ELLIPTIC ANALOGS OF MULTIZETAS AND THE ELLIPTIC DOUBLE SHUFFLE RELATIONS

PIERRE LOCHAK, NILS MATTHES, AND LEILA SCHNEPS

Abstract. We define an *elliptic generating series* whose coefficients, the *elliptic multizeta values*, are related to the elliptic analogs of multizeta values introduced by Enriquez as the coefficients of his elliptic associator; both sets of coefficients lie in $\mathcal{O}(\mathfrak{H})$, the ring of functions on the Poincaré upper half-plane $\mathfrak{H}$. The elliptic multizeta values generate a $\mathbb{Q}$-algebra $\mathcal{E}$ which is an elliptic analog of the algebra of multizetas. We show that the algebra $\mathcal{E}$ decomposes into a geometric and an arithmetic part, and study the precise relationship between the elliptic generating series and the elliptic associator defined by Enriquez. Working modulo $2\pi i$, we show that the elliptic multizeta values satisfy a double shuffle type family of algebraic relations similar to the double shuffle relations satisfied by multizetas. We prove that these elliptic double shuffle relations give all algebraic relations among elliptic multizetas if (a) the classical double shuffle relations give all algebraic relations among multizetas and (b) the elliptic double shuffle Lie algebra has a certain natural semi-direct product structure analogous to that established by Enriquez for the elliptic Grothendieck-Teichmüller Lie algebra.

1. Introduction

1.1. **Elliptic multizeta values.** An elliptic analog of the multizeta values first made an explicit appearance in Enriquez' article [15] under the name "analogues elliptiques des nombres multizetas". They arise as coefficients of his elliptic associator constructed in [14], which is closely related to the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) equation [8, 23] and to multiple elliptic polylogarithms [7, 23]; more recently, they have even found applications to computations in high energy physics [1]. Taking the regularized limit $\tau \to i\infty$ of elliptic multizetas, one retrieves the classical multiple zeta values [15, 25], which gives the explicit connection between the genus zero and genus one multiple zeta values. The idea of considering the graded $\mathbb{Q}$-algebra generated by these coefficients, was introduced in [2, 25, 26], which provide some explicit dimension results in depth 2.

Recall that the Drinfel'd associator Φ_{KZ} is a power series in two non-commutative variables, which is the generating series for the usual multiple zeta values. In analogy with this, Enriquez's elliptic associator takes the form of a pair of group-like power series in two non-commutative variables $\big(A(\tau), B(\tau)\big)$, whose coefficients are not real numbers but elements of $\mathcal{O}(\mathfrak{H})$[1]. We call the coefficients of $A(\tau)$ and $B(\tau)$ *A-elliptic multizetas* and *B-elliptic multizetas*, or A-EMZ's and B-EMZ's. The

2010 *Mathematics Subject Classification.* 11M32.

Key words and phrases. Multiple zetas, Grothendieck–Teichmüller theory, moulds.

[1]As usual, $\mathcal{O}(\mathfrak{H})$ denotes the ring of holomorphic functions of one variable τ running through the Poincaré upper half-plane. The functional dependence of elliptic multizetas is natural, as elliptic curves are parametrized by points in $\mathfrak{H}$. The elliptic values studied throughout this paper

acronym EMZ stands for *elliptic multiple zetas*; since they are functions of τ, we drop the word "values" in the elliptic situation.

In this paper, we introduce a third power series $E(\tau)$, called the *elliptic generating series*; its coefficients, which again lie in $\mathcal{O}(\mathfrak{H})$, are called *$E$-elliptic multizetas* or E-EMZ's. We show that $E(\tau)$ is group-like, so that like $A(\tau)$ and $B(\tau)$, its coefficients generate a $\mathbb{Q}$-algebra. We write $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ for the $\mathbb{Q}$-algebras generated by the E-EMZ's, the A-EMZ's and the B-EMZ's respectively.

For technical reasons linked to our use of Grothendieck-Teichmüller theory and Écalle's mould theory, we restrict our study of the above objects to their reductions modulo $2\pi i$ (in a specific sense): the three types of elliptic multiple zetas mod $2\pi i$ are called $\overline{E}$-EMZ's, $\overline{A}$-EMZ's and $\overline{B}$-EMZ's, and the $\mathbb{Q}$-algebras they generate are correspondingly called $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. In one of our main results, we give an explicit description of the structure of the third algebra, and show that the first two are equal and that they become equal to the third after adjunction of a single element. In view of this result, we refer to this algebra as the algebra of elliptic multiple zetas mod $2\pi i$. The $\overline{E}$-EMZ's (plus one element) form a new set of generators for this algebra. Our second main result is an explicit determination of a set of algebraic relations called the *elliptic double shuffle relations* satisfied by these elements. We give these relations in the form of two *elliptic double shuffle equations* satisfied by the power series $\overline{E}(\tau)$ which are very close to the usual double shuffle relations satisfied by the Drinfel'd associator Φ_{KZ}, but surprisingly more similar to their depth-graded version. We also show that assuming certain relatively standard conjectures from multiple zeta theory, the elliptic double shuffle relations form a complete set of algebraic relations for the elliptic multizeta values. The main tools used in the proofs of these results are Écalle's mould theory ([13] and [32]) and the mould theoretic results of the third-named author ([33])

1.2. **Precise description of the main results.** The most immediate and elementary fact concerning the algebras $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ is that they lie inside a $\mathbb{Q}$-algebra generated by distinct *arithmetic* and *geometric* parts. The arithmetic part is nothing other than the $\mathbb{Q}$-algebra $\mathcal{Z}[2\pi i]$, where $\mathcal{Z}$ is generated by the usual multizeta values viewed as constant functions inside $\mathcal{O}(\mathfrak{H})$. The geometric part, $\mathcal{E}^{\mathrm{geom}}$, is generated by the coefficients (in a precise sense explained in §2 of this paper) of a special automorphism $g(\tau)$ defined by Enriquez in section 5.1 of [15], whose coefficients lie in $\mathcal{O}(\mathfrak{H})$ and are realized as particular linear combinations of iterated integrals of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ (see [4, 24]). The key reason behind the decomposition into geometric and arithmetic parts is the fact that each of the three power series $E(\tau)$, $A(\tau)$ and $B(\tau)$ is equal to the image under the automorphism $g(\tau)$ of a power series (respectively denoted E, A and B) with coefficients in $\mathcal{Z}[2\pi i]$. This shows that all the coefficients of $E(\tau)$, $A(\tau)$ and $B(\tau)$ are algebraic expressions in elements of $\mathcal{E}^{\mathrm{geom}}$ and elements of $\mathcal{Z}[2\pi i]$.

Section §2 of the paper is devoted to the study of the structure and the transcendence and linear independence properties of elements of $\mathcal{E}^{\mathrm{geom}}$. The results are

are in particular different from Brown's "multiple modular values" [4], which actually are complex numbers.

given in Theorems 2.6 and Theorem 2.8 and its corollaries[2]. The specific result that we apply for the study of algebras of elliptic multizetas is the following.

Theorem (Cor. 2.9). *We have*

$$\mathcal{E}^{\mathrm{geom}} \cap \mathbb{C} = \mathbb{Q}.$$

Therefore, the $\mathbb{Q}$*-algebra generated inside* $\mathcal{O}(\mathfrak{H})$ *by* $\mathcal{E}^{\mathrm{geom}}$ *and* $\mathcal{Z}[2\pi i]$ *is isomorphic to the tensor product* $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$.

We deduce from this that the algebras $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ are naturally included in the tensor product $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$.

Our next main result determines the structure of $\mathcal{E}^{\mathrm{geom}}$ by connecting it with the bigraded Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ of the prounipotent radical of $\pi_1^{\mathrm{geom}}(MEM)$, where MEM denotes the Tannakian category of universal mixed elliptic motives [18]. More precisely, it is shown in [18], §22 that there is a monodromy representation of $\mathfrak{u}^{\mathrm{geom}}$ to the derivations of a free Lie algebra on two generators, whose image we denote by $\mathfrak{u}$. The proof of the next theorem makes crucial use of the previous one.

Theorem (Theorem 2.6). *There is a natural isomorphism*

$$\mathcal{E}^{\mathrm{geom}} \cong \mathcal{U}(\mathfrak{u})^{\vee},$$

where $\mathcal{U}(\mathfrak{u})^{\vee}$ *is the graded dual of the universal enveloping algebra of* $\mathfrak{u}$*. In particular,* $\mathcal{E}^{\mathrm{geom}}$ *is a commutative, graded, Hopf* $\mathbb{Q}$*-algebra.*

Section §3 is devoted to the definition of the elliptic generating series $E(\tau)$ and the ring $\mathcal{E}$ generated by its coefficients, and to the study of their properties mod $2\pi i$. Let $\overline{\mathcal{Z}}$ denote the quotient of $\mathcal{Z}[2\pi i]$ by the ideal generated by $2\pi i$, which is isomorphic to the quotient of the usual multizeta algebra $\mathcal{Z}$ by the ideal generated by $\zeta(2)$. By the linear independence theorem above (Cor 2.9), the ideal generated inside the tensor product $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$ by the constant $2\pi i$ (identified with $1 \otimes 2\pi i$) lies entirely inside the subring $\mathcal{Z}[2\pi i]$ (identified with $1 \otimes \mathcal{Z}[2\pi i]$), so the quotient of the tensor product by this ideal is isomorphic to $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. We show in Lemma 3.2 that $2\pi i \in \mathcal{E}$, and write $\overline{\mathcal{E}}$ for the quotient of $\mathcal{E}$ by the ideal generated by $2\pi i$. Since we saw above that $\mathcal{E} \subset \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$, we have $\overline{\mathcal{E}} \subset \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. The analogous inclusion holds for $\overline{\mathcal{B}}$. The case of $\mathcal{A}$ is slightly different because $A(\tau) = g(\tau) \cdot A$ where $A \equiv 0 \bmod 2\pi i$. To obtain a non-trivial $\mathbb{Q}$-algebra mod $2\pi i$ we set $A' = A^{1/2\pi i}$ and $A'(\tau) = g(\tau) \cdot A'$, and let $\overline{A'}(\tau)$ denote the reduction of $A'(\tau)$ mod $2\pi i$ and $\overline{\mathcal{A}}$ be the $\mathbb{Q}$-algebra generated by its coefficients. We then also have the inclusion $\overline{\mathcal{A}} \subset \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. Our first goal is to compare the four algebras $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$ and $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. The element $2\pi i\tau \in \mathcal{O}(\mathfrak{H})$ plays a special role in this comparison. It lies in $\mathcal{E}^{\mathrm{geom}}$ since it is the coefficient of ε_0 in $g(\tau)$ (see (2.5) below), so it lies in the tensor product $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, but it does not lie in $\mathcal{E}$ or $\mathcal{A}$, although it does lie in $\mathcal{B}$. Our main comparison result is the following.

Theorem (Cor. 3.1). *We have the equalities*

$$\overline{\mathcal{E}}[2\pi i\tau] = \overline{\mathcal{A}}[2\pi i\tau] = \overline{\mathcal{B}} = \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

---

[2]The second author subsequently generalized Theorem 2.8 to a linear independence result for iterated integrals of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ over the ring of quasimodular forms, cf. [27].

As noted above, the elliptic generating series $E(\tau)$ has the form $g(\tau)\cdot E$; thus its reduction mod $2\pi i$, $\overline{E}(\tau)$, has the form $g(\tau)\cdot\overline{E}$ where the reduction $\overline{E}$ of E has coefficients in $\overline{\mathcal{Z}}$. The key point of the proof is the fact that $\overline{E}$ is the image $\overline{\Phi}_{KZ}$ of the Drinfel'd associator mod $2\pi i$ under Enriquez' section Γ from the genus zero Grothendieck-Teichmüller group GRT to the genus one group GRT_{ell} (Theorem 3.3); in particular the coefficients of $\overline{E}$ generate all of $\overline{\mathcal{Z}}$. The ring $\overline{\mathcal{E}}$ is the $\mathbb{Q}$-algebra generated by the coefficients of $\overline{E}(\tau)$, which, as for $E(\tau)$, are thus all algebraic expressions in the coefficients of $g(\tau)$ and those of $\overline{E}$. The delicate part of the proof consists in showing that the coefficients of $\overline{E}(\tau)$ can be "untangled" to separately recover a set of generators of $\overline{\mathcal{Z}}$ and (together with $2\pi i\tau$) a set of generators of $\mathcal{E}^{\mathrm{geom}}$. The same arguments hold for $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, using a result from [25] to show that the coefficients of the arithmetic part generate all of $\overline{\mathcal{Z}}$ and the same untangling argument. In fact, the result can be framed for more general power series, as is done in Theorem 3.5.

The last main result in this article, contained in §4, concerns the algebraic relations satisfied by the $\overline{E}$-EMZ's. Based on the fact that $\overline{E}(\tau)=\Gamma(\overline{\Phi}_{KZ})$, the power series $\overline{E}(\tau)$ will satisfy equations that are transports by Γ of the usual double shuffle equations satisfied by $\overline{\Phi}_{KZ}$ (shuffle and stuffle relations, also known as double mélange in Racinet's terminology [31], and symmetrality/symmetrility in Écalle's [13]). These transported relations can be determined explicitly thanks to a major theorem of Écalle together with the results of [33]. We call them the *elliptic double shuffle equations*, and show that they are similar in nature to the well-known (extended) double shuffle relations for multizetas, but in fact closer to their graded version.

Rather than the double shuffle relations satisfied by the group-like power series $\overline{\Phi}_{KZ}$, it is equivalent but more convenient to give their Lie version, satisfied by the image ϕ_{KZ} of $\overline{\Phi}_{KZ}$ Drinfel'd associator obtained by reducing the coefficients modulo products; these are called the linearized double shuffle relations, or alternality/alternility in Écalle's terminology. In the same fashion, it is more convenient to give the Lie version of the elliptic double shuffle relations satisfied by the power series $\mathfrak{e}(\tau)$ of $\overline{E}(\tau)$ obtained by reducing the coefficients of $\overline{E}(\tau)$ modulo products. We saw that $\overline{E}(\tau)=g(\tau)\cdot\overline{E}$ where $\overline{E}=\Gamma(\overline{\Phi}_{KZ})$; in the Lie version, the power series $\mathfrak{e}(\tau)$ is equal to $r(\tau)\cdot\mathfrak{e}$ where $r(\tau)$ is the derivation $\log g(\tau)$ and $\mathfrak{e}=\gamma(\phi_{KZ})$ where γ is the Lie version of Enriquez' section from $\mathfrak{grt}$ to $\mathfrak{grt}_{ell}$.

In order to express the linearized elliptic double shuffle equations (and to prove them), it is essential to pass from the language of power series to the language of moulds, due to the important role played by denominators which cannot be expressed using only power series. The details, along with a small but self-contained introduction to mould definitions and notation, are given in §4. We content ourselves in this introduction with the following summary.

Recall that the fact that ϕ_{KZ} satisfies the linearized double shuffle relations can be expressed by saying that the mould associated to ϕ_{KZ} is alternal and its swap is alternil, while a mould satisfying the graded version of these relations would be alternal with swap that is also alternal, or *bialternal*. We say that a mould M satisfies the *linearized elliptic double shuffle relations* if it is Δ-*bialternal*, meaning that the mould $\Delta^{-1}(M)$ is bialternal, where Δ is a certain very simple mould operator (cf. (4.2) in §4.1). In this language, our main result can be phrased as follows.

Theorem (Theorem 4.4). *The mould associated to $\mathfrak{e}(\tau)$ is Δ-bialternal, i.e. satisfies the linearized elliptic double shuffle equations.*

The proof of the theorem relies on several difficult known results. The first is Écalle's major result ([13] but see [32] for a complete proof) showing that a certain crucial mould map ψ takes moulds that are alternal with alternil swap to moulds that are bialternal. The second, by the third-named author [33] shows that $\Delta \circ \psi$ coincides with the Lie version γ of Enriquez' section on $\mathfrak{grt}$. Since the elements of $\mathfrak{grt}$ satisfy the double shuffle relations thanks to a theorem of H. Furusho [17], so that their associated moulds are alternal with alternil swap, the two above results combine to show that Enriquez' section γ takes elements of $\mathfrak{grt}$ to power series whose associated moulds are Δ-bialternal. Since ϕ_{KZ} lies in $\mathfrak{grt}$, this shows that the power series $\mathfrak{e} = \gamma(\phi_{KZ})$ is Δ-bialternal. The final ingredient is a proof that the derivation $r(\tau) = \log g(\tau)$ respects Δ-bialternality, so that $\mathfrak{e}(\tau) = r(\tau) \cdot \mathfrak{e}$ is also Δ-bialternal.

In the final paragraphs of this article, we consider two further questions related to the results above. The first is the question of whether the elliptic double shuffle relations generate all algebraic relations between $\overline{E}$-EMZ's. We show in §4.2 that the elliptic double shuffle relations are a complete set in depth 2 (Prop. 4.6), thanks to the fact that depth 2 is too small for the real multiple zeta values to occur. In higher depth, we naturally encounter problems related to the unknown transcendence properties of the real multiple zeta values. However, we are able to show that they form a complete set under the following natural conjectures from multiple zeta theory:

(a) The double shuffle relations generate all algebraic relations among the multiple zeta values modulo $2\pi i$.
(b) The elliptic double shuffle Lie algebra $\mathfrak{ds}_{\rm ell}$ [33] is isomorphic to a semi-direct product $\mathfrak{ds}_{\rm ell} \cong \mathfrak{u} \rtimes \gamma(\mathfrak{ds})$, where $\mathfrak{ds}$ is the usual double shuffle Lie algebra and γ is the extension of Enriquez' section to $\mathfrak{ds}$ obtained by identifying it with the mould map $\Delta \circ \psi$ as above.

