On linearised and elliptic versions of the Kashiwara-Vergne Lie algebra

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Abstract

The goal of this article is to define a linearized or depth-graded version $\mathfrak{lkv}$, and a closely related elliptic version $\mathfrak{krv}_{ell}$, of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ originally constructed by Alekseev and Torossian as the space of solutions to the linearized Kashiwara-Vergne problem. We show how the elliptic Lie algebra $\mathfrak{krv}_{ell}$ is related to earlier constructions of elliptic versions $\mathfrak{grt}_{ell}$ and $\mathfrak{ds}_{ell}$ of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$ respectively. Based on the known relationships between the three Lie algebra $\mathfrak{grt}$, $\mathfrak{ds}$ and $\mathfrak{krv}$, we discuss the corresponding relationships between the linearized versions, and also between the elliptic versions.

1. Introduction

This article studies two Lie algebras closely related to the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$ defined in [AT]: firstly, a depth-graded (or "linearized") version $\mathfrak{lkv}$, and secondly, an elliptic version $\mathfrak{krv}_{ell}$. The results are motivated by the comparison of $\mathfrak{krv}$ with two other Lie algebras familiar from the theory of multiple zeta values: the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ and the double shuffle Lie algebra $\mathfrak{ds}$. Our definition of $\mathfrak{lkv}$ is an analog of the definition of the depth-graded (or linearized) double shuffle Lie algebra $\mathfrak{ls}$, whose structure has given rise to many results and conjectures, in particular the famous Broadhurst-Kreimer conjecture. Our definition of $\mathfrak{krv}_{ell}$ is an analog of the definition of the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ (cf. [S3]), which itself is related on the one hand to $\mathfrak{ls}$ and on the other to the elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{ell}$. We explore all the relations between these different objects. The main observation is that the spaces $\mathfrak{lkv}$ and $\mathfrak{krv}_{ell}$ are defined by identical sets of properties, only applied to a different class of objects. This situation, which exactly parallels the case of $\mathfrak{ds}$ and $\mathfrak{ds}_{ell}$, reveals a close and surprising relationship between the depth-graded and the elliptic versions of the Lie algebras $\mathfrak{ds}$ and $\mathfrak{krv}$, which remains invisible without the use of mould theory as a basic tool.

Like $\mathfrak{grt}$ and $\mathfrak{ds}$, the Lie algebra $\mathfrak{krv}$ is equipped with a depth filtration; we write gr for the associated graded. We show that in analogy with the known injective map $gr\,\mathfrak{ds} \to \mathfrak{ls}$, there is an injective map $gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv}$ (Proposition 1.7). We also show that there is an injective Lie morphism $\mathfrak{ls} \hookrightarrow \mathfrak{lkv}$, and that the parts of these spaces of depths $d = 1, 2, 3$ are isomorphic for all weights n (Theorem 1.8), which yields the dimensions of the bigraded parts of $\mathfrak{lkv}$ (and also $gr\,\mathfrak{krv}$) of depths $1, 2, 3$ in all weights, since these dimensions are well-known for $\mathfrak{ls}$.

Passing to the elliptic situation, we define the elliptic version $\mathfrak{krv}_{ell}$ as a subspace of derivations of the free Lie algebra on two generators, and prove that it is closed under the Lie bracket of derivations (Theorem 1.12). We also define an injective Lie morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$ (in Theorem 1.14; see footnote 1 below for an explanation of the dotted arrow) in analogy with the section map $\mathfrak{grt} \hookrightarrow \mathfrak{grt}_{ell}$ ([E1]) and the mould-theoretic double shuffle map $\mathfrak{ds} \to \mathfrak{ds}_{ell}$ ([S3]). Finally, although we were not able to prove the existence of an injection $\mathfrak{grt}_{ell} \hookrightarrow \mathfrak{krv}_{ell}$, we define a Lie subalgebra $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$ such that the following diagram commutes[1]

$$\begin{tikzcd}
\mathfrak{grt} \arrow[r, hook] \arrow[d] & \mathfrak{ds} \arrow[r, hook, dotted] \arrow[d] \arrow[dr] & \mathfrak{krv} \arrow[d, hook, dotted] \\
\widetilde{\mathfrak{grt}} \arrow[r, hook] & \mathfrak{ds}_{ell} \arrow[r, hook] & \mathfrak{krv}_{ell}
\end{tikzcd}$$

(see §1.3 below for more details and references for all these maps). The main technique used for the constructions in this article is the mould theory developed by J. Écalle, to which we provide a brief introduction in §3, with complements in §4.

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1.1. **Special types of derivations of $\mathfrak{lie}_2$.** Let $\mathfrak{lie}_2$ denote the degree completion of the free Lie algebra over $\mathbb{Q}$ on non-commutative variables x and y. The Lie algebra $\mathfrak{lie}_2$ has a weight grading by the degree (=weight) of the polynomials, and a depth grading by the y-degree (=depth) of the polynomials. We write $(\mathfrak{lie}_2)_n$ for the graded part of weight n, $(\mathfrak{lie}_2)^r$ for the graded part of depth r, and $(\mathfrak{lie}_2)_n^r$ for the intersection, which is finite-dimensional.

All the Lie algebras we will study in this article (the well-known ones $\mathfrak{krv}$, $\mathfrak{grt}$ and $\mathfrak{ds}$ as well as the linearized $\mathfrak{ls}$, and the spaces $\mathfrak{lkv}$ and $\mathfrak{krv}_{ell}$ that we introduce) can be viewed either as Lie subalgebras of particular subalgebras of the derivations of $\mathfrak{lie}_2$, equipped with the bracket of derivations, or as subspaces of $\mathfrak{lie}_2$ equipped with particular Lie brackets coming from the Lie bracket of derivations. Both ways of considering our spaces are natural and useful, and we go back and forth between them as convenient for our proofs.

Let $\mathfrak{Der}_2$ denote the algebra of derivations on $\mathfrak{lie}_2$. It is a Lie algebra under the Lie bracket given by the commutator of derivations. For $a, b \in \mathfrak{lie}_2$, we write $D_{b,a}$ for the derivation defined by $x \mapsto b$ and $y \mapsto a$. The bracket is explicitly given by

$$[D_{b,a}, D_{b',a'}] = D_{\tilde{b},\tilde{a}} \tag{1}$$

with

$$\tilde{b} = D_{b,a}(b') - D_{b',a'}(b), \qquad \tilde{a} = D_{b,a}(a') - D_{b',a'}(a). \tag{2}$$

• Let $\mathfrak{oDer}_2$ denote the Lie subalgebra of $\mathfrak{Der}_2$ of derivations $D = D_{b,a}$ that annihilate the bracket $[x, y]$ and such that neither $D(x)$ nor $D(y)$ have a linear term in x. The map $\mathfrak{oDer}_2 \to \mathfrak{lie}_2$ given by $D \mapsto D(x)$ is injective (see Corollary 4.3).

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[1] The existence of the dotted morphisms in this diagram are shown in [S1] (for the horizontal arrow) and §4.2 (for the vertical arrow). However, both of these definitions rely on a certain mould theoretic property of the map, given explicitly in (86), which was stated by Écalle in his basic mould theory text [Ec] (see the precise reference in the footnote to (86)), but whose proof has never been fully written down. Throughout this article, we use dotted maps to indicate that their definition relies on this property, whereas solid arrows indicate that the definition of the map does not rely on it.

• Let $\mathfrak{tder}_2$ denote the Lie subalgebra of $\mathfrak{der}_2$ of *tangential derivations*, which are the derivations $E_{a,b}$ for elements $a, b \in \mathfrak{lie}_2$ such that a has no linear term in x and b has no linear term in y, such that

$$E_{a,b}(x) = [x, a], \quad E_{a,b}(y) = [y, b].$$

The Lie bracket is explicitly given by

$$[E_{a,b}, E_{a',b'}] = E_{\tilde{a},\tilde{b}} \tag{3}$$

where

$$\tilde{a} = [a, a'] + E_{a,b}(a') - E_{a',b'}(a), \quad \tilde{b} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b). \tag{4}$$

• Let $\mathfrak{sder}_2$ denote the Lie subalgebra of $\mathfrak{tder}_2$ of *special tangential derivations*, i.e. derivations such that $E_{a,b}(x + y) = [x, a] + [y, b] = 0$.

• Let $\mathfrak{ider}_2$ be the Lie subalgebra of $\mathfrak{tder}_2$ of *Ihara derivations*, which are those that annihilate x, i.e. those of the form $d_b = E_{0,b}$. The derivation d_b is defined by its values on x and y

$$d_b(x) = 0, \qquad d_b(y) = [y, b]. \tag{5}$$

The Lie bracket on $\mathfrak{ider}_2$ is given by $[d_b, d_{b'}] = d_{\{b,b'\}}$, where $\{b, b'\}$ is the *Poisson* (or Ihara) bracket given by

$$\{b, b'\} = [b, b'] + d_b(b') - d_{b'}(b), \tag{6}$$

i.e. the second term of (4).

We have the following diagram showing the connections between these subspaces:

$$\begin{tikzcd}
\mathfrak{oder}_2 \arrow[r, hook] & \mathfrak{der}_2 \\
& \mathfrak{tder}_2 \arrow[u, hook] \\
\mathfrak{sder}_2 \arrow[ur, hook] \arrow[r, "\sim"] & \mathfrak{ider}_2 \arrow[u, hook]
\end{tikzcd} \tag{7}$$

The isomorphism between $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ is given in Lemma 4.12.

1.2. Definition of the Kashiwara-Vergne Lie algebra $\mathfrak{krv}$.

The free associative algebra $\mathrm{Ass}_2 = \mathbb{Q}\langle\langle x, y\rangle\rangle$ on non-commutative generators x, y (i.e. the ring of formal power series in x and y) is the completion with respect to the degree of the universal enveloping algebra of the free Lie algebra $\mathfrak{lie}_2$ on x and y.

Definition 1.1. The trace vector space $\mathfrak{tr}_2$ (cf. [AT]) is defined to be the quotient of Ass_2 by the equivalence relation given between words in x and y by $w \sim w'$ if w' can be obtained from w by a cyclic permutation of the letters of the word w, and extended linearly to polynomials. The natural projection is denoted

$$\mathrm{tr} : \mathrm{Ass}_2 \to \mathfrak{tr}_2.$$

For any polynomial $f \in \mathrm{Ass}_2$ with constant term c, we can decompose f in two ways as

$$f = c + f_x x + f_y y = c + x f^x + y f^y \tag{8}$$

for uniquely determined polynomials f_x, f_y, f^x, f^y in Ass_2.

Definition 1.2. The *divergence* map is given by

$$\mathrm{div}: \begin{array}{rcl} \mathfrak{tder}_2 & \longrightarrow & \mathfrak{tr}_2 \\ u = E_{a,b} & \longmapsto & \mathrm{tr}(a_x x + b_y y). \end{array}$$

Definition 1.3. The *Kashiwara-Vergne Lie algebra* $\mathfrak{krv}_2$ is defined to be the subspace of $\mathfrak{sder}_2$ of derivations $E_{a,b}$ such that there exists a one-variable power series $h(x) \in \mathbb{Q}[x]$ of degree ≥ 2 such that

$$\mathrm{div}(E_{a,b}) = tr\big(h(x+y) - h(x) - h(y)\big). \tag{9}$$

This definition comes from [AT], where it was shown that $\mathfrak{krv}_2$ is actually a Lie subalgebra of $\mathfrak{sder}_2$. This Lie algebra inherits a weight-grading from that of $\mathfrak{lie}_2$, for which $E_{a,b}$ is of weight n if b (and thus also a) is a Lie polynomial of homogeneous degree n. In particular, the weight 1 part of $\mathfrak{krv}_2$ is spanned by the single element $u = E_{y,x}$, and the weight 2 part is zero. In this article, we do not consider the weight 1 part of $\mathfrak{krv}_2$. For convenience, we set $\mathfrak{krv} = \oplus_{n\geq 3}(\mathfrak{krv}_2)_n$, where $(\mathfrak{krv}_2)_n$ denotes the weight graded part of $\mathfrak{krv}_2$ of weight n. We have

$$\mathfrak{krv}_2 = (\mathfrak{krv}_2)_1 \oplus \mathfrak{krv} = \mathbb{Q}[E_{y,x}] \oplus \mathfrak{krv}.$$

Because the other Lie algebras in the literature that are most often compared with the Kashiwara-Vergne Lie algebra have no weight 1 or weight 2 parts, it makes most sense to compare them with $\mathfrak{krv}$. Thus it is $\mathfrak{krv}$ that we study for the remainder of this article.

The Lie algebra $\mathfrak{krv}$ also inherits a depth filtration from the depth grading on $\mathfrak{lie}_2$, for which $E_{a,b}$ is of depth r if r is the smallest number of y's occurring in any monomial of b. We write $gr\,\mathfrak{krv}$ for the associated graded for this depth filtration, so that $gr\,\mathfrak{krv}$ is a Lie algebra that is bigraded for the weight and the depth; we write $gr_n^r\,\mathfrak{krv}$ for the part of weight n and depth r. Essentially, an element of $gr\,\mathfrak{krv}$ is a derivation $E_{\bar{a},\bar{b}} \in \mathfrak{sder}_2$ where $\bar{a}, \bar{b}$ are the lowest-depth parts (i.e. the parts of lowest y-degree) of elements $a, b \in \mathfrak{lie}_2$ such that $E_{a,b} \in \mathfrak{krv}$. If $\bar{b}$ is of homogeneous y-degree r, then $\bar{a}$ is of homogeneous y-degree $r+1$.

Example. The smallest element of $\mathfrak{krv}$ is in weight 3 and is given by $E_{a,b}$ with

$$a = [[x,y],y], \quad b = [x,[x,y]].$$

Since $\bar{a} = a$ and $\bar{b} = b$, this is also equal to $E_{\bar{a},\bar{b}} \in gr\,\mathfrak{krv}$. The next smallest element of $\mathfrak{krv}$ is in weight 5, and the depth-graded part $E_{\bar{a},\bar{b}}$ is given by

$$\bar{a} = [x,[x,[[x,y],y]]] - 2[[x,[x,y]],[x,y]], \qquad \bar{b} = [x,[x,[x,[x,y]]]].$$

1.3. **The Grothendieck-Teichmüller and double shuffle Lie algebras.** Recall that the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying the famous *pentagon relation*, equipped with the Poisson bracket (6). This algebra was first introduced by Y. Ihara in [I], with three defining relations, as a particular derivation algebra of $\mathfrak{lie}_2$ (via the association $b \mapsto d_b$ as in (5)); H. Furusho subsequently showed that the pentagonal relation implies the other two (cf. [F1]).

Recall also that the double shuffle Lie algebra $\mathfrak{ds}$ is the space of polynomials $b \in \mathfrak{lie}_2$ satisfying a particular set of conditions on the coefficients called the *stuffle relations*, studied in the first place by Racinet (cf. [R]), who gave a quite difficult proof that $\mathfrak{ds}$ is also a Lie algebra under the Poisson bracket (6). This proof was later somewhat streamlined by Furusho (cf. [F2], Appendix), and a recent preprint [EF] gives another proof with a different approach, identifying the space as a stabilizer. Putting together basic elements from Écalle's mould theory also yields a completely different and very simple proof of this result ([SS]).

There is a commutative triangle of injective Lie morphisms[2]:

(10)
$$\begin{tikzcd} \mathfrak{grt} \arrow[rr, hook] \arrow[dr, hook] & & \mathfrak{ds} \arrow[dl, hook, dotted] \\ & \mathfrak{krv} & \end{tikzcd}.$$

The existence of the injection $\mathfrak{grt} \to \mathfrak{ds}$ was proven in [F1]; it is given by $b(x,y) \mapsto b(x,-y)$. The existence of the injection $\mathfrak{grt} \to \mathfrak{krv}$ was proven in [AT]; it is given by $b(x,y) \mapsto b(z,y)$ where $z = -x-y$. Finally, the existence of the injection from $\mathfrak{ds}$ to $\mathfrak{krv}$ was proven in [S1] (using results from Écalle's mould theory including the statement (86), cf. footnote 1), and is given, of course, by $b(x,y) \mapsto b(z,-y)$. In particular, these morphisms respect the weight gradings and depth filtrations on all three spaces.

1.4. **The linearized Kashiwara-Vergne Lie algebra: main results.** For $i \geq 1$, set $C_i = ad(x)^{i-1}(y)$ for $i \geq 1$, and let $\mathfrak{lie}_C$ denote the degree completion of the Lie algebra freely generated over $\mathbb{Q}$ by $C_1, C_2, \ldots$. By Lazard elimination, $\mathfrak{lie}_C$ is free on the C_i and

$$\mathfrak{lie}_2 \simeq \mathbb{Q} \cdot x \oplus \mathfrak{lie}_C. \tag{11}$$

Thus, Lazard elimination shows that every polynomial $b \in \mathfrak{lie}_2$ having no linear term in x can be written uniquely as a Lie polynomial in the C_i.

Definition 1.4. Let the push-operator be defined on monomials in x, y by

$$push(x^{a_0}yx^{a_1}y\cdots yx^{a_r}) = x^{a_r}yx^{a_0}y\cdots yx^{a_{r-1}}. \tag{12}$$

The push is considered to act trivially on constants and powers of x^n, so we can extend it to all of Ass_2 by linearity. A polynomial b in x, y is said to be

- *push-invariant* if $push(b) = b$, and
- *push-neutral* if $b^r + push(b^r) + \cdots + push^r(b^r) = 0$ for all $r \geq 1$, where b^r denotes the depth r part of b. Finally, we say that b is
- *circ-neutral* if b^y is push-neutral in depths $r > 1$.

Definition 1.5. The *linearized Kashiwara-Vergne Lie algebra* $\mathfrak{lkv}$ is the space of elements $b \in \mathfrak{lie}_C$ of degree ≥ 3 such that

(i) b is push-invariant, and

(ii) b is circ-neutral.

Our first result on $\mathfrak{lkv}$ is that is a bigraded Lie algebra[3].

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[2]The dotted morphism is as in footnote 1.

[3]A similar result is shown in [FK], Theorem 3.23. With [FK] Remark 2.18 one can see directly that the depth > 1 part of $\mathfrak{lkv}$ forms a Lie algebra.

Proposition 1.6. *The space $\mathfrak{lkv}$ is bigraded by weight and depth, and forms a Lie algebra under the Poisson bracket defined in (6).*

In §1.5 below, we define a larger space, the elliptic Kashiwara-Vergne Lie algebra $\mathfrak{krv}_{ell}$, and show in Theorem 1.12 that it is a Lie algebra. Although it might be possible (albeit laborious) to prove Proposition 1.6 directly, it turns out to follow immediately from Theorem 1.12, due to the fact that there is a simple injection of $\mathfrak{lkv}$ into the larger space $\mathfrak{krv}_{ell}$ (see Proposition 1.13 following Theorem 1.12) whose image is easily identifiable as the intersection of two Lie subalgebras. For this reason, the proof of Proposition 1.6 can be found in Corollary 4.7 at the end of §4.1, following the proof of Theorem 1.12.

In §2, we show how we derive the definition of $\mathfrak{lkv}$ via a reformulation of the defining properties of $\mathfrak{krv}$, in the sense that the defining properties of $\mathfrak{lkv}$ are merely truncations of the two reformulated defining properties of $\mathfrak{krv}$ to their lowest-depth parts. The complete proof of the statement of Proposition 1.6 namely that $\mathfrak{lkv}$ is a Lie algebra, is deferred for convenience to §4.1 (Corollary 4.7), which contains all the necessary mould theory to spell out the proof. However, given the result of Proposition 1.6, the reformulation of the defining properties of $\mathfrak{lkv}$ in §2 suffices to prove the following result on $\mathfrak{lkv}$. The proof is given (modulo Proposition 1.6) in §2.3.

Proposition 1.7. *There is an injective Lie algebra morphism*

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv}.$$

We conjecture that these two spaces are in fact isomorphic.

In using this type of definition for $\mathfrak{lkv}$, we are following the analogous situation of the well-known double shuffle Lie algebra $\mathfrak{ds}$ and the associated linearized double shuffle space $\mathfrak{ls}$ studied in many articles (cf. for example [Br]). The bigraded linearized space $\mathfrak{ls}$ is defined as the set of Lie polynomials $f \in \mathfrak{lie}_2$ of weight $n \geq 3$ such that the polynomial $f_y y$, rewritten in the variables $y_i = x^{n-1}y$ for $n \geq 1$, is an element of the free Lie algebra on the y_i. One also adds the extra assumption that if f is of depth 1, then it is of odd weight, an assumption which is not needed for $\mathfrak{lkv}$ as it follows from the push-invariance condition in the definition. By its very construction, there is an injective Lie algebra homomorphism

$$gr\,\mathfrak{ds} \hookrightarrow \mathfrak{ls}, \tag{13}$$

and it is conjectured that these two spaces are isomorphic, but like for $\mathfrak{lkv}$, this is still an open question.

