles formed by $b_1 = b_3$, $b_2 = b_4$ respectively in B'_c. We now perform the diffeomorphism $(D_1 D_2)^3$ on B'_c, where $D_i = D_{c_i}$ $(i = 1, 2)$. Since $(D_1 D_2)^3$ commutes with D_i $(i = 1, 2)$, it preserves each circle c_i $(i = 1, 2)$, but it changes the orientations of these circles. The diffeomorphism thus twists the seams on B'_c as in the third picture of Figure 7.4. Then, returning to B_c by cutting the surface B'_c along c_1, c_2, we obtain another quilt on B_c (as in the last picure of Figure 7.4) having the same endpoints a_1, a_2 and others of seams on the boundary components as the original quilt on B_c. We then recast the original quilt Q/P on Σ by replacing the cylinder neighborhood B_c by this newly quilted cylinder. The resulting quilt is independent of the choice of b from the two boundary components of B_c up to isotopy, and is defined to be the *half-twist* "$D_c^{1/2}(Q)/P$" of the original quilt Q/P along the circle $c \in \mathcal{C}(P)$.

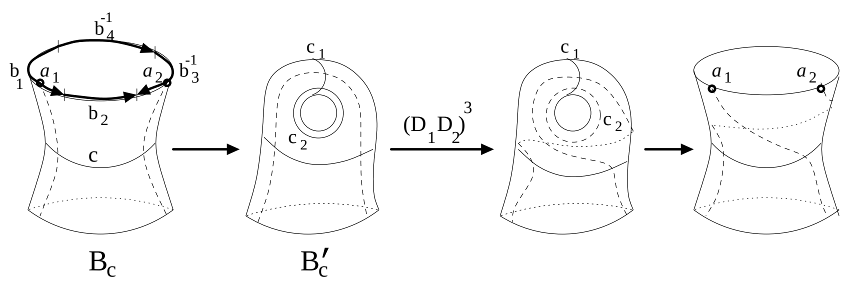


Figure 7.4

For any positive integer N, we shall write $D_c^{N/2}(Q)$ for the quilt obtained from Q by applying N times the half-twist $D_c^{1/2}$. Define also $D_c^{-N/2}(Q)$ so that $D_c^{N/2}(D_c^{-N/2}(Q)) = Q$ holds.

Quilts adjusted to circles.

Now we come to the stage of introducing a crucial procedure which plays an important role in the following arguments. This procedure deforms quilts along moves of underlying pants decompositions in unique ways. More precisely, given an A-move or S-move of pants decompositions (P, P') and a quilt Q over P, we shall define a quilt $Q_{P \to P'}$ over P' in a unique way, as the effect of that A-move or S-move on the quilt Q/P. Before introducing these A-moves and S-moves on quilts, we need to introduce the concept of *adjustment*.

Let P be a pants decomposition on Σ, let Q/P be a quilt, and let $c \in \mathcal{C}(P)$ be a circle of the pants decomposition.

Case 1. Suppose the neighborhood of c is of type $(0, 4)$, and let c' be a circle on Σ such that changing c to c' is an A-move on P. Let H denote the neighborhood of c; the circle c cuts H into two pants p_1 and p_2, and the seams of Q cut each pair of pants into two patches. The quilt Q is said to be *adjusted to* c' (or *to the pants decomposition obtained from* P *by replacing* c *by* c') if there is a simple closed curve in the homotopy class of c' such that c' intersects each of the four patches of H in exactly one segment.

Lemma 7.1. *Let Σ, Q, P and c be as above. Then there is a unique circle c_1 such that $c \to c_1$ is an A-move and Q is adjusted to c_1. If c' is any circle such that $c \to c'$ is an A-move, there exists a unique integer N such that the quilt $D_c^{N/2}(Q)$ over P is adjusted to c'.*

Proof. The typical situation is the quilt shown in the picture of Figure 7.5, which shows a quilt on the neighborhood of the horizontal circle c, which is adjusted to a vertical circle c_1. The key remark is that every other circle c' such that $c \to c'$ is an A-move (i.e. such that c' intersects c in two points with algebraic intersection 0 and lies on the neighborhood H), is obtained from c_1 by half-twists along c. But every such half-twist augments the number of segments in the intersection of the new circle with the patches of Q. This shows that that if c' is any circle such that $c \to c'$ is an A-move, then there is a unique N such that Q/P is adjusted to $D_c^{-N/2}(c')$. The second statement of the lemma follows immediately. □

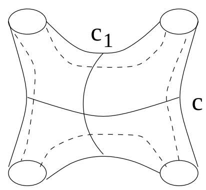


Figure 7.5

Case 2. Again we let Σ be a surface of type (g, r), P a pants decomposition and $c \in \mathcal{C}(P)$, but now we suppose that the neighborhood of c is of type $(1, 1)$. Let H denote this neighborhood; the circle c cuts H into a single pair of pants (of which c joins two legs). Consider the closure of seams of the quilt Q on the neighborhood H of c. There are two distinct situations, as this closure can form either one or two connected components of curves. Set the *associated quilt* $Q^\sharp$ to be $D_c^{1/2}(Q)$ if there is only one component, and otherwise set $Q^\sharp = Q$, so that the closure of seams of $Q^\sharp$ has two connected components, one of which forms a circle r. We call this r the *reference circle* of the quilt Q in H.

Now, let c' be a circle such that $c \to c'$ is an S-move on P, i.e. c' lies on H and $i(c, c') = 1$. We say that Q *is adjusted to* c' (or *to the pants decomposition obtained from P by replacing c by c'*), if $Q = Q^\sharp$ and the reference circle r is homotopic to c'.

Lemma 7.2. *Let Σ, Q/P, c be as above, and let $Q^\sharp$, r be the associated quilt, the reference curve for them respectively. Then there is a unique circle c_1 such that $c \to c_1$ is an S-move and $Q^\sharp$ is adjusted to c_1. If c' is any circle such that $c \to c'$ is an S-move, then there exists a unique integer M such that the quilt $D_c^M(Q^\sharp)$ over P is adjusted to c'.*

Proof. The first statement is obvious; c_1 must be homotopic to the reference circle r of $Q^\sharp$. For the second statement, we use the fact that every other circle c' on H such that $c \to c'$ is an S-move is obtained from c_1 by twists along c, i.e. there exists a unique integer M such that $c' = D_c^M(r)$. Thus the quilt $D_c^M(Q^\sharp)$ is adjusted to c'. □

A-moves and S-moves on quilts; associated pro-words.

Now we can proceed to the definition of A-moves and S-moves on quilts. Moreover, if we are given an element $F = (\lambda, f) \in \mathbb{I}\Gamma$ in addition to P, P', Q as above, we shall define an element $f_F(Q/P \to P')$ of $\hat{\Gamma}(\Sigma)$, the profinite completion of the mapping class group $\Gamma(\Sigma) \simeq \Gamma_{g,r}$.

Definition 7.2. Assume we are given a quilt Q/P and an A- or S-move (P, P') of pants decompositions replacing $c \in \mathcal{C}(P)$ by $c' \in \mathcal{C}(P')$.
(a) Case 1: Suppose (P, P') is an A-move. We define a quilt over P', denoted $Q_{P \to P'}$, to be the quilt whose patches consist of those obtained by cutting Σ along the circles of P' together with the seams of $D_c^{N/2}(Q)$, where N is as in the second statement of Lemma 7.1. For $F = (\lambda, f) \in \mathbb{I}\Gamma$, we define

$$f_F(Q/P \to P') := D_c^{N\mu} f(D_{c'}, D_c) \, (= D_c^{N\mu} f(D_c, D_{c'})^{-1}) \tag{7.1}$$

where $\mu = (\lambda - 1)/2$.
(b) Case 2: Suppose (P, P') is an S-move. We define a new quilt $Q_{P \to P'}$ over P' to be the image of $Q^\sharp$ by the mapping class $(D_c D_{c'} D_c) D_c^M$, where M is as in the statement of Lemma 7.2. We also define, for $F = (\lambda, f) \in \mathbb{I}\Gamma$,

$$f_F(Q/P \to P') := D_c^{N\mu - 8\rho_2} f(D_{c'}^2, D_c^2) D_{c'}^{8\rho_2} (D_c D_{c'} D_c)^{2\mu}, \tag{7.2}$$

where $\mu = (\lambda - 1)/2$, $\rho_2 = \rho_2(F)$, and $N = 2M + \varepsilon$ with $\varepsilon = 0, 1$ according as $Q^\sharp = Q$ or $D_c^{1/2}(Q)$. Note that N is chosen such that $D_c^{N/2}(Q)$ is adjusted to c'.

Remark. There is a certain Galois-theoretical reason for the necessity of putting $8\rho_2$ in exponents in the above definition of Case 2 (cf. [N2] §4 (4.11).)

Notation and Convention. Let $(P_0, P_1, \ldots, P_n)$ be a chain of pants decompositions of Σ such that (P_i, P_{i+1}) are A- or S-moves ($i = 0, \ldots, n-1$). For brevity, we call such $(P_0, P_1, \ldots, P_n)$ a chain of A/S-moves on Σ. For a given quilt Q_0 over P_0, we shall write $(Q_0)_{P_0 \to \cdots \to P_n}$ to designate $(...(((Q_0)_{P_0 \to P_1})_{P_1 \to P_2}) \cdots)_{P_{n-1} \to P_n}$, and define

$$f_F(Q_0/P_0 \to \cdots \to P_n) := f_F(Q_0/P_0 \to P_1) \cdots f_F(Q_{n-1}/P_{n-1} \to P_n), \tag{7.3}$$

for $F \in \mathbb{I}\Gamma$, where $Q_i = (Q_0)_{P_0 \to \cdots \to P_i}$ $(1 \leq i \leq n-1)$. We also introduce, for an A/S-move (P, P') replacing a circle $c \in \mathcal{C}(P)$ by $c' \in \mathcal{C}(P')$, a proword $f_F[P \to P']$ for $F = (\lambda, f) \in \mathbb{I}\Gamma$ by

$$f_F[P \to P'] := \begin{cases} f(D_{c'}, D_c), & (P, P') : \text{A-move}, \\ D_c^{-8\rho_2(F)} f(D_{c'}^2, D_c^2) D_{c'}^{8\rho_2(F)} (D_c D_{c'} D_c)^{\lambda - 1}, & (P, P') : \text{S-move}. \end{cases}$$

Note that $f_F[P \to P']$ belongs to a subgroup $\langle D_c, D_{c'} \rangle$ of $\hat{\Gamma}(\Sigma)$ generated by D_c, $D_{c'}$. We also observe that, for any quilt Q/P,

$$f_F(Q/P \to P') = D_c^N f_F[P \to P']$$

holds in both cases (a) and (b) of Definition 7.2, where N is the integer such that $D_c^{N/2}(Q)$ is adjusted to c'.

Lemma 7.3. (Back-tracking Lemma) *Let (P, P') be an A- or S-move of pants decompositions which replaces a circle $c \in \mathcal{C}(P)$ by another circle $c' \in \mathcal{C}(P')$. If (P, P') is an S-move, then denote by δ the circle bounding the neighborhood of c (and of c'). Let Q/P be a quilt, and N the unique integer such that $D_c^{N/2}(Q)$ is adjusted to c'. Suppose we are given a chain of A- or S-moves $\gamma = (P, P_1, \ldots, P_n)$ starting from P. Then,*

(i) *If (P, P') is an A-move, then $Q_{P\to P'\to P} = D_c^{N/2}(Q)$ and $f_F(Q/P \to P' \to P) = D_c^{N\mu}$ for all $F \in \mathbb{I}\Gamma$.*

(ii) *If (P, P') is an S-move, then $Q_{P\to P'\to P} = D_c^{N/2} D_\delta^{1/2}(Q)$ and $f_F(Q/P \to P' \to P) = D_c^{N\mu} D_\delta^{\mu}$ for all $F \in \mathbb{I}\Gamma$.*

(iii) *In either case, we have $f_F(Q/P \to P' \to P \overset{\gamma}{\dashrightarrow} P_n) \equiv f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n)$ for all $F \in \mathbb{I}\Gamma$ in the right coset space $\hat{\Gamma}(\Sigma)/\langle D_c \mid c \in \mathcal{C}(P_n)\rangle$.*