Conjecture (a) is a standard conjecture in multiple zeta theory (cf. [20]). It would imply strong transcendence results for multiple zeta values, and therefore seems out of reach at the moment. Conjecture (b), however, is purely algebraic, and may therefore be more tractable, although it still seems difficult. It would follow for example from Enriquez' generation conjecture ([14], §10) together with the conjecture that $\mathfrak{grt}_{\rm ell} \subset \mathfrak{ds}_{\rm ell}$ (an elliptic version of Furusho's theorem [17]).

The last question addressed in the paper concerns a family of algebraic relations satisfied by the A-EMZ's: the *Fay-shuffle relations*. These relations were described in depth two in [2, 26], where it was shown that for elliptic multizetas of depth two, the Fay-shuffle relations give a complete set of $\mathbb{Q}$-linear relations. The possible completeness of the relations in all depths (depending on conjectures such as those cited above), the precise comparison between the algebras $\overline{\mathcal{E}}$ and $\overline{\mathcal{A}}$, and above all the lifting of the questions considered here to the situation not modulo $2\pi i$ are topics for further research.

1.3. **Outline of the article.** The contents of this paper are organized as follows. In §2, we introduce the algebra $\mathcal{E}^{\mathrm{geom}}$ of *geometric elliptic multizetas*, describe their relation to iterated integrals of Eisenstein series, and prove the crucial linear independence of iterated Eisenstein integrals, as well as the relation between $\mathcal{E}^{\mathrm{geom}}$ and the Lie algebra $\mathfrak{u}$. In §3 we construct the elliptic generating series $E(\tau)$ and define the E-EMZ's to be its coefficients, and $\mathcal{E}$ to be the $\mathbb{Q}$-algebra they generate. Passing modulo $2\pi i$, we prove the main structural result $\overline{\mathcal{E}}[2\pi i\tau] \simeq \mathcal{E}^{\mathrm{geom}} \otimes \overline{\mathcal{Z}}$ and its analogs for $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. In §4, we study the elliptic double shuffle equations satisfied by the mod $2\pi i$ elliptic generating series $\overline{E}(\tau)$ (or more precisely, the linearized version satisfied by its Lie version $\mathfrak{e}(\tau)$), and give evidence for the completeness of the resulting system of algebraic relations between the $\overline{E}$-EMZ's. Finally, we study a family of relations satisfied by $\overline{A'}(\tau)$. The necessary background concerning moulds is briefly summarized in §4.1.

Acknowledgments. The authors are grateful to the referee for many useful comments. This paper was written while Nils Matthes was a PhD student at Universität Hamburg under the supervision of Ulf Kühn.

2. Geometric Elliptic Multiple Zeta Values

In the first two sections, we respectively recall the definition of a certain Lie algebra $\mathfrak{u}$ of derivations [29, 35] and of iterated integrals of Eisenstein series [4, 24].

In §2.3, we introduce the algebra of geometric elliptic multizetas, and prove that it is isomorphic to the graded dual of the universal enveloping algebra of $\mathfrak{u}$. The crucial step is a linear independence result for iterated integrals of Eisenstein series, which we prove (in slightly greater generality than needed) in §2.4.

2.1. **A family of special derivations.** We begin by fixing our notation. For a $\mathbb{Q}$-algebra A, let $\mathfrak{f}_2(A) = \mathrm{Lie}_A[\![x_1, y_1]\!]$ be the *completed* (with respect to the descending central series) free Lie algebra over A on two generators x_1, y_1 with Lie bracket $[\cdot,\cdot]$. Its (topological) universal enveloping algebra will be denoted by $\mathcal{U}(\mathfrak{f}_2)_A$, and $F_2(A) := \exp(\mathfrak{f}_2(A)) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is the set of exponentials of Lie series. Note that $\mathcal{U}(\mathfrak{f}_2)_A$ is canonically isomorphic to $A\langle\langle x_1, y_1\rangle\rangle$, the A-algebra of formal power series in non-commuting variables x_1, y_1. Moreover, $\mathcal{U}(\mathfrak{f}_2)_A$ is a complete Hopf A-algebra, whose (completed) coproduct Δ is uniquely determined by $\Delta(w) = w \otimes 1 + 1 \otimes w$, for $w \in \{x_1, y_1\}$. The group $F_2(A)$ can also be characterized as the set of group-like elements of $\mathcal{U}(\mathfrak{f}_2)_A$. Likewise, the Lie algebra $\mathfrak{f}_2(A) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is precisely the subset of Lie-like (or primitive) elements. If $A = \mathbb{Q}$, we will write $\mathfrak{f}_2$ instead of $\mathfrak{f}_2(\mathbb{Q})$ and likewise $\mathcal{U}(\mathfrak{f}_2)$ and F_2 instead of $\mathcal{U}(\mathfrak{f}_2)_A$ and $F_2(A)$. Now let $\mathrm{Der}(\mathfrak{f}_2)$ denote the Lie algebra of derivations of $\mathfrak{f}_2$, and define $\mathrm{Der}_0(\mathfrak{f}_2)$ as the subalgebra of those $D \in \mathrm{Der}(\mathfrak{f}_2)$ which (i) annihilate the bracket $[x_1, y_1]$:

$$D([x_1, y_1]) = 0$$

and (ii) are such that $D(y_1)$ contains no linear term in x_1. Since $\mathfrak{f}_2$ is free, the commutator of y_1 is $\mathbb{Q} \cdot y_1$, from which it follows easily that every derivation $D \in \mathrm{Der}_0(\mathfrak{f}_2)$ is uniquely determined by its value on x_1. Similarly, the only non-zero derivation $D \in \mathrm{Der}_0(\mathfrak{f}_2)$ which annihilates y_1 is the derivation ε_0 defined by $x_1 \mapsto y_1$, $y_1 \mapsto 0$.

We next recall the definition of a family of derivations, which was first considered in [35], also played an important role in [8], and was studied in detail in [29].

Definition 2.1. For $k \geq 0$, define a derivation $\varepsilon_{2k} \in \mathrm{Der}_0(\mathfrak{f}_2)$ by

$$\varepsilon_{2n}(x_1) = \mathrm{ad}(x_1)^{2n}(y_1),$$

and denote by

$$\mathfrak{u} = \mathrm{Lie}(\varepsilon_{2n}; n \geq 0) \subset \mathrm{Der}_0(\mathfrak{f}_2)$$

the Lie subalgebra generated by the ε_{2n}.

Note that $\varepsilon_2 = -\,\mathrm{ad}([x_1, y_1])$, and thus ε_2 is central in $\mathfrak{u}$.

We also define a Lie subalgebra $\mathfrak{u}' \subset \mathfrak{u}$ as the kernel of the canonical projection $\mathfrak{u} \to \mathbb{Q}\varepsilon_0$. Equivalently,

$$\mathfrak{u}' = \mathrm{Lie}(\mathrm{ad}^k(\varepsilon_0)(\varepsilon_{2n}); n \geq 1,\, k \geq 0). \tag{2.1}$$

As seen above, every ε_{2k} is uniquely determined by its value on x_1, while ε_0 is the only non-zero derivation $D \in \mathfrak{u}$, which annihilates y_1. From this, we get

Proposition 2.2. *The $\mathbb{Q}$-linear evaluation maps*

$$v_{x_1} : \mathfrak{u} \to \mathfrak{f}_2, \quad D \mapsto D(x_1),$$
$$v_{y_1} : \mathfrak{u}' \to \mathfrak{f}_2, \quad D \mapsto D(y_1),$$

are injective.

For the applications to elliptic multizetas, it will be more natural to scale the derivations ε_{2k} as follows:

$$\widetilde{\varepsilon}_{2k} := \begin{cases} \frac{2}{(2k-2)!}\varepsilon_{2k} & k > 0 \\ -\varepsilon_0 & k = 0. \end{cases}$$

In this way, $\widetilde{\varepsilon}_{2k}$ is the image of the Eisenstein generator $\mathbf{e}_{2k}$ under the monodromy representation $\mathfrak{u}^{\mathrm{geom}} \to \mathrm{Der}_0(\mathfrak{f}_2)$ (cf. [18], Theorem 22.3).

2.2. **Iterated Eisenstein Integrals.** In a sense to be made precise below, the derivation ε_{2k} naturally corresponds to integrals of Hecke-normalized Eisenstein series of weight $2k$ (for $\mathrm{SL}_2(\mathbb{Z})$), whereas commutators of ε_{2k} correspond to *iterated integrals of Eisenstein series.* These are special cases of *iterated Shimura integrals* (or *iterated Eichler integrals*) of modular forms introduced by Manin [24], and later generalized by Brown [4].[3]

For $k \geq 0$, let $G_{2k}(q)$ be the Hecke-normalized Eisenstein series, defined by $G_0(q) := -1$ and for $k \geq 1$

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n, \quad q = e^{2\pi i \tau}$$

Here, $\sigma_\ell(n) = \sum_{d|n} d^\ell$ denotes the ℓ-th divisor function, and the B_{2k} are the Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n \geq 1} B_{2n} \frac{z^{2n}}{(2n)!}.$$

[3]To be precise, Manin defined iterated Shimura integrals of cusp forms between base points on the upper half-plane (possibly cusps), and the extension to Eisenstein series (which requires a regularization procedure) is due to Brown.

Via the exponential map $\exp : \mathfrak{H} \to D^*$, $\tau \mapsto q = \exp(2\pi i\tau)$, from the upper half-plane to the punctured unit disc

$$D^* = \{q \in \mathbb{C}, 0 < |q| < 1\},$$

we may consider G_{2k} as a function of either variable q or τ, and we shall do so according to context.

Next, we define iterated integrals of Eisenstein series. More generally, if $f(q) = \sum_{n=0}^{\infty} a_n q^n$ is such that $a_0 = 0$, (e.g. f is a cusp form), then the definition of the indefinite integral $\int_\tau^{i\infty} f(\tau_1)\mathrm{d}\tau_1$ poses no problem, as by definition f vanishes at $i\infty$. This is not the case for the Eisenstein series G_{2k}, and consequently $\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1$ diverges. It can be regularized by setting, for $k \geq 1$,

$$\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1 := \int_\tau^{i\infty} \Big[G_{2k}(\tau_1) - G_{2k}^\infty\Big]\mathrm{d}\tau_1 - \int_0^\tau G_{2k}^\infty \mathrm{d}\tau_1,$$

where $G_{2k}^\infty = -\frac{B_{2k}}{4k}$ is the constant term in the Fourier expansion of G_{2k} (if $k = 0$, a similar method works). Note that the integral of G_{2k} so defined satisfies the differential equation $\mathrm{d}f(\tau) = -G_{2k}(\tau)\mathrm{d}\tau$. The definition of regularized iterated integrals of Eisenstein series in [4], which is a special case of Deligne's tangential base point regularization ([9], §15) generalizes this construction, and runs as follows.

Let $W = \mathbb{C}[\![q]\!]^{<1}$ be the $\mathbb{C}$-algebra of formal power series, which converge on $D = \{q \in \mathbb{C} \,|\, |q| < 1\}$. We may decompose $W = W^0 \oplus W^\infty$ with $W^0 = q\mathbb{C}[\![q]\!]$ and $W^\infty = \mathbb{C}$. For a power series $f \in W$, define f^0 to be its image in W^0 under the natural projection, and define $f^\infty \in W^\infty$ likewise. For example, in the case of the Eisenstein series $G_{2k}(q)$ with $k > 0$, we have

$$G_{2k}^\infty = -\frac{B_{2k}}{4k}, \quad G_{2k}^0(q) = \sum_{n \geq 1} \sigma_{2k-1}(n) q^n.$$

We denote by $T^c(W)$ the *shuffle algebra* on the $\mathbb{C}$-vector space W. As a $\mathbb{C}$-vector space, $T^c(W)$ is simply the graded (for the length of tensors) dual of the tensor algebra $T(W) = \bigoplus_{n \geq 0} W^{\otimes n}$. It is customary to write down elements of the dual space $(W^{\otimes n})^\vee$ using bar notation $[f_1|, \dots, |f_n]$. Moreover, $T^c(W)$ is naturally a commutative $\mathbb{C}$-algebra, whose product is the shuffle product $\sqcup\!\sqcup$, defined by

$$[f_1|\dots|f_r] \sqcup\!\sqcup [f_{r+1}|\dots|f_{r+s}] = \sum_{\sigma \in \Sigma_{r,s}} f_{\sigma^{-1}(1)} \cdots f_{\sigma^{-1}(r+s)},$$

where $\Sigma_{r,s}$ denotes the set of permutations σ on $\{1, \dots, r+s\}$, such that σ is strictly increasing on both $\{1, \dots r\}$ and on $\{r+1, \dots, r+s\}$.

Now define a map $R : T^c(W) \to T^c(W)$ by the formula

$$R[f_1|\dots|f_n] = \sum_{i=0}^{n} (-1)^{n-i} [f_1|\dots|f_i] \sqcup\!\sqcup [f_n^\infty|\dots|f_{i+1}^\infty].$$

Following [4], eq. (4.11), we can now make the

Definition 2.3. Given $f_1, \dots, f_n \in W$ as above, their *regularized iterated integral* is defined as

$$I(f_1, \dots, f_n; \tau) := (2\pi i)^n \sum_{i=0}^{n} \int_\tau^{i\infty} R[f_1|\dots|f_i]_{\mathrm{d}\tau} \int_\tau^0 [f_{i+1}^\infty|\dots|f_n^\infty]_{\mathrm{d}\tau},$$

where

$$\int_a^b [f_1|\dots|f_n]_{\mathrm{d}\tau} := \int\cdots\int_{a\le\tau_1\le\ldots\le\tau_n\le b} f_1(\tau_1)\dots f_n(\tau_n)\mathrm{d}\tau_1\dots\mathrm{d}\tau_n.$$

Remark 2.4. The reason for the $(2\pi i)^n$-prefactor is to preserve the rationality of the Fourier coefficients. More precisely, if $f_1,\dots,f_n$ have rational coefficients (i.e. $f_i\in W_{\mathbb{Q}}:=\mathbb{Q}[\![q]\!]^{<1}$), then $I(f_1,\dots,f_n;\tau)\in W_{\mathbb{Q}}[\log(q)]$, where $\log(q):=2\pi i\tau$.

As is the case for usual iterated integrals ([19], Sect. 2), regularized iterated integrals satisfy the differential equation

$$\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_0} I(f_1,\dots,f_n;\tau) = -f_1(\tau_0)I(f_2,\dots,f_n;\tau_0), \tag{2.2}$$

as well as the shuffle product formula

$$I(f_1,\dots,f_r;\tau)I(f_{r+1},\dots,f_{r+s};\tau) = \sum_{\sigma\in\Sigma_{r,s}} I(f_{\sigma(1)},\dots,f_{\sigma(r+s)};\tau). \tag{2.3}$$

The only case of interest for us will be when $f_1,\dots,f_n$ are given by Eisenstein series $G_{2k_1},\dots,G_{2k_n}$. In this case, we set

$$\mathcal{G}_{\underline{k}}(\tau) := I(G_{2k_1},\dots,G_{2k_n};\tau), \tag{2.4}$$

where $\underline{k}=(k_1,\dots,k_n)$ and likewise denote by

$$\mathcal{I}^{\mathrm{Eis}} := \mathrm{Span}_{\mathbb{Q}}\{\mathcal{G}_{\underline{k}}(\tau)\} \subset \mathcal{O}(\mathfrak{H})$$

the $\mathbb{Q}$-span of all iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ for all multi-indices $\underline{k}$ (including $\mathcal{G}_\emptyset := 1$ for the empty index). Note that $\mathcal{I}^{\mathrm{Eis}}$ is a $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ by (2.3), and that it contains $\mathbb{Q}[2\pi i\tau]$ as a subalgebra, since by (2.4) we have

$$\mathcal{G}_0(\tau) = 2\pi i\tau. \tag{2.5}$$

2.3. **The τ-evolution equation and the algebra of geometric elliptic multizetas.** We now put together the special derivations $\widetilde{\varepsilon}_{2k}$ and the iterated Eisenstein integrals into a single, formal series

$$g(\tau) := \sum_{\underline{k}} \mathcal{G}_{\underline{k}}(\tau)\widetilde{\varepsilon}_{\underline{k}}, \tag{2.6}$$

where the sum is over all multi-indices $\underline{k}\in\mathbb{Z}^n_{\ge 0}$, for all n (including $\underline{k}=\emptyset$ when $n=0$), and for $\underline{k}=(k_1,\dots,k_n)$, we define $\widetilde{\varepsilon}_{\underline{k}} := \widetilde{\varepsilon}_{2k_1}\circ\ldots\circ\widetilde{\varepsilon}_{2k_n}\in\mathcal{U}(\mathfrak{u})$, the universal enveloping algebra of $\mathfrak{u}$. From (2.2), it is clear that $g(\tau)$ satisfies the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau}g(\tau) = -\Big(\sum_{k\ge 0} G_{2k}(\tau)\widetilde{\varepsilon}_{2k}\Big)g(\tau),$$

and it follows that $g(\tau)$ is group-like, i.e. it is the exponential $g(\tau)=\exp(r(\tau))$ of a Lie series

$$r(\tau) \in \widehat{\mathfrak{u}}\otimes_{\mathbb{Q}}\mathcal{I}^{\mathrm{Eis}} \tag{2.7}$$

(here $\widehat{\mathfrak{u}}$ is the graded completion of $\mathfrak{u}$, and $\otimes$ denotes the completed tensor product).