The injective Lie algebra morphism (10) from $\mathfrak{ds}$ to $\mathfrak{krv}$ yields a corresponding bigraded injective map:

$$gr\,\mathfrak{ds} \hookrightarrow gr\,\mathfrak{krv} \tag{14}$$

(for the meaning of the dotted arrow, see footnote 1). Our next result shows that there is a Lie algebra map on the generalized spaces spaces $\mathfrak{ls}$ and $\mathfrak{lkv}$ (without any recourse to Écalle's theorem).

Theorem 1.8. *There is a bigraded Lie algebra injection on linearized spaces*

$$\mathfrak{ls} \hookrightarrow \mathfrak{lkv}. \tag{15}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}_n^r \simeq \mathfrak{lkv}_n^r.$$

Remark. If the conjectures $\mathfrak{ls} \simeq gr\,\mathfrak{ds}$ and $\mathfrak{lkv} \simeq gr\,\mathfrak{krv}$ hold, then the map in Theorem 1.8 is the same as the map (14), which would thus no longer need to be a dotted arrow. Without those conjectures, we can only say that the Lie injection (15) should extend the dotted arrow (14), fitting into a commutative diagram

$$\begin{tikzcd} gr\,\mathfrak{ds} \arrow[r, hook, dotted] \arrow[d, hook] & gr\,\mathfrak{krv} \arrow[d, hook] \\ \mathfrak{ls} \arrow[r, hook] & \mathfrak{lkv}. \end{tikzcd} \tag{16}$$

We observe also that the existence of the injective Lie morphism (15) was extended to the cyclotomic situation in [FK].

Theorem 1.8 will be proved in §3, using mould theory, to which we give a brief and elementary introduction in that section, with more advanced elements of the theory given in §4. Mould theory is also essential for all the proofs concerning the elliptic Kashiwara-Vergne Lie algebra defined in the next subsection.

Adding a variety of known results in the depth 2 and depth 3 situations to this result, we obtain the following corollary.

Corollary 1.9. *The following spaces are isomorphic for $n \geq 3$ and $r = 1, 2, 3$:*

$$gr_n^r\mathfrak{grt} \simeq gr_n^r\mathfrak{ds} \simeq gr_n^r\mathfrak{krv} \simeq \mathfrak{ls}_n^r \simeq \mathfrak{lkv}_n^r.$$

In particular, all of these spaces are zero when $r = 1$ or 3 and n is even, or when $r = 2$ and n is odd.

Proof. The dimensions of the spaces $gr_n^r\mathfrak{grt}$, $gr_n^r\mathfrak{ds}$ and $\mathfrak{ls}_n^r$ in depths are known to be equal to each other in depths $r \leq 3$ ([R], [G]). Indeed more is known than merely the dimensions:

• the spaces $gr_n^1\mathfrak{grt}$, $gr_n^1\mathfrak{ds}$ and $\mathfrak{ls}_n^1$ are all 0 when n is even and 1-dimensional generated by $ad(x)^{n-1}(y)$ when n is odd;

the spaces $gr_n^2\mathfrak{grt}$, $gr_n^2\mathfrak{ds}$ and $\mathfrak{ls}_n^2$ are all 0 when n is odd and spanned by the double Poisson brackets $\{ad(x)^{p-1}(y), ad(x)^{q-1}(y)\}$ for odd $p, q \leq 3$ with $p + q = n$ when n is even;

• the spaces $gr_n^3\mathfrak{grt}$, $gr_n^3\mathfrak{ds}$ and $\mathfrak{ls}_n^3$ are all 0 when n even and spanned by the triple brackets $\{ad(x)^{p-1}(y), \{ad(x)^{q-1}(y), ad(x)^{s-1}(y)\}\}$ with odd $p, q, s \geq 3$ and $p + q + s = n$ when n is odd.

(Note that the proof for $r = 3$ and odd n is much more difficult than the proof for $r = 2$, and was discovered by Goncharov [G]; as for the case $r \geq 4$, the analogous result is known to be false.) By Theorem 1.8, we see that $\mathfrak{lkv}_n^r \simeq \mathfrak{ls}_n^r$ for $r = 1, 2, 3$ Finally, since it is known that $\mathfrak{grt}$ injects into $\mathfrak{krv}$ (cf. [AT]), we have a corresponding injection $gr_n^r\mathfrak{grt} \hookrightarrow gr_n^r\mathfrak{krv}$, so by Proposition 1.7, shows that $gr_n^r\mathfrak{krv}$ is sandwiched between $gr_n^r\mathfrak{grt}$ and $\mathfrak{lkv}_n^r$, which are equal for $r = 1, 2, 3$. This concludes the proof. □

We conjecture that $\mathfrak{lkv}_n^r \simeq \mathfrak{ls}_n^r$ for all n, r, and calculations up to about $n = 15$ bear this conjecture out, but we were not able to prove the isomorphism for any other cases, not even the special case $n \not\equiv r \bmod 2$, where it is well-known that $gr_n^r\mathfrak{grt} = gr_n^r\mathfrak{ds} = \mathfrak{ls}_n^r = 0$ (cf. [IKZ], [Br] for classical proofs, or [S2] for the exposition of Écalle's mould-theoretic proof).

Let us end this subsection by giving the *mould-language reformulation* of the definition of $\mathfrak{lkv}$, which will allow us to connect it directly to the definition of the elliptic Kashiwara-Vergne Lie algebra defined in the next subsection, which cannot be defined directly in terms of elements of $\mathfrak{lie}_2$. The mould definition of $\mathfrak{lkv}$ clearly echoes the definition in terms of Lie elements given above; the equivalence is shown in detail in §3.

Definition 1.10. ***Mould-reformulated*** $\mathfrak{lkv}$.
To express the elements of $\mathfrak{lkv}$ *as moulds, we use the following notation. Let* $b \in \mathfrak{lie}_C$ *and for each* $r \geq 0$*, write the depth* r *part of* b *as*

$$b^r = \sum_{\underline{a}} k_{\underline{a}}\, C_{a_1} \cdots C_{a_r}, \tag{17}$$

where the sum runs over tuples $\underline{a} = (a_1, \ldots, a_r)$*,* $a_i \geq 1$*,. Let* $B^0 = 0$ *and for each* $r \geq 1$*, let* $B^r(u_1, \ldots, u_r)$ *be defined by*

$$B^r(u_1, \ldots, u_r) = \sum_{\underline{a}} k_{\underline{a}}\, u_1^{a_1-1} \cdots u_r^{a_r-1}. \tag{18}$$

for commutative variables $u_1, \ldots, u_r$*. The family* $B = (B_r)_{r\geq 0}$ *is known as the mould associated to* b*; we write* $B = ma(b)$ *(the "mould map"* ma *will be introduced in more detail in Lemma 3.3).*

We use this definition to reformulate the definition of $\mathfrak{lkv}$ in terms of moulds. Let $b \in \mathfrak{lkv}$ and let $B = ma(b)$. Set

$$\bar{B}^r(v_1, \ldots, v_r) := B^r(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2) \tag{19}$$

for commutative variables $v_1, \ldots, v_r$. We define the following properties on B resp. $\bar{B}$:

(i) B^r is *push-invariant* for $r \geq 1$, i.e.

$$B(u_0, u_1, \ldots, u_{r-1}) = B(u_1, \ldots, u_r) \tag{20}$$

where $u_0 = -u_1 - \cdots - u_r$, and

(ii) $\bar{B}^r$ is *circ-neutral* for $r > 1$, i.e.

$$\bar{B}^r(v_1, \ldots, v_r) + \bar{B}^r(v_2, \ldots, v_r, v_1) + \cdots + \bar{B}^r(v_r, v_1, \ldots, v_{r-1}) = 0. \tag{21}$$

We prove in §2 that these two conditions on $B = ma(b)$ are equivalent to the defining conditions of specialness and the divergence condition on b.

1.5. **The elliptic Kashiwara-Vergne Lie algebra.** The last section of this article is devoted to the study of the *elliptic Kashiwara-Vergne Lie algebra*. The definition of this algebra is based on that of the linearized Lie algebra $\mathfrak{lkv}$, differing only from Definition $5'$ by the denominator appearing in (22), which makes it impossible to express it directly in terms of Lie elements like Definition 1.5.

Definition 1.11. The *elliptic Kashiwara-Vergne vector space* $\mathfrak{krv}_{ell}$ is spanned by the elements $b \in \mathfrak{lie}_C$ such that writing the depth r part b^r as in (17) and the associated polynomial B^r as in (18), and setting

$$B^r_*(u_1,\ldots,u_r) = \frac{1}{u_1\cdots u_r(u_1+\cdots+u_r)} B^r(u_1,\ldots,u_r) \tag{22}$$

and

$$\bar{B}^r_* = B^r_*(v_r, v_{r-1}-v_r,\ldots,v_1-v_2),$$

we have

(i) B^r_* is push-invariant as in (20) for $r \geq 1$;

(ii) $\bar{B}^r_*$ is circ-neutral as in (21) for $r > 1$.

The first main result on $\mathfrak{krv}_{ell}$ is of course that it is a bigraded Lie algebra, but this comes from an injective map from $\mathfrak{krv}_{ell}$ into $\mathfrak{oder}_2$ rather than into $\mathfrak{sder}_2$ as for $\mathfrak{lkv}$.

Theorem 1.12. *(i) The space $\mathfrak{krv}_{ell}$ is bigraded for the weight and the depth.*

(ii) For each $b \in \mathfrak{krv}_{ell}$, there exists a unique polynomial $a \in \mathfrak{lie}_C$, called the partner *of b, such that $D_{b,a} \in \mathfrak{oder}_2$.*

(iii) The image of the injective linear map $b \mapsto D_{b,a}$ is a Lie subalgebra of $\mathfrak{oder}_2$; in other words $\mathfrak{krv}_{ell}$ is a Lie algebra under the Lie bracket

$$\langle b, b'\rangle = D_{b,a}(b') - D_{b',a'}(b) \tag{23}$$

coming from the bracket of derivations as in (1) and (2).

This theorem is proven in §4.1 (Theorem 4.4); it necessitates the introduction of some more complicated definitions and results from mould theory than those used in §3.

The following result is key to the comparison of $\mathfrak{lkv}$ and $\mathfrak{krv}_{ell}$, and to the proof that $\mathfrak{lkv}$ is a Lie algebra.

Proposition 1.13. *There is an injective linear map*

$$\begin{gathered}\mathfrak{lkv} \hookrightarrow \mathfrak{krv}_{ell}\\ b(x,y) \mapsto [x, b(x,[x,y])].\end{gathered} \tag{24}$$

Equivalently, the map can be defined on the family B^r of polynomials in commutative variables such that $B = (B_r)_{r\geq 0} = ma(b)$ (cf. (18)) by

$$B^r(u_1,\ldots,u_r) \mapsto u_1\cdots u_r(u_1+\cdots+u_r)B^r(u_1,\ldots,u_r).$$

In fact, this linear map is actually a Lie morphism.

Sketch of Proof. The first statement, concerning the existence of the linear map, follows from the definitions. Indeed, by Definition 1.11, $\mathfrak{krv}_{ell}$ is isomorphic to the space spanned by the polynomials in the commutative variables u_i that become push-invariant and circ-neutral (possibly after adding a constant) after division by $u_1\cdots u_r(u_1+\cdots+u_r)$. On the other hand, $\mathfrak{lkv}$ is isomorphic to space of polynomials that are themselves push-invariant and circ-neutral. Thus, multiplying by the factor $u_1\cdots u_r(u_1+\cdots+u_r)$ maps $\mathfrak{lkv}$ precisely to the subspace of $\mathfrak{krv}_{ell}$ consisting of polynomials that are divisible by $u_1\cdots u_r(u_1+\cdots+u_r)$, and so we have a linear map from $\mathfrak{lkv}$ to $\mathfrak{krv}_{ell}$. The fact that the map is a Lie morphism is proven in Corollary 4.8. $\square$

In two independent articles, H. Tsunogai [Ts] and B. Enriquez [E1] defined a Lie algebra that Enriquez calls the *elliptic Grothendieck-Teichmüller Lie algebra* $\mathfrak{grt}_{ell}$, based on the idea that just as Ihara had defined $\mathfrak{grt}$ as the algebra of derivations on $\mathfrak{lie}_2$ (identified with the braid Lie algebra on four strands) that extend to a particular type of derivation on the braid Lie algebra on five strands, $\mathfrak{grt}_{ell}$ is the Lie algebra of derivations on $\mathfrak{lie}_2$ (now identified with the genus one braid Lie algebra on two strands) that extend to a very particular type of derivation of the genus one braid Lie algebra on three strands. The construction of $\mathfrak{grt}_{ell}$ shows that it is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is a canonical surjection

$$s : \mathfrak{grt}_{ell} \to \mathfrak{grt}. \tag{25}$$

Let $\mathfrak{r}_{ell}$ denote the kernel. Enriquez [E1] showed that there also exists a Lie algebra morphism

$$\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell} \tag{26}$$

that is a section of (25), i.e. such that $\gamma \circ s = id$ on $\mathfrak{grt}$. Thus, there is a semi-direct product isomorphism

$$\mathfrak{grt}_{ell} \simeq \mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt}). \tag{27}$$

An elliptic version $\mathfrak{ds}_{ell}$ of the double shuffle Lie algebra $\mathfrak{ds}$ was constructed in [S3] using mould theory, and it is shown there that like $\mathfrak{grt}_{ell}$, $\mathfrak{ds}_{ell}$ is a Lie subalgebra of $\mathfrak{oder}_2$, and that there is an injective Lie morphism $\tilde{\gamma} : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ that makes the diagram

$$\begin{tikzcd}
\mathfrak{grt} \arrow[rr, hook] \arrow[d, "\gamma"'] & & \mathfrak{ds} \arrow[d, "\tilde{\gamma}"] \\
\mathfrak{grt}_{ell} \arrow[dr] & & \mathfrak{ds}_{ell} \arrow[dl] \\
& \mathfrak{oder}_2 & .
\end{tikzcd} \tag{28}$$

commute.

Our second main result on $\mathfrak{krv}_{ell}$ is an analog of the existence of γ and $\tilde{\gamma}$.

Theorem 1.14. *There is an injective Lie algebra morphism*

$$\hat{\gamma} : \mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}.$$

Based on the known injective Lie morphisms $\mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{krv}$ evoked in §1.3 above, we believe that there are corresponding injective Lie morphisms between the elliptic versions of these Lie algebras. However, we were not able to prove that $\mathfrak{grt}_{ell}$ as defined in [E1] injects into $\mathfrak{ds}_{ell}$ or $\mathfrak{krv}_{ell}$. To circumvent this difficulty, we define a Lie subalgebra $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{grt}_{ell}$, conjecturally isomorphic to $\mathfrak{grt}_{ell}$, as follows.

Definition 1.15. For $n \geq 0$, let $\delta_{2n} \in \mathfrak{oder}_2$ denote the derivation of $\mathfrak{lie}_2$ defined by

$$\delta_{2n}(x) = ad(x)^{2n}(y), \quad \delta_{2n}([x,y]) = 0.$$

Let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{oder}_2$ generated by the δ_{2n}.

Enriquez showed in [E1] that $\delta_{2n} \in \mathfrak{r}_{ell}$ for $n \geq 0$, so $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{r}_{ell}$. Let $\mathfrak{B}$ denote the normalization of $\mathfrak{b} \subset \mathfrak{r}_{ell}$ under the semi-direct action of $\gamma(\mathfrak{grt})$ on $\mathfrak{r}_{ell}$ of (27). We set

$$\widetilde{\mathfrak{grt}}_{ell} = \mathfrak{B} \rtimes \gamma(\mathfrak{grt}). \tag{29}$$

Our third main result on $\mathfrak{krv}_{ell}$ relates all these maps via a commutative diagram.

Theorem 1.16. *We have the following commutative diagram of injective Lie morphisms[4]:*

$$\begin{tikzcd} \mathfrak{grt} \arrow[r, hook] \arrow[d, hook] & \mathfrak{ds} \arrow[r, hook, dotted] \arrow[d, hook] & \mathfrak{krv} \arrow[d, hook, dotted] \\ \widetilde{\mathfrak{grt}}_{ell} \arrow[r, hook] \arrow[dr, hook] & \mathfrak{ds}_{ell} \arrow[r, hook] \arrow[d, hook] & \mathfrak{krv}_{ell} \arrow[dl, hook] \\ & \mathfrak{oder}_2. & \end{tikzcd} \tag{30}$$

1.6. **Outline of the article.** In §2, we reformulate the defining conditions of $\mathfrak{krv}$, which lead to the first definition of $\mathfrak{lkv}$ and the proof of Proposition 1.7. The next section, §3, gives a brief introduction to mould theory and a translation of the defining conditions of $\mathfrak{lkv}$ into that language, and uses mould theory to prove Theorem 1.8. Finally, the proofs of Theorems 1.12, Theorem 1.14 and Theorem 1.16 are given in the three subsections of §4.

2. Reformulation of the definition of $\mathfrak{krv}$ and definition of the linearized Lie algebra $\mathfrak{lkv}$

In this section, we give a convenient reformulation of the defining conditions of $\mathfrak{krv}$, which leads to a simple definition of the linearized version $\mathfrak{lkv}$ that passes easily into the language of moulds which will be essential for our subsequent proofs in §§3,4.

2.1. **The first defining condition of $\mathfrak{krv}$: specialness.** The first of the two defining conditions of $\mathfrak{krv}$ is that $\mathfrak{krv}$ lies in $\mathfrak{sder}_2$, i.e. elements of $\mathfrak{krv}$ are special tangential derivations having the form $E_{a,b}$ with $E_{a,b}(x) = [x, a]$, $E_{a,b}(y) = [y, b]$ and $[x, a] + [y, b] = 0$.

The following equivalent formulations of the property of specialness as properties of the polynomial b were given in [S1].

Proposition 2.1. [Schneps, [S1]] *Let $n \geq 3$ and let $b \in \mathfrak{lie}_C$; write $b = b_x x + b_y y = xb^x + yb^y$. Then the following are equivalent:*

(i) There exists a unique element $a \in \mathfrak{lie}_C$ such that $[x, a] + [y, b] = 0$;

(ii) b is push-invariant;

(iii) $b_y = b^y$.

[4]The dotted morphisms are as in footnote 1.

Thanks to this proposition, we can now reformulate the first defining condition of $\mathfrak{krv}$ as follows: the pair of polynomials $a, b \in \mathfrak{lie}_C$ satisfies $[x,a]+[y,b]=0$ if and only if b is push-invariant and a is its partner.

2.2. **The second defining condition of $\mathfrak{krv}$: divergence.** We now consider the second defining condition of $\mathfrak{krv}$, the divergence condition. Because $\mathfrak{krv}$ is weight-graded, we may restrict attention to derivations $E_{a,b}$ of homogeneous weight n, i.e. such that a and b are Lie polynomials of homogeneous degree $n \geq 3$. The second defining condition (9) then simplifies to the existence of a constant c such that

$$\mathrm{tr}(xa_x + yb_y) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big) \text{ in } \mathfrak{tr}_2.$$

Let us reformulate this as a condition only on b, just as we did for the first defining condition. Since $a \in \mathfrak{lie}_2$, its trace is zero and thus $\mathrm{tr}(xa_x) = \mathrm{tr}(a_x x) = -\mathrm{tr}(a_y y) = -\mathrm{tr}(ya_y)$, so

$$\mathrm{tr}(xa_x + yb_y) = \mathrm{tr}(yb_y - ya_y).$$

Since $E_{a,b} \in \mathfrak{sder}$, we have $[x,a] = [b,y]$. Expanding this in terms of the decompositions of a and b, we obtain

$$xa_x x + xa_y y - xa^x x - ya^y x = xb^x y + yb^y y - yb_x x - yb_y y,$$

from which we deduce that $a_y = b^x$ and $a^y = b_x$. Thus

$$\mathrm{tr}(yb_y - ya_y) = \mathrm{tr}(yb_y - yb^x) = \mathrm{tr}\big(y(b_y - b^x)\big).$$

From Proposition 2.1, we have $b_y = b^y$, so now, using the circularity of the trace, the divergence condition can be reformulated as

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n + y^n\big).$$

We use this to express it as a condition directly on $b^y - b^x$ as follows, using the push-operator defined in (12).

Definition 2.2. A polynomial $b \in Ass_2$ of homogeneous weight $n > 1$ is said to be *push-constant for the value* c if b does not contain the monomial y^n and for each $1 < r < n$, writing b^r for the depth r part of b, we have

$$\sum_{i=0}^{r} push^i(b^r) = c \sum_w w$$

where the sum in the right-hand factor is over all monomials of weight n and depth r. Equivalently, b is push-constant if it does not contain y^n and for all monomials $w \neq x^n$, we have

$$\sum_{v \in Push(w)} (b|v) = c$$

where $(b|v)$ denotes the coefficient of the monomial v in b, and $Push(w)$ is the list (with possible repetitions) $[w, push(w), \ldots, push^r(w)]$. If $c = 0$, then b is said to be *push-neutral*. If b is a scalar multiple of x^n, then b is push-neutral by default.