Proof. (i) Let (P, P') be an A-move. Note that by construction, the quilt $Q_{P\to P'}$ over P' is already adjusted to c. Thus the quilt $Q_{P\to P'\to P}$ is the quilt whose patches are obtained directly by cutting Σ along the circles of P and the seams of $Q_{P\to P'}$. This quilt over P is adjusted to c', and equal to Q outside the two pairs of pants of P whose closures contain c, so it is obtained in a unique way as $D_c^{N/2}(Q)$. The fact that no twist is necessary to adjust $Q_{P\to P'}$ back to c means that

$$f_F(Q_{P\to P'}/P' \to P) = f(D_c, D_{c'}),$$

by (7.1), hence that

$$\begin{aligned} f_F(Q/P \to P' \to P) &= f_F(Q/P \to P') f_F(Q_{P\to P'}/P' \to P) \\ &= D_c^{N\mu} f(D_{c'}, D_c) f(D_c, D_{c'}) = D_c^{N\mu} \end{aligned}$$

by (7.3) and relation (I). This proves (i).
(ii) Let (P, P') be an S-move changing c to c'. As in Definition 7.2 (b), let $N = 2M + \varepsilon$ where M is as in Lemma 7.2, and $\varepsilon = 0, 1$ according to whether $Q = Q^\sharp$, $Q \neq Q^\sharp$. Thus, as we saw, $D_c^{N/2}(Q)$ is adjusted to c'. Therefore, by Definition 7.2 (b), we have $Q_{P\to P'} = (D_c D_{c'} D_c) D_c^{N/2}(Q)$. Again, the quilt $Q_{P\to P'}$ over P' is already adjusted to c, because $D_c D_{c'} D_c$ maps the reference curve $r = c'$ of the quilt $D_c^{N/2}(Q)$ to c. Noting that $D_c D_{c'} D_c = D_{c'} D_c D_{c'}$, we get $Q_{P\to P'\to P} = (D_c D_{c'} D_c)^2 D_c^{N/2}(Q)$. Since, for any quilt Q', $(D_c D_{c'} D_c)^2(Q') = D_\delta^{1/2}(Q')$ holds, and since $D_\delta^{1/2}$ commutes with $D_c^{1/2}$, this implies that $Q_{P\to P'\to P} = D_c^{N/2} D_\delta^{1/2}(Q)$. Using Definition 7.2 (b) and (7.3), we compute

$$\begin{aligned} f_F(Q/P \to P' \to P) &= f_F(Q/P \to P') f_F(Q_{P\to P'}/P' \to P) \\ &= D_c^{N\mu - 8\rho_2} f(D_{c'}^2, D_c^2) D_{c'}^{8\rho_2} (D_c D_{c'} D_c)^{2\mu} D_{c'}^{-8\rho_2} f(D_c^2, D_{c'}^2) D_c^{8\rho_2} (D_{c'} D_c D_{c'})^{2\mu} \\ &= D_c^{N\mu} (D_c D_{c'} D_c)^{4\mu} = D_c^{N\mu} D_\delta^{\mu}. \end{aligned}$$

(iii) Suppose first that (P, P') is an A-move. By (i), $Q_{P\to P'\to P}$ is the quilt $D_c^{N/2}(Q)$ over P (which is adjusted to c'). If c remains unchanged through the chain γ, then, since each move adds the same factors to $f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n)$ and $f_F(D_c^{N/2}(Q)/P \overset{\gamma}{\dashrightarrow} P_n)$, they are equal (and commute with D_c). Thus,

$$f_F(Q/P \to P' \to P \overset{\gamma}{\dashrightarrow} P_n) = f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n) D_c^{N\mu}.$$

Suppose next that c is changed in $\gamma : P = P_0 \to \ldots \to P_n$, and let m be the smallest index (≥ 1) such that c is changed under the move $P_{m-1} \to P_m$ (say, to c''). Then, $Q^\flat := Q_{P \overset{\gamma}{\dashrightarrow} P_{m-1}}$ and $Q^* := Q_{P \to P' \to P \overset{\gamma}{\dashrightarrow} P_{m-1}}$ are related by $Q^* = D_c^{N/2}(Q^\flat)$, where N is the same as above. Proceeding to P_{m-1}, we then find a unique integer N' such that the quilt over P_{m-1} adjusted to c'' is given by

$$D_c^{N'/2}(Q^\flat) = D_c^{(N'-N)/2}\big(D_c^{N/2}(Q^\flat)\big) = D_c^{(N'-N)/2}(Q^*).$$

Thus, both of the quilts $(Q^*)_{P_{m-1} \to P_m}$ and $(Q^\flat)_{P_{m-1} \to P_m}$ are equal (we will denote them by Q^m). Then,

$$\begin{aligned}
&f_F(Q/P \to P' \to P \overset{\gamma}{\dashrightarrow} P_n) \\
&= f_F(Q/P \to P' \to P \dashrightarrow P_{m-1}) f_F(Q^*/P_{m-1} \to P_m) f_F(Q^m/P_m \dashrightarrow P_n) \\
&= f_F(Q/P \dashrightarrow P_{m-1}) D_c^{N\mu} \cdot D_c^{(N'-N)\mu} f_F[P_{m-1} \to P_m] f_F(Q^m/P_m \dashrightarrow P_n) \\
&= f_F(Q/P \dashrightarrow P_{m-1}) D_c^{N'\mu} f_F[P_{m-1} \to P_m] f_F(Q^m/P_m \dashrightarrow P_n) \\
&= f_F(Q/P \dashrightarrow P_{m-1}) f_F(Q^\flat/P_{m-1} \to P_m) f_F(Q^m/P_m \dashrightarrow P_n) \\
&= f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n).
\end{aligned}$$

This concludes the proof when the move (P, P') is an A-move. When the move (P, P') is an S-move, the proof goes almost exactly as above, except for the need to pay attention to the two circles c and δ. If these circles are unchanged under γ, then, $f_F(Q/P \to P' \to P \overset{\gamma}{\dashrightarrow} P_n)$ differs from $f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n)$ by the (right) factor $D_c^{N\mu} D_\delta^\mu$. If either of c or δ are changed under γ, then the corresponding factor disappears from the difference, and if both c and δ are changed under γ, then $f_F(Q/P \to P' \to P \overset{\gamma}{\dashrightarrow} P_n)$ coincides with $f_F(Q/P \overset{\gamma}{\dashrightarrow} P_n)$. $\square$

§8. Defining $\mathbb{\Gamma}$-actions on Dehn twists.

Using the procedure prepared in the previous section, we shall prove the following proposition, which allows us to specify images of individual Dehn twists under the eventual action of $\mathbb{\Gamma}$ to be defined later. As in §7, we fix a surface $\Sigma \cong \Sigma_{g,r}$ with $2 - 2g - r < 0$, and keep the notation on pants decompositions etc.

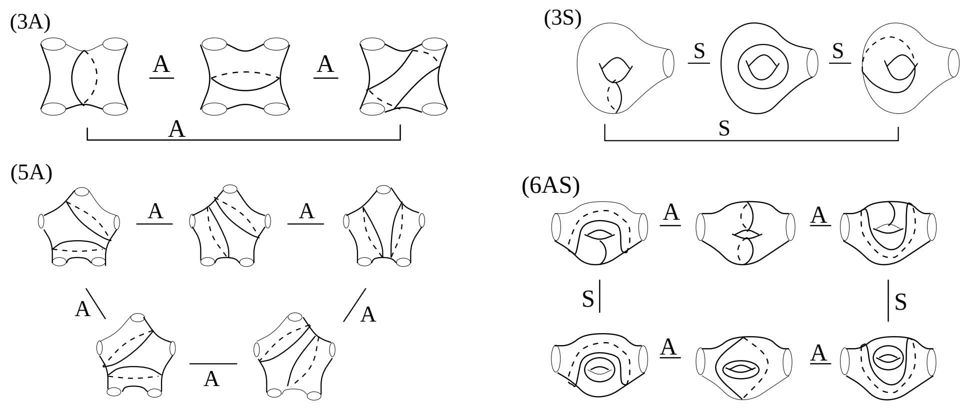
Proposition 8.1. *Let Q be a quilt on a pants decomposition P_0 of Σ, and let e be an arbitrary element of $\mathbb{S}(\Sigma)$. Pick a chain $\gamma = (P_0, \ldots, P_n)$ of A/S-moves of pants decompositions of Σ such that $e \in \mathcal{C}(P_n)$. Then, for each $F = (\lambda, f) \in \mathbb{\Gamma}$, the element*

$$F_{Q/P_0}(D_e) = f_F(Q/P_0 \to \cdots \to P_n)\, D_e^\lambda\, f_F(Q/P_0 \to \cdots \to P_n)^{-1}$$

is independent of the choice of γ.

The proof of Proposition 8.1 is based on two fundamental results. The first one, recalled from [HLS], claims the simple-connectedness of the simplicial complex whose vertices are the pants decompositions of Σ, whose edges are given by A/S-moves and whose faces are given by certain cycles given precisely in the theorem below.

Theorem 8.2. ([HLS]) *Any two chains of A/S-moves from a pants decomposition P to another P' can be deformed to each other via (a finite number of) successive replacements of subchains locally included in the diagrams (3A),(5A),(3S),(6AS) and commutativity squares (C) by their complementary chains in the same diagrams. Here, a commutativity square (C) means a rectangular cycle formed by two A/S-type replacements of circles supported on mutually disjoint subsurfaces.* □



The second necessary result is given in the following claim. Recall that $\mathcal{C}^*(P)$ denotes the union of the circles of a pants decomposition P and the circles parallel to boundary components (the latter circles are called *boundary circles.*)

Claim 8.3. *For cycles $(P_0, \cdots, P_n = P_0)$ of type (3A), (5A), (3S), (6AS) and (C) with a given quilt Q over P_0, there exist unique integers N_c ($c \in \mathcal{C}^*(P_0)$), such that*

$$Q = \Big(\prod_{c \in \mathcal{C}^*(P_0)} D_c^{N_c/2} \Big) (Q_{P_0 \to \cdots \to P_n}), \tag{8.3.1}$$

Furthermore, for every $F = (\lambda, f) \in \mathbb{I}\Gamma$ and for these N_c, we have

$$f_F(Q/P_0 \to \cdots \to P_n) \cdot \prod_{c \in \mathcal{C}^*(P_0)} D_c^{N_c \mu} = 1. \tag{8.3.2}$$

Proof. For commutativity squares (C), the statement simply holds essentially by virtue of Lemma 7.3. It is enough to consider each of the four cycles (3A), (5A), (3S), (6AS) as taking place on surfaces of the associated type (namely $(0,4)$ for (3A), $(1,1)$ for (3S), $(0,5)$ for (5A) and $(1,2)$ for (6AS), as in Theorem 8.2), because the cycles, performed on pants decompositions on larger topological surfaces Σ, leave everything outside of those subsurfaces fixed. So we only need to check the statements of the claim separately in each of the four cases, on surfaces of the corresponding type. Note furthermore the following simplification. If Q/P_0 is any quilt over P_0, then all other quilts over P_0 are obtained by (powers of) half-twists along the circles in $\mathcal{C}(P_0)$. If we apply such a set of half-twists to Q, obtaining a quilt Q', and then show (8.3.1) for Q', then applying the inverse of the product of half-twists to both sides of (8.3.1) gives (8.3.1) for Q with the same values of N_c. So it is enough to work with one chosen quilt over P_0.

(5A): We treat this case first, as it is particularly easy because no adjustment is needed. Name the five pants decompositions in (5A) consecutively as

$P = P_0, P_1, P_2, P_3, P_4$ and identify them with the figures in the (5A) part of the figure of Theorem 8.2, with P_0 in the upper left and the subsequent ones moving around the diagram clockwise. Let c_0 and c_2 denote the circles on P_0. The first A-move $P_0 \to P_1$ takes c_0 to c_1 so that $\mathcal{C}(P_1) = \{c_1, c_2\}$. The second move $P_1 \to P_2$ takes c_2 to c_3, so that $\mathcal{C}(P_2) = \{c_1, c_3\}$. The third move $P_2 \to P_3$ takes c_1 to c_4 so that $\mathcal{C}(P_3) = \{c_4, c_3\}$, and the fourth move $P_3 \to P_4$ takes c_3 to c_0, so $\mathcal{C}(P_4) = \{c_4, c_0\}$. Finally, $P_4 \to P_0$ takes c_4 to c_2.

The key point is that if we start with the quilt $Q/P_0 = Q_0/P_0$ whose seams are given by the "ridges", i.e. the edges of the figure, then this quilt is obviously adjusted to all five A-moves. Thus the successive quilts Q_i/P_i are all given by the seams of Q_0. In particular, we have

$$Q = Q_5 = Q_{P_0 \to \cdots \to P_0}.$$

This proves (8.3.1) with $N_{c_0} = N_{c_2} = 0$. Now, for (8.3.2) we have

$$f_F(Q/P_0 \overset{(5A)}{\dashrightarrow} P_0) = f(D_{c_0}, D_{c_1}) f(D_{c_2}, D_{c_3}) f(D_{c_1}, D_{c_4}) f(D_{c_3}, D_{c_0}) f(D_{c_4}, D_{c_2}).$$

The right-hand side belongs to the mapping class group $\Gamma_{0,5}$ of the sphere with five boundary components. This group maps surjectively to the mapping class group Γ_0^5 of the sphere with five punctures (by mapping the twists along boundary circles to 1), and the image of the right-hand side is 1 in Γ_0^5 by relation (III). Therefore, in $\Gamma_{0,5}$, we have

$$f_F(Q/P_0 \to \cdots \to P_0) = \prod_{i=1}^{5} D_{\epsilon_i}^{a_i}$$

for some integers a_i. But as usual, f_F is in the derived subgroup, and the D_{ϵ_i} form a free abelian subgroup of the abelianization of $\Gamma_{0,5}$, so we find $a_i = 0$ for $i = 1, 2, 3, 4, 5$ and

$$f_F(Q/P_0 \to \cdots \to P_0) = 1.$$

This proves (8.3.2) with $N_{c_0} = N_{c_2} = 0$.