Definition 2.5. Define the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ of geometric elliptic multizetas to be the $\mathbb{Q}$-algebra generated by the coefficients of $r(\tau)\cdot x_1$.

Equivalently, $\mathcal{E}^{\mathrm{geom}}$ is equal to the $\mathbb{Q}$-vector space linearly spanned by the coefficients of the series $g(\tau) \cdot e^{x_1}$, because the coefficients of each of the power series $r(\tau) \cdot x_1$ and $g(\tau) \cdot e^{x_1}$ can be written as algebraic expressions in the coefficients of the other. Also, note that since every derivation in $\mathfrak{u}$ is uniquely determined by its value on x_1, the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ is also the same as the $\mathbb{Q}$-algebra spanned by the coefficients of $g(\tau)$, viewed as a series in the monomials $\widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n}$.

We can now state the main result of §2.

Theorem 2.6. *For every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, there is an isomorphism*

$$\mathcal{U}(\mathfrak{u})^\vee \otimes_{\mathbb{Q}} A \cong \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} A$$

of A-algebras. In particular, $\mathcal{E}^{\mathrm{geom}}$ is a commutative, graded Hopf algebra in a natural way.

Proof. The main ingredient in the proof is that the iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are linearly independent over $\mathbb{C}$, as functions in τ. More precisely, by Corollary 2.9, proved in the next section, there is a natural isomorphism

$$\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A,$$

where $T^c(V_{\mathrm{Eis}})$ is the shuffle algebra on the $\mathbb{Q}$-vector space V_{Eis} spanned by all Eisenstein series G_{2k}, $k \geq 0$.

Assuming Corollary 2.9 for the moment, the proof of Theorem 2.6 proceeds as follows. Since the tensor algebra $T(V_{\mathrm{Eis}})$ is freely generated by one element in every even degree $2k \geq 0$, we get a canonical surjection $T(V_{\mathrm{Eis}}) \to \mathcal{U}(\mathfrak{u})$ of $\mathbb{Q}$-algebras, which induces by duality an injection

$$\iota : \mathcal{U}(\mathfrak{u})^\vee \hookrightarrow T^c(V_{\mathrm{Eis}}) \cong \mathcal{I}^{\mathrm{Eis}}.$$

On the other hand, choosing a (homogeneous) linear basis $\mathcal{B}$ of $\mathcal{U}(\mathfrak{u})$, the element $g(\tau)$ naturally defines a map

$$\begin{aligned} \widetilde{\iota} : \mathcal{U}(\mathfrak{u})^\vee &\hookrightarrow \mathcal{I}^{\mathrm{Eis}} \\ b^\vee &\mapsto b^\vee(g(\tau)), \end{aligned}$$

where $b^\vee \in \mathcal{B}^\vee$ are the dual basis elements. Clearly, the image of $\widetilde{\iota}$ does not depend on the choice of basis, and equals $\mathcal{E}^{\mathrm{geom}}$ by definition. On the other hand, it is easy to see that the maps $\iota, \widetilde{\iota} : \mathcal{U}(\mathfrak{u})^\vee \to \mathcal{I}^{\mathrm{Eis}}$ are equal, whence the result for $A = \mathbb{Q}$, and the general case follows simply by extension of scalars. Finally, it is well-known that the universal enveloping algebra of any graded Lie algebra has a natural structure of a (cocommutative) graded Hopf algebra, thus $\mathcal{U}(\mathfrak{u})^\vee$ is naturally a (commutative) graded Hopf algebra. □

2.4. **Linear independence.** In this subsection, we complete the proof of Theorem 2.6 by showing that the family of iterated Eisenstein integrals is linearly independent over $\mathbb{C}$, and that as a consequence $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} \mathbb{C} \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} \mathbb{C}$ as $\mathbb{C}$-algebras. Although these results can meanwhile also be deduced from [27], which proves linear independence of iterated integrals of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ over the (fraction field of the) ring of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$, we give a slightly different proof here in the special case of Eisenstein series which has the advantage that it works over a larger field of coefficients.

The main idea is to use the following general linear independence result.

Theorem 2.7 ([10]). *Let $(\mathcal{A}, \mathrm{d})$ be a differential algebra over a field k of characteristic zero, whose ring of constants* $\ker(\mathrm{d})$ *is precisely equal to k. Let $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. a subfield such that $\mathrm{d}\mathcal{C} \subset \mathcal{C}$), X any set with associated free monoid X^*. Suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution to the differential equation*

$$\mathrm{d}S = M \cdot S,$$

where $M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle$ is a homogeneous series of degree 1, with initial condition $S_1 = 1$, where S_1 denotes the coefficient of the empty word in the series S. The following are equivalent:

(i) *The family of coefficients $(S_w)_{w \in X^*}$ of S is linearly independent over $\mathcal{C}$.*

(ii) *The family $\{u_x\}_{x \in X}$ is linearly independent over k, and we have*

$$\mathrm{d}\mathcal{C} \cap \mathrm{Span}_k(\{u_x\}_{x \in X}) = \{0\}. \tag{2.8}$$

Using this theorem, we can now prove linear independence of iterated Eisenstein integrals.

Theorem 2.8. *The family $\{\mathcal{G}_{\underline{k}}(\tau)\}$ is linearly independent over* $\mathrm{Frac}(\mathbb{Z}[\![q]\!])$.

Proof. We will apply Theorem 2.7 with the following parameters:

- $k = \mathbb{Q}$, $\mathcal{A} = \mathbb{Q}[\log(q)](\!(q)\!)$ with differential $\mathrm{d} = q\frac{\partial}{\partial q}$, and $\mathcal{C} = \mathrm{Frac}(\mathbb{Z}[\![q]\!])$ (the latter is a differential field by the quotient rule for derivatives)
- $X = \{a_{2k}\}_{k \geq 0}$, $u_{a_{2k}} = -G_{2k}(q)$, hence

$$M(q) = -\sum_{k \geq 0} G_{2k}(q) a_{2k}.$$

With these conventions, it follows from (2.2) that the formal series

$$1 + \int_q^0 [M]_{\mathrm{d}\log q} + \int_q^0 [M|M]_{\mathrm{d}\log q} + \ldots \in \mathcal{O}(\mathfrak{H})\langle\langle X \rangle\rangle,$$

with the iterated integrals regularized as in Section 2.2, is a solution of the differential equation $\mathrm{d}S = M \cdot S$, with $S_1 = 1$. Consequently, the coefficient of the word $w = a_{2k_1} \ldots a_{2k_n}$ in S is equal to $\mathcal{G}(2k_1, \ldots, 2k_n; \tau)$. Moreover, since the $\mathbb{Q}$-linear independence of the Eisenstein series is well-known (cf. e.g. [34], VII.3.2), it remains to verify (2.8) in our situation.

To this end, assume that there exist $\alpha_{2k} \in \mathbb{Q}$, all but finitely many of which are equal to zero, such that

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) \in \mathrm{d}\mathcal{C}. \tag{2.9}$$

Clearing denominators, we may assume that $\alpha_{2k} \in \mathbb{Z}$. Furthermore, from the definition of $\mathrm{d} = q\frac{\partial}{\partial q}$, one sees that the image $\mathrm{d}\mathcal{C}$ of the differential operator d does not contain any constant except for zero. Therefore, the coefficient of the trivial word 1 in (2.9) vanishes; in other words

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) = \sum_{k \geq 1} \alpha_{2k} E_{2k}^0(q) \in q\mathbb{Q}[\![q]\!].$$

Now the differential d is invertible on $q\mathbb{Q}[\![q]\!]$, and inverting d is the same as integrating. Hence (2.9) is equivalent to

$$\sum_{k \geq 1} \alpha_{2k} \mathcal{G}_{2k}^0(\tau) \in \mathcal{C}, \quad \mathcal{G}_{2k}^0(\tau) := \int_q^0 E_{2k}^0(q_1) \frac{\mathrm{d}q_1}{q_1}. \tag{2.10}$$

But this is absurd, unless all the α_{2k} vanish, as we shall see now. Indeed, if $f \in \mathcal{C} = \mathrm{Frac}(\mathbb{Z}[\![q]\!])$, then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $f \in \mathbb{Z}[m^{-1}](\!(q)\!)$. This follows from the well-known inversion formula for power series. On the other hand, the coefficient of q^p in $\mathcal{G}^0_{2k}(\tau)$, for p a prime number, is given by

$$\frac{\sigma_{2k-1}(p)}{p} = \frac{p^{2k-1}+1}{p} \equiv \frac{1}{p} \mod \mathbb{Z}.$$

Thus, we must have $\frac{1}{p}\sum_{k\geq 1}\alpha_{2k} \in \mathbb{Z}[m^{-1}]$, for every prime number p, in particular $\sum_{k\geq 1}\alpha_{2k}$ is divisible by infinitely many primes (namely, at least all the primes which don't divide m), which implies $\sum_{k\geq 1}\alpha_{2k} = 0$.

Now assume that k_1 is the smallest positive, even integer with the property that $\alpha_{k_1} \neq 0$. Consider the coefficient of $q^{p^{k_1}}$ in $\mathcal{G}^0_{2k}(\tau)$, which is equal to

$$\frac{\sigma_{2k-1}(p^{k_1})}{p^{k_1}} = \frac{1}{p^{k_1}}\sum_{j=0}^{k_1} p^{j(2k-1)} \equiv \begin{cases} \frac{1}{p^{k_1}} \mod \mathbb{Z} & \text{if } 2k > k_1 \\ \frac{1}{p^{k_1}} + \frac{1}{p} \mod \mathbb{Z} & \text{if } 2k = k_1. \end{cases}$$

By (2.10), we have $\frac{\alpha_{k_1}}{p} + \frac{1}{p^{k_1}}\sum_{k\geq 1}\alpha_{2k} \in \mathbb{Z}[m^{-1}]$, and by what we have seen before, $\sum_{k\geq 1}\alpha_{2k} = 0$. Hence $\frac{\alpha_{k_1}}{p} \in \mathbb{Z}[m^{-1}]$, for every prime number p, which again implies $\alpha_{k_1} = 0$, in contradiction to our assumption $\alpha_{k_1} \neq 0$. Therefore, in (2.10), we must have $\alpha_{2k} = 0$ for all $k \geq 1$ and (2.8) is verified. $\square$

Corollary 2.9. *The iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are $\mathbb{C}$-linearly independent, and for every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, we have a natural isomorphism of A-algebras*

$$\psi_A : T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A \to \mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A$$
$$[G_{2k_1}|\ldots|G_{2k_n}] \mapsto \mathcal{G}_{\underline{k}}(\tau),$$

where $\underline{k} = (k_1,\ldots,k_n)$ and $V_{\mathrm{Eis}} = \mathrm{Span}_{\mathbb{Q}}\{G_{2k}(\tau) \,|\, k \geq 0\} \subset \mathcal{O}(\mathfrak{H})$.

Proof. Since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[\![q]\!])$, Theorem 2.8 shows in particular that the $\mathcal{G}_{\underline{k}}$ are linearly independent over $\mathbb{Q}$. Since the Eisenstein series G_{2k} have coefficients in $\mathbb{Q}$, it follows from the definition that $\mathcal{G}_{\underline{k}} \in \mathbb{Q}(\!(q)\!)[\log(q)]$, and elements of $W_{\mathbb{Q}}[\log(q)] = \mathbb{Q}(\!(q)\!)[\log(q)]$ are linearly independent over $\mathbb{Q}$, if and only they are so over $\mathbb{C}$.

For the second statement, it is clear that ψ_A is a homomorphism of $\mathbb{Q}$-algebras (since both sides are endowed with the shuffle product) and that it is surjective. The injectivity of ψ_A is just the A-linear independence of iterated Eisenstein integrals. $\square$

Corollary 2.10. *We have $\mathcal{I}^{\mathrm{Eis}} \cap \mathbb{C} = \mathbb{Q}$ and $\mathcal{E}^{\mathrm{geom}} \cap \mathbb{C} = \mathbb{Q}$. In particular, the $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ generated by $\mathcal{I}^{\mathrm{Eis}}$ and $\mathbb{C}$ is canonically isomorphic to $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof. If some linear combination of the $\mathcal{G}_{\underline{k}}$ with coefficients in $\mathbb{Q}$ were equal to $c \in \mathbb{C}$, then since $\mathcal{G}_{\emptyset} = 1$, this would give a linear relation

$$-c\mathcal{G}_{\emptyset} + \sum_{\underline{k}} a_{\underline{k}}\mathcal{G}_{\underline{k}} = 0,$$

so by Theorem 2.8 we must have $c = a_{\emptyset}$, i.e. $c \in \mathbb{Q}$. The second statement follows from the first, since by definition of $\mathcal{E}^{\mathrm{geom}}$, it lies inside $\mathcal{I}^{\mathrm{Eis}}$. $\square$

3. The generating series of elliptic multizetas

In the first part of this section we will recall the definition of the elliptic associator defined by B. Enriquez and use it to define a power series $E \in F_2(\mathcal{Z})$; we then set $E(\tau) = g(\tau)\cdot E$, where $g(\tau)$ is the automorphism studied in the previous section. We call $E(\tau)$ the *elliptic generating series*, and its coefficients the *elliptic multizetas*. We define $\mathcal{E}$ to be the $\mathbb{Q}$-algebra generated by the elliptic multizetas. This algebra is essentially the same as the one generated by the coefficients of the elliptic associator, but the elliptic multizetas themselves are different from those coefficients (which are called "analogues elliptiques de nombres multizetas" by Enriquez).

In the remainder of the section, we work modulo $2\pi i$. In particular, we consider the power series $\overline{\Phi}_{KZ}$ and $\overline{E}$ which are obtained from Φ_{KZ} and E by reducing the coefficients from $\mathcal{Z}$ to $\overline{\mathcal{Z}} = \mathcal{Z}/\langle (2\pi i)^2\rangle$.

In §3.2, we give an expression for $\overline{E}$ which relates it explicitly to the Drinfel'd associator $\overline{\Phi}_{KZ}$. In §3.3 we use this expression for $\overline{E}$ to prove the equality

$$\overline{\mathcal{E}}[2\pi i\tau] = \mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

These two results will allow us to compute the algebraic relations satisfied by the elliptic multizetas, as well as algebraic relations satisfied by Enriquez' elliptic multizetas, which are the coefficients of the elliptic associator (always modulo $2\pi i$). Because these results necessitate a very brief introduction to mould theory, we introduce them in §4.

3.1. **Definition of the elliptic generating series $E(\tau)$.** Throughout this section, we use the following change of variables: $a = y_1$ and $b = x_1$. This change of variables will be applied to all the expressions in x_1, y_1 encountered in the previous section, such as $g(\tau)\cdot y_1$, and we will also express other quantities studied by B. Enriquez in terms of a and b, in particular the elliptic associator. The purpose of this change of variables is for the application of mould theory in §4.

Let Ass_μ denote the set of genus zero associators $\Phi \in F_2(\mathbb{C})$ such that the coefficient of ab in Φ is equal to $\mu^2/24$ [11]. We will use the same elements t_{01}, t_{02}, t_{12} as in [14], but rewritten in the variables a, b:

$$t_{01} = Ber_b(-a),\quad t_{02} = Ber_{-b}(a),\quad t_{12} = [a,b], \tag{3.1}$$

where

$$Ber_x(y) = \frac{\mathrm{ad}(x)}{e^{\mathrm{ad}(x)}-1}(y),$$

so that $t_{01}+t_{02}+t_{12} = 0$. Recall that Enriquez showed that a section from Ass_μ to the set of elliptic associators is given by mapping $\Phi \in Ass_\mu$ to the elliptic associator (μ, Φ, A, B) defined by

$$A = \Phi(t_{01}, t_{12})e^{\mu\, t_{01}}\Phi(t_{01}, t_{12})^{-1}$$
$$B = e^{\mu\, t_{12}/2}\Phi(t_{02}, t_{12})e^{b}\Phi(t_{01}, t_{12})^{-1}$$

(this is denoted (μ, Φ, A_+, A_-) in [14]).

In this section we take $\mu = 2\pi i$, so $\mu^2/24 = -\zeta(2)$, and consider Φ_{KZ}, the Drinfeld associator, whose coefficients are the (shuffle-regularized) multizetas [16]. The Lie algebra $\mathfrak{f}_2 = \mathrm{Lie}[\![a,b]\!]$ is topologically generated by a and b, but since the operator Ber_b is invertible, we have

$$a = -Ber_b^{-1}(t_{01}) = \Big(\frac{e^{\mathrm{ad}(b)}-1}{\mathrm{ad}(b)}\Big)(-t_{01}), \tag{3.2}$$

so that we can just as well take t_{01} and b as generators. Similarly, we can take $e^{t_{01}}$ and e^b as generators of the group $F_2 = F_2(\mathbb{Q}) = \exp(\mathfrak{f}_2)$, which is a priori generated by e^a and e^b.