Example. The simplest example of a push-constant polynomial is the sum of all monomials of a given depth, for example

$$b = x^a y x^b y x^c + x^c y x^a y x^b + x^b y x^c y x^a + x^a y x^c y x^b + x^b y x^a y x^c + x^c y x^b y x^a.$$

More interesting push-constant polynomials can be obtained from elements $\psi \in \mathfrak{grt}$ by taking the projection of ψ onto the words ending in y and writing this as by. In this way we obtain for example:

$$b = 2x^2y^2 - \frac{11}{2}xyxy + \frac{9}{2}xy^2x - \frac{1}{2}yx^2y + 2yxyx - \frac{1}{2}y^2x^2.$$

The following proposition shows that the divergence condition comes down to requiring that $b^y - b^x$ be push-constant.

Proposition 2.3. *([S1]) Let b be a push-invariant Lie polynomial of homogeneous degree n. Then b satisfies the divergence condition*

$$\mathrm{tr}((b^y - b^x)y) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

if and only if $b^y - b^x$ is push-constant for the value nc. Furthermore, if this is the case then

$$c = \frac{1}{n}(b \,|\, x^{n-1}y). \tag{31}$$

Proof. Let w be a monomial of degree n and depth $r \geq 1$, and let C_w denote the list of words obtained from w by cyclically permuting the letters, so that C_w contains exactly n words (with possible repetitions). Let C_w^y denote the list obtained from C_w by removing all words ending in x, so that C_w^y contains exactly r words. Write $C_w^y = [u_1y, \ldots, u_ry]$. Then we have the equality of lists

$$[u_1, ..., u_r] = Push(u_1).$$

Let $c_w = \mathrm{tr}(w)$, i.e. c_w is the equivalence class of w, which is the set of the words in the list C_w, without repetitions: thus C_w is nothing other than $n/|c_w|$ copies of c_w. The divergence condition

$$\mathrm{tr}\big((b^y - b^x)y\big) = c\ \mathrm{tr}\big((x+y)^n - x^n - y^n\big)$$

translates as the following family of conditions for one word in each equivalence class c_w:

$$\sum_{v \in c_w} \big((b^y - b^x)y \,|\, v\big) = c|c_w|, \tag{32}$$

where each side is the coefficient of the class c_w in the trace, i.e. the sum of the coefficients of the words in c_w in the original polynomial.

If $r > 1$, we can choose a word $uy \in C_w$ that starts in y. Then from (32), the divergence condition on b implies that

$$\begin{aligned} c &= \frac{1}{|c_w|} \sum_{v \in c_w} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{v \in C_w} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{v \in C_w^y} \big((b^y - b^x)y \,|\, v\big) \\ &= \frac{1}{n} \sum_{u' \in Push(u)} ((b^y - b^x) \,|\, u'). \end{aligned}$$

This is exactly the definition of $b^y - b^x$ being push-constant for the value nc.

If $r=1$, then w is of depth 1, $|c_w|=n$ and $x^{n-1}y$ is the only word in c_w ending in y. Thus (32) comes down to

$$\big((b^y-b^x)y\,|\,x^{n-1}y\big)=nc.$$

But since b is a Lie polynomial, we have $(b|x^n)=(b^x|x^{n-1})=0$, so using $b^y=b_y$ (by Proposition 2.1), we also have

$$\begin{aligned}\big((b^y-b^x)y\,|\,x^{n-1}y\big)&=(b^y-b^x\,|\,x^{n-1})=(b^y|x^{n-1})\\&=(b_y|x^{n-1})=(b_yy|x^{n-1}y)=(b|x^{n-1}y),\end{aligned}$$

which proves that $nc=(b|x^{n-1}y)$ as desired. Note that this condition means that if b has no depth 1 part, then b^y-b^x is push-neutral. $\square$

We now have a new way of expressing $\mathfrak{krv}$, which is much easier to translate into the mould language.

Definition 2.4. Let $V_{\mathfrak{krv}}$ be the completion of the vector space spanned by polynomials $b\in\mathfrak{lie}_C$ of homogeneous degree $n\geq 3$ such that

(i) b is push-invariant, and
(ii) b^y-b^x is push-constant for the value $(b\,|\,x^{n-1}y)$,

equipped with the Lie bracket

$$\{b,b'\}=[b,b']+E_{a,b}(b')-E_{a',b'}(b)$$

where a and a' are the (unique) partners of b and b' respectively.

Indeed, since Propositions 2.1 and 2.3 show that

$$\begin{array}{ccc}\mathfrak{krv}&\xrightarrow{\sim}&V_{\mathfrak{krv}}\\E_{a,b}&\mapsto&b\end{array}\tag{33}$$

is an isomorphism of vector spaces and $\mathfrak{krv}$ is known to be a Lie subalgebra of $\mathfrak{sder}_2$, the bracket on $V_{\mathfrak{krv}}$ is inherited directly from this and makes $V_{\mathfrak{krv}}$ into a Lie algebra.

2.3. **The linearized Kashiwara-Vergne Lie algebra $\mathfrak{lkv}$.** Using the above isomorphism of $\mathfrak{krv}$ with the vector space $V_{\mathfrak{krv}}$ given by $E_{a,b}\mapsto b$, let us now consider the depth-graded versions of the defining conditions of $V_{\mathfrak{krv}}$, i.e. determine what these conditions say about the lowest-depth parts of elements $b\in V_{\mathfrak{krv}}$. The push-invariance is a depth-graded condition, so it restricts to the statement that the lowest depth part of b is still push-invariant; in particular, by Proposition 2.1 it admits of a unique partner $a\in\mathfrak{lie}_C$ such that $[x,a]+[y,b]=0$, i.e. such that the associated derivation $E_{a,b}$ lies in $\mathfrak{sder}_2$.

In the second condition, if b is of degree n and depth $r=1$ and b^1 denotes the lowest-depth part of b, then $(b^1)^y=x^{n-1}$, so the push-constance condition on b^1 is empty since $(b^1)^y=(b|x^{n-1}y)x^{n-1}$. If $r>1$, however, then $(b|x^{n-1}y)=0$ and so the push-constance condition on b^y-b^x is actually push-neutrality, which implies the push-neutrality of $(b^r)^y$ alone, since $(b^r)^y$ is the only part of the expression b^y-b^x of minimal depth $r-1$. These observations lead directly to the definition of the *linearized version* $\mathfrak{lkv}$ *of the Kashiwara-Vergne Lie algebra* given in Definition 1.5 above, and that by definition it is bigraded by weight and depth. The statement of

Proposition 1.6, that $\mathfrak{lkv}$ is a Lie algebra under the bracket coming from the bracket of derivations in $\mathfrak{sder}_2$, namely

$$\{b, b'\} = [b, b'] + E_{a,b}(b') - E_{a',b'}(b),$$

will be proved at the end of §4.1. Up to this fact, the proof of Proposition 1.7 now follows trivially from the equivalences above.

Proof of Proposition 1.7 assuming Proposition 1.6. The defining properties of the associated graded $gr\,\mathfrak{krv}$ are properties satisfied by the the lowest-depth parts of elements of $\mathfrak{krv}$. We identify $\mathfrak{krv}$ with $V_{\mathfrak{krv}}$ and use the version of its defining properties expressed in Definition 2.4. Since both the properties of being a Lie element and being push-invariant respect the depth, the same properties are satisfied by elements of $gr\,\mathfrak{krv}$. For the divergence, the argument in the paragraph preceding this proof shows that it implies no condition on the lowest-depth part if the depth is 1, and it implies the push-neutrality of the lowest-depth part if the depth is > 1. We do not know if this property along with being Lie and push-invariant, which together define $\mathfrak{lkv}$, are all that is implied on the lowest-depth part of an element of $\mathfrak{krv}$ by its defining properties, but we certainly know that they all hold for the lowest-depth part, and therefore we obtain the desired inclusion of vector spaces

$$gr\,\mathfrak{krv} \hookrightarrow \mathfrak{lkv}.$$

Furthermore, this inclusion is a Lie morphism as both spaces are equipped with the same Lie bracket, coming from $\mathfrak{sder}_2$. □

Remark. No examples of elements of $\mathfrak{lkv}$ that are not truncations to lowest depth of elements of $\mathfrak{krv}$ are known. It would be interesting to try to prove the equality of $\mathfrak{lkv}$ with $gr\,\mathfrak{krv}$ by starting with a polynomial $\mathfrak{lkv}$ of depth $r > 1$ and finding a way to construct a depth by depth lifting to an element of $\mathfrak{krv}$.

3. Rational and polynomial moulds

In this section, we introduce the language of moulds and reformulate the defining conditions of $\mathfrak{lkv}$ in this language. We end the section with the proof of Theorem 1.8 and its corollary in terms of moulds. We hope that this section and the next one, which explores the elliptic version of $\mathfrak{krv}$, will illustrate the way in which moulds are powerful tools in this context.

3.1. **Moulds and alternality.** For the purposes of this article, we are concerned only with rational function-valued moulds defined over the rationals. Écalle defines moulds with more general arguments and more general values, but in this article we will use the term *mould* merely to denote a collection $A = \big(A^r(u_1, \ldots, u_r)\big)_{r\geq 0}$ where each $A^r(u_1, u_2, ..., u_r) \in \mathbb{Q}(u_1, \ldots, u_r)$, i.e. each A^r is a rational function in r commutative variables $u_1, \ldots, u_r$ with coefficients in $\mathbb{Q}$. The rational function A^r is the *depth* r part of the mould. When the context is clear we sometimes drop the index and write $A(u_1, \ldots, u_r)$ instead of $A^r(u_1, \ldots, u_r)$ for the depth r part. In particular we have $A_0 = A(\emptyset) \in \mathbb{Q}$.

Moulds are equipped with addition and multiplication by scalars componentwise; thus they form a vector space. We write ARI for the subspace of (rational) moulds A with $A(\emptyset) = 0$ (keeping in mind that this ARI is only a very small subspace of the full space of moulds studied by Écalle). For convenience, we also define the vector space $\overline{ARI}$ of moulds defined exactly like ARI except on a set of commutative variables $v_1, v_2, \ldots$, i.e. $B \in \overline{ARI}$ means $B = \big(B^r\big)_{r\geq 0}$ with $B^r \in \mathbb{Q}(v_1, \ldots, v_r)$.

We say that a mould A is *concentrated in depth* r if $A^s = 0$ for all $s \neq r$, and we let $ARI^r \subset ARI$ be the subspace of moulds concentrated in depth r. Thus $ARI = \oplus_{r\geq 1} ARI^r$.

We now introduce Écalle's important *swap* operator on moulds[5].

Definition 3.1. The *swap* operator maps ARI to $\overline{ARI}$, and is defined by

$$swap(B)(v_1, \ldots, v_r) = B(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2)$$

for $B \in ARI$. The inverse operator mapping $\overline{ARI}$ to ARI (which we also denote by *swap*, as the context is clear according to whether *swap* is acting on a mould in ARI or one in $\overline{ARI}$) is given by

$$swap(C)(u_1, \ldots, u_r) = C(u_1 + \cdots + u_r, u_1 + \cdots + u_{r-1}, \ldots, u_1)$$

for $C \in \overline{ARI}$. Thus it makes sense to write $swap \circ swap = id$.

We also need to consider an important symmetry on moulds, based on the *shuffle* operator on tuples of commutative variables, which is defined by

$$Sh\big((u_1, \ldots, u_i)(u_{i+1}, \ldots, u_r)\big) = \Big\{(u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(r)}) \,|\, \sigma \in S_r^i\Big\},$$

where S_r^i is the subset of permutations $\sigma \in S_r$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots \sigma(r)$.

Definition 3.2. A mould $A \in ARI$ is alternal if in each depth $r \geq 2$ we have

$$\sum_{w \in Sh((u_1,\ldots,u_i)(u_{i+1},\ldots,u_r))} A^r(w) = 0 \quad \text{for} \quad 1 \leq i \leq \left[\frac{r}{2}\right].$$

By convention, the alternality condition is void in depth 1, i.e. all depth 1 moulds are considered to be alternal.

Example. In depth 4, there are two alternality conditions, given by

$$A(u_1, u_2, u_3, u_4) + A(u_2, u_1, u_3, u_4) + A(u_2, u_3, u_1, u_4) + A(u_2, u_3, u_4, u_1) = 0$$
$$A(u_1, u_2, u_3, u_4) + A(u_3, u_1, u_2, u_4) + A(u_3, u_4, u_1, u_2) + A(u_1, u_3, u_2, u_4)$$
$$+ A(u_1, u_3, u_4, u_2) + A(u_3, u_1, u_4, u_2) = 0$$

We write ARI_{al} for the subspace of ARI consisting of alternal moulds.

3.2. **Lie elements and alternal moulds.** Alternality is important because alternal polynomial moulds correspond to Lie polynomials in the sense given in the following lemma, whose statements are well-known: the first one is a direct consequence of Lazard elimination (cf. Bourbaki), and for complete elementary proofs of all the statements, see [SST] or [S2].

We write ARI^{pol} for the vector subspace of polynomial-valued moulds in ARI.

[5]For Écalle's original definitions of the mould operators *swap*, *push* and all the others used in this article, see [Ec] (2.4)-(2.11) and (2.55). Another basic reference for these operators is [S2], §2.4.

Lemma 3.3. *(i) The degree-completed free associative algebra* Ass_2 *on* x, y *(for the degree given by* $\deg x = \deg y = 1$*) can be decomposed as a direct sum*

$$\mathbb{Q}\langle\langle x, y\rangle\rangle = \mathbb{Q}x \oplus \mathbb{Q}\langle\langle C\rangle\rangle,$$

where $Ass_C = \mathbb{Q}\langle\langle C\rangle\rangle = \mathbb{Q}\langle\langle C_1, C_2, \ldots\rangle\rangle$ *is the degree-completion of the free non-commutative polynomial algebra on variables* $C_i = \mathrm{ad}_x^{i-1}(y)$ *for* $i \geq 1$*, for the degree given by* $deg\, C_i = i$.

(ii) For $r, n \geq 1$*, let* $Ass_C^{(r,n)}$ *denote the (finite-dimensional) subspace of* Ass_C *spanned by monomials* $C_{a_1} \cdots C_{a_r}$ *with* $a_1 + \cdots + a_r = n$*, and let* $ARI^{(r,n),pol}$ *denote the subspace of* ARI *consisting of polynomial moulds of degree* $n - r$ *concentrated in depth* r*. The map*

$$ma: \begin{array}{rl} Ass_C^{(r,n)} & \to ARI^{(r,n),pol} \\ C_{a_1} \cdots C_{a_r} & \longmapsto u_1^{a_1-1} \cdots u_r^{a_r-1} \end{array}$$

is a vector space isomorphism.

(iii) For each $r \geq 1$*, the map* ma *restricts to a (finite-dimensional) vector space isomorphism*

$$ma : \mathfrak{lie}_C^{(r,n)} \to ARI_{al}^{(r,n),pol},$$

where $\mathfrak{lie}_C^{(r,n)} = \mathfrak{lie}_C \cap Ass_C^{(r,n)}$.

Examples. The mould $ma(C_3) = ma([x,[x,y]])$ is the mould concentrated in depth 1 given by u_1^2. Similarly, $ma(C_2C_1 - C_1C_2) = ma([[x,y],y])$ is the mould concentrated in depth 2 given by $u_1^1u_2^0 - u_1^0u_2^1 = u_1 - u_2$.

Definition 3.4. Let β denote the backwards writing operator on words in x, y, meaning that $\beta(m)$ is obtained from a word m by writing it from right to left. The operator β extends to polynomials by linearity.

Let us give the translation of the restriction of the swap operator to polynomial-valued moulds directly in terms of elements of Ass_2 (cf. [R] or [S2]). Let $f \in Ass_C^r$, and write $f = xf^x + yf^y$. Set $g = \beta(yf^y)$, where β is the backwards operator of Definition 3.4. Thus all the monomials of g end in y. If we write g explicitly as

$$g = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1} y \cdots y x^{a_r} y, \tag{34}$$

then (as shown in [S2], (3.2.6)), $swap\big(ma(f)\big)$ is the mould concentrated in depth r given by

$$swap\big(ma(f)\big)(v_1, \ldots, v_r) = \sum_{\underline{a}} k_{\underline{a}}\, v_1^{a_1} ... v_r^{a_r}. \tag{35}$$

3.3. **Push-invariance and the first defining relation of $\mathfrak{krv}$.** Let us define the push-operator on moulds in ARI by

$$(push\, B)(u_1, \ldots, u_r) = B(u_0, u_1, ..., u_{r-1})$$

where $u_0 = -u_1 - u_2 - \cdots - u_r$. A mould $B \in ARI$ is push-invariant if $push\,(B) = B$ (in all depths).

The following proposition shows that this definition is precisely the translation into mould terms of the property of push-invariance for a Lie polynomial given in Definition 5 above.

Proposition 3.5. *Let $b \in \mathfrak{lie}_C$. Then b is a push-invariant polynomial if and only if $ma(b)$ is a push-invariant mould.*

Proof. If $b = y$, then $ma(b)$ is concentrated in depth 1 with value $ma(b)(u_1) = 1$, so these are both clearly push-invariant.

Now let $b \in (\mathfrak{lie}_C)_n^{r-1}$ with $n \geq r \geq 2$. We write

$$b = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1} y \cdots y x^{a_r}.$$

Let $f = yb$, so that $b = f^y$. Recalling that $y = C_1$, the associated moulds are related by the formula

$$ma(f)(u_1, ..., u_r) = ma(C_1 b) = u_1^0 ma(b)(u_2, ..., u_r) = ma(b)(u_2, ..., u_r). \tag{36}$$

Since $b \in (\mathfrak{lie}_C)_n$, we have $\beta(b) = (-1)^{n-1} b$. Set

$$g = \beta(yf^y) = \beta(yb) = (-1)^{n-1} by = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, x^{a_1} y ... y x^{a_r} y.$$

By (35), we have

$$swap\big(ma(f)\big)(v_1, \ldots, v_r) = (-1)^{n-1} \sum_{\underline{a}} k_{\underline{a}}\, v_1^{a_1} ... v_r^{a_r}.$$

Looking at

$$push(b)y = \sum_{\underline{a}} k_{\underline{a}}\, x^{a_r} y x^{a_1} y \cdots x^{a_{r-1}} y,$$

we see that $push(b)y$ is obtained from by by cyclically permuting the groups $x^{a_i} y$. Since $b = push(b)$ if and only if $k_{(a_1,\ldots,a_r)} = k_{(a_r,a_1,\ldots,a_{r-1})}$ for each $\underline{a}$, this is equivalent to

$$swap\big(ma(f)\big)(v_1, ..., v_r) = swap\big(ma(f)\big)(v_r, v_1, \ldots, v_{r-1}). \tag{37}$$

Using the definition of the *swap*, we rewrite (37) in terms of $ma(f)$ as

$$ma(f)(v_r, v_{r-1} - v_r, \ldots, v_1 - v_2) = ma(f)(v_{r-1}, v_{r-2} - v_{r-1}, \ldots, v_r - v_1) \tag{38}$$

We now make the change of variables $v_r = u_1 + ... + u_r, v_r - v_1 = u_r$, $v_1 - v_2 = u_{r-1}, \ldots, v_{r-2} - v_{r-1} = u_2$, $v_{r-1} = u_1$ in this equation, obtaining

$$ma(f)(u_1 + \cdots + u_r, -u_2 - \cdots - u_r, u_2, \ldots, u_{r-1}) = ma(f)(u_1, u_2, \ldots, u_r). \tag{39}$$

Finally, using relation (36), we write this in terms of $ma(b)$ as

$$ma(b)(-u_2 - \cdots - u_r, u_2, \ldots, u_{r-1}) = ma(b)(u_2, \ldots, u_r). \tag{40}$$

Making the variable change $u_i \mapsto u_{i-1}$ changes this to

$$ma(b)(-u_1 - \cdots - u_{r-1}, u_1, \ldots, u_{r-2}) = ma(b)(u_1, \ldots, u_{r-1}), \tag{41}$$

which is just the condition of mould push-invariance $ma(b)$ in depth $r - 1$. □

3.4. **Circc-neutrality and the second defining relation of $\mathfrak{krv}$.** Let us now show how to reformulate the second defining property of elements of $\mathfrak{krv}$ in terms of moulds.

Definition 3.6. Let *circ* be the mould operator[6] defined on moulds in $\overline{ARI}$ by

$$circ(B)(v_1,\ldots,v_r)=B(v_2,\ldots,v_r,v_1).$$

A mould $B\in\overline{ARI}$ is said to be *circ-neutral* if for $r>1$ we have

$$\sum_{i=0}^{r-1} circ^i(B)(v_1,\ldots,v_r)=0.$$

If B is a polynomial-valued mould of homogeneous degree n (i.e. the polynomial $B(v_1,\ldots,v_r)$ is of homogeneous degree $n-r$ for $1\le r\le n$), we say that B is *circ-constant* if

$$\sum_{i=0}^{r-1} circ^i(B)(v_1,\ldots,v_r)=c\Bigl(\sum_{\substack{a_1+\cdots+a_r=n-r\\ a_i\ge 0}} v_1^{a_1}\cdots v_r^{a_r}\Bigr)$$

for all $1<r\le n$, where $B(v_1)=cv_1^{n-1}$. (If $c=0$, then a circ-constant mould is circ-neutral.) Correspondingly, we also say that a polynomial $b\in Ass_C$ of homogeneous degree n is *circ-constant* if, setting $c=(b|x^{n-1}y)$, we have $b=b_0+\frac{c}{n}y^n$ where b_0^y is push-constant for the value c (cf. Definition 2.2). A polynomial-valued mould (resp. a polynomial in Ass_C) is said to be circ-constant if it is a sum of circ-constant homogeneous moulds (resp. polynomials).