(3A): We identify the 4-holed sphere with a square minus 3 holes as in Figure 8.1 below, and start from the choice of quilt $Q = Q_0/P_0$ drawn in the upper left part of Figure 8.1. Let us draw the quilts coming from the successive moves in the direction suggested by the arrows; seams of quilts are drawn by dotted lines. Then, one observes that: Q is adjusted to b, $Q_1 = Q_{P_0 \to P_1}$, $Q_2 = D_b^{-1/2}(Q_1)$ which is adjusted to c, $Q_3 = Q_{P_0 \to P_1 \to P_2}$, $Q_4 = D_c^{-1/2}(Q_3)$ which is adjusted to a and $Q_5 = Q_{P_0 \to P_1 \to P_2 \to P_0}$. By construction, the quilt Q_5 is exactly the quilt $Q_{P_0 \to P_1 \to P_2 \to P_0}$. Directly from Figure 8.1 showing (on the top left) the original quilt $Q = Q_0/P_0$ and (on the top right) Q_5/P_0, we see that

$$Q = D_a^{-1/2} (D_{\epsilon_1} D_{\epsilon_2} D_{\epsilon_3} D_{\epsilon_4})^{1/2} (Q_5)$$

where $\epsilon_1, \ldots, \epsilon_4$ denote the boundary circles, so this proves (8.3.1) with $N_a = -1$, $N_{\epsilon_i} = 1$ for $i = 1, 2, 3, 4$. By the definition of f_F, we find that

(8.3.3) $\quad f_F(Q/P_0 \to P_1 \to P_2 \to P_0) = f(D_b, D_a) D_b^{-\mu} f(D_c, D_b) D_c^{-\mu} f(D_a, D_c).$

Now, equation (1.5.1) from Lemma 1.5 says that

$$f(x,y)x^{\mu}f(z,x)z^{\mu}f(y,z)y^{\mu} = \omega^{\mu} \tag{8.3.4}$$

whenever x, y, z are elements of a group such that $\omega = xyz$ commutes with x, y and z. By the lantern equation in $\Gamma_{0,4}$, we know that

$$D_c D_a D_b = \prod_{i=1}^{4} D_{\epsilon_i},$$

and the right-hand side is central in $\Gamma_{0,4}$. Thus we can apply (8.3.4) with $x = D_c$, $y = D_a$, $z = D_b$, to obtain

$$f(D_c, D_a)D_c^{\mu} f(D_b, D_c)D_b^{\mu} f(D_a, D_b)D_a^{\mu} = \prod_{i=1}^{4} D_{\epsilon_i}^{\mu}.$$

Inverting this and substituting it into (8.3.3) give

$$f_F(Q/P_0 \to P_1 \to P_2 \to P_0) = D_a^{\mu} \prod_{i=1}^{4} D_{\epsilon_i}^{-\mu},$$

which is exactly (8.3.2) for $N_a = -1$, $N_{\epsilon_i} = 1$, $i = 1, 2, 3, 4$.

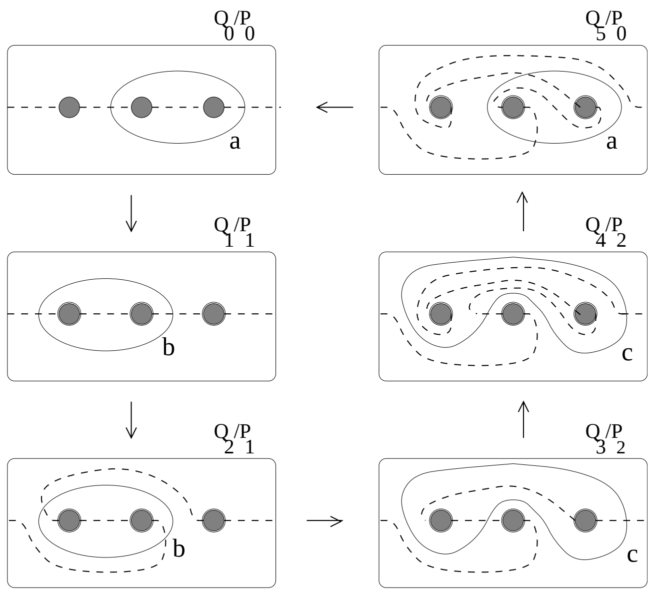


Figure 8.1

(3S): In this case, we consider the three-cycle of pants decompositions shown in Figure 8.2, starting with the quilt Q/P_0 indicated in the left most picture of Figure 8.3 (in which only the seams of the quilts are indicated).

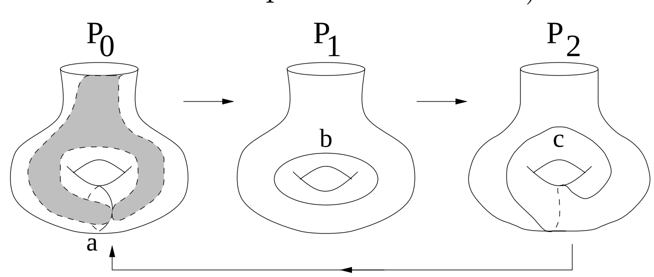


Figure 8.2

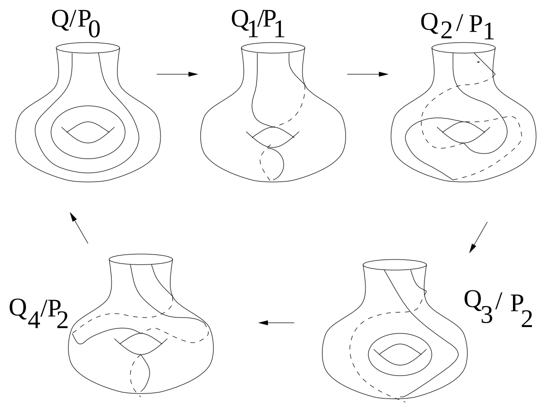


Figure 8.3

Let us trace what happens along $P_0 \to P_1 \to P_2 \to P_0$, following Figure 8.3. First Q/P_0 is adjusted to the circle b, hence $Q_1 = Q_{P_0 \to P_1}$ is defined by $Q_1 = (D_b D_a D_b)(Q)$ whose reference curve is a. Then, we take $Q_2 = D_b(Q_1)/P_1$ whose reference curve is then $c = D_b(a)$, i.e., Q_2 is adjusted to c. So $Q_3 = Q_{P_0 \to P_1 \to P_2}$ is defined by $Q_3 = (D_b D_c D_b)(Q_2)$ whose reference curve is b. Next, we take $Q_4 = D_c(Q_3)$ over P_2 whose reference curve is $D_c(b) = D_b D_a D_b^{-1}(b) = a$. This means Q_4/P_2 is adjusted to P_0, hence $Q_5 = Q_{P_0 \to P_1 \to P_2 \to P_0}$ is defined by $Q_5 = D_a D_c D_a(Q_4)$. Noticing that, if ϵ denotes the boundary curve of the one-holed torus, then $D_\epsilon = (D_a D_b D_a)^4$ holds, we compute $Q = D_\epsilon^{-1} D_a(Q_5)$. Thus, (8.3.1) holds for integers $N_\epsilon = -2$, $N_a = 2$. Moreover, by using relation (II) via (1.5.1) (exactly as in the case of (3A) above), we compute:

$$\begin{aligned} f_F(Q/P_0 \overset{3S}{\dashrightarrow} P_0) =& D_a^{-8\rho_2} f(D_b^2, D_a^2)(D_a D_b D_a)^{2\mu} D_b^{2\mu} f(D_c^2, D_b^2) \\ & \cdot (D_b D_c D_b)^{2\mu} D_c^{2\mu} f(D_a^2, D_c^2) D_a^{8\rho_2} (D_c D_a D_c)^{2\mu} \\ =& (D_a D_b D_a)^{6\mu} D_a^{-2\mu} (D_a D_b D_a)^{2\mu}. \end{aligned}$$

Note here that $(D_a D_b D_a)^2$, $(D_b D_c D_b)^2$ and $(D_c D_a D_c)^2$ are all equal and generate the center of $\hat{\Gamma}_{1,1}$. Thus, we obtain $f_F(Q/P_0 \to \ldots \to P_0) D_\epsilon^{-2\mu} D_a^{2\mu} = 1$ as desired in (8.3.2).

(6AS): The A/S-moves on pants decompositions are given in Figure 8.4. We start from the quilt shown over P_0, and move around the diagram clockwise.

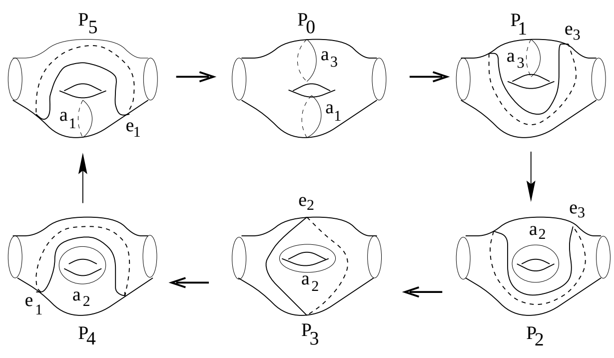


Figure 8.4

We start with the quilt decomposition Q/P_0 of Figure 8.5, in which only the seams of the quilts are shown (for simpler visualization). The corresponding pants decompositions are shown in the labels of the figures; they correspond to the pants decompositions shown in Figure 8.4.

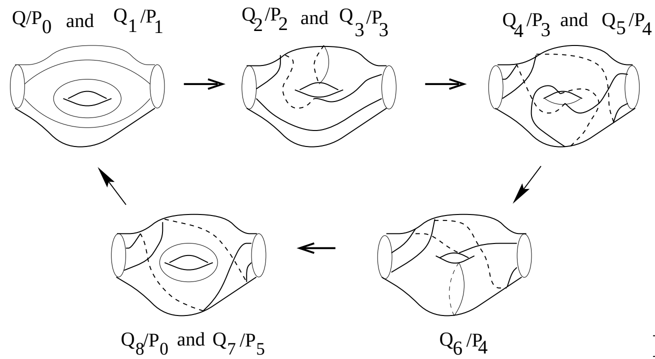


Figure 8.5

Let us take a close look at the effect of the successive moves of (6AS) on quilts and prowords in a step-by-step manner. The reader should move around Figures 8.4 and 8.5 while following the steps, also consulting Figures 8.6-8.8 which are alternative figures illustrating the identical step-by-step procedure.

First, the quilt Q is adjusted to the circle e_3, so $f_F(Q/P_0 \to P_1) = f(D_{e_3}, D_{a_1})$. One finds then that the quilt $Q_1 = Q_{P_0 \to P_1}$ is adjusted to a_2. Therefore,

$$f_F(Q/P_0 \to P_1 \to P_2) = f(D_{e_3}, D_{a_1}) D_{a_3}^{-8\rho_2} f(D_{a_2}^2, D_{a_3}^2) D_{a_2}^{8\rho_2} (D_{a_3} D_{a_2} D_{a_3})^{2\mu}.$$

The resultant quilt $Q_2 = (Q_1)_{P_1 \to P_2}$ is adjusted to e_2; hence we obtain

$$f_F(Q/P_0 \dashrightarrow P_3) = f_F(Q/P_0 \dashrightarrow P_2) f(D_{e_2}, D_{e_3}).$$

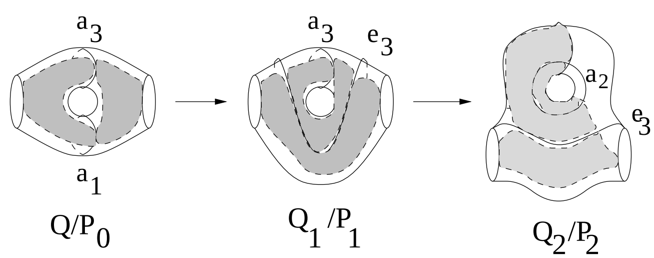


Figure 8.6

The quilt $Q_3 = (Q_2)_{P_2 \to P_3}$ is not easy to draw precisely, and in Figure 8.7, each of the RHS's shows two pairs of pants of P_3 where seams are suggested. Deformation of the quilt on each pairs of pants by $D_{e_2}^{1/2}$ yields in total $Q_4 = D_{e_2}(Q_3)$ over P_3 as described in the lower line of Figure 8.7. Then, Q_4 is adjusted to the curve $e_1 \in \mathcal{C}(P_4)$ (cf. also Figure 8.8).