Let us define an automorphism σ of $F_2(\mathcal{Z})$, where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra of multizetas, by

$$\begin{aligned}\sigma(e^{t_{01}}) &= \Phi_{KZ}(t_{01}, t_{12}) e^{t_{01}} \Phi_{KZ}(t_{01}, t_{12})^{-1}\\ \sigma(e^b) &= e^{\pi i t_{12}} \Phi_{KZ}(t_{02}, t_{12}) e^b \Phi_{KZ}(t_{01}, t_{12})^{-1}.\end{aligned}$$

We set

$$E = 1 - a + \sigma(a), \quad C = \exp(E - 1).$$

The automorphism σ extends to an automorphism of the completed enveloping algebra $\mathcal{U}(\mathfrak{f}_2)$, and restricts to an automorphism of $\mathfrak{f}_2$. Thus the power series $\sigma(a)$ is Lie-like, so $E - 1$ is Lie-like. Thus, by Lazard elimination, it can be expressed in the variables a and $c_i = \mathrm{ad}(a)^{i-1}(b)$, $i \geq 1$. From now on, we expand all group-like and Lie-like power series in these variables, and when we refer to the *coefficients* of such power series, we intend the coefficients of the power series in these variables. (This language is adapted to mould theory and will be useful in §4.) Up to degree 4, the explicit expansion of E is given by

$$E = 1 - \frac{i\pi}{2} c_3 + \frac{\pi^2}{6} c_4 + \frac{i\pi}{12} [c_1, c_3]. \tag{3.3}$$

We now recall the automorphism

$$g(\tau) = \sum_{\underline{k}} \mathcal{G}_{\underline{k}}(\tau) \widetilde{\varepsilon}_{\underline{k}} \tag{3.4}$$

defined in the previous section, and consider it as an automorphism of the group $F_2(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z})$. Acting on a, we find

$$g(\tau) \cdot a = a - \frac{1}{2\pi i} \mathcal{G}_2(\tau)\, \mathrm{ad}(a)^2(b) + \frac{3}{(2\pi i)^2} \mathcal{G}_{0,2}(\tau)\, \mathrm{ad}(b)^2(a) + \cdots$$

In [14], Enriquez studied the elliptic associator

$$\big(2\pi i, \Phi_{KZ}, A(\tau), B(\tau)\big) \tag{3.5}$$

where

$$A(\tau) = g(\tau) \cdot A, \quad B(\tau) = g(\tau) \cdot B.$$

In analogy with this, we set

$$E(\tau) = g(\tau) \cdot E = g(\tau)\big(1 - a + \sigma(a)\big), \quad C(\tau) = \exp\big(E(\tau) - 1\big).$$

As above, $g(\tau)$ extends to an automorphism of the universal enveloping algebra, so in particular it preserves the Lie algebra $\mathfrak{f}_2 \otimes_{\mathbb{Q}} (\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z})$. Thus $E(\tau) - 1$ is Lie-like, and $C(\tau)$ is group-like.

Definition 3.1. The Lie-like power series $E(\tau) - 1$ is called the *elliptic generating series*, and its coefficients are the *elliptic multizetas*. For $\underline{k} = (k_1, \ldots, k_r)$ we write $E(\underline{k})$ for the coefficient in $E(\tau) - 1$ of the monomial $c_{k_1} \cdots c_{k_r}$. The $\mathbb{Q}$-algebra generated by the elliptic multizetas $E(\underline{k})$ is denoted $\mathcal{E}$.

We can use $C(\tau)$ to obtain a vector space basis for $\mathcal{E}$.

Lemma 3.2. *The underlying vector space of $\mathcal{E}$ is spanned by the coefficients of $C(\tau)$. The element $2\pi i$ belongs to $\mathcal{E}$.*

Proof. Let $\mathcal{E}'$ denote the $\mathbb{Q}$-vector space generated by the coefficients of $C(\tau)$. Then $\mathcal{E}'$ is in fact a $\mathbb{Q}$-algebra, because $C(\tau)$ is a group-like power series so that the product of two of its coefficients can be written as a linear combination of such by using the (multiplicative) shuffle relations. Since $E(\tau) = 1 + \log(C(\tau))$, we see that the coefficients of $E(\tau)$ can be expressed as algebraic and thus linear combinations of the coefficients of $C(\tau)$, so that $\mathcal{E} \subset \mathcal{E}'$. Conversely, since $C(\tau) = \exp\big(E(\tau) - 1\big)$, the coefficients of $C(\tau)$ are algebraic combinations of those of $E(\tau)$, and therefore lie in $\mathcal{E}$, so $\mathcal{E}' \subset \mathcal{E}$, which completes the proof of the first statement. For the second statement, we first note from (3.4) that $E(\tau) = g(\tau) \cdot E = E + \sum_{\underline{k} \neq \emptyset} \mathcal{G}_{\underline{k}}(\tau)\tilde{\varepsilon}_{\underline{k}} \cdot E$. No $\tilde{\varepsilon}_{\underline{k}}$ acting on E can produce another c_3 term; in fact $\tilde{\varepsilon}_0(c_3) = 0$ and all other terms $\tilde{\varepsilon}_{\underline{k}}$ send c_3 to a term of higher weight. Thus, the coefficient of c_3 in $E(\tau)$ is the coefficient of c_3 in E, which is $-\pi i/2$ by (3.3). Therefore the $\mathbb{Q}$-vector space generated by πi lies in $\mathcal{E}$. □

3.2. **An expression for E modulo $2\pi i$.** From now until the end of this section, we work modulo $2\pi i$, in the sense that if a series has coefficients in $\mathcal{Z}$, we reduce these coefficients to the quotient $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ modulo the idea generated by $(2\pi i)^2$, or equivalently, by $\zeta(2)$. We use overlining to denote the reduced objects. The goal of the section is to obtain an expression for $\overline{E}$ that relates it directly to the reduced Drinfeld associator $\overline{\Phi}_{KZ}$.

In order to approach this result, we will move from the Lie algebra of derivations over to power series in a and b by using the map given by evaluation at a. This is important because it allows us to compare derivations with power series in a and b such as $\overline{\Phi}_{KZ}$.

Let v_a denote the linear map given by evaluation at a, i.e.

$$\begin{aligned} v_a : \mathrm{Der}_0(\mathfrak{f}_2) &\to \mathfrak{f}_2 \\ D &\mapsto D(a). \end{aligned} \tag{3.6}$$

Let the push-operator be defined to cyclically permute the powers of a between the letters b in a monomial:

$$push(a^{k_0}b \cdots ba^{k_r}) = a^{k_r}ba^{k_0}b \cdots a^{k_{r-1}}, \tag{3.7}$$

extended to polynomials and power series by linearity. A power series is said to be *push-invariant* if $push(p) = p$. It is shown in [33] that the restriction of v_a to the Lie subalgebra generated by $\mathrm{Der}_0(\mathfrak{f}_2) \setminus \mathbb{Q}\varepsilon_0$ is an injective linear map whose image is equal to the space of push-invariant Lie series $\mathfrak{f}_2^{\mathrm{push}} \subset \mathfrak{f}_2$. The map v_a transports the Lie bracket and exponential from $\mathrm{Der}_0(\mathfrak{f}_2)$ to $\mathfrak{f}_2^{\mathrm{push}}$ as follows:

$$\langle D(a), D'(a)\rangle = [D, D'](a), \quad \exp_a\big(D(a)\big) = 1 + \sum_{n\geq 1} \frac{1}{n!} D^n(a) \tag{3.8}$$

We have the useful identity

$$\exp(D) \cdot a = a + D(a) + \frac{1}{2}D^2(a) + \cdots = a - 1 + \exp_a(D(a)). \tag{3.9}$$

Let $\mathfrak{grt}_{\mathrm{ell}}$ be the elliptic Grothendieck-Teichmüller Lie algebra defined by B. Enriquez in [14]. Not surprisingly, this Lie algebra will be an essential tool in proving our results. Let us recall some of the basic facts concerning it. Firstly, Enriquez showed that there is a natural Lie morphism $\mathfrak{grt}_{\mathrm{ell}} \to \mathrm{Der}_0(\mathfrak{f}_2)$. It was further shown in [33] that this map is injective. We will identify $\mathfrak{grt}_{\mathrm{ell}}$ with its image in $\mathrm{Der}_0(\mathfrak{f}_2)$.

Enriquez also proved the following results. There is a canonical surjection $\mathfrak{grt}_{\mathrm{ell}} \to \mathfrak{grt}$. Let $\mathfrak{r}_{\mathrm{ell}}$ denote the kernel; then it is easy to see that $\mathfrak{u} \subset \mathfrak{r}_{\mathrm{ell}}$. Finally, Enriquez gave a section $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{\mathrm{ell}}$ of the canonical surjection, and $\mathfrak{grt}_{\mathrm{ell}}$ has the form of a semi-direct product

$$\mathfrak{grt}_{\mathrm{ell}} \cong \mathfrak{r}_{\mathrm{ell}} \rtimes \gamma(\mathfrak{grt}).$$

We write γ_a for the composition map $v_a \circ \gamma$, so that

$$\gamma_a : \mathfrak{grt} \to \mathfrak{f}_2^{\mathrm{push}}. \tag{3.10}$$

Let $\exp^{\odot}$ denote the ("twisted Magnus") exponential map $\exp^{\odot} : \mathfrak{grt} \to GRT$. Then we have the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Der}^*(\mathrm{Lie}[[x,y]]) & \leftarrow & \mathfrak{grt} & \overset{\gamma}{\to} & \mathfrak{grt}_{\mathrm{ell}} & \overset{v_a}{\longrightarrow} & \mathfrak{f}_2^{\mathrm{push}} \\ \exp\downarrow & & \exp^{\odot}\downarrow & & exp\downarrow & & \downarrow \exp_a \\ \mathrm{Aut}^*(\mathrm{Lie}[[x,y]]) & \leftarrow & GRT & \overset{\Gamma}{\to} & GRT_{\mathrm{ell}} & \overset{1-a+v_a}{\longrightarrow} & F_2, \end{array}$$

where Γ is the group homomorphism that makes the middle square commmute. The upper map $\mathfrak{grt} \to \mathrm{Der}^*(\mathrm{Lie}[[x,y]])$ in the left-hand square is the map that takes a Lie element $\psi \in \mathfrak{f}_2$ to the associated *Ihara derivation* D_ψ defined by

$$D_\psi(x) = 0, \quad D_\psi(y) = [\psi(x,y), y]. \tag{3.11}$$

Ihara [21, 22] studied these derivations in detail, and in particular, he showed that if $\Psi = \exp^{\odot}(\psi)$ and A_Ψ denotes the automorphism $\exp(D_\psi)$ of $\mathcal{U}(\mathrm{Lie}[[x,y]])$, then

$$A_\Psi(x) = x, \quad A_\Psi(y) = \Psi\ y\ \Psi^{-1}. \tag{3.12}$$

The lower horizontal map of the left-hand square is thus given by $\Psi \mapsto A_\Psi$. In analogy with γ_a, we set $\Gamma_a = v_a \circ \Gamma$.

We can now state the main result of this subsection.

Theorem 3.3. *Let $\overline{E}$ be obtained from E by reducing the coefficients from $\overline{\mathcal{Z}}$ to $\mathcal{Z}/\langle(2\pi i)^2\rangle$. Then*

$$\overline{E} = \Gamma_a(\overline{\Phi}_{KZ}).$$

Proof. Let $\psi \in \mathfrak{grt}$, and let $\Psi = \exp^{\odot}(\psi) \in GRT$. Then $\gamma(\psi) \in \mathfrak{grt}_{\mathrm{ell}} \subset \mathrm{Der}_0(\mathfrak{f}_2)$ and $\Gamma(\Psi) = \exp\big(\gamma(\psi)\big) \in GRT_{\mathrm{ell}} \subset \mathrm{Aut}^0(\mathfrak{f}_2)$. The proof is based on a result from [14], Lemma-Definition 4.6, which states that the automorphism $\Gamma(\Psi)$ acts as follows:

$$\begin{aligned} \Gamma(\Psi)(t_{01}) &= \Psi(t_{01},t_{12})t_{01}\Psi(t_{01},t_{12})^{-1} \\ \Gamma(\Psi)(b) &= \log\big(\Psi(t_{02},t_{12})e^b\Psi(t_{01},t_{12})^{-1}\big), \end{aligned} \tag{3.13}$$

where t_{01} is as in (3.1). Recall from (3.2) that we can take t_{01} and b as generators of $\mathfrak{f}_2$.

Recall that $\overline{\Phi}_{KZ} \in GRT \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. (This is the reason for which we work mod $2\pi i$, since the term $-\zeta(2)[x,y]$ in Φ_{KZ} means that it does not lie in GRT, preventing us from taking advantage of the results on $\mathfrak{grt}_{\mathrm{ell}}$.) Set $\phi_{KZ} = \log^{\odot}(\overline{\Phi}_{KZ})$, so that $\phi_{KZ} \in \mathfrak{grt} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. Let $\overline{\sigma}$ be the automorphism of $F_2(\overline{\mathcal{Z}})$ obtained from σ by reducing modulo $2\pi i$, i.e.

$$\begin{aligned} \overline{\sigma}(e^{t_{01}}) &= \overline{\Phi}_{KZ}(t_{01},t_{12})e^{t_{01}}\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1} = \overline{A}' \\ \overline{\sigma}(e^b) &= \overline{\Phi}_{KZ}(t_{02},t_{12})e^b\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1} = \overline{B}, \end{aligned}$$

where we set $A' = A^{\frac{1}{2\pi i}}$ and $\overline{A}'$ and $\overline{B}$ are the reductions of A' and B mod $2\pi i$.

Comparing with the values of $\Gamma(\overline{\Phi}_{KZ})$ from (3.13) on the generators t_{01}, b of $\mathfrak{f}_2$, we find that $\overline{\sigma} = \Gamma(\overline{\Phi}_{KZ})$, so $\log(\overline{\sigma}) = \gamma(\phi_{KZ})$. Evaluating on a, we set $\mathfrak{e}$ to be the element

$$\mathfrak{e} = \log(\overline{\sigma})(a) = v_a\big(\gamma(\phi_{KZ})\big) = \gamma_a(\phi_{KZ}), \tag{3.14}$$

so by (3.9), we have

$$\overline{\sigma}(a) = a - 1 + \exp_a(\mathfrak{e}) = a - 1 + \Gamma_a(\overline{\Phi}_{KZ}). \tag{3.15}$$

Since $E = 1 - a + \sigma(a)$, we have

$$\overline{E} = 1 - a + \overline{\sigma}(a) = \Gamma_a(\overline{\Phi}_{KZ}), \tag{3.16}$$

which concludes the proof. □

Corollary 3.4. *The $\mathbb{Q}$-algebra generated by the coefficients of $\overline{E}$ is all of $\overline{\mathcal{Z}}$.*

Proof. As remarked earlier (Lemma 3.2), the $\mathbb{Q}$-algebra linearly spanned by the coefficients of a group-like power series is equal to that multiplicatively generated by the coefficients of its log. Therefore in particular, since the coefficients of $\overline{\Phi}_{KZ}$ linearly span $\overline{\mathcal{Z}}$, the coefficients of ϕ_{KZ} multiplicatively generate the same ring. Similarly, the $\mathbb{Q}$-algebra generated by the coefficients of $\gamma(\phi_{KZ})$ (written in a basis of $\mathfrak{grt}$, say) is the same as the one linearly spanned by the coefficients of $\overline{E} = \Gamma(\overline{\Phi}_{KZ})$. But since the section map $\gamma_a : \mathfrak{grt} \to \mathfrak{f}_2^{\rm push}$ is injective and defined over $\mathbb{Q}$, it maps a linear basis of $\mathfrak{grt}$ to linearly independent elements of $\mathfrak{f}_2^{\rm push}$ with the same coefficients, so the coefficients of $\gamma_a(\phi_{KZ})$ again generate the same $\mathbb{Q}$-algebra as those of ϕ_{KZ}, which is $\overline{\mathcal{Z}}$. □

3.3. **Structure of the $\mathbb{Q}$-algebras $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$.** Since $E(\tau) = g(\tau) \cdot E$, the $\mathbb{Q}$-algebra $\mathcal{E}$ generated by the coefficients of $E(\tau)$ is contained in the $\mathbb{Q}$-algebra generated by $\mathcal{E}^{\rm geom}$ (the ring generated by the coefficients of $g(\tau)$) together with the algebra $\mathcal{Z}[2\pi i]$ of multizetas (generated by the coefficients of E), and the same holds for $A(\tau) = g(\tau) \cdot A$ and $B(\tau) = g(\tau) \cdot B$. Thanks to Corollary 2.10, the algebra generated inside $\mathcal{O}(\mathfrak{H})$ by $\mathcal{E}^{\rm geom}$ and $\mathcal{Z}[2\pi i]$ is isomorphic to their tensor product over $\mathbb{Q}$. By the same result, the ideal generated by $1 \otimes 2\pi i$ inside the tensor product lies in the arithmetic part $1 \otimes \mathcal{Z}[2\pi i]$, so the quotient of $\mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$ by this ideal is isomorphic to $\mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. The main result of this subsection will allow us to compare the three subrings $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ of $\mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. Recall that $\mathfrak{f}_2$ denotes the completed free Lie algebra on generators a, b, and that we have $\mathfrak{u} = \langle \varepsilon_0 \rangle \oplus \mathfrak{u}'$ where $\mathfrak{u}'$ is as defined in (2.1).