Example. Let $\psi\in\mathfrak{grt}$ be homogeneous of degree n. Then as we saw in the example following Definition 8, the polynomial ψ^y is push-constant, so $\psi^y y$ is circ-constant. For example if $n=5$, then $\psi^y y$ is given by

$$\psi^y y=x^4y-2x^3y^2+\frac{11}{2}x^2yxy-\frac{9}{2}xyx^2y+3yx^3y+2x^2y^3-\frac{11}{2}xyxy^2+\frac{9}{2}xy^2xy$$
$$-\frac{1}{2}yx^2y^2+2yxyxy-\frac{1}{2}y^2x^2y-xy^4+4yxy^3-6y^2xy^2+4y^3xy$$

which is easily seen to be circ-constant.

For an example of a circ-constant mould, we take $B=swap\bigl(ma(\psi)\bigr)$, which has the same coefficients as $\psi^y y$: it is given by

$$\begin{aligned}
B(v_1)&=v_1^4\\
B(v_1,v_2)&=-2v_1^3+\frac{11}{2}v_1^2v_2-\frac{9}{2}v_1v_2^2+3v_2^3\\
B(v_1,v_2,v_3)&=2v_1^2-\frac{11}{2}v_1v_2-\frac{1}{2}v_2^2+\frac{9}{2}v_1v_3+2v_2v_3-\frac{1}{2}v_3^2\\
B(v_1,v_2,v_3,v_4)&=-v_1+4v_2-6v_3+4v_4.
\end{aligned}$$

The following result proves that the circ-constance of a polynomial b and that of the associated mould $ma(b)$ are always connected as in the example above. By additivity, it suffices to prove the result for b a homogeneous polynomial of degree n, so that the circ-constance of b is relative to just one constant $c_n=c=(b|x^{n-1}y)$.

[6] This operator is denoted *pus* in [Ec]

Proposition 3.7. *Let $b \in Ass_C$ be of homogeneous weight $n \geq 3$. Then b is a circ-constant polynomial if and only if $swap\big(ma(b)\big)$ is a circ-constant mould, and b is circ-neutral if and only if $swap\big(ma(b)\big)$ is circ-neutral.*

Proof. Let β be the backwards-writing operator on Ass_C (cf. Definition 3.4). Write $b = xb^x + yb^y$, and let $g = \beta(yb^y) = \beta(b^y)y$. For $r \geq 1$, let g^r denote the depth r part of g. If we write the polynomial g^r as

$$g^r = \beta\big((b^y)^{r-1}\big)y = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, x^{a_1}y\cdots yx^{a_r}y, \tag{42}$$

then we saw in (34) and (35) that

$$swap\big(ma(b)\big)(v_1,\ldots,v_r) = \sum_{\underline{a}=(a_1,\ldots,a_r)} k_{\underline{a}}\, v_1^{a_1}\cdots v_r^{a_r}. \tag{43}$$

Observe that a polynomial is push-constant if and only it is also push-constant written backwards, so in particular, b^y is push-constant if and only if $\beta(b^y)$ is. Suppose that b is circ-constant, i.e. that b^y and thus $\beta(b^y)$ are push-constant for the value $c = (b|x^{n-1}y)$. In view of (42), this means that $\sum_{\underline{a}'} k_{\underline{a}'} = c$ when $\underline{a}'$ runs through the cyclic permutations of $\underline{a} = (a_1,\ldots,a_r)$ for every tuple $\underline{a}$, and this in turns means precisely that the mould $swap\big(ma(b)\big)$ is circ-constant. As for the circ-neutrality equivalence, it follows from the circ-constance, since circ-neutrality is nothing but circ-constance for the constant 0. $\square$

The notion of circ-constance will play a role later in §4.2. In this section we only need circ-neutrality. Indeed, we showed that a polynomial b lies in $\mathfrak{lkv}$, i.e. b is a Lie polynomial that is push-invariant and circ-neutral, if and only if the associated mould $ma(b)$ is alternal (by Lemma 3.3 (iii)), push-invariant (by Proposition 3.5) and its swap is circ-neutral (by Proposition 3.7). In other words, we have shown that ma gives a vector space isomorphism

$$ma : \mathfrak{lkv} \xrightarrow{\sim} ARI^{pol}_{al+push/circneut}, \tag{44}$$

where the right-hand space is the subspace of ARI of polynomial-valued moulds in ARI that are alternal and push-neutral with circ-neutral swap. In fact this map is an isomorphism

$$\mathfrak{lkv}^r_n \simeq ARI^r_{n-r} \cap ARI^{pol}_{al+push/circneut}, \tag{45}$$

of each bigraded piece, where in general we write ARI^r_d for the subspace of polynomial-valued moulds of homogeneous degree d concentrated in depth r.

We will show at the end of §4.1 below (see Corollary 4.7) that $ARI^{pol}_{al+push/circneut}$ is a Lie algebra under the ari-bracket, and thus by the compatibility (121) of the ari-bracket with the Poisson bracket given below, we will then be able to conclude that $\mathfrak{lkv}$ is also a Lie algebra, proving Proposition 1.6 of this paper.

3.5. **Proof of Theorem 1.8.** Recall the statement of Theorem 1.8.

Theorem 1.8. *There is a bigraded Lie algebra injection on linearized spaces*

$$\mathfrak{ls} \hookrightarrow \mathfrak{lkv}. \tag{46}$$

For all $n \geq 3$ and $r = 1, 2, 3$, the map is an isomorphism of the bigraded parts

$$\mathfrak{ls}^r_n \simeq \mathfrak{lkv}^r_n.$$

In order to prove this theorem, we first reformulate the statement in terms of moulds and give its proof. Let $ARI_{al/al}$ denote the space of moulds that are alternal and have alternal swap, and following Écalle's notation, let $ARI_{\underline{al}/\underline{al}}$ denote the subspace of $ARI_{al/al}$ of moulds that are even in depth 1. Directly from the definition of $\mathfrak{ls}$, we see that the map ma gives an isomorphism

$$ma : \mathfrak{ls} \xrightarrow{\sim} ARI^{pol}_{\underline{al}/\underline{al}}$$

onto the space of polynomial-valued moulds in $ARI_{\underline{al}/\underline{al}}$. Therefore, Theorem 1.8 can be stated very simply in terms of moulds as

$$ARI^{pol}_{\underline{al}/\underline{al}} \subset ARI^{pol}_{al+push/circneut}.$$

We will actually prove the more general result without the polynomial hypothesis.

Theorem 3.8. *There is an inclusion of mould subspaces*

$$ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push/circneut},$$

Moreover in depths $r \leq 3$, we have

$$ARI^r \cap ARI_{\underline{al}/\underline{al}} = ARI^r \cap ARI_{al+push/circneut}.$$

Proof. It is well-known that every alternal mould satisfies

$$A(u_1, \ldots, u_r) = (-1)^{r-1} A(u_r, \ldots, u_1)$$

(cf. [S2], Lemma 2.5.3) and that a mould that is al/al and even in depth 1 is also push-invariant (cf. [S2], Lemma 2.5.5). Thus in particular $ARI_{\underline{al}/\underline{al}} \subset ARI_{al+push}$. It remains only to show that a mould in $ARI_{\underline{al}/\underline{al}}$ is necessarily circ-neutral. In fact, since the circ-neutrality condition is void in depth 1, we will show that even a mould in $ARI_{al/al}$ is circ-neutral; the condition of evenness in depth 1 is there to ensure the push-invariance, but not needed for the circ-neutrality.

The first alternality relation on $swap(A)$ is given by

$$swap(A)(u_1, \ldots, u_r) + swap(A)(u_2, u_1, \ldots, u_r) + \cdots + swap(A)(u_2, \ldots, u_r, u_1) = 0.$$

Since $swap(A)$ is push-invariant, this is equal to

$$push^r swap(A)(u_1, \ldots, u_r) + push^{r-1} swap(A)(u_2, u_1, \ldots, u_r) + \cdots$$
$$+ push\, swap(A)(u_2, \ldots, u_r, u_1) = 0.$$

But explicitly considering the action of the push operator on each term shows that

$$\begin{aligned} push^i swap(A)&(u_2, \ldots, u_{r-i+1}, u_1, u_{r-i+2}, \ldots, u_r) \\ &= swap(A)(u_{r-i+2}, \ldots, u_r, u_0, u_2, \ldots, u_{r-i+1}) \\ &= circ^{r-i+1} swap(A)(u_0, u_2, \ldots, u_r) \end{aligned}$$

where $u_0 = -u_1 - \cdots - u_r$, so this sum is equal to

$$\sum_{i=0}^{r-1} circ^i swap(A)(u_0, u_2, \ldots, u_r) = 0,$$

which proves that $swap(A)$ is circ-neutral. This gives the inclusion.

Let us now prove the isomorphism in the cases $r = 1, 2, 3$. The case $r = 1$ is trivial since the alternality conditions are void in depth 1. A polynomial-valued mould concentrated in depth 1 is a scalar multiple of u_1^d, which is automatically in

$ARI_{al/al}$, and lies in $ARI_{\underline{al}/\underline{al}}$ if and only if d is even. Such a mould is automatically alternal and the circ-neutral condition is void; it is push-invariant thanks to the evenness of d. This shows that in depth 1, both spaces are generated by moulds u_1^d for even d, and are thus isomorphic.

Now consider the case $r = 2$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 2. The circ-neutral property of the swap is explicitly given in depth 2 by $swap(A)(v_1, v_2) + swap(A)(v_2, v_1) = 0$. But this is also the alternality condition on $swap(A)$, so $A \in ARI_{al/al}$. The isomorphism in depth 2 is thus trivial.

Finally, we consider the case $r = 3$. Let $A \in ARI^{pol}_{al+push/circneut}$ be concentrated in depth 3, and let $B = swap(A)$. Again, we only need to show that B is alternal, which in depth 3 means that B must satisfy the single equation

$$B(v_1, v_2, v_3) + B(v_2, v_1, v_3) + B(v_2, v_3, v_1) = 0. \tag{47}$$

The circ-neutrality condition on B is given by

$$B(v_1, v_2, v_3) + B(v_3, v_1, v_2) + B(v_2, v_3, v_1) = 0. \tag{48}$$

It is enough to show that B satisfies the equality

$$B(v_1, v_2, v_3) = B(v_3, v_2, v_1), \tag{49}$$

since applying this to the middle term of (48) immediately yields the alternality property (47) in depth 3. So let us show how to prove (49).

We rewrite the push-invariance condition in the v_i, which gives

$$\begin{aligned} B(v_1, v_2, v_3) &= B(v_2 - v_1, v_3 - v_1, -v_1) && (50)\\ &= B(v_3 - v_2, -v_2, v_1 - v_2) && (51)\\ &= B(-v_3, v_1 - v_3, v_2 - v_3). && (52) \end{aligned}$$

Making the variable change exchanging v_1 and v_3, this gives

$$\begin{aligned} B(v_3, v_2, v_1) &= B(v_2 - v_3, v_1 - v_3, -v_3) && (53)\\ &= B(v_1 - v_2, -v_2, v_3 - v_2) && (54)\\ &= B(-v_1, v_3 - v_1, v_2 - v_1). && (55) \end{aligned}$$

By (50), the term $B(v_2 - v_1, v_3 - v_1, -v_1)$ is circ-neutral with respect to the cyclic permutation of v_1, v_2, v_3, so we have

$$B(v_2 - v_1, v_3 - v_1, -v_1) = -B(v_3 - v_2, v_1 - v_2, -v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). \tag{56}$$

But the circ-neutrality of B also lets us cyclically permute the three arguments of B, so we also have

$$-B(v_3 - v_2, v_1 - v_2, -v_2) = B(-v_2, v_3 - v_2, v_1 - v_2) + B(v_1 - v_2, -v_2, v_3 - v_2).$$

Using (50) and substituting this into the right-hand side of (56) yields

$$\begin{aligned} B(v_1, v_2, v_3) &= B(-v_2, v_3 - v_2, v_1 - v_2)\\ &\quad + B(v_1 - v_2, -v_2, v_3 - v_2) - B(v_1 - v_3, v_2 - v_3, -v_3). && (57) \end{aligned}$$

Now, exchanging v_1 and v_2 in (55) gives

$$B(v_3, v_1, v_2) = B(-v_2, v_3 - v_2, v_1 - v_2),$$

and doing the same with (53) gives

$$B(v_3, v_1, v_2) = B(v_1 - v_3, v_2 - v_3, -v_3).$$

Substituting these two expressions as well as (54) into the right-hand side of (57), we obtain the desired equality (49). This concludes the proof of Theorem 1.8. □

Remark. We conjecture that the inclusion of Theorem 3.8 is an isomorphism. But even the proof of the simple equality (49) is surprisingly complicated in depth 3, let alone in higher depth. Computer calculation does lead to the general conjecture:

Conjecture. *If $A \in ARI_{al+push/circneut}$ and $B = swap(A)$, then for all $r > 1$, we have*

$$B(v_1, \ldots, v_r) = (-1)^{r-1} B(v_r, \ldots, v_1). \tag{58}$$

The identity (58) would also yield the following useful partial result, which is the mould analog for $\mathfrak{lkv}$ of a result that is well-known for $\mathfrak{ls}$, namely that the bigraded part $\mathfrak{ls}_n^r = 0$ when $n \not\equiv r \bmod 2$.

Lemma 3.9. *Fix $1 \le r \le n$. Let $A \in ARI_{n-r}^r \cap ARI_{al+push/circneut}^{pol}$ and let $B = swap(A)$. Assume that B satisfies (58). Then if $n - r$ is odd, $A = 0$.*

Proof. Let $mantar$ denote the operator on moulds in ARI (resp. $\overline{ARI}$) defined by

$$mantar(A)(u_1, \ldots, u_r) = (-1)^{r-1} A(u_r, \ldots, u_1) \tag{59}$$

(resp. the same expression with v_i instead of u_i). It is easy to check the following identity of operators noted by Écalle:

$$neg \circ push = mantar \circ swap \circ mantar \circ swap,$$

where

$$neg(A)(u_1, \ldots, u_r) = A(-u_1, \ldots, -u_r). \tag{60}$$

Let $A \in ARI_{al+push/circneut}$; then A is push-invariant, so applying the left-hand operator to A gives $neg(A)$. Assuming (58) for $B = swap(A)$, i.e. assuming that $B = mantar(B)$, we see that applying the right-hand operator to A fixes A since on the one hand $swap \circ swap = id$ and on the other, $mantar(A) = A$ for all alternal moulds (cf. [S2], Lemma 2.5.3). Thus A must satisfy $neg(A) = A$, i.e. if $A \neq 0$ then the degree $d = n - r$ of A must be even. □

This implies the following result, which is the analogy for $\mathfrak{lkv}$ of the similar well-known result on $\mathfrak{ls}$.

Corollary 3.10. *If the swaps of all elements of $ARI_{al+push/circneut}^{pol}$ are mantar-invariant, then $ARI_d^r \cap ARI_{al+push/circneut}^{pol} = 0$ whenever d is odd, i.e. by (45),*

$$\mathfrak{lkv}_n^r = 0 \quad \text{when } n \not\equiv r \bmod 2$$

4. The elliptic Kashiwara-Vergne Lie algebra

In this section we follow the procedure of [S3] for the double shuffle Lie algebra to define a natural candidate for the elliptic Kashiwara-Vergne Lie algebra, closely related to the linearized Kashiwara-Vergne Lie algebra, and give some of its properties.

4.1. Definition of the elliptic Kashiwara-Vergne Lie algebra.

4.1.1. *The Kashiwara-Vergne Lie algebra.* Let Δ be the mould operator given by

$$\Delta(A)(u_1,\ldots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)A(u_1,\ldots,u_r) \tag{61}$$

for $r\geq 1$, and let ARI^{Δ} denote the space of rational-function moulds A such that $\Delta(A)$ is a polynomial mould (i.e. the denominator of the rational function A is "at worst" $u_1\cdots u_r(u_1+\cdots+u_r)$). We write ARI_p^{Δ} for the space of moulds in $ARI^{\Delta}\cap ARI_p$, where p may represent any (or no) properties on moulds in ARI; we will consider properties p such as for example al, $push$, combinations of these etc.

Recall that earlier we used the notation $ARI_{a/b}$ for the space of moulds having property a and whose swaps have property b; for example, $ARI_{al/al}$ denotes the space of alternal moulds with alternal swap. In this section, following Écalle, we introduce a slightly more general notation ARI_{a*b} to denote the space of moulds having property a and whose swap has property b up to adding on a constant-valued mould; thus, we write ARI_{al*al} for the space of alternal moulds whose swaps are alternal up to adding on a constant-valued mould. An example of a mould in ARI_{al*al} is the mould $\Delta^{-1}(A)$, where A is the polynomial mould concentrated in depth 3 given by

$$A(u_1,u_2,u_3) = -\frac{1}{4}u_1^3u_2+\frac{1}{4}u_1^3u_3-\frac{1}{4}u_1^2u_2^2+\frac{1}{2}u_1^2u_3^2+\frac{1}{4}u_1u_3^3-\frac{1}{4}u_2^2u_3^2-\frac{1}{4}u_2u_3^3$$
$$-\frac{1}{12}u_1^2u_2u_3+\frac{1}{6}u_1u_2^2u_3-\frac{1}{12}u_1u_2u_3^2.$$

It is easy to check that $\Delta^{-1}(A)$ is alternal, but its swap is not alternal unless one adds on the constant 1/3.

Definition 4.1. The mould-version elliptic Kashiwara-Vergne vector space is the subspace of polynomial-valued moulds

$$\Delta\big(ARI^{\Delta}_{al+push*circneut}\big).$$

The elliptic Kashiwara-Vergne vector space is the subspace $\mathfrak{krv}_{ell}\subset\mathfrak{lie}_C$ such that

$$ma\big(\mathfrak{krv}_{ell}\big) = \Delta\big(ARI^{\Delta}_{al+push*circneut}\big). \tag{62}$$

The operator Δ trivially respects push-invariance of moulds, so the space $\mathfrak{krv}_{ell}$ lies in the space $\mathfrak{lie}_C^{push}$ of push-invariant elements of $\mathfrak{lie}_C$. We will now show that the subspace $\mathfrak{krv}_{ell}$ is actually a Lie subalgebra of $\mathfrak{lie}_C^{push}$, which is itself a Lie algebra thanks to the following lemma, of which a more explicit version (with a formula for the partner) is proved in [S3] (Lemma 2.1.1).

Lemma 4.2. *Let $b\in\mathfrak{lie}_C$. Then $b\in\mathfrak{lie}_C^{push}$ if and only if there exists a unique element $a\in\mathfrak{lie}_C$ (the partner of b), such that if $D_{b,a}$ is the derivation of $\mathfrak{lie}_2$ defined by $x\mapsto b$, $y\mapsto a$, then $D_{b,a}$ annihilates $[x,y]$.*

By identifying $\mathfrak{lie}_C^{push}$ with the space of derivations that annihilate $[x,y]$, this lemma shows that $\mathfrak{lie}_C^{push}$ is a Lie algebra under the bracket of derivations. We state this as a corollary.

Corollary 4.3. *The map $b\mapsto D_{b,a}$ gives an isomorphism*

$$\partial:\mathfrak{lie}_C^{push}\to\mathfrak{oder}_2 \tag{63}$$

whose inverse is $D_{b,a} \mapsto D_{b,a}(x) = b$*, and this becomes a Lie isomorphism when* $\mathfrak{lie}_C^{push}$ *is equipped with the Lie bracket*

$$\langle b, b' \rangle = [D_{b,a}, D_{b',a'}](x) = D_{b,a}(b') - D_{b',a'}(b). \tag{64}$$

Thus we know that $\mathfrak{lie}_C^{push}$ is a Lie algebra and it contains the elliptic Kashiwara-Vergne space $\mathfrak{krv}_{ell}$ as a subspace. This leads to our first main result on $\mathfrak{krv}_{ell}$.

Theorem 4.4. *The subspace* $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ *is a Lie subalgebra.*

In order to prove this theorem, we will make essential use of mould theory, and in particular, of the *ari*-bracket[7] defined by Écalle that makes ARI into a Lie algebra ARI_{ari}. The hairiest definitions and proofs have been relegated to Appendix 1, in order to streamline the exposition of the next paragraph, which contains some basic elements of mould theory that will lead to the proof of the theorem in 4.1.3.