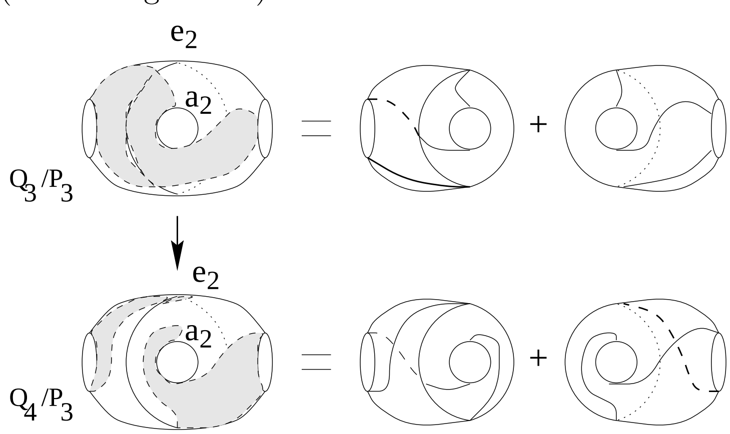


Figure 8.7

Thus,

$$f_F(Q/P_0 \dashrightarrow P_4) = f_F(Q/P_0 \dashrightarrow P_3) D_{e_2}^{2\mu} f(D_{e_1}, D_{e_2}).$$

In order to move to P_5, we have to deform $Q_5 = Q_{P_0 \dashrightarrow P_4}$ to $Q_6 = D_{a_2}^{-1}(Q_5)$ to be adjusted to a_1. Then,

$$f_F(Q/P_0 \dashrightarrow P_5) = f_F(Q/P_0 \dashrightarrow P_4) D_{a_2}^{-2\mu-8\rho_2} f(D_{a_1}^2, D_{a_2}^2) D_{a_1}^{8\rho_2} (D_{a_1} D_{a_2} D_{a_1})^{2\mu}.$$

It turns out that $Q_7 = Q_{P_0 \dashrightarrow P_5}$ is adjusted to a_3 as it is, so that it follows that

$$f_F(Q/P_0 \dashrightarrow P_0) = f_F(Q/P_0 \dashrightarrow P_5) f(D_{a_3}, D_{e_1}).$$

Let $Q_8 = Q_{P_0 \dashrightarrow P_0}$. From the last picture of Figure 8.8, we find $Q = D_{\epsilon_1}^{-1/2} D_{\epsilon_2}^{-1/2}(Q_8)$ where ϵ_i $(i = 1, 2)$ denote the boundary circle of the $\Sigma_{1,2}$. In particular, the claim (8.3.1) holds.

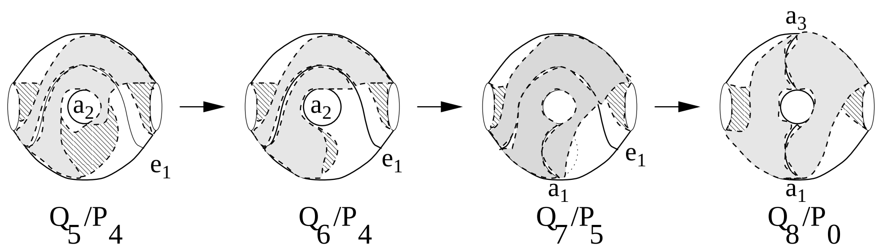


Figure 8.8

To show (8.3.2), we shall prove that

$$f_F(Q/P_0 \to P_1 \to P_2 \to P_3 \to P_4 \to P_5 \to P_0) = D_{\epsilon_1}^{\mu} D_{\epsilon_2}^{\mu}.$$

We use the above detail of the moves of (6AS) on quilts to compute

$$\begin{aligned} f_F(Q/P_0 \to \cdots \to P_0) &= f_F(Q/P_0 \to P_1) f_F(Q_1/P_1 \to P_2) f_F(Q_2/P_2 \to P_3) \cdot \\ &\quad \cdot f_F(Q_3/P_3 \to P_4) f_F(Q_5/P_4 \to P_5) f_F(Q_7/P_5 \to P_0) \\ &= f(D_{e_3}, D_{a_1}) D_{a_3}^{-8\rho_2} f(D_{a_2}^2, D_{a_3}^2) D_{a_2}^{8\rho_2} (D_{a_3} D_{a_2} D_{a_3})^{2\mu} \cdot \\ &\quad \cdot f(D_{e_2}, D_{e_3}) D_{e_2}^{2\mu} f(D_{e_1}, D_{e_2}) D_{a_2}^{-2\mu-8\rho_2} f(D_{a_1}^2, D_{a_2}^2) \cdot \\ &\quad \cdot D_{a_1}^{8\rho_2} (D_{a_1} D_{a_2} D_{a_1})^{2\mu} \cdot f(D_{a_3}, D_{e_1}). \end{aligned}$$

After cancellation of the terms $D_{a_2}^{8\rho_2}$ and $D_{a_2}^{-8\rho_2}$, which commute with the terms lying between them, the relation to be proved becomes

$$(\mathrm{R}') \qquad \begin{aligned} &f(D_{e_3}, D_{a_1}) D_{a_3}^{-8\rho_2} f(D_{a_2}^2, D_{a_3}^2) (D_{a_3} D_{a_2} D_{a_3})^{2\mu} \\ &f(D_{e_2}, D_{e_3}) D_{e_2}^{2\mu} f(D_{e_1}, D_{e_2}) D_{a_2}^{-2\mu} f(D_{a_1}^2, D_{a_2}^2) D_{a_1}^{8\rho_2} \\ &(D_{a_1} D_{a_2} D_{a_1})^{2\mu} f(D_{a_3}, D_{e_1}) D_{\epsilon_1}^{-\mu} D_{\epsilon_2}^{-\mu} = 1. \end{aligned}$$

This is indeed a consequence of our defining relations for $\mathbb{F}$. First we prepare some notation. Recalling that $D_{a_1}, D_{a_2}, D_{a_3}$ satisfy braid relations, introduce: $x_{12} = D_{a_1}^2$, $x_{23} = D_{a_2}^2$, $x_{34} = D_{a_3}^2$, $x_{ij} = D_{a_{j-1}} \cdots D_{a_{i+1}} D_{a_i}^2 D_{a_{i+1}}^{-1} \cdots D_{a_{j-1}}^{-1}$ $(1 \le i < j \le 4)$ and $\mathbf{x}_{45} = (D_{a_1} D_{a_2} D_{a_1})^2$, $\mathbf{x}_{51} = (D_{a_3} D_{a_2} D_{a_3})^2$. Then, the doughnut relation (given explicitly in Theorem 9.2 below) shows $D_{e_1} = \mathbf{x}_{45}^2$, $D_{e_3} = \mathbf{x}_{51}^2$. On the other hand, a simple chase of twisting shows $D_{e_2} = D_{a_3} D_{a_2} D_{a_1} D_{a_2}^{-1} D_{a_3}^{-1}$ so that $D_{e_2}^2 = x_{14}$.

Now, we shall rewrite $f(D_{e_3}, D_{a_1})$, $f(D_{e_2}, D_{e_3})$, $f(D_{e_1}, D_{e_2})$, $f(D_{a_3}, D_{e_1})$ by using the relation (IV). Let A_3 be the subgroup of $B_3 = \langle \tau_1, \tau_2 \rangle$ generated by $\{\tau_1, \tau_2^2\}$

with a single relation $[\tau_2^2, \tau_1\tau_2^2\tau_1] = 1$ (cf. §2). One can construct homomorphisms of A_3 into $\Gamma_{1,2}$ by letting the images of $\{\tau_1, \tau_2^2\}$ be $\{D_{a_1}, \mathbf{x}_{51}\}$, $\{D_{e_2}, \mathbf{x}_{51}\}$, $\{D_{e_2}, \mathbf{x}_{45}\}$ $\{D_{a_3}, \mathbf{x}_{45}\}$ respectively. Then, the relation (IV) reads in $\hat{\Gamma}_{1,2}$:

$$\begin{aligned} f(D_{e_3}, D_{a_1}) &= x_{34}^{2\rho_2} x_{12}^{-2\rho_2} f(\mathbf{x}_{51}, x_{12}) \mathbf{x}_{51}^{-4\rho_2}, \\ f(D_{e_2}, D_{e_3}) &= \mathbf{x}_{51}^{4\rho_2} f(x_{14}, \mathbf{x}_{51}) x_{14}^{2\rho_2} x_{23}^{-2\rho_2}, \\ f(D_{e_1}, D_{e_2}) &= x_{23}^{2\rho_2} x_{14}^{-2\rho_2} f(\mathbf{x}_{45}, x_{14}) \mathbf{x}_{45}^{-4\rho_2}, \\ f(D_{a_3}, D_{e_1}) &= \mathbf{x}_{45}^{4\rho_2} f(x_{34}, \mathbf{x}_{45}) x_{34}^{2\rho_2} x_{12}^{-2\rho_2}. \end{aligned}$$

After the above formulae input into LHS of (R′), in middle part of the result we find the following consecutive terms appearing:

$$\mathbf{x}_{51}^{\mu} f(x_{14}, \mathbf{x}_{51}) x_{14}^{\mu} f(\mathbf{x}_{45}, x_{14}) \mathbf{x}_{45}^{\mu}.$$

This part turns out to be $f(\mathbf{x}_{45}, \mathbf{x}_{51}) x_{23}^{\mu} (D_{\epsilon_1} D_{\epsilon_2})^{\mu}$ by relation (II), because of (1.5.1) and $x_{14}\mathbf{x}_{51}\mathbf{x}_{45} = (D_{a_3} D_{a_2} D_{a_1})^4 D_{a_2}^2 = D_{\epsilon_1} D_{\epsilon_2} D_{a_2}^2$. Taking this together with $x_{23}x_{24} = \mathbf{x}_{51} x_{34}^{-1}$, $x_{13}x_{23} = \mathbf{x}_{45} x_{12}^{-1}$, $x_{12}x_{13} = \mathbf{x}_{45} x_{23}^{-1}$, $x_{24}x_{34} = \mathbf{x}_{51} x_{23}^{-1}$ into account, we finally see that the LHS of (R′) equals to

$$f(x_{23}x_{24}, x_{12}) f(x_{23}, x_{34}) f(x_{12}x_{13}, x_{24}x_{34}) f(x_{12}, x_{23}) f(x_{34}, x_{13}x_{23}),$$

which is trivial by Drinfeld's form of (III) in $\hat{B}_4$ (cf. §1). This completes the proof of Claim 8.3. □

8.4. *Proof of Proposition 8.1.* The proof of Proposition 8.1 is based on repeated applications of Claim 8.3 and the Back-Tracking Lemma 7.3. Recall that in the notation of the statement of the proposition, we are given a quilt decomposition Q/P_0 and an arbitrary circle e on a surface Σ, and we choose a pants decomposition P_n containing e and a chain of A/S-moves $\gamma = (P_0, \ldots, P_n)$ taking P_0 to P_n. To prove Proposition 8.1, we must show that up to multiplication on the right by an element of $\hat{\Gamma}_{g,n}^m$ commuting with D_e, the quantity

$$f_F(Q/P_0 \to \cdots \to P_n) \tag{8.4.1}$$

is independent of the choices of P_n and of γ.

In steps 1 to 3 below, we suppose the choice of P_n fixed and show that the quantity (8.4.1) is (essentially) independent of the choice of the sequence of A/S-moves γ; in step 4 we show that it is (essentially) independent also of the choice of P_n. The goal of the first three steps is to compute the effect on (8.4.1) when a cycle of type (3A), (3S), (5A), (6AS) or (C)

$$P_i \to R_1 \to \cdots \to R_m \to P_i$$

is inserted at a pants decomposition P_i of a fixed chain $\gamma : P_0 \to \ldots \to P_n$. We assume, for $j = 0, \ldots, n-1$, the move $P_j \to P_{j+1}$ replace the circle $a_j \in \mathcal{C}(P_j)$ by $a_j' \in \mathcal{C}(P_{j+1})$.

Note that, since we know that any chain of moves from P_0 to P_n can be obtained from γ by successive replacements of parts through such 4 types of cycles, by the

Back-Tracking Lemma 7.3, we are reduced to the above situation. By the same Lemma 7.3, after supplementing the chain $P_n \to P_{n-1} \to P_n$ to γ if necessary, we may assume $i < n$ without loss of generality.

Let us set up the notation used in these first three steps. Let $C_0(P_i)$ be the set of circles in P_i concerned by the above cycle $P_i \to R_1 \dashrightarrow R_m \to P_i$ including the circles bounding the subsurface on which the cycle lives, and let $C_0'(P_i)$ be equal to $C_0(P_i)$ if $a_i \notin C_0(P_i)$ and to $C_0(P_i) \setminus \{a_i\}$ otherwise.

Step 1. Under the assumption that $P_i \to P_{i+1}$ is a move in the sequence γ such that the quilt $Q^i := Q_{P_0 \to \cdots \to P_i}$ is adjusted to P_{i+1}, there exist integers N_c ($c \in C_0'(P_i)$) such that

$$
\begin{aligned}
&f_F(Q/P_0 \to \cdots \to P_i \to P_{i+1}) \prod_{c \in C_0'(P_i)} D_c^{N_c \mu} \\
&= f_F(Q/P_0 \to \cdots \to P_i \to R_1 \to \cdots \to R_m \to P_i \to P_{i+1})
\end{aligned} \tag{8.4.2}
$$

for all $F = (\lambda, f) \in \mathbb{I}\Gamma$ with $\mu = (\lambda - 1)/2$.