Let $\mathrm{Der}_0'(\mathfrak{f}_2)$ be the kernel of the linear map $\mathrm{Der}^0(\mathfrak{f}_2) \to \mathbb{Q}[\varepsilon_0]$ which annihilates every element of a basis of $\mathrm{Der}_0(\mathfrak{f}_2)$ except ε_0.

Theorem 3.5. *Let $f \in \mathfrak{f}_2$ have the property that the evaluation map $v_f : \mathrm{Der}_0'(\mathfrak{f}_2) \to \mathfrak{f}_2$ given by $v_f(D) = D(f)$ is injective. Let $F = \Gamma(\overline{\Phi}_{KZ}) \cdot f$, and let $\mathfrak{Z}$ be the $\mathbb{Q}$-subalgebra of $\overline{\mathcal{Z}}$ generated by the coefficients of F. Let*

$$F(\tau) = g(\tau) \cdot F.$$

Let $\mathcal{F}$ be the $\mathbb{Q}$-algebra generated by the coefficients of $F(\tau)$. Then

$$\mathcal{F}[2\pi i\tau] = \mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \mathfrak{Z} \subset \mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

Proof. Recall from (2.6) and (3.19) that the automorphism $g(\tau)$ satisfies $g(\tau) = \exp\big(r(\tau)\big)$ for a derivation $r(\tau) \in \widehat{\mathfrak{u}} \otimes_{\mathbb{Q}} \mathcal{E}^{\rm geom}$. Since we also have $\Gamma(\overline{\Phi}_{KZ}) = \exp\big(\gamma(\phi_{KZ})\big)$, we can write

$$F(\tau) = \exp\big(r(\tau)\big)\cdot\exp\big(\gamma(\phi_{KZ})\big)\cdot f = \exp\big(ch_{[\,,\,]}(r(\tau),\gamma(\phi_{KZ})\big)\cdot f,$$

where $ch_{[\,,\,]}$ denotes the Campbell-Hausdorff law on the Lie algebra of derivations of $\mathfrak{f}_2$ with the usual bracket. In analogy with (3.8), we define

$$\langle D(f), D'(f)\rangle = [D,D'](f), \quad \exp_f\big(D(f)\big) = 1 + \sum_{n\geq 1}\frac{1}{n!}D^n(f),$$

and we have

$$\exp(D)\cdot f = f + D(f) + \frac{1}{2}D^2(f) + \cdots = f - 1 + \exp_f\big(D(f)\big).$$

In particular this gives

$$F(\tau) = \exp\big(ch_{[\,,\,]}(r(\tau),\gamma(\phi_{KZ})\big)\cdot f = f - 1 + \exp_f\big(ch_{[\,,\,]}(r(\tau),\gamma(\phi_{KZ})\cdot f\big).$$

Letting $\tilde{F}(\tau) = F(\tau) - f + 1$, we have

$$\tilde{F}(\tau) = \exp_f\big(ch_{[\,,\,]}(r(\tau),\gamma(\phi_{KZ})\big)\cdot f,$$

so

$$\begin{aligned}\log_f\big(\tilde{F}(\tau)\big) &= ch_{[\,,\,]}(r(\tau),\gamma(\phi_{KZ})\cdot f\big)\\ &= r(\tau)\cdot f + \gamma(\phi_{KZ})\cdot f + \frac{1}{2}[r(\tau),\gamma(\phi_{KZ})]\cdot f + \cdots\\ &= r(\tau)\cdot f + \gamma(\phi_{KZ})\cdot f + \frac{1}{2}\langle r(\tau)\cdot f, \gamma(\phi_{KZ})\cdot f\rangle + \cdots \qquad (3.17)\end{aligned}$$

We rewrite this as

$$\log_f\big(\tilde{F}(\tau)\big) = r(\tau)\cdot f + \gamma(\phi_{KZ})\cdot f + s(\tau) \qquad (3.18)$$

where $s(\tau)$ is the sum of all the bracketed terms in the right-hand side of (3.18). The coefficients of the power series $F(\tau)$, $\tilde{F}(\tau)$ and $\log_f\big(\tilde{F}(\tau)\big)$ all generate the same $\mathbb{Q}$-algebra, so $\mathcal{F}$ is generated by the right-hand side of (3.18).

We will need to use a linear basis of $\mathfrak{u}$ that is adapted to the depth grading. Recall that $\mathfrak{u} = \langle\varepsilon_0\rangle \oplus \mathfrak{u}'$ where $\mathfrak{u}'$ is defined in (2.1). Let $u_0 = \varepsilon_0$. For each $r \geq 1$, let $\mathfrak{u}'_r$ denote the subspace of derivations $D \in \mathfrak{u}'$ such that $v_f(D)$ is of homogeneous b-degree r. Let u_i, $i \geq 1$ denote a linear basis for $\mathfrak{u}'$ that is depth-graded, in the sense that each basis element u_i lies in some $\mathfrak{u}'_r$. Let $V = v_f(\mathfrak{u}')$, and for each $r \geq 1$, let $V_r = v_f(\mathfrak{u}'_r)$, so that V_r is a space of Lie polynomials in a, b with homogeneous degree r in b. The images $v_i = v_f(u_i)$ with $u_i \in \mathfrak{u}'_r$ form a basis of $V_r \subset \mathfrak{f}_2^{\rm push}$, since v_f is injective on $\mathfrak{u}'$ by hypothesis. The u_i for $i \geq 0$ form a basis for $\mathfrak{u}$, so since $r(\tau) \in \widehat{\mathfrak{u}} \otimes_{\mathbb{Q}} \mathcal{E}^{\rm geom}$, we can expand it as

$$r(\tau) = \sum_{i\geq 0} r_i u_i \qquad (3.19)$$

in the basis u_i with coefficients $r_i \in \mathcal{E}^{\rm geom}$. By (2.6) and (2.5), we have $r_0 = \mathcal{G}_0(\tau) = 2\pi i\tau$. The r_i generate the same ring as the coefficients of $g(\tau)$, namely all of $\mathcal{E}^{\mathrm{geom}}$. In view of (3.19), we have

$$r(\tau)\cdot f=\sum_{i\geq 0} r_i v_i \in V\otimes_{\mathbb{Q}} \mathcal{E}^{\mathrm{geom}}. \tag{3.20}$$

Let us write $\mathfrak{nz}$ for the vector space of *new multizetas* obtained by taking the vector space quotient of $\overline{\mathcal{Z}}$ by the vector subspace spanned by $\mathbb{Q}$ and by the ideal of $\overline{\mathcal{Z}}$ generated by products z_1z_2 of elements $z_1,z_2\in\overline{\mathcal{Z}}\setminus\mathbb{Q}$. Let $\mathcal{MZ}$ denote the $\mathbb{Q}$-algebra of *motivic multizetas* defined by Goncharov (in which $\zeta^{\mathfrak{m}}(2)=0$), which is graded for the weight. Let $\mathfrak{nmz}$ denote the quotient of the space $\mathcal{MZ}_{>0}$ of positive weight elements by products. We have the sequence of inclusions

$$\mathfrak{nz}^{\vee}\subset\mathfrak{nmz}^{\vee}\subset\mathfrak{grt}, \tag{3.21}$$

where the first is the dual injection arising from the surjection $\mathcal{MZ}\to\overline{\mathcal{Z}}$ and the second is the dual injection arising from the fact that Goncharov's motivic multizetas satisfy the associator relations. Note that these are all subspaces of $\mathfrak{f}_2$. Since Goncharov's motivic multiple zeta values in $\mathcal{MZ}$ satisfy the associator relations and $\mathcal{MZ}$ surjects onto $\overline{\mathcal{Z}}$, the Lie series ϕ_{KZ} lies in the vector space $\mathfrak{nz}^{\vee}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$, so by (3.21), it can also be considered as lying in the larger vector space $\mathfrak{nmz}^{\vee}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$; thus in particular we also have

$$\gamma(\phi_{KZ})\cdot f\in v_f\big(\gamma(\mathfrak{nmz}^{\vee})\big)\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

It follows from Brown's result in [5] that the Lie algebra $\mathfrak{nmz}^{\vee}$ is identified with the fundamental Lie algebra of the category of mixed Tate motives over $\mathbb{Z}$, which is free on one generator in each odd weight ≥ 3. In [18], a category of universal mixed elliptic motives is defined, and it is shown that the fundamental Lie algebra of that category has a monodromy representation in $\mathrm{Der}_0(\mathfrak{f}_2)$ whose image Π is isomorphic to a semi-direct product $\Pi\cong\mathfrak{u}'\rtimes\gamma(\mathfrak{nmz}^{\vee})$. Thus for derivations $D_1\in\mathfrak{u}'$ and $D_2\in\gamma(\mathfrak{nmz}^{\vee})$, we have $[D_1,D_2]\in\mathfrak{u}'$. Indeed, if R is any $\mathbb{Q}$-algebra, then for derivations $D_1\in\mathfrak{u}'\otimes_{\mathbb{Q}}R$ and $D_2\in\gamma(\mathfrak{nmz}^{\vee})\otimes_{\mathbb{Q}}R$, we have $[D_1,D_2]\in\mathfrak{u}'\otimes_{\mathbb{Q}}R$. In particular, if we take $R=\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$, $D_1=r(\tau)$ and $D_2=\gamma(\phi_{KZ})$, we see by (3.8) that

$$\langle r(\tau)\cdot f,\gamma(\phi_{KZ})\cdot f\rangle=[r(\tau),\gamma(\phi_{KZ})](f)\in v_f(\mathfrak{u}')\otimes_{\mathbb{Q}}R=V\otimes_{\mathbb{Q}}R.$$

Thus the bracket $\langle r(\tau)\cdot f,\gamma(\phi_{KZ})\cdot f\rangle$ lies in $V\otimes_{\mathbb{Q}}R$, so the whole of the bracketed term $s(\tau)$ of $\log_f\big(\tilde{F}(\tau)\big)$ lies in $V\otimes_{\mathbb{Q}}R$. We saw in (3.20) that the term $r(\tau)\cdot f$ also lies in $V\otimes_{\mathbb{Q}}R$, while $\gamma(\phi_{KZ})\cdot f$ lies in $\mathfrak{nmz}^{\vee}\otimes_{\mathbb{Q}}R$.

The $\mathbb{Q}$-algebra $\mathcal{F}$ lies inside $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\mathfrak{Z}$ since all the coefficients of $F(\tau)$ are algebraic expressions in the coefficients of $g(\tau)$ and those of F. To prove the desired result that $\mathcal{F}[2\pi i\tau]=\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\mathfrak{Z}$, it thus suffices to prove separately that $\mathcal{F}\supset\mathfrak{Z}$ and $\mathcal{F}\supset\mathcal{E}_0^{\mathrm{geom}}$.

We start with $\mathfrak{Z}$. Since by hypothesis, the evaluation map v_f is injective on $\mathrm{Der}_0'(\mathfrak{f}_2)$, it is in particular injective on the subspace $\mathfrak{u}'\rtimes\gamma(\mathfrak{nmz}^{\vee})$. We have $V=v_f(\mathfrak{u}')$, and we let $W=v_f\big(\gamma(\mathfrak{nmz}^{\vee})\big)$; thus, the direct sum $V\oplus W$ is the underlying vector space of $v_f\big(\mathfrak{u}'\rtimes\gamma(\mathfrak{nmz}^{\vee})\big)$.

Let $v_1,v_2,\ldots$ denote a basis of V and $w_1,w_2,\ldots$ a basis of W, which together form a basis of $V\oplus W$. Writing $\log_f\big(\tilde{F}(\tau)\big)$ in this basis, the coefficients of the basis elements generate $\mathcal{F}$ by definition. But by (3.18), $\log_f\big(\tilde{F}(\tau)\big)$ decomposes as a sum of two terms, $\gamma(\phi_{KZ})\cdot f\in W\otimes_{\mathbb{Q}}R$ and $r(\tau)\cdot f+s(\tau)\in V\otimes_{\mathbb{Q}}R$. Therefore, the coefficient of any basis element w_i in $\log_f\big(\tilde{F}(\tau)\big)$ is equal to the coefficient of w_i in $\gamma(\phi_{KZ})\cdot f$. This proves that $\mathcal{F}\supset\mathfrak{Z}$.

It remains to show that $\mathcal{F}\supset\mathcal{E}_0^{\rm geom}$. We use a similar reasoning to the above, with an additional argument to deal with the $s(\tau)$ term. The coefficient of a basis element v_i in $\mathfrak{e}(\tau)$ is a sum r_i+s_i, where r_i (resp. s_i) is the coefficient of v_i in $r(\tau)$ (resp. $s(\tau)$). We will prove that $\mathcal{F}\supset\mathcal{E}_0^{\rm geom}$ by showing by induction on the depth r that $\mathcal{F}$ contains each individual coefficient r_i for $i\ge 1$.

For the base case $r=1$, the depth 1 part of $r(\tau)\cdot f+s(\tau)$ comes entirely from $r(\tau)\cdot f$, since the sum $s(\tau)$ of bracketed terms has no depth 1 part. Thus, all the coefficients r_i of basis elements $v_i\in V_1$ occur as coefficients of $r(\tau)\cdot f+s(\tau)$, and therefore they lie in $\mathcal{F}$.

Now fix $r>1$ and assume that that all the r_j that are the coefficients in $r(\tau)\cdot f$ of basis elements $v_j\in V_s$ with $s<r$ lie in $\mathcal{F}$, and consider a basis element $v_i\in V_r$. Its coefficient in $r(\tau)\cdot f+s(\tau)$ is r_i+s_i. But since $s(\tau)$ is a sum of brackets, the coefficient s_i is an algebraic expression in elements of $\mathfrak{Z}$ and coefficients r_j of $r(\tau)\cdot f$ corresponding to basis elements v_j of depth $<r$. Thus by the induction hypothesis together with the inclusion $\mathfrak{Z}\subset\mathcal{F}$, we have $s_i\in\mathcal{F}$, and thus $r_i\in\mathcal{F}$. This shows that all the coefficients r_i of $r(\tau)\cdot f$ lie in $\mathcal{F}$, and thus $\mathcal{E}_0^{\rm geom}\subset\mathcal{F}$ as desired. This concludes the proof. □

Corollary 3.1. *We have the following equalities:*

$$\overline{\mathcal{E}}[2\pi i\tau]=\overline{\mathcal{A}}[2\pi i\tau]=\overline{\mathcal{B}}=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

Proof. The coefficients of $\overline{E}$ generate all of $\overline{\mathcal{Z}}$, since $\overline{E}$ is the image of the Drinfel'd associator by an isomorphism defined over $\mathbb{Q}$. It follows from [25], Theorem 5.4.2 that the coefficients of $\overline{A}'$ also generate $\overline{\mathcal{Z}}$, as do those of $\overline{B}$.

We apply Theorem 3.5 as follows. For $\overline{\mathcal{E}}$, we let $f=a$, so $F=\overline{E}$ and $F(\tau)=\overline{E}(\tau)$. Therefore the theorem implies that $\overline{\mathcal{E}}[2\pi i\tau]=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. For $\overline{\mathcal{B}}$, we take $f=b$; a standard argument then shows that since derivations in $\mathrm{Der}_0(\mathfrak{f}_2)$ annihilate $[a,b]$, these derivations are determined by their values on b; thus v_b is injective on all of $\mathrm{Der}_0(\mathfrak{f}_2)$. Setting $F=\overline{B}$ and $F(\tau)=\overline{B}(\tau)$, we again apply the theorem to conclude that $\overline{\mathcal{B}}[2\pi i\tau]=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. However, in this case we observe that $2\pi i\tau$ already lies in $\overline{\mathcal{B}}$ since explicit computation of the beginning of the series shows that it occurs as the coefficient of a. Thus $\overline{\mathcal{B}}=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$.

Finally, for $\overline{\mathcal{A}}$, we set $f=a$. The hypothesis on the injectivity of the evaluation map v_a holds since the kernel of v_a is exactly ε_0. We set $F=\overline{A}'$ and $F(\tau)=g(\tau)\cdot A'$. Recalling that the reduced algebra $\overline{\mathcal{A}}$ is defined as the $\mathbb{Q}$-algebra generated by the coefficients of $\overline{A}'(\tau)$, once again by the theorem we find that $\overline{\mathcal{A}}[2\pi i\tau]=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. This concludes the proof. □

4. The elliptic double shuffle and push-neutrality relations

In this section we use mould theory to explore and compare algebraic relations between the elliptic multizetas (coefficients of $\overline{E}(\tau)$), and algebraic relations between Enriquez' "analogues elliptiques de nombres multizetas".

Our main result on elliptic multizetas arises as a corollary of the preceding theorem and the main result of [33]. We show that $\overline{E}(\tau)$ satisfies a certain double family of algebraic relations called the *elliptic double shuffle relations*, related to the familiar double shuffle properties of Φ_{KZ}. In fact, they bear a close relation to the graded double shuffle relations studied for example in [3]. We show that if one assumes certain standard conjectures in multizeta and Grothendieck-Teichmüller theory, the elliptic double shuffle relations can be expected to form a *complete* set of algebraic relations for the elliptic multizetas mod $2\pi i$. We investigate these relations in detail in depth 2.