4.1.2. *A few facts about moulds.* In this paragraph we give a few brief reminders about some of the basic operators of mould theory and their connections with the familiar situation of $\mathfrak{lie}_2$; a very concise but self-contained exposition with full definitions is given in Appendix 1, and a complete exposition with proofs can be found in Chapters 2 and 3 of [S2]. In this section, we content ourselves with giving a list of mould operators that generalize the some of the most frequently considered operators on $\mathfrak{lie}_2$ such as the usual and the Poisson bracket, Ihara and special derivations, and the bracket $\langle\, , \,\rangle$ on $\mathfrak{lie}_C^{push}$. It is important to make the following two observations: (i) all these operators given in mould-theoretic terms can be applied to a much wider class of moulds than merely polynomial-valued moulds, which permits a number of proofs of results on polynomial-valued moulds (and thus polynomials in x, y) that are not accessible otherwise; (ii) there are some very important mould operators that are not translations of anything that can be phrased in the polynomial situation; this is where the real richness of mould theory comes into play. We do not use any of these in this section, but some of them will play a key role in the next subsection (see 4.2.4).

Recall from Lemma 3.3 (iii) that in fixed depth r and weight n, we have a linear isomorphism of finite-dimensional vector spaces

$$ma : \mathfrak{lie}_C^{(r,n)} \to ARI_{al}^{(r,n),pol}.$$

The precise definitions of all the Lie brackets and derivations below are given in Appendix 1.

• There is a Lie bracket lu on ARI satisfying

$$ma\big([f,g]\big) = lu\big(ma(f), ma(g)\big)$$

[7]The fact that *ari* really is a Lie bracket was stated by Écalle and has been used constantly in the mould literature. However, it appears that no complete proof of this fact was ever written down (as is the case with many of Écalle's statements). The full detailed proof has finally been given by Furusho and Komiyama, cf. [FK], Prop. 1.12. For the purposes of this article, in which *ari* is applied only to the space ARI^Δ of rational moulds with denominator at worst Δ, complete proofs that *ari* restricted to ARI^Δ is a Lie bracket were given in [E2], Prop. 4.2, and in [S3] (where the result follows from the definition of the *Dari*-bracket given there as a bracket of derivations on polynomial moulds, and then of the *ari*-bracket as the transport of the *Dari*-bracket by the vector space isomorphism Δ).

for $f, g \in \mathfrak{lie}_C$. We write ARI_{lu} for the Lie algebra ARI with this bracket.

• For each mould $A \in ARI$, there is a derivation $arit(A)$ of ARI_{lu} that corresponds to the Poisson or Ihara derivation on $\mathfrak{lie}_C$ in the sense that

$$arit\big(ma(f)\big) \cdot ma(g) = -ma\big(d_f(g)\big).$$

• There is a Lie bracket ari on ARI given by

$$ari(A, B) = lu(A, B) - arit(A) \cdot B + arit(B) \cdot A$$

that corresponds to the Poisson or Ihara bracket on $\mathfrak{lie}_C$ in the sense that

$$ari\big(ma(f), ma(g)\big) = ma\big(\{f, g\}\big). \tag{65}$$

We write ARI_{ari} for the Lie algebra with this Lie bracket.

• There is a Lie bracket $\overline{ari}$ on $\overline{ARI}$ which satisfies the following relation with the ari-bracket in the special case where A and B are both push-invariant moulds:

$$\overline{ari}\big(swap(A), swap(B)\big) = swap\big(ari(A, B)\big). \tag{66}$$

• There is a third Lie bracket on ARI, the $Dari$-bracket, which is obtained by transfer by the Δ-operator given in (61), i.e. it is given by

$$Dari(A, B) = \Delta\Big(ari\big(\Delta^{-1}(A), \Delta^{-1}(B)\big)\Big). \tag{67}$$

This means that Δ gives an isomorphism of Lie algebras

$$\Delta : ARI_{ari} \xrightarrow{\sim} ARI_{Dari}, \tag{68}$$

where ARI_{Dari} denotes the Lie algebra given by the vector space ARI equipped with the $Dari$ Lie bracket.

• For each mould $A \in ARI$, there is an associated derivation $Darit(A)$ of ARI_{lu} that preserves ARI^{pol} if A is polynomial-valued and satisfies the following property: the $Dari$-bracket of (67) can also be defined by

$$Dari(A, B) = Darit(A) \cdot B - Darit(B) \cdot A. \tag{69}$$

The definition of the derivation $Darit$ is given explicitly in Appendix 1, equation (117).

We end this section by comparing the $Dari$-bracket to the bracket $\langle\, ,\, \rangle$ on $\mathfrak{lie}_C^{push}$ given in Corollary 4.3.

Proposition 4.5. *The map*

$$ma : \mathfrak{lie}_C^{push} \to ARI_{Dari},$$

is a Lie algebra morphism, i.e. the Lie brackets $\langle\ ,\ \rangle$ and $Dari$ are compatible in the sense that

$$ma\big(\langle b, b'\rangle\big) = Dari\big(ma(b), ma(b')\big).$$

Proof. The key point is the following non-trivial result, which is one of the main results of [BS]: if D_1 and D_2 lie in $\mathfrak{oder}_2$, then the map

$$\begin{aligned} \Psi : \mathfrak{oder}_2 &\to ARI_{ari} \\ D &\mapsto \Delta^{-1}\big(ma\big(D(x)\big)\big), \end{aligned} \tag{70}$$

is an injective Lie morphism, i.e.

$$\Delta^{-1}\Big(ma\big([D_1,D_2](x)\big)\Big) = ari\Big(\Delta^{-1}\big(ma(D_1(x))\big), \Delta^{-1}\big(ma(D_2(x))\big)\Big)$$

(see Theorem 3.5 of [BS]). Applying Δ to both sides of this and using (67), this is equivalent to

$$ma\big([D_1,D_2](x)\big) = Dari\Big(ma\big(D_1(x)\big), ma\big(D_2(x)\big)\Big), \tag{71}$$

which in turn means that

$$ma : \mathfrak{oder}_2 \to ARI_{Dari} \tag{72}$$

is a Lie algebra morphism. We saw in Corollary 4.3 that we have a Lie isomorphism $\mathfrak{lie}_C^{push} \xrightarrow{\sim} \mathfrak{oder}_2$ when $\mathfrak{lie}_C^{push}$ is equipped with the Lie bracket (64), so by composition, we have an injective Lie morphism

$$b \overset{\partial}{\mapsto} D_{b,a} \overset{\Psi}{\mapsto} \Delta^{-1}\big(ma(D_{b,a}(x))\big) \overset{\Delta}{\mapsto} ma(b)$$

(where ∂ is as in Corollary 4.3 and Ψ is as in (70)) is an injective Lie morphism $\mathfrak{lie}_C^{push} \to ARI_{Dari}$, which proves the result. □

4.1.3. *Proof that $\mathfrak{krv}_{ell}$ is a Lie algebra.* This subsection is devoted to the proof of Theorem 4.4, i.e. that the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the bracket $\langle\ ,\ \rangle$.

From Proposition 4.5, ma gives an injective Lie algebra morphism

$$\mathfrak{lie}_C^{push} \to ARI_{Dari}.$$

Thus it is equivalent to prove that the image of the subspace $\mathfrak{krv}_{ell} \subset \mathfrak{lie}_C^{push}$ is closed under the $Dari$-bracket. Since we saw above that

$$\Delta^{-1} : ARI_{Dari} \to ARI_{ari},$$

it is equivalent to show that $ARI^{\Delta}_{al+push*circneut}$ is a Lie subalgebra of ARI_{ari}.

Let $b \in \mathfrak{lie}_C$ be push-invariant and let $D_{b,a} = \partial(b)$ where $\partial : \mathfrak{lie}_C^{push} \to \mathfrak{oder}_2$ is as in (63). It is shown[8] in [BS], Prop. B.1 that for all $b' \in \mathfrak{lie}_C$, we have

$$ma\big(D_{b,a}(b')\big) = Darit\big(ma(b)\big)\big(ma(b')\big). \tag{73}$$

Thus when $b \in \mathfrak{lie}_C^{push}$ and $B = ma(b)$, $Darit\big(ma(b)\big) = Darit(B)$ is simply the mould form of $D_{b,a}$. The derivation $D_{b,a}$ extends to all of $\mathbb{Q}\langle x,y\rangle$ and restricts to $\mathbb{Q}\langle C\rangle$ since by Lemma 4.2, both $b(x,y)$ and $a(x,y)$ lie in $\mathfrak{lie}_C \subset \mathbb{Q}\langle C\rangle$. Since any polynomial mould B' is of the form $B' = ma(b')$ with $b' \in \mathbb{Q}\langle C\rangle$, (73) shows that $Darit(B)\cdot B' = ma\big(D_{b,a}(b')\big) \in \mathbb{Q}\langle C\rangle$, so $Darit(B)$ preserves ARI^{pol}. Furthermore, since it is the mould form of $D_{b,a}$, we have $Darit(B)\cdot ma([x,y]) = 0$ and $Darit(B)\cdot ma(y) = ma(a)$. If $b, b' \in \mathfrak{lie}_C$ and $B = ma(b)$, $B' = ma(b')$, then by (69) and (73),

[8] Note that the notation is slightly different there; we recover this statement by setting $F = b'$, $U = b$, $D_U = D_{b,a}$ and taking care to note that the definition of $Darit_U$ in that article is the conjugation of the definition (124) used here by dar, i.e. it is (124) without the dar terms.

we have

$$
\begin{aligned}
Dari(B,B') &= Darit(B)\cdot B' - Darit(B')\cdot B\\
&= ma\big(D_{b,a}(b') - D_{b',a'}(b)\big)\\
&= ma\big([D_{b,a}, D_{b',a'}](x)\big)\\
&= ma\big(\langle b,b'\rangle\big). \qquad (74)
\end{aligned}
$$

We use $Darit$ and $Dari$ to prove the desired result in three steps as follows.

Step 1. Since $\mathfrak{lie}_C^{push}$ is the space of push-invariant Lie polynomials, we have

$$ma(\mathfrak{lie}_C^{push}) = ARI_{al+push}^{pol}.$$

But we saw in Proposition 4.5 that $\mathfrak{lie}_C^{push}$ is a Lie algebra under $\langle\ ,\ \rangle$, so $ARI_{al+push}^{pol}$ is a Lie algebra under $Dari$.

Step 2. The space $ARI_{al+push}^{\Delta}$ is a Lie algebra under ari. Indeed, the definition of Δ shows that this operator does not change the properties of push-invariance or alternality, i.e. $\Delta^{-1}(ARI_{al+push}) = ARI_{al+push}$. Restricted to polynomial-valued moulds, we have $\Delta^{-1}(ARI_{al+push}^{pol}) = ARI_{al+push}^{\Delta}$. Since Δ is an isomorphism from ARI_{ari} to ARI_{Dari} by virtue of (68) and $ARI_{al+push}^{pol}$ is a Lie subalgebra of ARI_{Dari} by Step 1, its image $ARI_{al+push}^{\Delta}$ under Δ^{-1} is thus a Lie subalgebra of ARI_{ari}.

Step 3. We can now complete the proof of Theorem 4.4 by showing that the space $ARI_{al+push*circneut}^{\Delta}$ is a Lie algebra under ari. For this, we need the following lemma, whose proof is deferred to the end of Appendix 1.

Lemma 4.6. *The space $\overline{ARI}_{circneut}$ of circ-neutral moulds $A \in \overline{ARI}$ forms a Lie algebra under the $\overline{ari}$-bracket.*

Given this, it is an easy matter to conclude. Let A,B lie in $ARI_{al+push*circneut}^{\Delta}$, and let us show that $ari(A,B)$ lies in the same space. By Step 2, we know that $ari(A,B) \in ARI_{al+push}^{\Delta}$, so we only need to show that $swap\big(ari(A,B)\big)$ is *circ-neutral. But we will show that in fact this mould is actually circ-neutral. To see this, let A_0 and B_0 be the constant-valued moulds such that $swap(A)+A_0$ and $swap(B)+B_0$ are circ-neutral. By Lemma 4.6, we have

$$\overline{ari}\big(swap(A)+A_0, swap(B)+B_0\big) \in \overline{ARI}_{circneut}.$$

Using the identity $swap\big(ari(M,N)\big) = \overline{ari}\big(swap(M), swap(N)\big)$, valid whenever M and N are push-invariant moulds (cf. [S], (2.5.6)), as well as the fact that constant-valued moulds are both push and swap invariant, we have

$$
\begin{aligned}
&\overline{ari}\big(swap(A)+A_0, swap(B)+B_0\big) = \overline{ari}\big(swap(A+A_0), swap(B+B_0)\big)\\
&\quad = swap\cdot ari(A+A_0, B+B_0)\\
&\quad = swap\cdot ari(A,B) + swap\cdot ari(A,B_0) + swap\cdot ari(A_0,B) + swap\cdot ari(A_0,B_0)\\
&\quad = swap\cdot ari(A,B)
\end{aligned}
$$

since the definition of the ari-bracket shows that $ari(C,M)=0$ whenever C is a constant-valued mould. Thus $swap\cdot ari(A,B)$ is circ-neutral, which completes the proof of Theorem 4.4. $\square$

The following easy corollary, which uses the result just shown (in Step 3 of the proof of Theorem 4.4) that

$$ARI^{\Delta}_{al+push*circneut}$$

is a Lie algebra under the *ari*-bracket, provides the promised proof of Proposition 1.6 stating that $\mathfrak{lkv}$ is a Lie algebra.

Corollary 4.7. *The subspace* $ARI^{pol}_{al+push/circneut}$ *is a Lie algebra under the ari-bracket, and the space* $\mathfrak{lkv}$ *is a Lie algebra under the Poisson bracket.*

Proof. Clearly

$$ARI^{\Delta}_{al+push/circneut} \subset ARI^{\Delta}_{al+push*circneut} \tag{75}$$

is a Lie subalgebra, since the *ari*-bracket of moulds without constant correction also has no constant correction. Since ARI^{pol} is a Lie subalgebra of ARI, the intersection

$$ARI^{pol}_{al+push/circneut} = ARI^{pol} \cap ARI^{\Delta}_{al+push/circneut} \subset ARI^{\Delta}_{al+push*circneut} \tag{76}$$

is also a Lie algebra, proving the first statement of the corollary. Thus, by (65), the space

$$\mathfrak{lkv} = ma^{-1}\big(ARI^{pol}_{al+push/circneut}\big)$$

a Lie algebra under the Poisson bracket, which completes the proof. □

Corollary 4.8. *The linear map* Δ *gives a Lie algebra morphism*

$$\Delta : ARI^{pol}_{al+push/circneut} \to \Delta(ARI^{\Delta}_{al+push*circneut}), \tag{77}$$

which induces a Lie algebra morphism

$$\mathfrak{lkv} \to \mathfrak{krv}_{ell}.$$

Proof. For the first statement, composing the inclusion map in (76) with the operator Δ, considered as an injective linear map on moulds gives an injective linear map

$$ARI^{pol}_{al+push/circneut} \hookrightarrow \Delta(ARI^{\Delta}_{al+push*circneut}).$$

It is shown in Step 3 of the proof of Theorem 4.4 (in §4.1.3) that $ARI^{\Delta}_{al+push*circneut}$ is a Lie algebra under the *ari*-bracket, and in Corollary 4.7 that $ARI^{pol}_{al+push/circneut}$ is a Lie subalgebra of it. A basic property of the linear map Δ is that it transforms the *ari*-bracket into the *Dari*-bracket (cf. (67), or for more detail (122) in Appendix 1), so the space $\Delta(ARI^{\Delta}_{al+push*circneut})$ is a Lie algebra under the *Dari*-bracket. Thus the map in (77) is a Lie algebra morphism from a Lie subalgebra of ARI_{ari} to a Lie subalgebra of ARI_{Dari}.

Finally, by (44) we have

$$ma(\mathfrak{lkv}) = ARI^{pol}_{al+push/circneut} \subset ARI_{ari}$$

and by (62) we have

$$ma(\mathfrak{krv}_{ell}) = \Delta(ARI^{\Delta}_{al+push*circneut}) \subset ARI_{Dari},$$

so (77) translates directly under ma^{-1} to a Lie morphism $\mathfrak{lkv} \to \mathfrak{krv}_{ell}$. □

4.2. **The map from $\mathfrak{krv} \to \mathfrak{krv}_{ell}$.** In this subsection we prove our next main result on the elliptic Kashiwara-Vergne Lie algebra, which is analogous to known results on the elliptic Grothendieck-Teichmüller Lie algebra of [E1] and the elliptic double shuffle Lie algebra of [S3]. The subsection 4.3 below is devoted to connections between these three situations.

Theorem 4.9. *There is an injective Lie algebra morphism*[9]

$$\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell} \tag{78}$$

The proof constructs the morphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ in four main steps as follows.

Step 1. We first consider a twisted version of the Kashiwara-Vergne Lie algebra, or rather of the associated polynomial space $V_{\mathfrak{krv}}$ of Definition 2.4, via the map

$$\nu : V_{\mathfrak{krv}} \xrightarrow{\sim} W_{\mathfrak{krv}} \tag{79}$$

$$f \mapsto \nu(f), \tag{80}$$

where ν is the automorphism of Ass_2 defined by

$$\nu(x) = z = -x - y, \quad \nu(y) = y. \tag{81}$$

In paragraph 4.2.1, we prove that $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson or Ihara bracket, and give a description of $W_{\mathfrak{krv}}$ via two properties, the "twisted" versions of the two defining properties of $V_{\mathfrak{krv}}$ given in Definition 2.4.

Step 2. To describe this step we first need a definition.

Definition 4.10. *(i) Let teru be the operator defined on moulds in ARI as follows: teru(A) is equal to A in depths* 0 *and* 1*, and for depths $r > 1$, we have*

$$teru(A)(u_1, \ldots, u_r) = \tag{82}$$
$$A(u_1, \ldots, u_r) + \frac{1}{u_r}\Big(A(u_1, \ldots, u_{r-2}, u_{r-1} + u_r) - A(u_1, \ldots, u_{r-2}, u_{r-1})\Big).$$

(ii) A mould $A \in ARI$ is said to satisfy the senary relation *(cf. (3.64) in §3.5 of [Ec]) if*

$$teru(A) = push \circ mantar \circ teru \circ mantar(A), \tag{83}$$

and the twisted senary relation *if $pari(A)$ satisfies the senary relation, where*

$$pari(A)(u_1, \ldots, u_r) = (-1)^r A(u_1, \ldots, u_r). \tag{84}$$

(iii) We define the mould subspace $ARI^{pol}_{al+tsen/circconst}$ (resp. $ARI^{pol}_{al+tsen*circconst}$) to be the subspace of alternal polynomial-valued moulds $A \in ARI$ such that $swap(A)$ is circ-constant (resp. up to adding a constant mould) and A satisfies the twisted senary relation. Observe that if $swap(A) \in \overline{ARI}$ is a polynomial-valued mould of homogeneous degree n which is circ-constant up to addition of a constant-valued mould, then the constant-valued mould is uniquely determined as being the mould whose only non-zero value is the constant value c/n in depth n, where c is given by

$$swap(A)(v_1) = cv_1^{n-1}.$$

[9] The dotted map again refers back to footnote 1.

In paragraph 4.2.2, we study the mould space $ma\big(W_{\mathfrak{krv}}\big)$. Thanks to the compatibility of the *ari*-bracket with the Poisson bracket (see (65), or for more detail see Appendix 1 especially (121)), this space is a Lie subalgebra of ARI_{ari}. Just as we reformulated the defining properties of $\mathfrak{lkv}$ in mould terms in §3, proving that $ma(\mathfrak{lkv}) = ARI^{pol}_{al+push/circneut}$, in Step 2 we will reformulate the defining properties of $W_{\mathfrak{krv}}$ in mould terms and show that

$$ma\big(W_{\mathfrak{krv}}\big) = ARI^{pol}_{al+tsen*circconst}. \tag{85}$$

Step 3. For this part we need to introduce Écalle's mould *pal* and its inverse *invpal*, which lie in the Lie group *GARI* associated to the Lie algebra ARI_{ari}, and study the inverse adjoint operators $Ad_{ari}(pal)$ and $Ad_{ari}(invpal)$ on ARI_{ari}. The statement of the result by Écalle discussed in footnote 1[10] is that adjoint action of the mould $pal \in GARI$ gives a Lie algebra morphism of ARI_{ari} which restricts to an isomorphism of Lie subalgebras

$$Ad_{ari}(pal) : ARI_{push} \overset{\sim}{\dashrightarrow} ARI_{sen}, \tag{86}$$

where ARI_{push} denotes the *push*-invariant moulds, and ARI_{sen} denotes the moulds satisfying the senary relation (92).

Letting Ξ denote the map

$$Ad_{ari}(invpal) \circ pari : ARI_{ari} \to ARI_{ari},$$

we show in Step 3, relying on the statement (86), that Ξ gives an injective Lie morphism

(87)

$$ARI^{pol}_{al+tsen*circconst} \xrightarrow{pari} ARI^{pol}_{al+sen*circconst} \overset{Ad_{ari}(invpal)}{\hookrightarrow} ARI^{\Delta}_{al+push*circneut}$$

of subalgebras of ARI_{ari}, where the dotted arrow indicates as usual that Écalle's statement (86) has not been proved in the literature.