Proof of Step 1. Let $Q^j = Q_{P_0 \to \cdots \to P_j}$ for $j = 1, \ldots, n$. By assumption, Q^i is already adjusted to P_{i+1}, so we have

$$
\begin{aligned}
&f_F(Q/P_0 \to \cdots \to P_i \to P_{i+1}) \\
&= f_F(Q/P_0 \to \cdots \to P_i) f_F\big(Q^i/P_i \to P_{i+1}\big) \\
&= f_F(Q/P_0 \to \cdots \to P_i) f_F[P_i \to P_{i+1}].
\end{aligned} \tag{8.4.3}
$$

Now let us compute the right-hand side of (8.4.2). If we set Q' to be the quilt $Q_{P_0 \dashrightarrow P_i \to R_1 \dashrightarrow R_m \to P_i}$, then by Claim 8.3, we have

$$
Q' = (Q^i)_{P_i \to R_1 \to \cdots \to R_m \to P_i} = \prod_{c \in C_0(P_i)} D_c^{N_c/2}(Q^i)
$$

for some integers N_c ($c \in C_0(P_i)$). Therefore, we get

$$
\begin{aligned}
&f_F(Q/P_0 \to \cdots \to P_i \to R_1 \to \cdots \to R_m \to P_i \to P_{i+1}) \\
&= f_F\big(Q/P_0 \dashrightarrow P_i\big) f_F\big(Q^i/P_i \to R_1 \dashrightarrow R_m \to P_i\big) f_F\big(Q'/P_i \to P_{i+1}\big) \\
&= f_F\big(Q/P_0 \to \cdots \to P_i\big) \prod_{c \in C_0(P_i)} D_c^{N_c \mu}\, f_F(Q'/P_i \to P_{i+1}) \\
&= f_F\big(Q/P_0 \to \cdots \to P_i\big) \prod_{c \in C_0(P_i)} D_c^{N_c \mu} \cdot D_{a_i}^{M_i \mu} f_F[P_i \to P_{i+1}],
\end{aligned} \tag{8.4.4}
$$

where $D_{a_i}^{M_i/2}$ is the twist necessary on Q' to adjust it to P_{i+1}.

Now, if the circle a_i does not lie in $C_0(P_i)$, then the quilt Q' is adjusted to P_{i+1}, so $M_i = 0$. Furthermore $D_c^{N_c \mu}$ commutes with both $D_{a_i'}$ and D_{a_i} for each $c \in C_0(P_i)$. Therefore (8.4.4) can be written as

$$
f_F\big(Q/P_0 \to \cdots \to P_i\big) f_F[P_i \to P_{i+1}] \prod_{c \in C_0(P_i)} D_c^{N_c \mu},
$$

which is equal to the left hand side of (8.4.2) by virtue of (8.4.3). If $a_i \in C_0(P_i)$, then since Q^i was adjusted to P_{i+1}, we must have $M_i = -N_{a_i}$ in order to readjust Q' back to P_{i+1}. Thus (8.4.4) can be written as

$$\begin{aligned} & f_F\big(Q/P_0 \to \cdots \to P_i\big) \prod_{\substack{c \in C_0(P_i) \\ c \neq a_i}} D_c^{N_c \mu} f_F[P_i \to P_{i+1}] \\ & = f_F\big(Q/P_0 \to \cdots \to P_i\big) f_F[P_i \to P_{i+1}] \prod_{c \in C_0'(P_i)} D_c^{N_c \mu} \end{aligned}$$

since the D_c along c in $C_0'(P_i)$ commute with D_a. By (8.4.3) we again see that this is equal to the left hand side of (8.4.2). This completes the proof of step 1.

Step 2. As in step 1, suppose that the quilt Q^i is adjusted to P_{i+1}. Then there exists an element $B \in \hat{\Gamma}(\Sigma)$ commuting with D_e such that

$$\begin{aligned} & f_F(Q/P_0 \to \cdots \to P_i \to P_{i+1} \to \cdots \to P_n)B \\ & = f_F(Q/P_0 \dashrightarrow P_i \to R_1 \dashrightarrow R_m \to P_i \to P_{i+1} \dashrightarrow P_n). \end{aligned} \tag{8.4.5}$$

Proof of Step 2. Let Q' be as in the proof of step 1, and let $\widetilde{Q} = (Q')_{P_i \to P_{i+1}}$, $\widetilde{Q}^k = \widetilde{Q}_{P_{i+1} \dashrightarrow P_k}$ $(i+1 < k \leq n)$. By definition, $f_F(Q/P_0 \dashrightarrow P_i \to P_{i+1} \dashrightarrow P_n)$ is equal to

$$f_F(Q/P_0 \to \cdots \to P_{i+1}) f_F(Q^{i+1}/P_{i+1} \to \cdots \to P_n),$$

while, by step 1, the right hand side of (8.4.5) is equal to

$$f_F(Q/P_0 \to \cdots \to P_{i+1}) \prod_{c \in C_0'(P_i)} D_c^{N_c \mu} \cdot f_F(\widetilde{Q}/P_{i+1} \to \cdots \to P_n) \tag{8.4.6}$$

for some integers N_c $(c \in C_0'(P_i))$. Now, we observe:

$$\begin{aligned} & f_F(Q^{i+1}/P_{i+1} \to \cdots \to P_n) \\ & = f_F(Q^{i+1}/P_{i+1} \to P_{i+2}) \cdots f_F(Q^{n-1}/P_{n-1} \to P_n) \\ & = D_{a_{i+1}}^{M_{i+1}\mu} f_F[P_{i+1} \to P_{i+2}] \cdots D_{a_{n-1}}^{M_{n-1}\mu} f_F[P_{n-1} \to P_n] \end{aligned} \tag{8.4.7}$$

and

$$\begin{aligned} & f_F(\widetilde{Q}/P_{i+1} \to \cdots \to P_n) \\ & = f_F(\widetilde{Q}/P_{i+1} \to P_{i+2}) \cdots f_F(\widetilde{Q}^{n-1}/P_{n-1} \to P_n) \\ & = D_{a_{i+1}}^{M'_{i+1}\mu} f_F[P_{i+1} \to P_{i+2}] \cdots D_{a_{n-1}}^{M'_{n-1}\mu} f_F[P_{n-1} \to P_n], \end{aligned} \tag{8.4.8}$$

where M_j (resp. M_j') is an integer so that $D_{a_j}^{M_j}(Q^j)$ (resp. $D_{a_j}^{M_j'}(\widetilde{Q}^j)$) is adjusted to P_{j+1} $(i+1 \leq j \leq n-1)$. Since $\widetilde{Q} = (Q')_{P_i \to P_{i+1}}$ and $Q' = \prod_{c \in C_0(P_i)} D_c^{N_c/2}(Q^{i+1})$, we have

$$\begin{cases} M_j' = M_j, & \text{if } a_j \notin C_0'(P_i),\ j \in \{i+1, \ldots, n-1\}, \\ M_j' = N_{a_j} + M_j, & \text{if } a_j \in C_0'(P_i),\ j \in \{i+1, \ldots, n-1\}. \end{cases} \tag{8.4.9}$$

Then, letting C_0'' denote the subset of circles of $C_0'(P_i)$ which are not equal to any a_j for $i+1 \le j \le n-1$ and using (8.4.7-9), we may rewrite the right-hand portion of (8.4.6) as

$$\begin{aligned}
&\prod_{c \in C_0'(P_i)} D_c^{N_c \mu} f(\widetilde{Q}/P_{i+1} \to \cdots \to P_n) \\
(8.4.10) \qquad &= D_{a_{i+1}}^{M_{i+1}\mu} f_F[P_{i+1} \to P_{i+2}] \cdots D_{a_{n-1}}^{M_{n-1}\mu} f_F[P_{n-1} \to P_n] \prod_{c \in C_0''} D_c^{N_c \mu} \\
&= f(Q^{i+1}/P_{i+1} \to \cdots \to P_n) \prod_{c \in C_0''} D_c^{N_c \mu}.
\end{aligned}$$

Thus, if B is set to be $\prod_{c \in C_0''} D_c^{N_c \mu}$, then the proword (8.4.6), i.e., the right-hand side of (8.4.5), can be written as

$$f_F(Q/P_0 \dashrightarrow P_{i+1}) f_F(Q^{i+1}/P_{i+1} \dashrightarrow P_n)B = f_F(Q/P_0 \dashrightarrow P_n)B.$$

This shows the equality (8.4.5). It remains only to show that B commutes with D_e. Let j_0 be the largest index such that $e = a'_{j_0}$. If $j_0 \ge i+1$ then we saw that the twists along circles C_0'' commute with D_e. If $j_0 \le i$, then e lies in P_i and so do the circles of C_0'', so again the corresponding twists commute. This concludes the proof of step 2.

Step 3. In this step we show that the statement of step 2 remains true even when Q^i/P_i is not necessarily adjusted to P_{i+1}.

Proof of Step 3. By the Back-Tracking Lemma 7.3, we have

$$\begin{aligned}
&f_F(Q/P_0 \dashrightarrow P_i \to R_1 \dashrightarrow R_m \to P_i \to P_{i+1} \dashrightarrow P_n) \\
&\quad = f_F(Q/P_0 \dashrightarrow P_i \to P_{i+1} \to P_i \to R_1 \to \cdots \to R_m \to P_i \to P_{i+1} \dashrightarrow P_n),
\end{aligned}$$

where the quilt $Q_{P_0 \dashrightarrow P_i \to P_{i+1} \to P_i}$ is adjusted to P_{i+1}. Then by (8.4.5) in step 2, this is equal to

$$f_F(Q/P_0 \to \cdots \to P_i \to P_{i+1} \to P_i \to P_{i+1} \to \cdots \to P_n)B$$

for some element B commuting with D_e. Finally by Lemma 7.3 again, this is equal to $f_F(Q/P_0 \to \cdots \to P_n)B$ as desired.

The three preceding steps show that if we fix a choice of pants decomposition P_n containing e and a chain of A/S-moves $\gamma = (P_0, \ldots, P_n)$, then the quantity $f(Q/P_0 \to \cdots \to P_n)$ is independent of the choice of γ up to multiplication on the right by a factor B commuting with D_e. In the following step, we show that it is also independent of the choice of P_n containing e.

Step 4. Suppose that P_n and P'_m are two pants decompositions containing e, and that $\gamma = (P_0, \ldots, P_n)$ and $\gamma' = (P_0 = P'_0, P'_1, \ldots, P'_m)$ are chains of A/S-moves. Then there is a chain δ of A/S-moves taking P'_m to P_n such that none of the moves is on the circle e. So the chain δ' given by composing γ' with δ is a chain from P_0 to P_n. Thus by step 3, for some B commuting with D_e, we have

$$\begin{aligned}
&f_F(Q/P_0 \to \cdots \to P_n)B = f_F(Q/P_0 \to \cdots \to P'_m \to \cdots \to P_n) \\
&= f_F(Q/P_0 \to \cdots \to P'_m) f_F(Q_{P_0 \to \cdots \to P'_m}/P'_m \to \cdots \to P_n).
\end{aligned}$$

But the factor $f_F(Q_{P_0\to\cdots\to P'_m}/P'_m \to \cdots \to P_n)$ also commutes with D_e, since each A/S-move in the sequence $\delta : P'_m \to P_n$ takes a circle disjoint from e to another circle disjoint from e, so that all the corresponding twists commute. Thus, $f_F(P_0 \to \cdots P_n)$ differs $f_F(P_0 = P'_0 \to \cdots \to P'_m)$ only by multiplication on the right by an element commuting with D_e. This concludes the proof of step 4 and thus of Proposition 8.1. $\square$

§9. $\mathbb{I}\Gamma$-actions on $\hat{\Gamma}_{g,r}$.

We keep the notation of §§7,8. Recall that given a quilt decomposition Q/P_0 over Σ, we introduced in Proposition 8.1 a well-defined action of $\mathbb{I}\Gamma$ on the Dehn twists D_c ($c \in \mathbb{S}(\Sigma)$). Using the fact that Dehn twists D_c ($c \in \mathbb{S}^*$) form a set of generators of the mapping class group $\Gamma_{g,r}$, we shall now prove that this action extends to a group automorphism of $\hat{\Gamma}_{g,r}$.

Proposition 9.1. *For $F = (\lambda, f) \in \mathbb{I}\Gamma$ and for a quilt Q over a pants decomposition P_0 on Σ, define the action F_{Q/P_0} on the Dehn twists D_c by the formula of Proposition 8.1 for $c \in \mathbb{S}(\Sigma)$ and by $F_{Q/P_0}(D_c) = D_c^\lambda$ for $c \in \mathbb{S}^*(\Sigma) \setminus \mathbb{S}(\Sigma)$. Then, F_{Q/P_0} extends to an automorphism of $\hat{\Gamma}_{g,r}$.*

Since we know (by §8) the action of (λ, f) on all Dehn twists, it suffices to give a set of relations between those twists forming a presentation of $\Gamma_{g,r}$, and show that the action of F_{Q/P_0} associated to the element (λ, f) and the quilt Q/P_0 respects these relations. The presentation is given in the theorem below, due to S. Gervais, with an improvement by Feng Luo.