In §4.3 we turn our attention to the power series $A(\tau)$ that forms part of Enriquez' elliptic KZB associator [14]. Since we want to work modulo $2\pi i$ and $A(\tau) \equiv 0$ mod $2\pi i$, we first define a power series $\mathfrak{a}(\tau)$ that is closely related to $A(\tau)$ but not trivial mod $2\pi i$. The goal of the section is to display a double family of relations satisfied by $\mathfrak{a}(\tau)$. The first is just the usual shuffle, but the second is very different from the second shuffle relation satisfied by $\overline{E}(\tau)$; we call it the family of *push-neutrality relations* (of *Fay relations*). We show that these are related to the Fay-shuffle relations studied in [26].

4.1. **A very brief introduction to moulds.**

We recall some notions from Ecalle's theory of moulds [12, 13] that we will need in order to study algebraic relations between elliptic multizetas. Besides the original references, a more detailed introduction to moulds can be found in [32].

4.1.1. *Moulds and bialternality.* In this article, we use the term 'mould' to refer only to rational-function valued moulds with coefficients in $\mathbb{Q}$. Thus, a mould is a family of functions

$$\{P(u_1, \ldots, u_r) \mid r \geq 0\}$$

with $P(u_1, \ldots, u_r) \in \mathbb{Q}(u_1, \ldots, u_r)$. In particular $P(\emptyset)$ is a constant. The *depth* r part of a mould is the function $P(u_1, \ldots, u_r)$ in r variables. By defining addition and scalar multiplication of moulds in the obvious way, i.e. depth by depth, moulds form a $\mathbb{Q}$-vector space that we call $Moulds$. We write $Moulds_{pol}$ for the subspace of polynomial-valued moulds. The vector space ARI is the subspace of $Moulds$ consisting of moulds P with constant term $A(\emptyset) = 0$, and ARI_{pol} is again the subspace of polynomial-valued moulds in ARI.

The standard mould multiplication mu is given by

$$mu(P, Q)(u_1, \ldots, u_r) = \sum_{i=0}^{r} P(u_1, \ldots, u_i) Q(u_{i+1}, \ldots, u_r). \tag{4.1}$$

For simplicity, we write $P\,Q = mu(P, Q)$. This multiplication defines a Lie algebra structure on ARI with Lie bracket lu defined by $lu(P, Q) = mu(P, Q) - mu(Q, P)$.

We now introduce four operators on moulds. The Δ-operator on moulds is defined as follows: if $P \in ARI$, then

$$\Delta(P)(u_1, \ldots, u_r) = u_1 \cdots u_r (u_1 + \cdots + u_r) P(u_1, \ldots, u_r). \tag{4.2}$$

The dar-operator is defined by

$$dar(P)(u_1, \ldots, u_r) = u_1 \cdots u_r \, P(u_1, \ldots, u_r). \tag{4.3}$$

The $push$-operator is defined by

$$push(B)(u_1, \ldots, u_r) = B(u_2, \ldots, u_r, -u_1 - \cdots - u_r). \tag{4.4}$$

Finally, the $swap$ operator is defined by

$$swap(A)(v_1, \ldots, v_r) = A(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2). \tag{4.5}$$

Here the use of the alphabet $v_1, v_2, \dots$ instead of $u_1, \dots, u_r$ is purely a convenient way to distinguish a mould from its swap.

The main property on moulds that we will need to consider is *alternality*. A mould P is said to be *alternal* if for all $r > 1$ and for $1 \leq i \leq [r/2]$, we have

$$\sum_{\mathbf{u} \in sh((u_1,\dots,u_i),(u_{i+1},\dots,u_r))} P(\mathbf{u}) = 0, \tag{4.6}$$

where the set of r-tuples $sh\big((u_1,\dots,u_i),(u_{i+1},\dots,u_r)\big)$ is the set

$$\big\{(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(r)}) \,\big|\, \sigma \in S_r \text{ such that } \sigma(1) < \cdots < \sigma(i),\ \ \sigma(i{+}1) < \cdots < \sigma(r)\big\}.$$

The mould $swap(A)$ is alternal if it satisfies the same property (4.6) in the variables v_i.

We write ARI^{al} for the space of alternal moulds in ARI, and $ARI^{al/al}$ for the space of moulds which are alternal and whose swap is also alternal. We also consider moulds which are alternal and whose swap is alternal up to addition of a constant-valued mould. The space of these moulds is denoted ARI^{al*al} and we call them *bialternal*.

We say that a mould P is Δ-bialternal if $\Delta^{-1}(P)$ is bialternal, and we write $ARI^{\Delta\text{-}al*al}$ for the space of such moulds.

4.1.2. *From power series to moulds.* Let $c_i = \mathrm{ad}(a)^{i-1}(b)$ for $i \geq 1$ as in §3.1. Let the depth of a monomial $c_{i_1}\cdots c_{i_r}$ be the number r of c_i in the monomial; the depth forms a grading on the formal power series ring $\mathbb{Q}\langle\langle C\rangle\rangle = \mathbb{Q}\langle\langle c_1, c_2, \dots\rangle\rangle$ on the free variables c_i. Similarly, we write $L[\![C]\!] = \mathrm{Lie}[\![c_1, c_2, \dots]\!]$ for the corresponding free Lie algebra. By Lazard elimination, we have an isomorphism

$$\mathbb{Q}a \oplus L[\![C]\!] \cong \mathfrak{f}_2 = \mathrm{Lie}[\![a,b]\!].$$

Following Écalle, let ma denote the standard vector space isomorphism from $\mathbb{Q}\langle\langle C\rangle\rangle$ to the space $(Moulds)^{pol}$ defined by

$$\begin{aligned} ma : \mathbb{Q}\langle\langle C\rangle\rangle &\xrightarrow{\sim} (Moulds)^{pol} \\ c_{k_1}\cdots c_{k_r} &\mapsto (-1)^{k_1+\cdots+k_r-r} u_1^{k_1-1}\cdots u_r^{k_r-1} \end{aligned} \tag{4.7}$$

on monomials, extended by linearity to all power series.

It is well-known that $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if p is a Lie series, i.e. $p \in \mathrm{Lie}[\![C]\!]$. The alternality property on moulds is analogous to these shuffle relations, that is a series $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if $ma(p)$ is alternal (see e.g. [32], §2.3 and Lemma 3.4.1]). Writing ARI^{al} for the subspace of alternal moulds and ARI^{al}_{pol} for the subspace of alternal polynomial-valued moulds, this shows that the map ma restricts to a Lie algebra isomorphism

$$ma : \mathrm{Lie}[\![C]\!] \xrightarrow{ma} ARI^{al}_{lu,pol}.$$

Finally, we recall that for any mould $P \in ARI$, Écalle defines a derivation $arit(P)$ of the Lie algebra ARI_{lu}. We do not need to recall the definition of $arit$ here (but it is given in §4.4 below where we prove a technical lemma). For now it is enough to know that when restricted to polynomial-valued moulds, it is related to the Ihara derivations (3.11) via the morphism ma:

$$ma\big(D_f(g)\big) = -arit\big(ma(f)\big)\cdot ma(f).$$

For each $P \in ARI$, we also define the derivation

$$arat(P) = -arit(P) + \mathrm{ad}(P), \tag{4.8}$$

where $\mathrm{ad}(P) \cdot Q = lu(P, Q)$.

4.1.3. *Reminders on the elliptic double shuffle Lie algebra* $\mathfrak{ds}_{\mathrm{ell}}$. We end this subsection by recalling the definition and a few facts about the elliptic double shuffle Lie algebra $\mathfrak{ds}_{\mathrm{ell}}$ from [33].

Definition 4.1. The *elliptic double shuffle Lie algebra* $\mathfrak{ds}_{\mathrm{ell}}$ is the subspace of $\mathfrak{f}_2$ such that

$$ma\big(\mathfrak{ds}_{\mathrm{ell}}\big) = ARI_{pol}^{\Delta\text{-}al*al},$$

i.e. $\mathfrak{ds}_{\mathrm{ell}}$ consists of the Lie power series $f \in \mathfrak{f}_2$ such that $ma(f)$ is Δ-bialternal.

The following results are shown in [33].

Proposition 4.2. *The space* $\mathfrak{ds}_{\mathrm{ell}}$ *satisfies the following properties.*

(i) $\mathfrak{ds}_{\mathrm{ell}} \subset \mathfrak{f}_2^{\mathrm{push}}$*, where* $\mathfrak{f}_2^{\mathrm{push}}$ *has been defined in Section 3.2;*

(ii) $\mathfrak{ds}_{\mathrm{ell}}$ *is a Lie algebra under the bracket* $\langle\, , \rangle$ *on* $\mathfrak{f}_2^{\mathrm{push}}$ *defined in* (3.8).

(iii) There is a Lie algebra inclusion

$$\widetilde{\mathfrak{grt}}_{\mathrm{ell}} \subset \mathfrak{ds}_{\mathrm{ell}},$$

where $\widetilde{\mathfrak{grt}}_{\mathrm{ell}}$ *is the Lie subalgebra of* $\mathfrak{grt}_{\mathrm{ell}}$ *generated by* $\gamma(\mathfrak{grt})$ *and* $\mathfrak{u}$.

Remark 4.3. In [6], a Lie algebra called $\mathfrak{pls}$ (for "polar linearized shuffle") is introduced, which is essentially equivalent to $\mathfrak{ds}_{\mathrm{ell}}$. It is also shown that $\mathfrak{u}$ embeds into $\mathfrak{pls}$ ([6], Proposition 4.6) and, moreover, it is asked whether the equality $\mathfrak{u} = \mathfrak{pls}$ holds. Proposition 4.2.(iii) implies that $\mathfrak{ds}_{\mathrm{ell}}$ is, in fact, much larger than $\mathfrak{u}$. More precisely, Enriquez ([14], §7) has shown that $\mathfrak{u}$ lies in the kernel of the surjection $\mathfrak{grt}_{\mathrm{ell}} \to \mathfrak{grt}$ from which it follows that the image $\gamma(\mathfrak{grt}) \subset \widetilde{\mathfrak{grt}}_{\mathrm{ell}}$ of $\mathfrak{grt}$ under the splitting γ is disjoint from $\mathfrak{u}$. In particular, the Lie algebra $\mathfrak{u}$ cannot equal $\mathfrak{ds}_{\mathrm{ell}}$.

4.2. **The elliptic double shuffle relations.** We can now give the elliptic double shuffle property of $\overline{E}(\tau)$. It is in fact phrased more directly as a property on $\mathfrak{e}(\tau) = \log_a\big(\overline{E}(\tau)\big)$, or rather, on the mould version of this power series

$$\mathfrak{e}_m(\tau) = ma\big(\mathfrak{e}(\tau)\big).$$

Theorem 4.4. *The mould* $\mathfrak{e}_m(\tau)$ *is* Δ*-bialternal, i.e.* $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ *is a bialternal mould.*

Proof. We saw in the proof of Theorem 3.5 that $\mathfrak{e}(\tau) = \mathfrak{e} + r_a(\tau) + s(\tau)$ where $\mathfrak{e} \in \gamma(\mathfrak{grt}) \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$ and $r_a(\tau) + s(\tau) \in \mathfrak{u} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. Therefore, $\mathfrak{e}(\tau) \in \widetilde{\mathfrak{grt}}_{\mathrm{ell}}$ by the definition of $\widetilde{\mathfrak{grt}}_{\ell}$, and since $\widetilde{\mathfrak{grt}}_{\mathrm{ell}} \subset \mathfrak{ds}_{\mathrm{ell}}$ by Proposition 4.2 (iii), we also have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{\mathrm{ell}} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. But this is equivalent to

$$\mathfrak{e}_m(\tau) \in ARI_{pol}^{\Delta\text{-}al*al}.$$

□

We conjecture that the elliptic double shuffle relations form a complete set of algebraic relations between the elliptic multizetas modulo $2\pi i$. This statement really breaks down into two statements, one concerning the arithmetic part $\overline{\mathcal{Z}}$ of $\overline{\mathcal{E}}$ and the other the geometric part $\mathcal{U}(\mathfrak{u})^\vee$. We show that indeed, the result follows from two conjectures: the first one a standard conjecture from multizeta theory, and the second a similar conjecture from elliptic multizeta theory. Due to the fact that it is much easier to work in the geometric situation than the arithmetic situation (as there are no problems of transcendence), we are actually able to prove that the elliptic double shuffle relations are complete in depth 2, without any recourse to conjectures (see Proposition 4.6).

The first conjecture amounts to the inclusions in (3.21) being all isomorphisms as well as the standard conjecture that the inclusion $\mathfrak{grt} \subset \mathfrak{ds}$ (proved by Furusho in [17]) is actually also an isomorphism. We simply state the conjecture

Conjecture 1: $$\mathfrak{nz}^\vee \cong \mathfrak{ds}.$$

This is equivalent to conjecturing that the double shuffle relations suffice to generate all the algebraic relations satisfied by multizetas [20].

The second conjecture amounts to the existence of a canonical semi-direct product structure on the elliptic double shuffle Lie algebra $\mathfrak{ds}_{\rm ell}$. This is inspired by Enriquez result that the elliptic Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_{\rm ell}$ is isomorphic to a semi-direct product $\mathfrak{r}_{\rm ell} \rtimes \gamma(\mathfrak{grt})$ where $\mathfrak{r}_{\rm ell}$ is a certain Lie ideal of $\mathfrak{grt}_{\rm ell}$ containing $\mathfrak{u}$. Analogously, we have

Conjecture 2: $$\mathfrak{u} \rtimes \gamma(\mathfrak{ds}) \cong \mathfrak{ds}_{\rm ell}.$$

This conjecture is closely related to Enriquez' "generation conjecture" for $\mathfrak{grt}_{\rm ell}$ [14], namely that $\mathfrak{u} \cong \mathfrak{r}_{\rm ell}$. If Enriquez' conjecture were true, then the left hand side of our Conjecture 2 would be isomorphic to $\mathfrak{grt}_{\rm ell}$, and Conjecture 2 would reduce to showing that $\mathfrak{grt}_{\rm ell} \cong \mathfrak{ds}_{\rm ell}$ (the elliptic analog of Furusho's theorem [17]).

One can also merge Conjectures 1 and 2 into a single conjecture, thereby extending (3.21) to the elliptic setting. The elliptic analog of $\mathfrak{nmz}^\vee$ is the elliptic motivic fundamental Lie algebra, which is conjecturally isomorphic to its image $\Pi = V \rtimes \mathfrak{nmz}^\vee$ in the derivation algebra $\mathrm{Der}_0(\mathfrak{f}_2)$ (cf. the proof of Theorem 3.5). Then we get inclusions

$$V \rtimes \mathfrak{nz}^\vee \subset V \rtimes \mathfrak{nmz}^\vee \cong \Pi \subset \widetilde{\mathfrak{grt}}_{\rm ell}, \tag{4.9}$$

which conjecturally are all equalities. Note that the first equality would also follow from Conjecture 1 above.

Proposition 4.5. *If Conjectures 1 and 2 are true, then the elliptic double shuffle relations generate all algebraic relations between elliptic multizetas.*

Proof. By Conjecture 1, we would have $\overline{\mathcal{Z}} \cong \mathcal{U}(\mathfrak{ds})^\vee$, so since $\mathcal{E}^{\rm geom} \cong \mathcal{U}(\mathfrak{u})^\vee \cong \mathcal{U}(V)^\vee$ by Theorem 2.6, we would have

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(V)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee.$$

It is known that the underlying vector space of the universal enveloping algebra $\mathcal{U}(R \rtimes L)$ of a semi-direct product of Lie algebras $R \rtimes L$ is the space $\mathcal{U}(R) \otimes_{\mathbb{Q}} \mathcal{U}(L)$; in fact $\mathcal{U}(R \rtimes L)$ is a Hopf algebra equipped with the smash product ([28]) and with the standard coproduct for which elements of $R \rtimes L$ are primitive. The dual $\mathcal{U}(R \rtimes L)^\vee$ has underlying $\mathbb{Q}$-algebra $\mathcal{U}(R)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(L)^\vee$ (and is equipped with the smash coproduct).

Thus by Conjecture 2, we would have the isomorphism of $\mathbb{Q}$-algebras

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(\mathfrak{u})^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee \cong \mathcal{U}(\mathfrak{ds}_{\rm ell})^\vee.$$

Now, for any Lie algebra $\mathfrak{g}$ defined over $\mathbb{Q}$ and any $\mathbb{Q}$-algebra R, if f is an element of $\mathfrak{g}\otimes_{\mathbb{Q}} R$, then the subring of R generated by the coefficients of f (in a linear basis of $\mathfrak{g}$) generate a subring of R which is necessarily isomorphic to a quotient of $\mathcal{U}(\mathfrak{g})^\vee$; in other words, the coefficients of f satisfy relations that are imposed by the fact that f lies in the Lie algebra $\mathfrak{g}$, and possibly others. If this quotient is actually isomorphic to $\mathcal{U}(\mathfrak{g})^\vee$, this signifies that the coefficients do not satisfy any further algebraic relations than those imposed on them by the fact that f lies in $\mathfrak{g}$.