Step 4. The final step is to compose (87) with the Lie morphism $\Delta : ARI_{ari} \to ARI_{Dari}$, obtaining an injective Lie morphism

$$ARI^{pol}_{al+tsen*circconst} \to \Delta\big(ARI^{\Delta}_{al+push*circneut}\big),$$

where the left-hand space is a subalgebra of ARI_{ari} and the right-hand one of ARI_{Dari}. Since the right-hand space is equal to $ma(\mathfrak{krv}_{ell})$, the desired injective Lie morphism from $\mathfrak{krv}$ to $\mathfrak{krv}_{ell}$ is obtained by composing all the maps described

[10]This statement and the senary relation itself can be found in [Ec], (3.51)-(3.58), in the situation of a general flexion unit. To give the dictionary between the notation for the general case and the special case studied in this article: $\mathfrak{E}$ denotes a general flexion unit and $adari(\mathfrak{es})$ the corresponding adjoint action, while in our situation we take the flexion unit $\mathfrak{E}(u_1) = 1/u_1$ and the adjoint action is then $ad_{ari}(pal)$ (or $adari(pal)$). If a mould M is *push*-invariant, Écalle uses the term $\mathfrak{E}$-push-invariant to indicate the property of the mould $adari(\mathfrak{es}) \cdot M$ coming from transporting the push-invariance of M; in other words, $adari(\mathfrak{es}) \cdot M$ is $\mathfrak{E}$-push-invariant if and only if M is push-invariant. In (3.53)-(3.54), Écalle gives the key statement that a mould is $\mathfrak{E}$-push-invariant if and only if it satisfies the senary relation (3.58) for the flexion unit $\mathfrak{E}$. In our situation this means that $ad_{ari}(pal) \cdot M$ satisfies the senary relation (92) (corresponding to the flexion unit $\mathfrak{E}(u_1) = 1/u_1$, and the notation $teru := \mathfrak{E}\text{-}ter$) if and only if M is push-invariant.

above, as shown in the following diagram:

(88)
$$
\begin{tikzcd}
\mathfrak{krv} \arrow[d, "\text{by (33)}"'] & & \\
V_{\mathfrak{krv}} \arrow[d, "\text{by (79)}"', "\nu"] & & \\
W_{\mathfrak{krv}} \arrow[d, "\text{by (85)}"', "ma"] & & \mathfrak{krv}_{ell} \\
ARI^{pol}_{al+tsen*circconst} \arrow[r, dotted, "\Xi", "\text{by (87)}"'] & ARI^{\Delta}_{al+push*circneut} \arrow[r, "\Delta"] & \Delta\big(ARI^{\Delta}_{al+push*circneut}\big) \arrow[u, "ma^{-1}", "\text{by (62)}"']
\end{tikzcd}
$$

4.2.1. *Step 1: The twisted space $W_{\mathfrak{krv}}$.*

Proposition 4.11. *Let $W_{\mathfrak{krv}} = \nu(V_{\mathfrak{krv}})$. Then $W_{\mathfrak{krv}}$ is a Lie algebra under the Poisson bracket.*

Proof. The key point is the following lemma on derivations.

Lemma 4.12. *Conjugation by ν induces an isomorphism of Lie algebras*

$$
\begin{aligned}
\mathfrak{sder}_2 &\overset{\sim}{\to} \mathfrak{ider}_2 \\
E_{a,b} &\mapsto d_{\nu(b)}.
\end{aligned}
\tag{89}
$$

Proof. Recall that $E_{a,b} \in \mathfrak{sder}_2$ maps $x \mapsto [x,a]$ and $y \mapsto [y,b]$, and $d_{\nu(b)} \in \mathfrak{ider}_2$ is the Ihara derivation defined by $x \mapsto 0$, $y \mapsto [y, \nu(b)]$ (cf. §1.1).

Let us first show that $d_{\nu(b)}$ is the conjugate of $E_{a,b}$ by ν, i.e. $d_{\nu(b)} = \nu \circ E_{a,b} \circ \nu$ (since ν is an involution). It is enough to show they agree on x and y, so we compute

$$\nu \circ E_{a,b} \circ \nu(x) = \nu \circ E_{a,b}(z) = 0 = d_{\nu(b)}(x)$$

and

$$\nu \circ E_{a,b} \circ \nu(y) = \nu \circ E_{a,b}(y) = \nu\big([y,b]\big) = [y, \nu(b)] = d_{\nu(b)}(y).$$

This shows that $\nu \circ E_{a,b} \circ \nu$ is indeed equal to $d_{\nu(b)}$. To show that $d_{\nu(b)}$ lies in $\mathfrak{ider}_2$, we check that $d_{\nu(b)}(z)$ is a bracket of z with another element of $\mathfrak{lie}_2$:

$$d_{\nu(b)}(z) = \nu \circ E_{a,b} \circ \nu(z) = \nu \circ E_{a,b}(x) = \nu([x,a]) = [z, \nu(a)].$$

The same argument goes the other way to show that conjugation by ν maps an element of $\mathfrak{ider}_2$ to an element of $\mathfrak{sder}_2$, which yields the isomorphism (89) as vector spaces. To see that it is also an isomorphism of Lie algebras, it suffices to note that conjugation by ν preserves the Lie bracket of derivations in $\mathfrak{der}_2$, i.e.

$$\nu \circ [D_1, D_2] \circ \nu = [\nu \circ D_1 \circ \nu, \nu \circ D_2 \circ \nu],$$

since ν is an involution. Since the Lie brackets on $\mathfrak{sder}_2$ and $\mathfrak{ider}_2$ are just restrictions to those subspaces of the Lie bracket on the space of all derivations, conjugation by ν carries one to the other. □

We use the lemma to complete the proof of Proposition 4.11. Write

$$\mathfrak{krv}^{\nu} = \{\nu \circ E \circ \nu \,|\, E \in \mathfrak{krv}\} \subset \mathfrak{ider}_2.$$

By restricting the isomorphism (89) to the subspace $\mathfrak{krv} \subset \mathfrak{sder}_2$, we obtain a commutative diagram of isomorphisms of vector spaces

$$\begin{tikzcd} \mathfrak{krv} \arrow[r] \arrow[d] & \mathfrak{krv}^\nu \arrow[d] \\ V_{\mathfrak{krv}} \arrow[r, "\nu"] & W_{\mathfrak{krv}} \end{tikzcd}$$

where the left-hand vertical arrow is the isomorphism (33) mapping $E_{a,b} \mapsto b$, and the right-hand vertical map sends an Ihara derivation d_f to f. Equipping $W_{\mathfrak{krv}}$ with the Lie bracket inherited from $\mathfrak{krv}^\nu$ makes this into a commutative diagram of Lie isomorphisms. But this bracket is nothing other than the Poisson bracket since $\mathfrak{krv}^\nu \subset \mathfrak{ider}_2$. □

We now give a characterization of $W_{\mathfrak{krv}}$ by two defining properties which are the twists by ν of those defining $V_{\mathfrak{krv}}$. Recall that β is the the backwards operator given in Definition 3.4.

Proposition 4.13. *The space $W_{\mathfrak{krv}}$ is the space spanned by polynomials $b \in \mathfrak{lie}_C$, of homogeneous degree $n \geq 3$, such that*

(i) $b_y - b_x$ is anti-palindromic*, i.e. $\beta(b_y - b_x) = (-1)^{n-1}(b_y - b_x)$, and*

(ii) $b + \frac{c}{n} y^n$ is circ-constant*, where $c = (b|x^{n-1}y)$.*

Proof. Let $f = \nu(b)$, so that $f \in V_{\mathfrak{krv}}$. Then the property that $b_y - b_x$ is anti-palindromic is precisely equivalent to the push-invariance of f (this is proved as the equivalence of properties (iv) and (v) of Theorem 2.1 of [S1]). This proves (i).

For (ii), we note that since $f \in V_{\mathfrak{krv}}$, $f^y - f^x$ is push-constant for the value $c = (f|x^{n-1}y) = (-1)^{n-1}(b|x^{n-1}y)$. We have

$$b(x, y) = xb^x(x, y) + yb^y(x, y),$$

so

$$f(x, y) = b(z, y) = zb^x(z, y) + yb^y(z, y) = -xb^x(z, y) - yb^x(z, y) + yb^y(z, y).$$

Thus since $f(x, y) = xf^x(x, y) + yf^y(x, y)$, this gives

$$f^x = -b^x(z, y) \quad \text{and} \quad f^y = -b^x(z, y) + b^y(z, y),$$

so

$$f^y - f^x = b^y(z, y) = \nu(b^y).$$

Thus to prove the result, it suffices to prove that the following statement: if $g \in Ass_C$ is a polynomial of homogeneous degree n that is push-constant for $(-1)^{n-1}c$, then $\nu(g)$ is circ-constant for c, since taking $g = f^y - f^x$ then shows that $\nu(g) = b^y$ is circ-constant for c. The proof of this statement is straightforward using the substitution $z = -x - y$ (but see the proof of Lemma 3.5 in [S1] for details). To complete the proof of (ii), we note that when $f \in V_{\mathfrak{krv}}$ is of even degree n we have $c = 0$. In fact this follows from Corollary 1.9, which states that $\mathfrak{lkv}_n^1 = 0$ when n is even; this means that there are no elements in $\mathfrak{krv}$ of even weight n and depth 1, so there are no such elements in $V_{\mathfrak{krv}}$. Since c is the coefficient of the depth 1 term $x^{n-1}y$, we have $c = 0$ when n is even. This completes the proof of (ii). □

4.2.2. *Step 2: The mould version $ma(W_{\mathfrak{krv}})$.* The space $ma(W_{\mathfrak{krv}})$ is closed under the *ari*-bracket by (65), since $W_{\mathfrak{krv}}$ is closed under the Poisson bracket.

Let $b \in W_{\mathfrak{krv}}$ and let $B = ma(b)$. Then since b is a Lie polynomial, B is an alternal polynomial mould. Let us give the mould reformulations of properties (i) and (ii) of Proposition 4.13. The second property is easy since we already showed, in Proposition 3.7, that a polynomial b is circ-constant if and only if $swap(B)$ is circ-constant.

Expressing the first property in terms of moulds is more complicated and calls for an identity discovered by Écalle. We need to use the mould operator *mantar* defined in (59), as well as the mould operator *pari* defined by

The operator *pari* extends the operator $y \mapsto -y$ on polynomials to all moulds, and *mantar* extends the operator $f \mapsto (-1)^{n-1}\beta(f)$.

Lemma 4.14. *Let $b \in \mathfrak{lie}_C$. Then the following are equivalent:*

(1) $b_y - b_x$ is anti-palindromic;

(2) if $B = ma(b)$, then $pari(B)$ satisfies the senary relation

$$teru \circ pari(B) = push \circ mantar \circ teru \circ pari(B). \tag{90}$$

(Note that since b is a Lie element, B and $pari(B)$ are alternal and thus *mantar*-invariant, so we can drop the right-hand *mantar* from the senary relation (83).)

Proof. It suffices to prove the statement for an element b of homogeneous degree n. The statement is a consequence of the following result, proved in Proposition A.3 of the Appendix of [S1]. Let $\tilde{b} \in \mathfrak{lie}_C$ and let $\tilde{B} = ma(\tilde{b})$. Write $\tilde{b} = \tilde{b}_x x + \tilde{b}_y y$ as usual. Then for each depth r part $(\tilde{b}_x + \tilde{b}_y)^r$ of the polynomial $\tilde{b}_x + \tilde{b}_y$ $(1 \le r \le n-1)$, the anti-palindromic property

$$(\tilde{b}_x + \tilde{b}_y)^r = (-1)^{n-1}\beta(\tilde{b}_x + \tilde{b}_y)^r \tag{91}$$

translates directly to the following relation on $\tilde{B}$:

$$teru(\tilde{B})(u_1, \ldots, u_r) = push \circ mantar \circ teru(\tilde{B})(u_1, \ldots, u_r). \tag{92}$$

Let us deduce the equivalence of (1) and (2) from that of (91) and (92). Let $\tilde{b}$ be defined by $\tilde{b}(x, y) = b(x, -y)$. This implies that $(b_x)^r = (-1)^r(\tilde{b}_x)^r$, $(b_y)^r = (-1)^{r-1}(\tilde{b}_y)^r$, and $\tilde{B} = pari(B)$. Thus $b_y - b_x$ is anti-palindromic if and only if $\tilde{b}_y + \tilde{b}_x$ is, i.e. if and only if (91) holds for $\tilde{b}$, which is the case if and only if (92) holds for $\tilde{B}$, which is equivalent to (90) with $\tilde{B} = pari(B)$. This proves the lemma. □

Corollary 4.15. *We have the isomorphism of Lie algebras*

$$ma : W_{\mathfrak{krv}} \xrightarrow{\sim} ARI^{pol}_{al+tsen*circconst} \subset ARI_{ari}\ .$$

Proof. By Proposition 4.13, the space $W_{\mathfrak{krv}}$ is the space of Lie polynomials b satisfying (i) $b_y - b_x$ is antipalindromic and (ii) $b + (c/n)y^n$ is circ-constant, where $c = (b|x^{n-1}y)$. By Lemma 4.14, property (i) is equivalent to the fact that $pari(B)$ satisfies the senary relation (83). By Proposition 3.7 the fact that b is circ-constant is equivalent to $swap(B)$ being circ-constant (and the remark at the end of Definition 4.10 shows that the constant is necessarily unique and the same). But by definition 4.10, $ARI^{pol}_{al+tsen*circconst}$ is precisely the space of alternal polynomial moulds satisfying precisely these two mould properties. □

4.2.3. *Mould background: Exponential maps from ARI to GARI.* The next stage of our proof, the construction of a Lie algebra morphism

$$ARI^{pol}_{al+tsen*circconst} \dashrightarrow ARI^{\Delta}_{al+push*circneut}, \tag{93}$$

is the most difficult, and requires some further definitions from mould theory. In order to keep it simple, we will make use of the following general scheme.

Let $\mathfrak{g}$ denote a vector space equipped with a grading in weights $n \geq 1$, equipped with a pre-Lie law $p(f,g)$ that respects the grading in the sense that if f is of weight n and g of weight m then $p(f,g)$ is of weight $n+m$. Then $\mathfrak{g}$ is also automatically equipped with

- a Lie bracket $[f,g] := p(f,g) - p(g,f)$;
- an exponential map exp_p mapping $\mathfrak{g}$ into the set of group-like elements of the completed universal enveloping algebra of the Lie algebra $\mathfrak{g}$, and its inverse map log_p;
- the group law $*$ making $G := exp_p(\mathfrak{g})$ into a group, given by

$$exp_p(f) * exp_p(g) = exp_p\big(ch_{[,]}(f,g)\big).$$

- an adjoint action of G on the graded completion $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ defined for each element $H \in G$ by letting $h = log_p(H)$ and setting

$$Ad_{[,]}(H) \cdot f = exp\big(ad(h)\big) \cdot f = \sum_{n \geq 0} \frac{1}{n!} ad(h)^n \cdot f \in \widehat{\mathfrak{g}},$$

where $ad(h) \cdot f = [h,f]$.

When $\mathfrak{g} = ARI$ equipped with the grading by the depth, we have $\mathfrak{g} = \widehat{\mathfrak{g}}$. We have seen that ARI can be equipped with various pre-Lie laws and Lie brackets. The underlying set of the associated group will always be the set $GARI$ of all moulds with constant term 1, just as ARI is the space of all moulds with constant term 0. (The same holds for $\overline{ARI}$ and $\overline{GARI}$.)

Écalle has studied a large family of different pre-Lie laws on ARI and $\overline{ARI}$, together with all their attendant structures as in the list above. The only ones we need here are the pre-Lie laws

$$\begin{aligned} preari(A,B) &= arit(B) \cdot A + mu(A,B) \qquad \text{on } ARI \\ \overline{preari}(A,B) &= \overline{arit}(B) \cdot A + mu(A,B) \qquad \text{on } \overline{ARI}, \end{aligned}$$

where $arit$ (resp. $\overline{arit}$) are the derivations of ARI_{lu} (resp. $\overline{ARI}_{lu}$) defined in Appendix 1. We will not use these pre-Lie laws in and of themselves, but in the next paragraph we will be using their associated adjoint actions Ad_{ari} and $Ad_{\overline{ari}}$.

We end this paragraph by defining, for any mould $Q \in \overline{GARI}$, an automorphism $\overline{ganit}(Q)$ of the Lie algebra $\overline{ARI}_{lu}$[11]. Set $\mathbf{v} = (v_1, \ldots, v_r)$, and let $W_{\mathbf{v}}$ denote the

[11]The explicit expression given below does not show immediately why $\overline{ganit}(Q)$ is an automorphism. However, this can be seen by using Écalle's explicit definition of the operator $\overline{anit}$ given in Appendix 1: for every $A \in \overline{ARI}$, the operator $\overline{anit}(A)$ is a derivation of $\overline{ARI}_{lu}$ (see Appendix A.1 of [S2]). Then $\overline{preani}(A,B) = \overline{anit}(B) \cdot A - mu(A,B)$ is a pre-Lie law on $\overline{ARI}$. Let $exp_{\overline{ani}}$ be the associated exponential map. The explicit formula for $\overline{ganit}$ shows that we have

$$\overline{ganit}\big(exp_{ani}(A)\big) = exp\big(\overline{anit}(A)\big)$$

set of decompositions $d_{\mathbf{v}}$ of $\mathbf{v}$ into chunks

$$d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1 \cdots \mathbf{a}_s\mathbf{b}_s \tag{94}$$

for $s \geq 1$, where with the possible exception of $\mathbf{b}_s$, the $\mathbf{a}_i$ and $\mathbf{b}_i$ are non-empty. Thus for instance, when $r = 2$ there are two decompositions in $W_{\mathbf{v}}$, namely $\mathbf{a}_1 = (v_1, v_2)$ and $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2)$, and when $r = 3$ there are four decompositions, three for $s = 1$: $\mathbf{a}_1 = (v_1, v_2, v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1, v_2)(v_3)$, $\mathbf{a}_1\mathbf{b}_1 = (v_1)(v_2, v_3)$, and one for $s = 2$: $\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 = (v_1)(v_2)(v_3)$.

Écalle's explicit expression for $\overline{ganit}(Q)$ is given by

$$\big(\overline{ganit}(Q)\cdot P\big)(\mathbf{v}) = \sum_{\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s \in W_{\mathbf{v}}} Q(\lfloor \mathbf{b}_1)\cdots Q(\lfloor \mathbf{b}_s)\ P(\mathbf{a}_1\cdots\mathbf{a}_s), \tag{95}$$

where if $\mathbf{b}_i$ is the chunk $(v_k, v_{k+1}, \ldots, v_{k+l})$, then we use the notation

$$\lfloor \mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1}). \tag{96}$$

4.2.4. *Mould background: The special mould pal and Écalle's fundamental identity.* We are now ready to introduce the fundamental identity of Écalle, which is the key to the construction of the desired map (93).

Definition 4.16. Let constants $c_r \in \mathbb{Q}$, $r \geq 1$, be defined by setting $f(x) = 1 - e^{-x}$ and expanding $f_*(x) = \sum_{r\geq 1} c_r x^{r+1}$, where $f_*(x)$ is the *infinitesimal generator* of $f(x)$, defined by

$$f(x) = \Big(exp\big(f_*(x)\frac{d}{dx}\big)\Big)\cdot x.$$

Let *lopil* be the mould in $\overline{ARI}_{\overline{ari}}$ defined by $lopil(v_1) = -\frac{1}{2v_1}$ and for $r \geq 2$ by the simple expression

$$lopil(v_1, \ldots, v_r) = c_r\,\frac{v_1 + \cdots + v_r}{v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r} \tag{97}$$

Set $pil = exp_{\overline{ari}}(lopil)$ where $exp_{\overline{ari}}$ denotes the exponential map associated to $\overline{preari}$, and set $pal = swap(pil)$.

The mould *lopil* is easily seen to be both alternal and circ-neutral. It is also known (although surprisingly difficult to show) that the mould $lopal = log_{ari}(pal)$ is alternal (cf. [Ec2], or [S2], Chap. 4.). Thus the moulds *pil* and *pal* are both exponentials of alternal moulds; this is called being *symmetral*. The inverses of *pal* (in $GARI$) and *pil* (in $\overline{GARI}$) are given by

$$invpal = exp_{ari}(-lopal), \qquad invpil = exp_{\overline{ari}}(-lopil).$$

The key maps we will be using in our proof are the adjoint operators associated to *pal* and *pil*, given by

$$Ad_{ari}(pal) = exp\big(ad_{ari}(lopal)\big), \qquad Ad_{\overline{ari}}(pil) = exp\big(ad_{\overline{ari}}(lopil)\big), \tag{98}$$

where $ad_{ari}(P)\cdot Q = ari(P,Q)$. The inverses of these adjoint actions are given by

$$Ad_{ari}(invpal) = exp\big(ad_{ari}(-lopal)\big), \quad Ad_{\overline{ari}}(invpil) = exp\big(ad_{\overline{ari}}(-lopil)\big). \tag{99}$$

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for all $A \in \overline{ARI}$, so as the exponential of a derivation, it is an automorphism. A direct proof from the definition can be found in [K], Thm. 3.7. Observe also that if *ani* denotes the Lie bracket on $\overline{ARI}$ given by $\overline{preani}(A,B) - \overline{preani}(B,A)$ and *gani* denotes the corresponding multiplication on $exp_{ani}(\overline{ARI})$ given by the Campbell-Hausdorff formula, then $ganit(A) \circ ganit(B) = 1$ if and only if $gani(A,B) = 1$.