Theorem 9.2. (Gervais [Ge], Feng Luo [FL]) *The mapping class group $\Gamma_{g,r}$ has a presentation by the (infinitely many) generators D_c ($c \in \mathbb{S}^*(\Sigma)$) subject to the relations of the following four types:*

(C) $D_aD_b = D_aD_b$ if $i(a, b) = 1$.

(B) $D_c = D_aD_bD_a^{-1}$ if $i(a, b) = 1$ and $c = D_a(b)$.

(L) $D_{b_1}D_{b_2}D_{b_3} = D_{a_1}D_{a_2}D_{a_3}D_{a_4}$ for circles b_i, a_j ($i = 1, 2, 3,\ j = 1, 2, 3, 4$) located as in Figure 9.1.

(D) $(D_aD_bD_a)^4 = D_d$ for circles a, b, d located as in Figure 9.2. $\square$

The relations (C), (B), (L), (D) above are called the *commutativity relations*, *braid relations*, *lantern relations* and *doughnut relations* respectively.

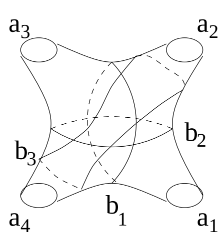


Figure 9.1

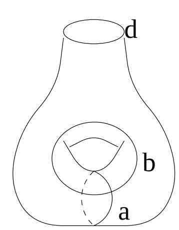


Figure 9.2

Proof of Proposition 9.1. Let $P = P_0$. We shall prove first that $F_{Q/P}$ preserves the relations (C),(B),(L),(D) respectively, and then at the end show that this suffices to ensure that $F_{Q/P}$ extends to an automorphism of the profinite completion $\hat{\Gamma}_{g,r}$ of $\Gamma_{g,r}$ even though (C), (B), (L), (D) give a presentation of the discrete group. The argument is a variation of [HLS] §4 Step 2 refined for quilts.

(C): This is almost clear. One can take a pants decomposition P' with $\mathcal{C}(P') \ni a, b$ together with a chain $(P_0, \ldots, P_n = P')$ of A/S-moves. Then, after simple observation of our definition of F_{Q/P_0}, the commutativity of $F_{Q/P_0}(D_a)$ and $F_{Q/P_0}(D_b)$ follows from that of D_a and D_b.

(B): Pick a chain $(P_0, \ldots, P_n = P')$ of A/S-moves such that $a \in \mathcal{C}(P')$ and no circle of $\mathcal{C}(P')$ except for a intersects b. Then, $F_{Q/P_0}(D_a) = \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P')\big)(D_a^\lambda)$. Let $p \in \Pi(P')$ be the pair of pants such that $a, b \subset \bar{p} \cong \Sigma_{1,1}$ and let r be the reference curve on $\bar{p}$ of the quilt $Q' = Q_{P_0 \dashrightarrow P'}$. Then, there exists a unique integer N such that $b = D_a^N(r)$, and we have

$$F_{Q/P_0}(D_b) = \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_a^{(2N+\varepsilon)\mu - 8\rho_2} f(D_b^2, D_a^2)\big)(D_b^\lambda),$$

where $\varepsilon = 0, 1$ according to whether the number of connected components of (closure of) seams of Q' in $\bar{p}$ is two or one. On the other hand, since $c = D_a(b)$, it follows that $c = D_a^{N+1}(r)$. Therefore, in a similar way, we have

$$F_{Q/P_0}(c) = \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_a^{(2N+\varepsilon+2)\mu - 8\rho_2} f(D_c^2, D_a^2)\big)(D_c^\lambda).$$

Then we see that

$$\begin{aligned}
&F_{Q/P_0}(D_a) F_{Q/P_0}(D_b) F_{Q/P_0}(D_a^{-1}) \\
&= \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_a^\lambda D_a^{(2N+\varepsilon)\mu - 8\rho_2} f(D_b^2, D_a^2)\big)(D_b^\lambda) \\
&= f_F(Q/P_0 \dashrightarrow P') D_a^{(2N+\varepsilon+2)\mu - 8\rho_2} D_a f(D_b^2, D_a^2) D_a^{-1} D_a D_b^\lambda D_a^{-1} D_a f(D_a^2, D_b^2) D_a^{-1} \cdot \\
&\quad D_a^{-(2N+\varepsilon+2)\mu + 8\rho_2} f(Q/P_0 \dashrightarrow P')^{-1} \\
&= f_F(Q/P_0 \dashrightarrow P') D_a^{(2N+\varepsilon+2)\mu - 8\rho_2} f(D_c^2, D_a^2) D_c^\lambda f(D_a^2, D_c^2) D_a^{-(2N+\varepsilon+2)\mu + 8\rho_2} \\
&= \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_a^{(2N+\varepsilon+2)\mu - 8\rho_2} f(D_c^2, D_a^2)\big)(D_c^\lambda) \\
&= F_{Q/P_0}(D_c).
\end{aligned}$$

(L): Let $(P_0, \ldots, P_n = P')$ be a chain of A/S-moves such that $\mathcal{C}(P')$ contains $a_1, \ldots, a_4$ and b_1 as in Figure 9.1. Then,

$$F_{Q/P_0}(D_{b_1}) = f_F(Q/P_0 \dashrightarrow P') D_{b_1}^\lambda f_F(Q/P_0 \dashrightarrow P')^{-1}.$$

Let $Q' = Q_{P_0 \dashrightarrow P'}$ and let N be the integer such that $D_{b_1}^{N/2}(Q')$ is adjusted to b_2. Then, from the definition, we find that

$$F_{Q/P_0}(D_{b_2}) = \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_{b_1}^{N\mu} f(D_{b_2}, D_{b_1})\big)(D_{b_2}^\lambda).$$

The seams of $Q'' = D_{b_1}^{N/2}(Q')$ must be given by the "ridges" of Figure 9.1, since only these seams correspond to a quilt adjusted to both b_1 and b_2. Then the quilt $D_{b_2}^{-1/2}(Q'')$ is adjusted to b_3. So we have

$$F_{Q/P_0}(D_{b_3}) = \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P') D_{b_1}^{N\mu} f(D_{b_2}, D_{b_1}) D_{b_2}^{-\mu} f(D_{b_3}, D_{b_2})\big)(D_{b_3}^\lambda).$$

Using this, one can check that

$$\begin{aligned}
&F_{Q/P_0}(D_{b_1})F_{Q/P_0}(D_{b_2})F_{Q/P_0}(D_{b_3})\\
&= f_n D_{b_1}^{\lambda} f(D_{b_2}, D_{b_1}) D_{b_2}^{1+\mu} f(D_{b_3}, D_{b_2}) D_{b_3}^{\lambda} f(D_{b_2}, D_{b_3}) D_{b_2}^{\mu} f(D_{b_1}, D_{b_2}) f_n^{-1}\\
&= f_n D_{b_1}^{\lambda} \cdot f(D_{b_2}, D_{b_1}) D_{b_2}^{1+\mu} f(D_{b_3}, D_{b_2}) D_{b_3}^{1+\mu} f(D_{b_1}, D_{b_3})\\
&\qquad \cdot D_{b_1}^{-\mu}(D_{a_1}D_{a_2}D_{a_3}D_{a_4})^{\mu} f_n^{-1}\\
&= f_n D_{b_1}^{\lambda} D_{b_1}^{-1-\mu}(D_{a_1}D_{a_2}D_{a_3}D_{a_4})^{1+\mu} D_{b_1}^{-\mu}(D_{a_1}D_{a_2}D_{a_3}D_{a_4})^{\mu} f_n^{-1}\\
&= (D_{a_1}D_{a_2}D_{a_3}D_{a_4})^{\lambda} = F_{Q/P_0}(D_{a_1})F_{Q/P_0}(D_{a_2})F_{Q/P_0}(D_{a_3})F_{Q/P_0}(D_{a_4}),
\end{aligned}$$

where $f_n = f_F(Q/P_0 \dashrightarrow P')D_{b_1}^{N\mu}$.

In the above calculations, we made use of formula (1.5.2).

(D): Let $(P_0, \ldots, P_n = P')$ be a chain of A/S-moves such that $\mathcal{C}(P') \ni a, d$ of Figure 9.2, and let r be the reference curve of $Q' = Q_{P_0 \dashrightarrow P_n}$ in the closure of the pair of pants p bounded by a, d. If N is the integer such that $b = D_a^N(r)$, then

$$\begin{cases} F_{Q/P_0}(D_a) &= \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P_n)\big)(D_a^{\lambda}),\\ F_{Q/P_0}(D_b) &= \mathrm{Inn}\big(f_F(Q/P_0 \dashrightarrow P_n)D_a^{(2N+\varepsilon)\mu-8\rho_2} f(D_b^2, D_a^2)\big)(D_b^{\lambda}). \end{cases}$$

Here again $\varepsilon = 0, 1$ according as the number of connected components of (closure of) seams of Q' in $\bar{p}$ is two or one. Using this, we compute:

$$\begin{aligned}
&F_{Q/P_0}(D_a)F_{Q/P_0}(D_b)F_{Q/P_0}(D_a)\\
&= f_n D_a^{2\mu} f(D_a D_b^2 D_a^{-1}, D_a^2) D_a D_b^{\lambda} f(D_a^2, D_b^2) D_a^{\lambda} f_n^{-1}\\
&= f_n D_a^{2\mu} \cdot f(D_a D_b^2 D_a^{-1}, D_a^2) D_a D_b^{2\mu} D_a^{-1} f(D_b^2, D_a D_b^2 D_a^{-1}) D_a D_b D_a^{\lambda} f_n^{-1}\\
&= f_n f(D_b^2, D_a^2) D_b^{-2\mu} \rho^{2\mu+1} D_a^{2\mu} f_n^{-1}\\
&= f_n f(D_b^2, D_a^2) \rho^{2\mu+1} f_n^{-1},
\end{aligned}$$

where $f_n = f_F(Q/P_0 \dashrightarrow P_n)D_a^{(2N+\varepsilon)\mu-8\rho_2}$ and $\rho = D_a D_b D_a$. Here in the last equality, we also used $\rho D_a = D_b \rho$. Then, since ρ^2 is a central element of $\langle D_a, D_b\rangle$, we obtain

$$\big(F_{Q/P_0}(D_a)F_{Q/P_0}(D_b)F_{Q/P_0}(D_a)\big)^2 = \rho^{4\mu+2},$$

and hence $\big(F_{Q/P_0}(D_a)F_{Q/P_0}(D_b)F_{Q/P_0}(D_a)\big)^4 = \rho^{8\mu+4} = D_d^{\lambda}$ as desired. Thus, the proof that the action F_{Q/P_0} preserves the relations of type (C),(B),(L),(D) is completed.

To conclude the proof by showing that $F_{Q/P}$ extends to an automorphism of the profinite group $\hat{\Gamma}_{g,r}$, let $\mathfrak{F}_{\mathbb{S}^*}$ denote the free discrete group generated by the infinite set $\mathbb{S}^*(\Sigma)$ and let R denote the normal subgroup generated by the discrete words corresponding to (C), (B), (L), (D). Then Theorem 9.2 states that $\Gamma(\Sigma) = \mathfrak{F}_{\mathbb{S}^*}/R$. The profinite completion functor is right exact, so we may regard $\hat{\Gamma}(\Sigma)$ as the quotient of $\hat{\mathfrak{F}}_{\mathbb{S}^*}$ modulo the closure of the image of R. Thus, the above argument shows that F_{Q/P_0} gives an endomorphism of $\hat{\Gamma}(\Sigma)$. Since we already know that $\mathbb{\Gamma}$ forms a group, to conclude that $F_{Q/P}$ gives an automorphism, it suffices to show the following

Lemma 9.3. *Let Q/P_0 be a quilt on a surface Σ. Then, for any $F, F' \in \mathbb{I}\Gamma$, we have*

$$(FF')_{Q/P_0}(x) = F_{Q/P_0}(F'_{Q/P_0}(x)) \qquad (x \in \hat{\Gamma}(\Sigma)).$$

Proof. If we put $F = (\lambda, f)$, $F' = (\lambda', f')$, then $FF' = (\lambda\lambda', f(x,y)f'(x^\lambda, f^{-1}y^\lambda f))$. From this and the definition of f_F given in (7.1) and (7.2) of Definition 7.2, it follows immediately that for any A- or S- move (P, P') and quilt Q/P, we have

$$f_{FF'}(Q/P \to P') = F_{Q/P}(f_{F'}(Q/P \to P'))f_F(Q/P \to P') \tag{9.1}$$

holds. To prove the lemma, we need only consider the case $x = D_c$ for $c \in \mathbb{S}(\Sigma)$. Now, let $(P_0, \ldots, P_n)$ be a chain of A/S-moves on Σ such that $c \in \mathcal{C}(P_n)$. It suffices then to show that

$$f_{FF'}(Q/P_0 \dashrightarrow P_n) = F_{Q/P_0}(f_{F'}(Q/P_0 \dashrightarrow P_n))f_F(Q/P_0 \dashrightarrow P_n) \tag{9.2}$$

Let us argue by induction on n. When $n = 0$, this is nothing but (9.1). Assume $n \geq 1$ and let $Q_1 = Q_{P_0 \to P_1}$. Then, by the induction hypothesis (and (9.1)), we obtain

$$\begin{aligned} \text{LHS of (9.2)} = &F_{Q/P_0}(f_{F'}(Q/P_0 \to P_1))f_F(Q/P_0 \to P_1) \\ &\cdot F_{Q_1/P_2}(f_{F'}(Q_1/P_2 \dashrightarrow P_n))f_F(Q_1/P_2 \dashrightarrow P_n). \end{aligned}$$

Now, putting together the formula in the statement of Proposition 8.1 and formula (7.3) from §7, we find that for any Dehn twist D_e with ($e \in \mathbb{S}^*(\Sigma)$), and any chain $(P_0, P_1, \ldots, P_n)$ of A- and S- moves such that $e \in \mathcal{C}(P_n)$, we have

$$\begin{aligned} F_{Q/P_0}(D_e) &= f_F(Q/P_0 \to \cdots \to P_n)D_e^\lambda f_F(Q/P_0 \to \cdots \to P_n)^{-1} \\ &= f_F(Q/P_0 \to P_1)f_F(Q_1/P_1 \to P_2)\cdots f_F(Q_{n-1}/P_{n-1} \to P_n)D_e^\lambda \cdot \\ &\quad \cdot f_F(Q_{n-1}/P_{n-1} \to P_n)^{-1}\cdots f_F(Q_1/P_1 \to P_2)^{-1}f_F(Q/P_0 \to P_1)^{-1} \\ &= f_F(Q/P_0 \to P_1)F_{Q_1/P_1}(D_e)f_F(Q/P_0 \to P_1)^{-1}. \end{aligned}$$

The expression

$$F_{Q/P_0}(x) = f_F(Q/P_0 \to P_1)F_{Q_1/P_1}(x)f_F(Q/P_0 \to P_1)^{-1}$$

deduced from the first and last terms of the previous group of equalities then holds for all x in $\hat{\Gamma}_{g,r}$, which concludes the proof of Lemma 9.4. □

Thus the proof of Proposition 9.1 is completed.