In our case, we have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{\rm ell} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, and the coefficients of $\mathfrak{e}(\tau)$, together with $2\pi i\tau$, generate $\overline{\mathcal{E}}[2\pi i\tau]$, which by the conjectures is isomorphic to $\mathcal{U}(\mathfrak{ds}_{\rm ell})^\vee$, implying that the coefficients of $\mathfrak{e}(\tau)$ do not satisfy any other algebraic relations than those imposed by the fact that $\mathfrak{e}(\tau)$ lies in $\mathfrak{ds}_{\rm ell}$, i.e. is Δ-bialternal. $\square$

Explicit elliptic double shuffle relations. Let us take a closer look at what the Δ-bialternality properties are. The first property is that $\mathfrak{e}_m(\tau)$ is Δ-alternal, i.e. that $\Delta^{-1}(\mathfrak{e}_m(\tau))$ is alternal. But Δ trivially preserves alternality, so this is equivalent to saying that $\mathfrak{e}_m(\tau)$ is alternal, i.e. that for each $r > 1$,

$$(EDS.1) \qquad \sum_{u\in sh\big((u_1,\ldots,u_k),(u_{k+1},\ldots,u_r)\big)} \mathfrak{e}_m(\tau)(u) = 0$$

for $1 \le k \le [r/2]$. This condition is equivalent to the statement that the power series $\mathfrak{e}(\tau)$ is a Lie series.

The new relations on $\mathfrak{e}_m(\tau)$ are the second set, which say that up to adding on a constant-valued mould, the swap of the mould $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ is also alternal, where the swap-operator is defined in (4.5). This alternality is given by the equalities for $r > 1$

$$(EDS.2) \qquad \sum_{v\in sh\big((v_1,\ldots,v_k),(v_{k+1},\ldots,v_r)\big)} swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big)(v) = 0$$

for $1 \le k \le [r/2]$.

The swapped mould is given explicitly by

$$swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big) = \frac{1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}\,\mathfrak{e}_m(\tau)(v_r, v_{r-1}-v_r,\ldots,v_1-v_2).$$

Thus the alternality conditions in (EDS.2) are all sums of rational functions with denominators that are products of terms of the form v_i and $(v_i - v_j)$, which sum to zero. Therefore, by multiplying through by the common denominator

$$v_1\cdots v_r \prod_{i>j}(v_i - v_j),$$

the second elliptic shuffle equation can be expressed as a family of polynomial conditions on the mould $swap(\mathfrak{e}_m(\tau))$.

Elliptic double shuffle relations in depth 2. Let us work this out explicitly in depth 2. The usual alternality condition reduces to

$$(EDS.1\text{-depth } 2) \qquad \mathfrak{e}_m(\tau)(u_1,u_2) + \mathfrak{e}_m(\tau)(u_2,u_1) = 0.$$

The swap alternality condition reads

$$\frac{1}{v_1(v_1-v_2)v_2}swap(\mathfrak{e}_m(\tau))(v_1,v_2)+\frac{1}{v_1(v_2-v_1)v_2}swap(\mathfrak{e}_m(\tau))(v_2,v_1)=0,$$

which, clearing denominators, reduces simply to

$$swap(\mathfrak{e}_m(\tau))(v_1,v_2)-swap(\mathfrak{e}_m(\tau))(v_2,v_1)=0.$$

Since $swap(\mathfrak{e}_m(\tau))(v_1,v_2)=e_m(v_2,v_1-v_2)$, this is given by the relation

$$\mathfrak{e}_m(\tau)(v_2,v_1-v_2)=\mathfrak{e}_m(\tau)(v_1,v_2-v_1)$$

directly on $\mathfrak{e}_m(\tau)$. Applying the depth 2 swap operator from $\overline{ARI}$ to ARI (given by $v_1\mapsto u_1+u_2$, $v_2\mapsto u_1$), we transform this relation into

$$\mathfrak{e}_m(\tau)(u_1,u_2)=\mathfrak{e}_m(\tau)(u_1+u_2,-u_2).$$

Finally, $\mathfrak{e}_m(\tau)$ is of odd degree, so by the depth 2 version of (EDS.1), we have $\mathfrak{e}_m(\tau)(-u_2,-u_1)=\mathfrak{e}_m(\tau)(u_1,u_2)$, which gives

$$\mathfrak{e}_m(\tau)(u_1,u_2)=\mathfrak{e}_m(\tau)(u_2,-u_1-u_2). \quad (EDS.2\text{-depth } 2)$$

Note that this is nothing other than $\mathfrak{e}_m(\tau)(u_1,u_2)=push\big(\mathfrak{e}_m(\tau)\big)(u_1,u_2)$ where the push-operator is defined in (4.4). Thus in depth 2, the Δ-bialternality conditions correspond to alternality and push-invariance of $\mathfrak{e}_m(\tau)$ (which in turn correspond to the fact that $\mathfrak{e}(\tau)$ is a Lie series that is push-invariant in depth 2 in the sense of power series, as in (3.7)). This simple reformulation is special to depth 2; the Δ-bialternal property does not lend itself so easily to a direct expression as a property of $\mathfrak{e}(\tau)$ in higher depths.

We end this subsection by showing that the conjecture that the Δ-bialternal relations are sufficient holds in depth 2.

Proposition 4.6. *The relations (EDS.1) and (EDS.2) in odd degrees are the only relations satisfied by $\mathfrak{e}_m(\tau)$ in depth 2.*

Proof. We can prove this result without recourse to any conjectures, essentially because depth 2 is too small to contain any of the arithmetic part of $\mathfrak{e}_m(\tau)$ (we qualify this statement below), and the geometric part $V=v_a(\mathfrak{u})$ is well-understood in depth two. We know that $\mathfrak{e}(\tau)\in\mathfrak{ds}_{\rm ell}\subset\mathfrak{f}_2^{push}$. The graded dimensions of $\mathfrak{f}_2$ in depth 2 are given by

$$dim(\mathfrak{f}_2^{\rm push})^2_n=\left\lfloor\frac{n-5}{6}\right\rfloor+1. \tag{4.10}$$

Now the depth two part of $\mathfrak{ds}_{\rm ell}\supset V\rtimes\gamma(\mathfrak{nz}^\vee)$ is contained in the depth two part of V, since $\gamma(\mathfrak{nz}^\vee)$ is of depth ≥ 3. Thus

$$\dim\big(\mathfrak{ds}_{\rm ell}\big)^2_n=\dim V^2_n=\begin{cases}\lfloor\frac{n-5}{6}\rfloor+1 & \text{if } n \text{ is odd} \geq 5\\ 0 & \text{otherwise.}\end{cases} \tag{4.11}$$

Indeed, the last equality follows from the fact that in depth 2, V is spanned by the $[\varepsilon_{2j},\varepsilon_{2k}](a)$ with $j<k$, $j,k\neq 1$, which are all of odd weight, and the fact that, as shown in [29], the only relations between these $\lfloor\frac{n-3}{4}\rfloor$ brackets come from period polynomials, whose number is given by $\lfloor\frac{n-7}{4}\rfloor-\lfloor\frac{n-5}{6}\rfloor$. Thus $V^2=\mathfrak{ds}^2_{\rm ell}=(\mathfrak{f}_2^{push})^2$, so the Lie relation (EDS.1) and the push-invariance relation (EDS.2) suffice to characterize elements of $\mathfrak{ds}_{\rm ell}$ in depth 2. □

Depth 2 elements of $\mathfrak{ds}_{\rm ell}$ in low weights:

- in weight 5,

$$ma\big([\varepsilon_0,\varepsilon_4](a)\big)=2u_1^3+3u_1^2u_2-3u_1u_2^2-2u_2^3.$$

- in weight 7,

$$ma\big([\varepsilon_0,\varepsilon_6](a)\big)=2u_1^5+5u_1^4u_2+2u_1^3u_2^2-2u_1^2u_2^3-5u_1u_2^4-2u_2^5.$$

- in weight 9,

$$ma\big([\varepsilon_0,\varepsilon_8](a)\big)=2u_1^7+7u_1^6u_2+9u_1^5u_2^2+5u_1^4u_2^3-5u_1^3u_2^4-9u_1^2u_2^5-7u_1u_2^62u_2^7.$$

- in weight 11,

$$ma\big([\varepsilon_0,\varepsilon_{10}](a)\big)=8u_1^9+36u_1^8u_2+74u_1^7u_2^2+91u_1^6u_2^3+41u_1^5u_2^4-41u_1^4u_2^5$$
$$-91u_1^3u_2^6-74u_1^2u_2^7-36u_1u_2^8-8u_2^9$$
$$ma\big([\varepsilon_4,\varepsilon_6](a)\big)=-2u_1^7u_2^2-7u_1^6u_2^3-5u_1^5u_2^4+5u_1^4u_2^5+7u_1^3u_2^6+2u_1^2u_2^7.$$

4.3. **The elliptic associator and the push-neutrality relations mod $2\pi i$.**

Definition 4.7. Let $\mathfrak{a}$ be the power series with coefficients in $\overline{\mathcal{Z}}$ given by

$$\mathfrak{a}=\frac{1}{2\pi i}\log(A)\ \mathrm{mod}\ 2\pi i=\log(\overline{A}')=\overline{\Phi}_{KZ}(t_{01},t_{12})t_{01}\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1},$$

and let $\mathfrak{a}(\tau)=g(\tau)\cdot\mathfrak{a}$.

The coefficients of $\mathfrak{a}(\tau)$ generate the $\mathbb{Q}$-algebra $\overline{\mathcal{A}}$ of $\overline{A}$-EMZ's. In this paragraph we will consider certain relations satisfied by the coefficients of $\mathfrak{a}(\tau)$, different from the linearized elliptic double shuffle relations satisfied by $\mathfrak{e}(\tau)$. The first family of relations on the coefficients of $\mathfrak{a}(\tau)$ is the usual family of *alternality* relations, but the second is the family of *push-neutrality* relations. These relations are related (mod $2\pi i$) to the *Fay-shuffle relations* introduced in [26], and studied explicitly in depth 2. We show that in depth 2, the push-neutrality relations are identical to the Fay-shuffle relations. We also show that even in depth 2 and mod $2\pi i$, the alternality and push-neutrality relations are strictly weaker than the linearized elliptic double shuffle relations.

We will give our relations in terms of mould theory (but see Corollary 4.11 for a translation into power series terms at the end). For this we recall the *push* and *dar*-operators defined in (4.4) and (4.3). We will say that a mould B is *push-neutral* if

$$B(u_1,\ldots,u_r)+push(B)(u_1,\ldots,u_r)+\cdots+push^r(B)(u_1,\ldots,u_r)=0 \qquad (4.12)$$

for all $r\geq 1$, where *push* denotes the push-operator on moulds defined in (4.4).

Theorem 4.8. *Let $\mathfrak{a}_m(\tau)=ma\big(\mathfrak{a}(\tau)\big)$. Then $\mathfrak{a}_m(\tau)$ is alternal and $dar^{-1}\big(\mathfrak{a}_m(\tau)\big)$ is push-neutral in depth $r>1$.*

Proof. Recall the derivation *arat* defined in (4.8). For any $P\in ARI$, set

$$\mathrm{Darit}(P)=dar\circ arat\big(\Delta^{-1}(P)\big)\circ dar^{-1}. \qquad (4.13)$$

It is shown in [33] that the map

$$\begin{aligned} \mathrm{Der}_0(\mathfrak{f}_2) &\hookrightarrow \mathrm{Der}(ARI_{lu}) \\ D &\mapsto \mathrm{Darit}\big(ma(v_a(D))\big) \end{aligned} \tag{4.14}$$

is an injective Lie morphism, so that we have

$$ma\big(D(f)\big) = \mathrm{Darit}\big(ma(v_a(D))\big) \cdot ma(f). \tag{4.15}$$

Let $\mathfrak{a}_m = ma(\mathfrak{a})$, $\mathfrak{a}_m(\tau) = ma\big(\mathfrak{a}(\tau)\big)$, and $r_m(\tau) = ma\big(r_a(\tau)\big)$. Under the map (4.14), we have $r(\tau) \mapsto \mathrm{Darit}\big(r_m(\tau)\big)$, so

$$ma\big(r(\tau)\cdot\mathfrak{a}\big) = \mathrm{Darit}\big(r_m(\tau)\big)\cdot \mathfrak{a}_m.$$

Since

$$\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a} = \sum_{n\geq 0}\frac{1}{n!} r(\tau)^n\cdot\mathfrak{a}, \tag{4.16}$$

we have

$$\mathfrak{a}_m(\tau) = \sum_{n\geq 0}\frac{1}{n!}\mathrm{Darit}\big(r_m(\tau)\big)^n\cdot\mathfrak{a}_m. \tag{4.17}$$

Let $\overline{\sigma}$ denote the automorphism of $\mathfrak{f}_2$ defined in §3.2. We have

$$\mathfrak{a} = \overline{\sigma}(t_{01}).$$

Recall from §3.2 that $\overline{\sigma} = \gamma(\phi_{KZ})$, where $\phi_{KZ} = \log_a\big(\overline{\Phi}_{KZ}\big)$.

The derivation $\gamma(\phi_{KZ})$ lies in $\mathrm{Der}_0(\mathfrak{f}_2)$, so $\gamma(\phi_{KZ})\cdot t_{01} \in \mathfrak{f}_2$; thus $\mathfrak{a}$ is a Lie series. Since $r(\tau) \in \mathrm{Der}_0(\mathfrak{f}_2)$, we have $r(\tau)^n\cdot\mathfrak{a} \in \mathfrak{f}_2$ for all $n \geq 1$, so by (4.16), $\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a} \in \mathfrak{f}_2$, which means that $\mathfrak{a}_m(\tau)$ is alternal. This settles the first property of $\mathfrak{a}_m(\tau)$ stated in the theorem.

Let us consider the second property. Since $\gamma(\phi_{KZ}) \in \mathrm{Der}_0(\mathfrak{f}_2)$, it annihilates t_{12}. Therefore, setting $t'_{01} = t_{01} + \frac{1}{2}t_{12}$, we have

$$\mathfrak{a} = \gamma(\phi_{KZ})\cdot t_{01} = \gamma(\phi_{KZ})\cdot t'_{01}. \tag{4.18}$$

Set $T'_{01} = ma(t'_{01})$, and set

$$\mathfrak{z} = ma\Big(v_a\big(\gamma(\phi_{KZ})\big)\Big) = ma\big(\gamma_a(\phi_{KZ})\big).$$

Then by (4.15), the equality (4.18) translates into moulds as

$$\mathfrak{a}_m = \mathrm{Darit}(\mathfrak{z})\cdot T'_{01}.$$

To complete the proof of the second property, we will use the following lemma, whose proof is deferred to the final subsection of this paper.

Lemma 4.9. *Let $A \in ARI$. If A is push-neutral, then $arat(P)\cdot A$ is push-neutral for all $P \in ARI$. If $dar^{-1}A$ is push-neutral, then $dar^{-1}\cdot\mathrm{Darit}(P)\cdot A$ is push-neutral for all $P \in ARI$.*

It is easy to see that if A is a push-invariant mould, then $dar^{-1}A$ is push-neutral, since

$$\begin{aligned} &dar^{-1}(A)(u_1,\dots,u_r) + push\big(dar^{-1}(A)\big)(u_1,\dots,u_r) + \cdots + push^r\big(dar^{-1}(A)\big)(u_1,\dots,u_r) \\ &\qquad = \left(\frac{1}{u_1\cdots u_r} + \frac{1}{u_2\cdots u_0} + \cdots + \frac{1}{u_0u_1\cdots u_{r-1}}\right)A(u_1,\dots,u_r) \\ &\qquad = \left(\frac{u_0+u_1+\cdots+u_r}{u_0u_1\cdots u_r}\right)A(u_1,\dots,u_r) = 0, \end{aligned}$$

where $u_0 = -u_1 - \cdots - u_r$. By Proposition 4.10 below, $dar^{-1}T'_{01}$ is push-neutral and by Lemma 4.9, so is

$$dar^{-1}\mathfrak{a}_m = dar^{-1} \cdot \mathrm{Darit}(\mathfrak{z}) \cdot T'_{01}.$$

To show that $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral we use the same lemma again. Since $dar^{-1}\mathfrak{a}_m$ is push-neutral, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big) \cdot \mathfrak{a}_m$, and then successively, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big)^n \cdot \mathfrak{a}_m$ for all $n \geq 1$. Thus $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral by (4.17). This proves the theorem. □

The following proposition was used in the proof of Theorem 4.8.

Proposition 4.10. *The mould*

$$ma([t'_{01}, a]) = -\sum_{n=2}^{\infty} \frac{B_n}{n!} ma([\mathrm{ad}^n(b)(a), a]) \tag{4.19}$$

is push-neutral.