These adjoint actions produce remarkable transformations of certain mould properties into others, and form the heart of much of Écalle's theory of multizeta values. Écalle's *fundamental identity*[12] relates the two adjoint actions of (98). Valid for all push-invariant moulds R, it is given by

$$swap \cdot Ad_{ari}(pal) \cdot R = \overline{ganit}(pic) \cdot Ad_{\overline{ari}}(pil) \cdot swap(R), \tag{100}$$

where $pic \in \overline{GARI}$ is defined by $pic(v_1, \dots, v_r) = 1/v_1 \cdots v_r$.

For our purposes, it is useful to give a slightly modified version of this identity. Let $poc \in \overline{GARI}$ be the mould defined by

$$poc(v_1, \dots, v_r) = \frac{1}{v_1(v_1 - v_2) \cdots (v_{r-1} - v_r)}. \tag{101}$$

Then it is shown in [B] that $\overline{ganit}(pic)$ and $\overline{ganit}(poc)$ are inverse automorphisms of $\overline{ARI}_{lu}$ (see [B], Lemmas 4.36 (esp. (4.15) and 4.37; see also the end of footnote 11). Thus, we can rewrite the above identity (100) as

$$\overline{ganit}(poc) \cdot swap \cdot Ad_{ari}(pal) \cdot R = Ad_{\overline{ari}}(pil) \cdot swap(R), \tag{102}$$

and letting $N = Ad_{ari}(pal) \cdot R$, i.e. $R = Ad_{ari}(invpal) \cdot N$, we rewrite it in terms of N as

$$Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap(N) = swap \cdot Ad_{ari}(invpal) \cdot N, \tag{103}$$

this identity being valid whenever $R = Ad_{ari}(invpal) \cdot N$ is push-invariant.

4.2.5. *Step 3: Construction of the map* Ξ. In this section we finally arrive at the main step of the construction of our map $\mathfrak{krv} \to \mathfrak{krv}_{ell}$, namely the construction of the map Ξ given in the following proposition.

Proposition 4.17. *The operator* $\Xi = Ad_{ari}(invpal) \circ pari$ *gives an injective Lie morphism of Lie subalgebras of* ARI_{ari}*:*

$$\Xi : ARI^{pol}_{al+tsen*circconst} \dashrightarrow ARI^{\Delta}_{al+push*circneut}. \tag{104}$$

Proof. We have already shown that both spaces are Lie subalgebras of ARI_{ari}, the first in Corollary 4.15 and the second in 4.1.3. Furthermore, since $pari$ and $Ad_{ari}(invpal)$ are both invertible and respect the ari-bracket, the proposed map is indeed an injective map of Lie subalgebras. Thus it remains only to show that the image of $ARI^{pol}_{al+tsen*circconst}$ under Ξ really lies in $ARI^{\Delta}_{al+push*circneut}$. We will show separately that if $B \in ARI^{pol}_{al+tsen*circconst}$ and $A = \Xi(B)$, then

(i) A is push-invariant,

(ii) A is alternal,

(iii) $swap(A)$ is circ-neutral up to addition of a constant-valued mould,

(iv) $A \in ARI^{\Delta}$.

[12] This identity, given in [Ec], is proved in [S2], Theorem 4.5.2; the proof relies among other things on a basic fact of mould theory stated by Écalle and used constantly in the mould literature, namely that the operator $\overline{ganit}(pic)$ transforms alternal moulds in $\overline{ARI}$ to alternil moulds. A full proof of this fact was not written down until the recent article [K] by N. Komiyama, see Corollary 3.25.

Proof of (i). This statement follows directly on Écalle's senary property (86) mentioned in footnote 1. Using this, since B satisfies (90), $\tilde{B} = pari(B)$ satisfies (92), so $Ad_{ari}(invpal)(\tilde{B}) = \Xi(B) = A$ is push-invariant.

Proof of (ii). The subspace of alternal moulds ARI_{al} is closed under ari (cf. [SS]), so $exp_{ari}(ARI_{al})$ forms a subgroup of $GARI_{gari}$, which we denote by $GARI^{as}_{gari}$ (the superscript *as* stands for *symmetral*). The *pal* is known to be symmetral (cf. [Ec2], or in more detail [S2], Theorem 4.3.4). Thus, since $GARI^{as}_{gari}$ is a group, the *gari*-inverse mould *invpal* is also symmetral. Therefore the adjoint action $Ad_{ari}(invpal)$ on ARI restricts to an adjoint action on the Lie subalgebra ARI_{al} of alternal moulds. If B is alternal, then $pari(B)$ is alternal, and so $A = \Xi(B)$ is alternal. This completes the proof of (ii).

For the assertions (iii) and (iv), we will make use of Écalle's fundamental identity in the version (103) given in 4.2.4, with $N = pari(B)$ (recall that (103) is valid whenever $Ad_{ari}(invpal)\cdot N$ is push-invariant, which is the case for $pari(B)$ thanks to (i) above). The key point is that the operators $\overline{ganit}(poc)$ and $Ad_{\overline{ari}}(pil)$ on the left-hand side of (103) are better adapted to tracking the circ-neutrality and the denominators than the right-hand operator $Ad_{ari}(invpal)$ considered directly.

Proof of (iii). Let $b \in W_{\mathfrak{krv}}$, and assume that b is of homogeneous degree n. Let $B = ma(b)$. Then by Corollary 4.15, $swap(B)$ is circ-constant, and even circ-neutral if n is even.

We need to show that $swap\cdot Ad_{ari}(invpal)\cdot pari(B)$ is *circ-neutral. To do this, we use (103) with $N = pari(B)$, and in fact show the result on the left-hand side, which is equal to

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari\cdot swap(B)$$

(noting that *pari* commutes with *swap*). We prove that this mould is *circ-neutral in three steps. First we show that the operator $\overline{ganit}(poc)\cdot pari$ changes a circ-constant mould into one that is circ-neutral (Proposition 4.18). Secondly, we show that the operator $Ad_{\overline{ari}}(invpil)$ preserves the property of circ-neutrality (Proposition 4.20). Finally, we show that if M is a mould that is not circ-constant but only *circ-constant, and if M_0 is the (unique) constant-valued mould such that $M+M_0$ is circ-constant, then

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(M) + M_0$$

is circ-neutral. Using (103), this will show that $swap\cdot Ad_{ari}(invpal)\cdot M$ is *circ-neutral when $Ad_{ari}(invpil)\cdot M$ is push-invariant.

Proposition 4.18. *Fix $n \geq 3$, and let $M \in \overline{ARI}$ be a circ-constant polynomial-valued mould of homogeneous degree n. Then $\overline{ganit}(poc)\cdot pari(M)$ is circ-neutral.*

Notation for the proof of Proposition 4.18.

Let $\mathbf{v} = (v_1, \ldots, v_r)$, and let $\mathbf{W_v}$ be the set of decompositions $d_\mathbf{v}$ of $\mathbf{v}$ into chunks $d_\mathbf{v} = \mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s$ as in (94). For any decomposition $d_\mathbf{v}$, we let its $\mathbf{b}$-part be the unordered set $\{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$, its $\mathbf{a}$-part the unordered set $\{\mathbf{a}_1, \ldots, \mathbf{a}_s\}$, and we write $|\mathbf{a}|$ for the number of letters in the $\mathbf{a}$-part, i.e. $|\mathbf{a}| = |\mathbf{a}_1| + \cdots + |\mathbf{a}_s|$.

Let

$$\mathbf{W} = \coprod_i \mathbf{W}_{\sigma_r^i(\mathbf{v})},$$

where the $\sigma_r^i(\mathbf{v})$ are the cyclic permutations of $\mathbf{v} = (v_1, \ldots, v_r)$, and let $\mathbf{W}^{\mathbf{b}}$ denote the subset of decompositions in $\mathbf{W}$ having identical $\mathbf{b}$-part, so that we have

$$\mathbf{W} = \coprod_{\mathbf{b}=(b_1,\ldots,b_s)} \mathbf{W}^{\mathbf{b}}. \tag{105}$$

Let $\mathbf{w} = (v_{i+1}, \ldots, v_r, v_1, \ldots, v_i)$ be any cyclic permutation of $\mathbf{v} = (v_1, \ldots, v_r)$; let $d_{\mathbf{w}} = \mathbf{a_1 b_1} \cdots \mathbf{a_s b_s}$ be a decomposition of $\mathbf{w}$, and let $\mathbf{b} = \{\mathbf{b_1}, \ldots, \mathbf{b_s}\}$ be its $\mathbf{b}$-part. We will list all the elements of $\mathbf{W}^{\mathbf{b}}$, i.e. all decompositions of all cyclic permutations of $\mathbf{v}$ having $\mathbf{b}$-part equal to $\mathbf{b}$. Let $\mathbf{a} = \{\mathbf{a_1}, \ldots, \mathbf{a_s}\}$ be the $\mathbf{a}$-part of $d_{\mathbf{w}}$. Then there exists a decomposition of a cyclic permutation of $\mathbf{v}$ having $\mathbf{b}$-part equal to $\mathbf{b}$ if and only if the cyclic permutation begins with a letter $v_k \in \mathbf{a}$; for such a cyclic permutation, there is exactly one decomposition with $\mathbf{b}$-part $\mathbf{b}$, obtained by cyclically shifting the pieces of the decomposition $d_{\mathbf{w}}$.

Example. Let $\mathbf{w} = (v_3, v_4, v_5, v_6, v_7, v_1, v_2)$ and consider the decomposition

$$\mathbf{w} = \mathbf{a_1 b_1 a_2 b_2 a_3} = (v_3, v_4)(v_5)(v_6)(v_7, v_1)(v_2).$$

Then $\mathbf{b} = \{v_1, v_5, v_7\}$ and $\mathbf{a} = \{v_2, v_3, v_4, v_6\}$. The only cyclic permutations of $\mathbf{v} = (v_1, \ldots, v_7)$ admitting the $\mathbf{b}$-part $\{v_1, v_5, v_7\}$ are the ones starting with $v_k \in \mathbf{a}$, and for each one, there is a unique decomposition determined by $\mathbf{b}$:

$$\begin{cases} (v_2, v_3, v_4)(v_5)(v_6)(v_7, v_1) \\ (v_3, v_4)(v_5)(v_6)(v_7, v_1)(v_2) \\ (v_4)(v_5)(v_6)(v_7, v_1)(v_2, v_3) \\ (v_6)(v_7, v_1)(v_2, v_3, v_4)(v_5). \end{cases}$$

Let the *ordered* $\mathbf{a}$-*part* of a decomposition $d_{\mathbf{w}}$ of a cyclic permutation $\mathbf{w}$ of $\mathbf{v} = (v_1, \ldots, v_r)$ be the word $\mathbf{a_1} \cdots \mathbf{a_s}$ of the decomposition $d_{\mathbf{w}}$. Then by the above, there are exactly $|\mathbf{a}|$ decompositions in $\mathbf{W}^{\mathbf{b}}$, and their ordered $\mathbf{a}$-parts are given by

$$\{\sigma^j_{|\mathbf{a}|}(\mathbf{a_1} \cdots \mathbf{a_s}) \,|\, j = 0, \ldots, |\mathbf{a}| - 1\} \tag{106}$$

i.e. the cyclic permutations of the letters of $\mathbf{a_1} \cdots \mathbf{a_s}$.

Proof of Proposition 4.18. Let $c = \big(M(v_1) \,|\, v_1^{n-1}\big)$, and let $N = pari(M)$, so that N is a polynomial mould of fixed homogeneous degree n, with $N(v_1) = -cv_1^{n-1}$. Since M is circ-constant for c (cf. Definition 3.6 for the definition), we have

$$N(v_1, \ldots, v_r) + \cdots + N(v_r, v_1, ..., v_{r-1}) = (-1)^r c \sum_{\substack{e_1+\cdots+e_r=n-r \\ e_i \geq 0}} v_1^{e_1} \cdots v_r^{e_r}. \tag{107}$$

By the explicit formula (95), we have

$$\big(\overline{ganit}(poc) \cdot N\big)(v_1, \ldots, v_r) = \sum_{\mathbf{W_v}} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) N(\mathbf{a}_1 \cdots \mathbf{a}_s), \tag{108}$$

so adding up over the cyclic permutations of $\mathbf{v}$, we have

$$\begin{aligned}
\sum_{i=0}^{r-1}\big(\overline{ganit}(poc)\cdot N\big)\big(\sigma_r^i(\mathbf{v})\big) &= \sum_{\mathbf{W}} poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\
&= \sum_{\mathbf{b}=\{\mathbf{b}_1,\ldots,\mathbf{b}_s\}}\ \sum_{\mathbf{W}^{\mathbf{b}}} poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s)N(\mathbf{a}_1\cdots\mathbf{a}_s)\\
&= \sum_{\mathbf{b}=\{\mathbf{b}_1,\ldots,\mathbf{b}_s\}} poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s)\sum_{j=0}^{|\mathbf{a}|-1} N\big(\sigma_{|\mathbf{a}|}^j(\mathbf{a}_1\cdots\mathbf{a}_s)\big)
\end{aligned}$$

$$= c\sum_{\mathbf{b}=\{\mathbf{b}_1,\ldots,\mathbf{b}_s\}}(-1)^{|\mathbf{a}|}poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s)\sum_{\substack{e_1+\cdots+e_{|\mathbf{a}|}=n-|\mathbf{a}|\\ e_j\geq 0}} v_{i_1}^{e_1}\cdots v_{i_{|\mathbf{a}|}}^{e_{|\mathbf{a}|}}, \tag{109}$$

where the first equality is the definition of $\overline{ganit}(poc)$, the second equality follows directly from (105), the third follows directly from (106), and the last equality from (107) using the notation $\{\mathbf{a}_1,\ldots,\mathbf{a}_s\}=\{v_{i_1},\ldots,v_{i_{|\mathbf{a}|}}\}$ for the subset $\mathbf{a}$ of $\{v_1,\ldots,v_r\}$.

If $c=0$, i.e. if M is a circ-neutral mould, the expression (109) is trivially equal to zero in all depths $r>1$, proving Proposition 4.18 in the case where M is circ-neutral. In order to deal with the case where M is circ-constant for a value $c\neq 0$, we use a trick and subtract off a known mould that is also circ-constant for c. For $A\subset\{1,\ldots,n\}$, let S_d^A denote the sum of all monomials of degree d in the letters v_i, $i\in A$.

Lemma 4.19. *For $n>1$ and any constant c, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1,\ldots,v_r)=\frac{c}{r}S_{n-r}^{\{1,\ldots,r\}}$$

for $1\leq r\leq n$. Then T_c^n is circ-constant and $\overline{ganit}(poc)\cdot pari(T_c^n)$ is circ-neutral.

The proof of this lemma is surprisingly long and technical, so we have relegated it to Appendix 2. Using the result, we can now finish the proof of Proposition 4.18. Indeed we have $M(v_1)=T_c^n(v_1)=cv_1^{n-1}$, so the mould $M-T_c^n$ is circ-neutral and thus $\overline{ganit}(poc)\cdot pari(M-T_c^n)$ is also circ-neutral. But Lemma 4.19 shows that $\overline{ganit}(poc)\cdot pari(T_c^n)$ is itself circ-neutral, so we have

$$\sum_{i=1}^{r} circ^i\big(\overline{ganit}(poc)\cdot pari(M)\big)=\sum_{i=1}^{r} circ^i\big(\overline{ganit}(poc)\cdot pari(T_c^n)\big)=0$$

and thus $\overline{ganit}(poc)\cdot pari(M)$ is also circ-neutral, completing the proof of Proposition 4.18. □

We now proceed to the second step, showing that the operator $Ad_{\overline{ari}}(invpil)$ preserves circ-neutrality.

Proposition 4.20. *If $M\in\overline{ARI}$ is circ-neutral then $Ad_{\overline{ari}}(invpil)\cdot M$ is also circ-neutral.*

Proof. By (99), we have

$$Ad_{\overline{ari}}(invpil)=exp\big(ad_{\overline{ari}}(-lopil)\big)=\sum_{n\geq 0}\frac{(-1)^n}{n}ad_{\overline{ari}}(lopil)^n. \tag{110}$$

The definition of *lopil* in (97) shows that *lopil* is trivially circ-neutral. Thus, since M is circ-neutral, $ad_{\overline{ari}}(lopil)\cdot M = \overline{ari}(lopil, M)$ is also circ-neutral by Lemma 4.6, and successively so are all the terms $ad_{\overline{ari}}(lopil)^n(M)$. Thus $Ad_{\overline{ari}}(invpil)\cdot M$ is circ-neutral. $\square$

Finally, we now assume that $swap(B)$ is a *circ-constant polynomial-valued mould in $\overline{ARI}$ of homogeneous degree n. Let B_0 be the (unique) constant-valued mould such that $swap(B)+B_0$ is circ-constant. Then by Propositions 4.18 and 4.20, the mould

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari\big(swap(B)+B_0\big)$$

is circ-neutral. This mould breaks up as the sum

$$Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(swap(B)) + Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(B_0),$$

but the operator $Ad_{\overline{ari}}(invpil)\cdot\overline{ganit}(poc)$ preserves constant-valued moulds (cf. [S2], Lemma 4.6.2 for the proof). Thus

$$\begin{aligned}Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari\big(swap(B)+B_0\big) &=\\ &Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot pari(swap(B))+B_0\end{aligned}$$

is circ-neutral, or equivalently,

$$Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot pari(swap(B))$$

is *circ-neutral. However, using the fact that *pari* trivially commutes with *swap* and also the fact that by Proposition 4.17 (i) (relying on Écalle's assertion (**??**)) $Ad_{ari}(invpal)\cdot pari(B)$ is push-invariant, we can apply (103) to find that

$$\begin{aligned}Ad_{\overline{ari}}(invpil)\cdot ganit(poc)\cdot swap\big(pari(B)\big) &= swap\cdot Ad_{ari}(invpal)\cdot pari(B)\\ &= swap\cdot \Xi(B).\end{aligned}$$

Thus $swap\cdot\Xi(B)$ is *circ-neutral, which concludes the proof of (iii).

Proof of (iv). We will again use Écalle's assertion (86) and the equality (103); this time we will study the left-hand side of (103) to to track the denominators that appear in the right-hand side. By (103), if B is a polynomial-valued mould satisfying the senary relation, and if $A=\Xi(B)=Ad_{ari}(invpal)\cdot pari(B)$, then A lies in ARI^{Δ} if and only if

$$swap\cdot Ad_{\overline{ari}}(invpil)\cdot \overline{ganit}(poc)\cdot swap\big(pari(B)\big)\in ARI^{\Delta}. \tag{111}$$

We will prove that this is the case, by studying the denominators that are produced, first by applying $\overline{ganit}(poc)$ to a polynomial-valued mould, and then by applying $Ad_{\overline{ari}}(invpil)$. The first result is that the denominators introduced by applying $\overline{ganit}(poc)$ are at worst of the form $(v_1-v_2)\cdots(v_{r-1}-v_r)$.