§10. Proofs of Theorems 1.3 and 1.4.

In this section, we will settle the last two theorems stated in §1. By virtue of Proposition 9.1 and Lemma 9.3, given $\Sigma = \Sigma_{g,r}$ and a quilt Q/P_0 on it, we can define a representation in the profinite Teichmüller modular group:

$$\rho^\Sigma_{Q/P_0} : \mathbb{I}\Gamma \longrightarrow \operatorname{Aut}\hat{\Gamma}(\Sigma) \qquad (F \mapsto F_{Q/P_0}).$$

Moreover, filling n boundary components of $\Sigma \cong \Sigma_{g,r}$ (where $n \leq r$) by marked disks, we get a surjection of $\hat{\Gamma}(\Sigma_{g,r})$ onto $\hat{\Gamma}(\Sigma^n_{g,m})$ $(m+n=r)$ whose kernel is generated by the Dehn twists along those n boundary circles. Since each D_c $(c \in \mathbb{S}^* \setminus \mathbb{S})$ is acted on by $(\lambda, f) \in \mathbb{I}\Gamma$ in the form of $D_c \mapsto D_c^{\lambda}$, we see that the above representation in $\hat{\Gamma}(\Sigma_{g,r})$ also induces naturally a representation in $\hat{\Gamma}(\Sigma^n_{g,m})$. This settles Theorem 1.3 for all types of surfaces $\Sigma^n_{g,m}$.

When a given surface has marked points, as above, we may regard those marked points as reduction of the same number of boundary components of another surface without marked points. In this respect, the notion of quilts also can make sense for a surface with marked points in the obvious manner. Through the above $\mathbb{I}\Gamma$-compatible surjection $\hat{\Gamma}(\Sigma_{g,r}) \twoheadrightarrow \hat{\Gamma}(\Sigma^n_{g,m})$ $(m+n=r)$, we may often reduce our issues on $\mathbb{I}\Gamma$-actions on $\hat{\Gamma}(\Sigma^n_{g,m})$ to those on $\hat{\Gamma}(\Sigma_{g,r})$. (Note that $\Gamma(\Sigma)$ designates pure mapping class groups, not permuting marked points.) In particular, in the following discussions, we may assume $\Sigma = \Sigma_{g,r}$ without loss of generality.

Before going on to discuss Theorem 1.4, we shall make a remark on how $\rho^{\Sigma}_{Q/P}$ varies with respect to the change of quilts. Suppose we are given two quilt-decompositions Q/P, Q'/P' of the surface Σ. Take a chain $(P = P_0, P_1, \ldots, P_n = P')$ of A/S-moves and let $Q_n = Q_{P_0 \dashrightarrow P_n}$ over $P_n = P'$. The formulas given in (7.3) and the statement of Proposition 8.1 imply the equality

$$\rho^{\Sigma}_{Q_n/P_n}(F) = \mathrm{Inn}\Big(f_F(Q/P_0 \dashrightarrow P_n)^{-1}\Big) \circ \rho^{\Sigma}_{Q/P}(F) \tag{10.1}$$

for $F \in \mathbb{I}\Gamma$. (Note in particular that $F_{Q_n/P_n}(D_e) = D_e^{\lambda}$ follows for $e \in \mathcal{C}(P_n)$.) Now, since Q' and Q_n are both quilts over $P' = P_n$, there exist integers N_c $(c \in \mathcal{C}^*(P'))$ such that $Q' = \prod_{c \in \mathcal{C}^*(P')} D_c^{N_c/2}(Q_n)$, giving

$$\rho^{\Sigma}_{Q'/P'}(F) = \mathrm{Inn}\left(\prod_{c \in \mathcal{C}^*(P')} D_c^{-N_c\mu}\right) \circ \rho^{\Sigma}_{Q_n/P'}(F) \tag{10.2}$$

Putting (10.1) and (10.2) together, we find

$$\rho^{\Sigma}_{Q'/P'}(F) = \mathrm{Inn}\left(\prod_{c \in \mathcal{C}^*(P')} D_c^{-N_c\mu} \cdot f_F(Q_0/P_0 \to \cdots P_n)^{-1}\right) \circ \rho^{\Sigma}_{Q/P}(F) \tag{10.3}$$

for $F \in \mathbb{I}\Gamma$. From this, we especially see that, given any (compact oriented) surface Σ (with boundary components and marked points allowed), the representations $\rho^{\Sigma}_{Q/P}$'s for quilts Q/P on Σ give a single exterior representation:

$$\rho^{\Sigma} : \mathbb{I}\Gamma \longrightarrow \mathrm{Out}\,\hat{\Gamma}(\Sigma). \tag{10.4}$$

We call this ρ^{Σ} the *canonical exterior representation* of $\mathbb{I}\Gamma$ in the profinite Teichmüller modular group $\hat{\Gamma}(\Sigma)$.

Now, let Q/P be a quilt on Σ and suppose $\Sigma' \subset \Sigma$ is a connected subsurface consisting of the closure of some pairs of pants in P. Then, on Σ', we have a quilt Q'/P' naturally induced from Q/P by restriction. Pick any $F \in \mathbb{I}\Gamma$. For any simple closed curve $c \in \mathbb{S}^*(\Sigma')$, the process for defining $\rho^{\Sigma'}_{Q'/P'}(F)(D_c)$ inside Σ' is identical to the one defining $\rho^{\Sigma}_{Q/P}(F)(D_c)$, and concerns uniquely circles lying inside Σ'. From this observation, we deduce that $\rho^{\Sigma}_{Q/P}(F)$ preserves the image of $\hat{\Gamma}(\Sigma') \to \hat{\Gamma}(\Sigma)$, and makes the diagram

$$\begin{array}{ccc} \hat{\Gamma}(\Sigma') & \longrightarrow & \hat{\Gamma}(\Sigma) \\ {\scriptstyle \rho^{\Sigma'}_{Q'/P'}(F)}\downarrow & & \downarrow{\scriptstyle \rho^{\Sigma}_{Q/P}(F)} \\ \hat{\Gamma}(\Sigma') & \longrightarrow & \hat{\Gamma}(\Sigma) \end{array}$$

commute. Thus Theorem 1.4 is settled.

§11. Standard $\mathbb{I}\Gamma$-action.

Let $\Sigma = \Sigma_{g,r}$ be a compact oriented surface of genus g with r boundary components $\epsilon_1, \ldots, \epsilon_r$. We shall consider a standard pants decomposition P of Σ such that $\mathcal{C}(P)$ consists of the circles a_1, $d_{\pm i}$ $(2 \le i \le g)$, e_j $(1 \le j \le g)$, $k_2, \ldots, k_{r-1}$ indicated in Figure 11.1.

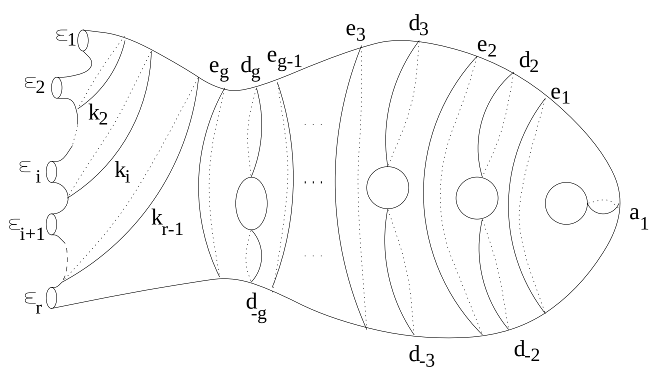


Figure 11.1

Also, let Q be a quilt over P defined by "ridges" of the figure dividing each pair of pants into front and back patches.

By a result of Dehn-Lickorish (cf. [Mu]), the pure mapping class group $\hat{\Gamma}(\Sigma)$ is generated by the Dehn twists along the simple closed curves $a_1, \ldots, a_{2g}$, $d_{\pm i}$ $(2 \le i \le g)$, e_j $(1 \le j \le g)$, $\epsilon_1, \ldots, \epsilon_r$ $h_1, \ldots, h_r$ and u_{ij} $(1 \le i \ne j \le n)$ indicated in Figure 11.2. The purpose of this section is to give the representation $\rho^{\Sigma}_{Q/P} : \mathbb{I}\Gamma \to \mathrm{Aut}\,\hat{\Gamma}(\Sigma)$ in a more compact form, namely to explicitly compute the images of this finite number of generators under $\rho^{\Sigma}_{Q/P}$ in terms of $(\lambda, f) \in \mathbb{I}\Gamma$. As mentioned in the previous section, by filling some of the boundary components by marked disks, one can reduce the $\mathbb{I}\Gamma$-action on $\hat{\Gamma}_{g,r}$ to that on $\hat{\Gamma}^n_{g,m}$ $(m+n=r)$ easily. So, knowing the action for $\hat{\Gamma}_{g,r}$ essentially gives the standard $\mathbb{I}\Gamma$-actions for all types of the profinite Teichmüller modular groups $\hat{\Gamma}^n_{g,m}$.

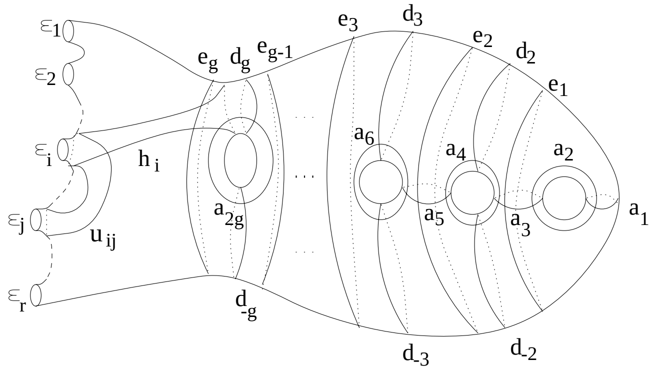


Figure 11.2

Before stating the main result, we introduce another system of circles on Σ and express their Dehn twists by our previous generators.

Lemma 11.1. *Let Σ' be the genus zero subsurface of Σ cut out by the circles $\epsilon_0 := d_{-g}$ and $\epsilon_\infty := d_g$, and express it as in Figure 11.3 (so that ϵ_∞ is enlarged to the rectangular rim). Define the circles v_{ij} $(0 \le i < j \le r)$ as illustrated in Figure 11.3, and put $h_i = u_{0i}$ $(i = 1, \dots, r)$. Then, the Dehn twist $D_{v_{ij}}$ is given by*

$$D_{v_{ij}} = (D_{\epsilon_i} \cdots D_{\epsilon_j})^{1+i-j} (D_{u_{i,i+1}})(D_{u_{i,i+2}} D_{u_{i+1,i+2}}) \cdots (D_{u_{ij}} \cdots D_{u_{j-1,j}})$$

for $0 \le i < j \le r$.