Proof. It is enough to show the push-neutrality of $f_n := ma([\mathrm{ad}^n(b)(a), a])$ for all $n \geq 2$ separately. Using the definition of ma (cf. Section 4.1), we see that

$$ma(\mathrm{ad}^n(b)(a)) = -\sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} u_k \in \mathbb{Q}[u_1, \ldots, u_n]. \tag{4.20}$$

Now in depth n, the operator $\mathrm{ad}(a)$ on $\mathbb{Q}\langle\langle C \rangle\rangle$ corresponds to multiplication by $-(u_1 + \ldots + u_n)$. Consequently,

$$\begin{aligned} ma([\mathrm{ad}^n(b)(a), a]) &= -ma([a, \mathrm{ad}^n(b)(a)]) \\ &= -(u_1 + \ldots + u_n) \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} u_k \\ &= -\sum_{j,k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} u_j u_k. \end{aligned} \tag{4.21}$$

On the other hand, by the definition of the push-operator (4.4), we have $push(f_n) = -\sum_{j,k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} u_{j+1} u_{k+1}$, where the indices are to be taken mod n (so that $u_{k+n} = u_k$). Using the elementary fact that $\sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} = 0$ for $n \geq 2$, it is now clear that

$$\sum_{i=0}^{n-1} push^i(f_n) = 0, \tag{4.22}$$

i.e. f_n is push-neutral for all $n \geq 2$, as was to be shown. □

We end this subsection by studying these relations more explicitly in depth 2 and comparing them with the elliptic double shuffle relations on $\mathfrak{e}_m(\tau)$. The alternality relation is of course the same:

$$(FS.1) \qquad \mathfrak{a}_m(\tau)(u_1, u_2) + \mathfrak{a}_m(\tau)(u_2, u_1) = 0.$$

The push-neutrality relation in depth 2 is given by

$$(FS.2) \qquad \frac{1}{u_1 u_2} \mathfrak{a}_m(\tau)(u_1, u_2) + \frac{1}{u_2 u_0} \mathfrak{a}_m(\tau)(u_2, u_0) + \frac{1}{u_0 u_1} \mathfrak{a}_m(\tau)(u_0, u_1) = 0$$

where $u_0=-u_1-u_2$. Multiplying by the common denominator $u_0u_1u_2$ yields the polynomial relation

$$u_0\mathfrak{a}_m(\tau)(u_1,u_2)+u_1\mathfrak{a}_m(\tau)(u_2,u_0)+u_2\mathfrak{a}_m(\tau)(u_0,u_1)=0.$$

It was shown in [26] that the dimension of the space of polynomials in u_1,u_2 of odd degree d satisfying (FS.1) and (FS.2) is given by $\lfloor \frac{d}{3}\rfloor+1$. In terms of the weight $n=d+2$ of the corresponding polynomials in $\mathfrak{f}_2$, this is

$$\left\lfloor \frac{n-2}{3}\right\rfloor+1.$$

In weight 5, for example, there are two independent such polynomials:

$$u_1^2u_2-u_1u_2^2 \quad\text{and}\quad u_1^3-u_2^3.$$

In weight 7, there are again two independent polynomials, given by

$$u_1^4u_2-u_1u_2^4 \quad\text{and}\quad u_1^5+u_1^3u_2^2-u_1^2u_2^3-u_2^5.$$

In weight 9, the space is three-dimensional, given by

$$\begin{array}{l} u_1^7-2u_1^4u_2^3+2u_1^3u_2^4-u_2^7\\ u_1^6u_2-u_1u_2^6\\ u_1^5u_2^2+u_1^4u_2^3-u_1^3u_2^4-u_1^2u_2^5. \end{array}$$

Finally, we work out the case of weight 11, where the dimension is four:

$$\begin{array}{l} u_1^9+3u_1^5u_2^4-u_1^4u_2^5-u_2^9\\ u_1^8u_2-u_1u_2^8\\ u_1^7u_2^2-u_1^5u_2^4+u_1^4u_2^5-u_1^2u_2^7\\ u_1^6u_2^3+u_1^5u_2^4-u_1^4u_2^5-u_1^3u_2^6 \end{array}$$

Observe that these dimensions are significantly bigger than those given by the elliptic double shuffle equations (EDS.1) and (EDS.2) in depth 2. This is explained by the fact that the vector space generated by the coefficients of $\mathfrak{a}_m(\tau)$ in a given weight and depth is not equal to the one generated by the analogous coefficients of $\mathfrak{e}_m(\tau)$.

Under the conjecture $\overline{\mathcal{Z}}\cong\mathcal{U}(\mathfrak{grt})^\vee$, the $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$ is isomorphic to $\mathcal{U}(\mathfrak{grt}_{\mathrm{ell}})^\vee$, and thus inherits a natural bigrading dual to that of $\mathfrak{grt}_{\mathrm{ell}}$. Together with products of elements of $\overline{\mathcal{E}}$ of smaller depth and weight (including $\mathcal{G}_0$), the coefficients of $\mathfrak{e}_m(\tau)$ in a given weight n and depth d span the bigraded part $\overline{\mathcal{E}}_n^d$, whereas those of $\mathfrak{a}_m(\tau)$ do not.

For example, in weight 5 and depth 2, the coefficients of $\mathfrak{e}_m(\tau)$ generate the 1-dimensional space $\langle 2\mathcal{G}_{0,4}+\mathcal{G}_0\mathcal{G}_4\rangle$. The bigraded subspace $\overline{\mathcal{E}}_5^2$ is spanned by $\mathcal{G}_2^2$, $\mathcal{G}_0\mathcal{G}_4$ and $\mathcal{G}_{0,4}$, but it is also spanned by the two products $\mathcal{G}_2^2$ and $\mathcal{G}_0\mathcal{G}_4$ and the single coefficient $2\mathcal{G}_{0,4}+\mathcal{G}_0\mathcal{G}_4$ of $\mathfrak{e}_m(\tau)$ in weight 5 and depth 2.

The weight 5, depth 2 coefficients of $\mathfrak{a}_m(\tau)$, however, do not lie in $\overline{\mathcal{E}}_5^2$. They span the 2-dimensional subspace $\langle -\frac{1}{12}\mathcal{G}_0\mathcal{G}_2+\frac{3}{2}\mathcal{G}_0\mathcal{G}_4+3\mathcal{G}_{0,4}-\frac{1}{360}\mathcal{G}_0^2+\frac{1}{2}\mathcal{G}_2^2, \frac{1}{240}\mathcal{G}_0^2-2\mathcal{G}_{0,4}-\mathcal{G}_0\mathcal{G}_4\rangle$ of $\overline{\mathcal{E}}$.

We end this subsection with a power series statement of the alternality and push-neutrality relations on $\mathfrak{a}_m(\tau)$.

Corollary 4.11. *The power series* $\mathfrak{A} = [a, \mathfrak{a}(\tau)]$ *is push-neutral in the sense that, if* $\mathfrak{A}^r$ *denotes the depth* r *part of* $\mathfrak{A}$ *for* $r > 1$*, then*

$$A^r + push(A^r) + \cdots + push^r(A^r) = 0$$

where $push$ *denotes the push-operator on power series defined in* (3.7)*.*

Proof. By Theorem 4.8, the mould $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral. Consider the operator

$$-\Delta(A)(u_1, \ldots, u_r) = u_1 \cdots u_r(-u_1 - \ldots - u_r)A(u_1, \ldots, u_r).$$

Since the factor $u_1 \ldots u_r(-u_1 - \ldots - u_r)$ is push-invariant, the mould $-\Delta(A)$ is push-neutral if A is. Therefore in particular $-\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)$ is push-neutral. But this mould is given by

$$\begin{aligned} -\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)(u_1, \ldots, u_r) &= -(u_1 + \cdots + u_r)\, \mathfrak{a}_m(\tau)(u_1, \ldots, u_r) \\ &= ma\big([a, \mathfrak{a}(\tau)]\big)(u_1, \ldots, u_r), \end{aligned}$$

where the last equality is a standard identity (see Appendix A of [30] or (3.3.1) of [32]). Therefore the mould $ma([a, \mathfrak{a}(\tau)])$ is a push-neutral mould, i.e. $[a, \mathfrak{a}(\tau)]$ is push-neutral as a power series. □

4.4. **Proof of Lemma 4.9.** In order to prove this lemma, we need to have recourse to the complete formula for the action of *arat*. We first recall Écalle's formula for *arit* (cf. [13] or [32]), which is given as

$$\big(arit(P) \cdot A\big)(w) = \sum_{\substack{w=abc \\ c\neq\emptyset}} A(a\lceil c)P(b) - \sum_{\substack{w=abc \\ a\neq\emptyset}} A(a\rceil c)P(b),$$

where if the word $u = (u_1, \ldots, u_r)$ is decomposed into three chunks as $u = abc$, $a = (u_1, \ldots, u_i)$, $b = (u_{i+1}, \ldots, u_{i+j})$, $c = (u_{i+j+1}, \ldots, u_r)$, then we use Écalle's notation

$$\begin{aligned} a\rceil &= (u_1, \ldots, u_{i-1}, u_i + u_{i+1} + \cdots + u_{i+j}) \\ \lceil c &= (u_{i+1} + \cdots + u_{i+j+1}, u_{i+j+2}, \ldots, u_r). \end{aligned}$$

Moreover

$$\mathrm{ad}(P) \cdot A = mu(P, A) - mu(A, P)$$

where mu is the mould multiplication defined in (4.1); these correspond precisely to the 'missing' terms $a = \emptyset$ and $c = \emptyset$, so that $arat(P) \cdot A$ actually has the simpler expression

$$\big(arat(P) \cdot A\big)(w) = \sum_{w=abc} \big(A(a\lceil c)P(b) - A(a\rceil c)P(b)\big). \tag{4.23}$$

Now let A be push-neutral, and let $P \in ARI$. We need to show that (4.23) is push-neutral. In fact we will show that the two terms

$$\sum_{w=abc} A(a\lceil c)P(b) \quad \text{and} \quad \sum_{w=abc} A(a\rceil c)P(b) \tag{4.24}$$

of (4.23) are separately push-neutral.

Because the push-neutrality relations take place in fixed depth, we may assume that A is concentrated in depth s and P in depth t, with $s + t = r$. We will prove the push-neurality of the first term in (4.24); the proof for the second term is completely analogous.

Therefore the decompositions $w = abc$ we need to consider are those of the form

$$w = abc = (u_1, \ldots, u_i)(u_{i+1}, \ldots, u_{i+t})(u_{i+t+1}, \ldots, u_r),$$

and we can rewrite the first term of (4.24) as

$$\sum_{i=0}^{r-t} A(u_1, \ldots, u_i, u_{i+1} + \cdots + u_{i+t+1}, u_{i+t+2}, \ldots, u_r) P(u_{i+1}, \ldots, u_{i+t}).$$

The k-th power of the push-operator acts by $u_i \mapsto u_{i-k}$, with indices considered modulo $(r+1)$. The push-neutrality condition thus reads

$$\sum_{k=0}^{r} \sum_{i=0}^{r-t} A(u_{1-k}, \ldots, u_{i-1-k}, u_{i-k}, u_{i+1-k} + \cdots + u_{i+t+1-k}, u_{i+t+2-k}, \ldots, u_{r-k}) \cdot P(u_{i+1-k}, \ldots, u_{i+t-k}) = 0.$$

We will show that the coefficients of each term $P(u_{m+1}, \ldots, u_{m+t})$ sums to zero due to the push-neutrality of A. In fact it is enough to show that the coefficient of $P(u_1, \ldots, u_t)$ sums to zero, as all the other terms are obtained from this one by applying powers of the push-operator.

The terms containing $P(u_1, \ldots, u_t)$ are those for which the index $k = i$, so that $k \in \{0, \ldots, r - t = s\}$, and we must show that the sum

$$\sum_{k=0}^{s} A(u_{r-k+2}, \ldots, u_r, u_0, u_1 + \cdots + u_{t+1}, u_{t+2}, \ldots, u_{r-k})$$

vanishes, where $u_0 = -u_1 - \cdots - u_r$ and we have shifted some of the indices modulo $(r+1)$ in order to make them positive. Note now that

$$u_1 + \cdots + u_{t+1} = -u_0 - u_{t+2} - \cdots + u_r.$$

As a result the last sum runs over the $(s+1)$ cyclic permutations of $u_{t+2}, \ldots, u_r, u_0$ and $-u_{t+2} - \cdots - u_r - u_0$, so it is equal to the sum over the push$_s$-orbit of just one term, say the one with $k = s$, i.e. to

$$\sum_{k=0}^{s} A(u_{t+2}, \ldots, u_r, u_0),$$

which indeed vanishes since A is push-neutral. This concludes the proof of Lemma 4.9. $\square$

References

[1] J. Broedel, C. R. Mafra, N. Matthes, and O. Schlotterer. Elliptic multiple zeta values and one-loop superstring amplitudes. *J. High Energy Phys.*, (7):112, front matter+41, 2015.

[2] J. Broedel, N. Matthes, and O. Schlotterer. Relations between elliptic multiple zeta values and a special derivation algebra. *J. Phys. A*, 49(15):155–203, 49, 2016.

[3] F. Brown. Depth-graded motivic multiple zeta values. `arXiv:1301.3053`.

[4] F. Brown. Multiple modular values for $\mathrm{SL}_2(\mathbb{Z})$. `arXiv:1407.5167`.

[5] F. Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[6] F. Brown. Zeta elements in depth 3 and the fundamental Lie algebra of the infinitesimal Tate curve. *Forum Math. Sigma*, 5, 2017.

[7] F. Brown and A. Levin. Multiple elliptic polylogarithms. `arXiv:1110.6917`.

[8] D. Calaque, B. Enriquez, and P. Etingof. Universal KZB equations: the elliptic case. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 165–266. Birkhäuser Boston, Inc., Boston, MA, 2009.
[9] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.
[10] M. Deneufchâtel, G. H. E. Duchamp, V. H. N. Minh, and A. I. Solomon. Independence of hyperlogarithms over function fields via algebraic combinatorics. In *Algebraic informatics*, volume 6742 of *Lecture Notes in Comput. Sci.*, pages 127–139. Springer, Heidelberg, 2011.
[11] V. G. Drinfel′d. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.
[12] J. Ecalle. ARI/GARI, la dimorphie et l'arithmétique des multizêtas: un premier bilan. *J. Théor. Nombres Bordeaux*, 15(2):411–478, 2003.
[13] J. Ecalle. The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles. In *Asymptotics in dynamics, geometry and PDEs; generalized Borel summation. Vol. II*, volume 12 of *CRM Series*, pages 27–211. Ed. Norm., Pisa, 2011. With computational assistance from S. Carr.
[14] B. Enriquez. Elliptic associators. *Selecta Math. (N.S.)*, 20(2):491–584, 2014.
[15] B. Enriquez. Analogues elliptiques des nombres multizétas. *Bull. Soc. Math. France*, 144(3):395–427, 2016.
[16] H. Furusho. The multiple zeta value algebra and the stable derivation algebra. *Publ. Res. Inst. Math. Sci.*, 39(4):695–720, 2003.
[17] H. Furusho. Double shuffle relation for associators. *Ann. Math.*, 174(1):341–360, 2011.
[18] R. Hain and M. Matsumoto. Universal Mixed Elliptic Motives. to appear in *J. Inst. Math. Jussieu*, 2018.
[19] R. M. Hain. The geometry of the mixed Hodge structure on the fundamental group. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 247–282. Amer. Math. Soc., Providence, RI, 1987.
[20] K. Ihara, M. Kaneko, and D. Zagier. Derivation and double shuffle relations for multiple zeta values. *Compos. Math.*, 142(2):307–338, 2006.
[21] Y. Ihara. The Galois representation arising from $\mathbf{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 299–313. Springer, New York, 1989.
[22] Y. Ihara. On the stable derivation algebra associated with some braid groups. *Israel J. Math.*, 80(1-2):135–153, 1992.
[23] A. Levin and G. Racinet. Towards multiple elliptic polylogarithms. `arXiv:math/0703237`.
[24] Y. I. Manin. Iterated integrals of modular forms and noncommutative modular symbols. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 565–597. Birkhäuser Boston, Boston, MA, 2006.
[25] N. Matthes. *Elliptic multiple zeta values*. PhD thesis, Universität Hamburg, 2016.
[26] N. Matthes. Elliptic double zeta values. *J. Number Theory*, 171:227–251, 2017.
[27] N. Matthes. On the algebraic structure of iterated integrals of quasimodular forms. *Algebra Number Theory*, 11(9):2113–2130, 2017.
[28] R. Molnar. Semi-direct products of Hopf algebras. *J. Alg*, 47:29–51, 1977.
[29] A. Pollack. Relations between derivations arising from modular forms. Master's thesis, Duke University, 2009.
[30] G. Racinet. *Séries génératrices non commutatives de polyzêtas et associateurs de Drinfeld*. PhD thesis, École normale supérieure Paris, 2000.
[31] G. Racinet. Doubles mélanges des polylogarithmes multiples aux racines de l'unité. *Publ. Math. Inst. Hautes Études Sci.*, (95):185–231, 2002.
[32] L. Schneps. ARI, GARI, Zig and Zag: An introduction to Ecalle's theory of moulds. `arXiv:1507.01534`.
[33] L. Schneps. Elliptic multiple zeta values, Grothendieck-Teichmüller and mould theory. `arXiv:1506.09050`.
[34] J.-P. Serre. *Cours d'arithmétique*, volume 2 of *Collection SUP: "Le Mathématicien"*. Presses Universitaires de France, Paris, 1970.
[35] H. Tsunogai. On some derivations of Lie algebras related to Galois representations. *Publ. Res. Inst. Math. Sci.*, 31(1):113–134, 1995.

P.L.: CNRS and Institut Mathématique de Jussieu, Université P. et M. Curie, 4 place Jussieu, F-75252 Paris Cedex 05
E-mail address: pierre.lochak@imj-prg.fr

N.M.: Fachbereich Mathematik (AZ) Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg
E-mail address: nils.matthes@uni-hamburg.de

Current address: Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
E-mail address: nils.oliver.matthes.454@m.kyushu-u.ac.jp

L.S.: CNRS and Institut Mathématique de Jussieu, Université P. et M. Curie, 4 place Jussieu, F-75252 Paris Cedex 05
E-mail address: leila.schneps@imj-prg.fr