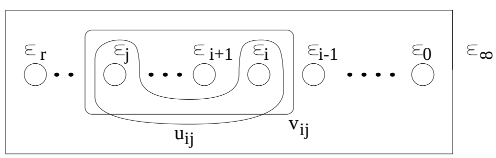


Figure 11.3

Proof. The proof is given by a simple induction by iterative use of the lantern relation (Theorem 9.2 (L)). Note that we may deform Figure 9.1 as:

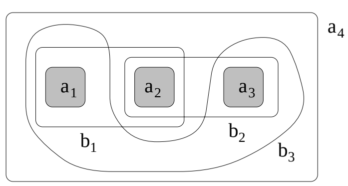


Figure 11.4

where the lantern relation claims $D_{b_1} D_{b_2} D_{b_3} = D_{a_1} D_{a_2} D_{a_3} D_{a_4}$. $\square$

Now we shall state the theorem giving an explicit formulation of the standard action of $\mathbb{I}\Gamma$ on $\hat{\Gamma}_{g,r}$. Recall that $w_1 = 1$, $w_i = (D_{a_1} \cdots D_{a_{i-1}})^i$ for $i > 1$.

Theorem 11.2. *Notations being as above, the action of $\rho^{\Sigma}_{Q/P}(F)$ $(F = (\lambda, f) \in \mathbb{I}\Gamma)$ on the Dehn twist generators of $\hat{\Gamma}(\Sigma)$ can be written explicitly as follows:*

(1) $D_{d_i} \mapsto D^{\lambda}_{d_i},\ D_{d_{-i}} \mapsto D^{\lambda}_{d_{-i}},\ D_{e_j} \mapsto D^{\lambda}_{e_j},\ D_{k_i} \mapsto D^{\lambda}_{k_i},\ D_{\epsilon_i} \mapsto D^{\lambda}_{\epsilon_i}.$

(2) $D_{a_{2i-1}} \mapsto w_{2i-1}^{4\rho_2} f(D^2_{a_{2i-1}}, w_{2i-1}) D^{\lambda}_{a_{2i-1}} f(w_{2i-1}, D^2_{a_{2i-1}}) w_{2i-1}^{-4\rho_2},$

$D_{a_{2i}} \mapsto w_{2i}^{-4\rho_2} f(D^2_{a_{2i}}, w_{2i}) D^{\lambda}_{a_{2i}} f(w_{2i}, D^2_{a_{2i}}) w_{2i}^{4\rho_2}.$

(3) $D_{h_i} \mapsto \mathrm{Inn}(\mathcal{F}_i)(D^{\lambda}_{h_i})$, *where $\mathcal{F}_i$ is given by*

$\mathcal{F}_i = f(D_{v_{0,r-1}}, D_{v_{1,r}}) \cdots f(D_{v_{0,i}}, D_{v_{1,i+1}}) \cdot D^{\mu}_{v_{1,i}} f(D_{h_i}, D_{v_{1,i}}).$

(4) $D_{u_{ij}} \mapsto \mathrm{Inn}(\mathcal{F}_{ij})(D_{u_{ij}}^{\lambda})$, *where* $\mathcal{F}_{ij}$ *is given by*

$$\mathcal{F}_{ij} = \prod_{s=0}^{j-i-2} f(D_{v_{j-2-s,j-1}}, D_{v_{1,j-2-s}}) \cdot f(D_{v_{ij}}, D_{v_{1,j-1}}) D_{v_{i,j-1}}^{-\mu} f(D_{u_{ij}}, D_{v_{i,j-1}}).$$

Here, $\mu = (\lambda - 1)/2$, $\rho_2 = \rho_2(F)$, *and we understand symbols in these formulae for all possible indices which make sense. In the case* $j = i+1$, *the product in* s *of (4) is understood to be trivial, and* $v_{i,j-1} = \epsilon_i$, $v_{ij} = u_{ij}$.

Proof. (1) is clear from the definition of $\rho_{Q/P}^{\Sigma}$, because these curves are elements of $\mathcal{C}(P)$. For (2), we first consider the action on a_2. The obvious S-move $P \to P'$ replacing a_1 by a_2 yields

$$f_F(Q/P \to P') = D_{a_1}^{-8\rho_2} f(D_{a_2}^2, D_{a_1}^2) D_{a_2}^{8\rho_2} (D_{a_1} D_{a_2} D_{a_1})^{2\mu}.$$

This settles the formula for $\rho_{Q/P}^{\Sigma}(F)(a_2)$. Next we consider the action on a_{2i} ($i \geq 2$). Note that, by Theorem 1.4, it suffices only to investigate the action of $\rho_{Q/P}^{\Sigma}$ locally on involved circles. We shall consider the local chain

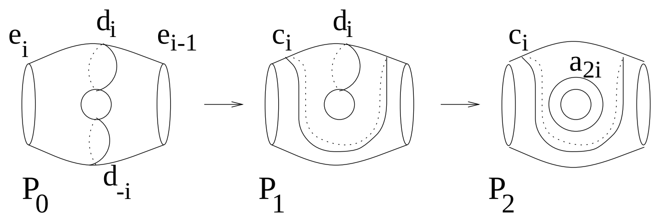


Figure 11.5

Then, it follows that

$$f_F(Q/P_0 \dashrightarrow P_2) = f(D_{c_i}, D_{d_{-i}}) D_{d_i}^{-8\rho_2} f(D_{a_{2i}}^2, D_{d_i}^2) D_{a_{2i}}^{8\rho_2} (D_{d_i} D_{a_{2i}} D_{d_i})^{2\mu}.$$

Noticing that $D_{c_i} = (D_{d_i} D_{a_{2i}} D_{d_i})^4$, we apply the relation (IV) to the first factor of the above (by putting $\tau_1 := D_{-d_i}$, $\tau_2^2 := (D_{d_i} D_{a_{2i}} D_{d_i})^2$), and then substitute notation (as in the paragraph following §8 (R′)) by $D_{d_{-i}}^2 = x_{12}$, $D_{a_{2i}}^2 = x_{23}$, $D_{d_i}^2 = x_{34}$, $(D_{d_{-i}} D_{a_{2i}} D_{d_{-i}})^2 = \mathbf{x}_{45}$ and $(D_{d_i} D_{a_{2i}} D_{d_i})^2 = \mathbf{x}_{51}$. Then, we obtain:

$$f_F(Q/P_0 \to P_1 \to P_2) = x_{12}^{-2\rho_2} x_{34}^{-2\rho_2} f(\mathbf{x}_{51}, x_{12}) f(x_{23}, x_{34}) \mathbf{x}_{51}^{-4\rho_2} x_{23}^{4\rho_2} \mathbf{x}_{51}^{\mu}.$$

To simplify this expression, recall the pentagon relation (III), given by

$$f(\mathbf{x}_{51}, x_{12}) f(x_{23}, x_{34}) f(\mathbf{x}_{45}, \mathbf{x}_{51}) f(x_{12}, x_{23}) f(x_{34}, \mathbf{x}_{45}) = 1.$$

Writing $f(\mathbf{x}_{45}, \mathbf{x}_{51}) = g(\mathbf{x}_{51}, \mathbf{x}_{45})^{-1} g(\mathbf{x}_{45}, \mathbf{x}_{51})$, the pentagon breaks into two pieces as

(11.1)
$$\begin{aligned} f(\mathbf{x}_{51}, x_{12}) f(x_{23}, x_{34}) g(\mathbf{x}_{51}, \mathbf{x}_{45})^{-1} &= f(\mathbf{x}_{45}, x_{34}) f(x_{23}, x_{12}) g(\mathbf{x}_{45}, \mathbf{x}_{51})^{-1} \\ &= \omega^{-\rho_2(F)} f(\tau_2^2, \tau_1 \tau_3), \end{aligned}$$

the last equality being a consequence of relation (III'_{bis}) (Proposition 5.4). We rewrite (11.1) as

(11.2) $$f(\mathbf{x}_{51}, x_{12}) f(x_{23}, x_{34}) = f(\tau_2^2, \tau_1 \tau_3) g(\mathbf{x}_{51}, \mathbf{x}_{45}) \omega^{-\rho_2(F)}$$

Now, the subpiece of Σ shown in the left-hand part of Figure 11.5 has topological type $(1,2)$, so its mapping class group is isomorphic to $\Gamma_{1,2}$. We can identify the group $\hat{B}_4$ with a subgroup of $\hat{\Gamma}_{1,2}$ via $\tau_1 \mapsto D_{d_{-i}}$, $\tau_2 \mapsto D_{a_{2i}}$, $\tau_3 \mapsto D_{d_i}$ and $\omega \mapsto D_{e_i} D_{e_{i-1}}$, so that (11.2) can be written

$$f(\mathbf{x}_{51}, x_{12}) f(x_{23}, x_{34}) = f(D_{a_{2i}}^2, D_{d_i} D_{d_{-i}}) g(\mathbf{x}_{51}, \mathbf{x}_{45}) (D_{e_i} D_{e_{i-1}})^{-\rho_2}.$$

From this and the fact that $D_{d_i} D_{d_{-i}} = w_{2i}$, we can compute $\rho_{Q/P_0}(F)(D_{a_{2i}}) = f_F(Q/P_0 \dashrightarrow P_2) D_{a_{2i}}^{\lambda} f_F(Q/P_0 \dashrightarrow P_2)^{-1}$ in desired form. Next, we shall consider the following local chain around the circle a_{2i-1}.

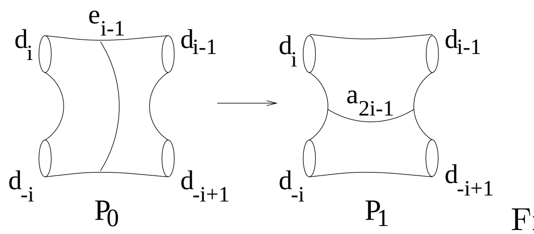


Figure 11.6

Then, using the relation (IV) (by putting $\tau_1 := D_{a_{2i-1}}$, $\tau_2^2 := w_{2i-1}$) and the fact that $D_{e_{i-1}} = w_{2i-1}^2$, we obtain

$$\begin{aligned} f_F(Q/P_0 \to P_1) &= f(D_{a_{2i-1}}, D_{e_{i-1}}) = f(D_{a_{2i-1}}, w_{2i-1}^2) \\ &= w_{2i-1}^{4\rho_2} f(D_{a_{2i-1}}^2, w_{2i-1})(D_{a_{2i-1}} w_{2i-1})^{-4\rho_2} D_{a_{2i-1}}^{4\rho_2}. \end{aligned}$$

From this the desired formula for $\rho_{Q/P_0}(F)(a_{2i-1})$ follows.

For (3),(4): We shall consider the subsurface Σ' introduced in Lemma 11.1, and let Q/P_0 denote the initial quilt on Σ' which can be illustrated as in Figure 11.5 with seams being dotted lines.

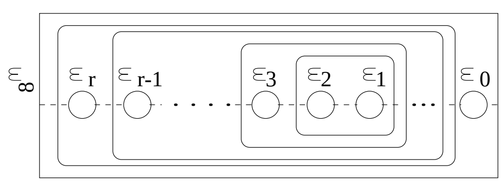


Figure 11.7

(3): Starting from Q/P_0 on Σ', we define successive A-moves $P_s \to P_{s+1}$ replacing $v_{1,r-s}$ by $v_{0,r-1-s}$ for $s = 0, \ldots, r-i-1$. Then, on the chain $P_0 \to \ldots \to P_{r-i}$, quilts are always adjusted to given pants decompositions so that

$$f(Q/P_0 \dashrightarrow P_{r-i}) = f(D_{v_{0,r-1}}, D_{v_{1,r}}) \cdots f(D_{v_{0,i}}, D_{v_{1,i+1}}).$$

In order to make $Q_{P_0 \dashrightarrow P_{r-i}}$ adjusted to h_i, we have to apply a half-twist along $v_{1,i}$ to it (cf. Figure 11.8). The formula (3) follows immediately from this observation.

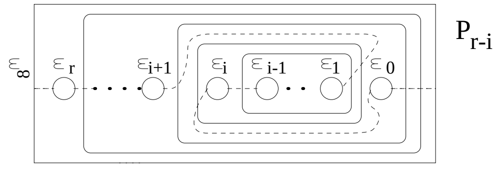


Figure 11.8

(4): In this case, we first trace succesive A-moves $P_s \to P_{s+1}$ which replace $v_{1,j-2-s}$ by $v_{j-2-s,j-1}$ $(s = 0, \ldots j-i-2)$, and then move along $P_{j-i-1} \to P_{j-i}$ replacing

$v_{1,j-1}$ by v_{ij}. In the above process, quilts are always adjusted to given pants decompositions so that

$$f(Q/P_0 \dashrightarrow P_{j-i}) = \prod_{s=0}^{j-i-2} f(D_{v_{j-2-s,j-1}}, D_{v_{1,j-2-s}}) \cdot f(D_{v_{ij}}, D_{v_{1,j}}).$$

Then, to make the quilt $Q_{P_0 \dashrightarrow P_{j-i}}$ adjusted to u_{ij}, we apply a negative half twist along $v_{i,j-1}$ (cf. Figure 11.9).

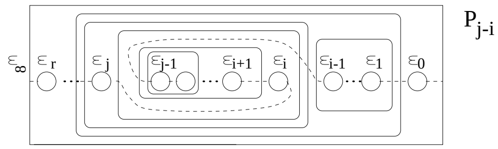


Figure 11.9

This conclude the formula of (4), and thus settles the proof of Theorem 11.2. □

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