Lemma 4.21. [[B], Prop. 4.38] *Let* $M\in\overline{ARI}^{pol}$. *Then*

$$swap\cdot\overline{ganit}(poc)\cdot M\in ARI^{\Delta}.$$

Proof. The explicit expression for $\overline{ganit}(Q)$ given in (95) shows that the only denominators that can occur in $\overline{ganit}(poc)\cdot M$ come from the factors

$$poc(\lfloor \mathbf{b}_1)\cdots poc(\lfloor \mathbf{b}_s) \tag{112}$$

for all decompositions $d_{\mathbf{v}} = \mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_2$ of $\mathbf{v} = (v_1,\ldots,v_r)$ into chunks as in (94), and

$$\lfloor\mathbf{b}_i = (v_k - v_{k-1}, v_{k+1} - v_{k-1}, \ldots, v_{k+l} - v_{k-1})$$

(for $k > 1$) as in (96). Since *poc* is defined as in (101), the only factors that can appear in (112) are $(v_l - v_{l-1})$ where v_l is a letter in one of $\mathbf{b}_i$, and these factors appear in each term with multiplicity one. Since the sum ranges over all possible decompositions, the only letter of $\mathbf{v}$ that never belongs to any $\mathbf{b}_i$ is v_1; all the other factors $(v_i - v_{i-1})$ appear. Thus $(v_1 - v_2)(v_2 - v_3)\cdots(v_{r-1} - v_r)$ is a common denominator for all the terms in the sum defining $\overline{ganit}(poc)\cdot M$. The swap of this common denominator is equal to $u_2\cdots u_r$, so this term is a common denominator for $swap\cdot\overline{ganit}(poc)\cdot M$, which proves the lemma. $\square$

Lemma 4.22. *Let $M, N \in \overline{ARI}_{*circneut}$ be two moulds such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}. Then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^{Δ}.*

Proof. In Proposition A.1 of the Appendix of [BS], it is shown that if M and N are alternal moulds in $\overline{ARI}$ such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}, then $swap\big(\overline{ari}(M,N)\big)$ also lies in ARI^{Δ}. In fact, it is shown in Proposition A.2 of that appendix that alternal moulds M whose swap lies in ARI^{Δ} satisfy the following property: setting

$$\check{M}(v_1,\ldots,v_r) = v_1(v_1 - v_2)\cdots(v_{r-1} - v_r)v_r\, M(v_1,\ldots,v_r),$$

we have

$$\check{M}(0, v_2,\ldots,v_r) = \check{M}(v_2,\ldots,v_r,0). \tag{113}$$

In fact, the proof that $swap\big(\overline{ari}(M,N)\big)$ lies in ARI^{Δ} does not use the full alternality of M and N, but only (113). Therefore, the same proof goes through when M and N are *circ-neutral moulds such that $swap(M)$ and $swap(N)$ lie in ARI^{Δ}, as long as we check that every *circ-neutral mould M such that $swap(M) \in ARI^{\Delta}$ satisfies (113).

To check this, let M be such a mould; by additivity, we may assume that M is concentrated in a single depth $r > 1$. This means that there is a constant C_M such that

$$M(v_1,\ldots,v_r) + M(v_2,\ldots,v_r,v_1) + \cdots + M(v_r,v_1,\ldots,v_{r-1}) = C_M,$$

which we can also write as

$$\frac{\check{M}(v_1,\ldots,v_r)}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r} + \frac{\check{M}(v_2,\ldots,v_r,v_1)}{v_2(v_2-v_3)\cdots(v_{r-1}-v_r)(v_r-v_1)v_1} +$$

$$\cdots + \frac{\check{M}(v_r,v_1,\ldots,v_{r-1})}{v_r(v_r-v_1)\cdots(v_{r-2}-v_{r-1})v_{r-1}} = C_M$$

where the numerators are polynomials. If we multiply the entire equality by v_1 and set $v_1 = 0$, only the first two terms do not vanish, and they yield precisely the desired relation (113). $\square$

Corollary 4.23. *If $P \in \overline{ARI}$ is a *circ-neutral mould such that $swap(P) \in ARI^{\Delta}$, then also*

$$swap\cdot Ad_{\overline{ari}}(invpil)\cdot P \in ARI^{\Delta}. \tag{114}$$

Proof. The mould *lopil* is circ-neutral and $swap{\cdot}lopil \in ARI^{\Delta}$ by (97). Therefore by Lemma 4.22, we have $swap \cdot \overline{ari}(lopil, P) \in ARI^{\Delta}$. In fact, applying Lemma 4.22 successively shows that $swap \cdot ad_{\overline{ari}}(lopil)^n(P) \in ARI^{\Delta}$ for all $n \geq 1$. Since $Ad_{\overline{ari}}(invpil) \cdot P$ is obtained by summing these terms by (110), we obtain (114). □

We can now complete the proof of (iv) of Proposition 4.17. Recall that $B \in ARI^{pol}_{al+sen*circconst}$ and $A = \Xi(B)$. By Lemma 4.21 we have

$$swap \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in ARI^{\Delta}. \tag{115}$$

Since *swap* and *pari* commute, the mould in (115) is equal to

$$swap \cdot \overline{ganit}(poc) \cdot pari \cdot swap(B) \in ARI^{\Delta}. \tag{116}$$

By Proposition 4.18, the mould $P := \overline{ganit}(poc) \cdot pari \cdot swap(B)$ is *circ-neutral. By (116), $swap(P)$ lies in ARI^{Δ}. Therefore we can apply Corollary 4.23 to conclude that

$$swap \cdot Ad_{\overline{ari}}(invpil) \cdot P = swap \cdot Ad_{\overline{ari}}(invpil) \cdot \overline{ganit}(poc) \cdot swap \cdot pari(B) \in ARI^{\Delta}.$$

Applying (103) with $N = pari(B)$, we finally find that

$$Ad_{ari}(invpal) \cdot pari(B) = \Xi(B) \in ARI^{\Delta},$$

which completes the proof of (iv).

We have thus finished proving Proposition 4.17. Backtracking, this means we have completed the details of Step 3 of the proof of Theorem 4.9. Step 4, the final step in the proof, is very easy and was explained completely just before paragraph 4.2.1. Thus we have now completed the proof of Theorem 4.9, i.e. we have completed the construction of the injective Lie algebra morphism $\mathfrak{krv} \hookrightarrow \mathfrak{krv}_{ell}$. □

4.3. **Relations with elliptic Grothendieck-Teichmüller and double shuffle.** The final result in this paper is the proof of Theorem 1.16. In fact, this result is simply a consequence of putting together the results of the previous sections with known results. Indeed, the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{grt} \arrow[r, hook] \arrow[d, hook] & \mathfrak{ds} \arrow[d, hook] \\
\widetilde{\mathfrak{grt}}_{ell} \arrow[r, hook] \arrow[dr, hook] & \mathfrak{ds}_{ell} \arrow[d, hook] \\
& \mathfrak{oder}_2
\end{tikzcd} \tag{117}$$

where $Ad_{ari}(invpal) : \mathfrak{ds} \to \mathfrak{ds}_{ell}$ is the right-hand vertical map is shown in [S3].

Let $b = b(x, y) \in \mathfrak{ds}$. By (10), the injective map $\mathfrak{ds} \hookrightarrow \mathfrak{krv}$ sends b to the derivation of Lie$[x, y]$ given by $y \mapsto \hat{b}(x, y) := b(-x-y, -y)$ and $[x, y] \mapsto 0$ (which determines the value of the derivation on x uniquely). If $b(x, y) \in \mathfrak{ds}$, then $b(x, -y)$ lies in $W_{\mathfrak{krv}}$ and $b(z, -y)$ lies in $V_{\mathfrak{krv}}$, so this map unpacks to

$$\mathfrak{ds} \xrightarrow{y \mapsto -y} W_{\mathfrak{krv}} \xrightarrow{x \mapsto z} V_{\mathfrak{krv}} \longrightarrow \mathfrak{krv},$$

where the last map comes from (33). We can thus construct a commutative square

(118)
$$\begin{tikzcd} \mathfrak{ds} \arrow[r, dotted] \arrow[d] & \mathfrak{krv} \arrow[d, dotted] \\ \mathfrak{ds}_{ell} \arrow[r, hook, equal] & \mathfrak{krv}_{ell} \end{tikzcd}$$

given in detail by

(119)
$$\begin{tikzcd} \mathfrak{ds} \arrow[r, hook, dotted, "y\mapsto -y"] \arrow[d, "ma"'] & W_{\mathfrak{krv}} \simeq \mathfrak{krv} \arrow[d, "ma"] \\ ARI^{pol}_{\underline{al}*\underline{il}} \arrow[r, hook, dotted, "pari"] \arrow[d, "Ad_{ari}(invpal)"'] & ARI^{pol}_{al+tsen*circconst} \arrow[d, dotted, "Ad_{ari}(invpal)\circ pari"] \\ ARI^{\Delta}_{\underline{al}*\underline{al}} \arrow[r, hook, equal] \arrow[d, "\Delta"'] & ARI^{\Delta}_{al+push*circneut} \arrow[d, "\Delta"] \\ \Delta(ARI^{\Delta}_{\underline{al}*\underline{al}}) \arrow[r, hook, equal] \arrow[d, "ma^{-1}"'] & \Delta(ARI^{\Delta}_{al+push*circneut}) \arrow[d, "ma^{-1}"] \\ \mathfrak{ds}_{ell} \arrow[r, hook, equal] & \mathfrak{krv}_{ell}. \end{tikzcd}$$

The second line of this diagram is the direct mould translation of the top line, as the left-hand space is exactly $ma(\mathfrak{ds})$, the right-hand space is $ma(W_{\mathfrak{krv}})$ by (85), and the map *pari* restricted to polynomials is nothing other than $y \mapsto -y$. The proof of the vertical morphism

$$Ad_{ari}(invpal) : ARI^{pol}_{\underline{al}*\underline{il}} \to ARI^{\Delta}_{\underline{al}*\underline{al}}$$

has two parts. The fact that $Ad_{ari}(invpal)$ maps ARI_{al*il} to $ARI_{\underline{al}*\underline{al}}$ is one of the fundamental results of Écalle's mould theory, and follows directly from Écalle's fundamental identity (100) (see [S2], Theorem 4.6.1). The fact that restricted to ARI^{pol}_{al*il}, the operator $Ad_{ari}(invpal)$ produces denominators at worst Δ was proved in [B], Thm. 4.35.

The vertical morphism

$$Ad_{ari}(invpal) \circ pari : ARI^{pol}_{al+tsen*circconst} \dashrightarrow ARI^{\Delta}_{al+push*circneut}$$

follows directly from Écalle's statement (86). (see footnote 1). Since *pari* is an involution, this proves that the horizontal injection in the third line of the diagram is simply an inclusion.

Finally, the last line of the diagram, which not rely on Écalle's senary statement, comes from the definitions $\mathfrak{ds}_{ell} = \Delta(ARI^{\Delta}_{al*al})$ ([S3]) and $\mathfrak{krv}_{ell} = \Delta(ARI^{\Delta}_{al+push*circneut})$ by Definition 4.1.

This diagram shows that the diagram (117) above can be completed by the diagram (118) to the commutative diagram of Theorem 1.16.

5. Appendix 1: Some facts on moulds

In this appendix, we introduce some mould definitions used in some of our proofs, and give the proof of Lemma 4.6.

Let ARI be the vector space of moulds with constant term 0. There are three different Lie brackets that one can put on the space ARI. We begin by introducing the standard mould multiplication that Écalle denotes $mu(A,B)$:

$$mu(A,B)(u_1,\ldots,u_r)=\sum_{i=0}^{r}A(u_1,\ldots,u_i)B(u_{i+1},\ldots,u_r).$$

The associated Lie bracket lu is defined by $lu(A,B)=mu(A,B)-mu(B,A)$. We write ARI_{lu} for ARI viewed as a Lie algebra for the lu-bracket. The identical formulas yield a multiplication and Lie algebra (also called mu and lu) on $\overline{ARI}$. If f and g are power series in Ass_C and $A=ma(f)$, $B=ma(g)$, then mu is a mould translation of the usual non-commutative multiplication, and lu the usual Lie bracket:

$$mu(A,B)=ma(fg),\qquad lu(A,B)=ma\big([f,g]\big).$$

In order to define Écalle's ari-bracket, we first introduce three derivations of ARI_{lu} associated to a given mould $A\in ARI$. It is non-trivial to prove that these operators are actually derivations (cf. [S2], Prop. 2.2.1).

Definition 5.1. [Ec] Let $B\in ARI$. Then the derivation $amit(B)$ of ARI_{lu} is given by

$$\big(amit(B)\cdot A\big)(u_1,\ldots,u_r)=\sum_{0\le i<j<r}A(u_1,\ldots,u_i,u_{i+1}+\cdots+u_{j+1},u_{j+2},\ldots,u_r)B(u_{i+1},\ldots,u_j),$$

and the derivation $anit(B)$ is given by

$$\big(anit(B)\cdot A\big)(u_1,\ldots,u_r)=\sum_{0<i<j\le r}A(u_1,\ldots,u_{i-1},u_i+\cdots+u_j,u_{j+1},\ldots,u_r)B(u_{i+1},\ldots,u_j).$$

We also have corresponding derivations $\overline{amit}(B)$ and $\overline{anit}(B)$ of $\overline{ARI}_{lu}$ for $B\in\overline{ARI}$, given by the formulas

$$\big(\overline{amit}(B)\cdot A\big)(v_1,\ldots,v_r)=\sum_{0\le i<j<r}A(v_1,\ldots,u_i,v_{j+1},\ldots,v_r)B(v_{i+1}-v_{j+1},\ldots,v_j-v_{j+1}),$$

$$\big(\overline{anit}(B)\cdot A\big)(v_1,\ldots,v_r)=\sum_{0<i<j\le r}A(v_1,\ldots,v_i,v_{j+1},\ldots,v_r)B(v_{i+1}-v_i,\ldots,v_j-v_i).$$

Finally, Écalle defines the derivation $arit(B)$ on ARI_{lu} by

$$arit(B)=amit(B)-anit(B),$$

and the ari-bracket on ARI by

$$ari(A,B)=arit(B)\cdot A-arit(A)\cdot B+lu(A,B),\tag{120}$$

as well as the derivation $\overline{arit}$ on $\overline{ARI}_{lu}$ and the bracket $\overline{ari}$ on $\overline{ARI}$ by the same formulas with overlines.

Remark. The definitions of $amit$, $anit$, $arit$ and ari are generalizations to all moulds of familiar derivations of Ass_C. Indeed, if $f,g\in Ass_C$ and $A=ma(f)$, $B=ma(g)$, then

$$amit(B)\cdot A=ma\big(D_g^l(f)\big)$$

where D_g^l is defined by $x \mapsto 0$, $y \mapsto gy$,

$$anit(B) \cdot A = ma\big(D_g^r(f)\big)$$

where D_g^r is defined by $x \mapsto 0$, $y \mapsto yg$, and thus

$$arit(B) \cdot A = ma\big(-d_g(f)\big)$$

where d_g is the Ihara derivation $x \mapsto 0$, $y \mapsto [y, g]$ (see (5)), and

$$ari(A, B) = ma\big([f, g] + d_f(g) - d_g(f)\big) = ma\big(\{f, g\}\big). \tag{121}$$

corresponds to the Ihara or Poisson Lie bracket (6) on $\mathfrak{lie}_C$. (See [S2], Corollary 3.3.4).

We now pass to the *Dari*-bracket, which is the Lie bracket on ARI obtained by transfer by the Δ-operator given in (61): it is given by

$$Dari(A, B) = \Delta\Big(ari\big(\Delta^{-1}(A), \Delta^{-1}(B)\big)\Big). \tag{122}$$

This means that Δ gives an isomorphism of Lie algebras

$$\Delta : ARI_{ari} \xrightarrow{\sim} ARI_{Dari}. \tag{123}$$

It is shown in [S3], Prop. 3.2.1 that we have a second definition for the *Dari*-bracket, which is more complicated but sometimes very useful in certain proofs. Let dar denote the mould operator defined by $dar(A)(u_1, \ldots, u_r) = u_1 \cdots u_r\, A(u_1, \ldots, u_r)$. We begin by introducing, for each $A \in ARI$, an associated derivation $Darit(A)$ of ARI_{lu} by the following formula:

$$Darit(A) = dar \circ \Big(-arit\big(\Delta^{-1}(A)\big) + ad\big(\Delta^{-1}(A)\big)\Big) \circ dar^{-1}, \tag{124}$$

where $ad(A) \cdot B = lu(A, B)$. Then $Dari$ corresponds to the bracket of derivations, in the sense that

$$Dari(A, B) = Darit(A) \cdot B - Darit(B) \cdot A. \tag{125}$$

We are now armed to attack Lemma 4.6, whose statement we recall.

Lemma 4.6. *The space $\overline{ARI}_{circneut}$ of circ-neutral moulds $A \in \overline{ARI}$ forms a Lie algebra under the $\overline{ari}$-bracket.*

Proof. Let $A, B \in \overline{ARI}_{circneut}$. We need to show that

$$\sum_{i=1}^{r} \overline{ari}(A, B)(v_i, \ldots, v_r, v_1, \ldots, v_{i-1}) = 0,$$

where the formula for the $\overline{ari}$-bracket on $\overline{ARI}$ is given as in (120) by the expression

$$\begin{aligned} \overline{ari}(A, B) &= lu(A, B) + \overline{arit}(B) \cdot A - \overline{arit}(A) \cdot B \\ &= lu(A, B) + \overline{amit}(B) \cdot A - \overline{anit}(B) \cdot A - \overline{amit}(A) \cdot B + \overline{anit}(A) \cdot B. \end{aligned}$$

We will show that this expression is circ-neutral because in fact, each of the five terms in the sum is individually circ-neutral. Let us start by showing this for the first term, $lu(A, B)$.

Let σ denote the cyclic permutation of $\{1, \ldots, r\}$ defined by

$$\sigma(i) = i + 1 \ \text{ for } \ 1 \leq i \leq r - 1, \quad \sigma(r) = 1.$$

By additivity, since the circ-neutrality property is depth-by-depth, we may assume that A is concentrated in depth s and B in depth t, with $s \le t$, $s+t=r$. In this simplifed situation, we have

$$lu(A,B)(v_1,\ldots,v_r) = A(v_1,\ldots,v_s)B(v_{s+1},\ldots,v_r) - B(v_1,\ldots,v_t)A(v_{t+1},\ldots,v_r).$$

If $s,t>1$, we have

$$\begin{aligned}
&\sum_{i=0}^{r-1} lu(A,B)(v_{\sigma^i(1)},\ldots,v_{\sigma^i(r)})\\
&= \sum_{i=0}^{r-1}\Big(A(v_{\sigma^i(1)},\ldots,v_{\sigma^i(s)})B(v_{\sigma^i(s+1)},\ldots,v_{\sigma^i(r)}) - B(v_{\sigma^i(1)},\ldots,v_{\sigma^i(t)})A(v_{\sigma^i(t+1)},\ldots,v_{\sigma^i(r)})\Big)\\
&= \sum_{i=0}^{r-1}\Big(A(v_{\sigma^i(1)},\ldots,v_{\sigma^i(s)})B(v_{\sigma^i(s+1)},\ldots,v_{\sigma^i(r)}) - A(v_{\sigma^{i+t}(1)},\ldots,v_{\sigma^{i+t}(s)})B(v_{\sigma^{i+t}(s+1)},\ldots,v_{\sigma^{i+t}(r)})\Big)\\
&= 0
\end{aligned}$$

as the terms cancel out pairwise.

We now prove that the second term

$$\big(\overline{amit}(B)\cdot A\big)(v_1,\ldots,v_r) = \sum_{i=1}^{s} A(v_1,\ldots,v_{i-1},v_{i+t},\ldots,v_r)B(v_i-v_{i+t},\ldots,v_{i+t-1}-v_{i+t})$$

is circ-neutral. Fix $j \in \{1,\ldots,s\}$ and consider the term

$$A(v_1,\ldots,v_{j-1},v_{j+t},\ldots,v_r)B(v_j-v_{j+t},\ldots,v_{j+t-1}-v_{j+t}).$$

Thus for each of the other terms

$$A(v_1,\ldots,v_{i-1},v_{i+t},\ldots,v_r)B(v_i-v_{i+t},\ldots,v_{i+t-1}-v_{i+t})$$

in the sum, with $i \in \{1,\ldots,s\}$, there is exactly one cyclic permutation, namely σ^{j-i}, that maps this term to

$$A(v_{\sigma^{j-i}(1)},\ldots,v_{\sigma^{j-i}(i-1)},v_{\sigma^{j-i}(i+t)},\ldots,v_{\sigma^{j-i}(r)})B(v_j-v_{j+t},\ldots,v_{j+t-1}-v_{j+t}).$$

For fixed $j \in \{1,\ldots,s\}$, the values of $k = j-i \bmod s$ as i runs through $\{1,\ldots,s\}$ are exactly $\{0,\ldots,s-1\}$. Therefore, the coefficient of the term $B(v_j-v_{j+t},\ldots,v_{j+t-1}-v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B)\cdot A$ is equal to

$$\sum_{k=0}^{s-1} A(v_{\sigma^k(1)},\ldots,v_{\sigma^k(i-1)},v_{\sigma^k(i+t)},\ldots,v_{\sigma^k(r)}),$$

which is zero due to the circ-neutrality of A. Thus the coefficient of the term $B(v_j-v_{j+t},\ldots,v_{j+t-1}-v_{j+t})$ in the sum of the cyclic permutations of $\overline{amit}(B)\cdot A$ is zero, and this holds for $1 \le j \le s$, so the entire sum is 0, i.e. $\overline{amit}(B)\cdot A$ is circ-neutral.

Example. $s=3, t=2, r=5$. We have

$$\begin{aligned}
\big(\overline{amit}(B)\cdot A\big)(v_1,v_2,v_3,v_4,v_5) &= A(v_4,v_5,v_6)B(v_1-v_4,v_2-v_4,v_3-v_4)\\
&\quad + A(v_1,v_5,v_6)B(v_2-v_5,v_3-v_5,v_4-v_5)\\
&\quad + A(v_1,v_2,v_6)B(v_3-v_6,v_4-v_6,v_5-v_6).
\end{aligned} \tag{126}$$

For $(\overline{amit}(B)\cdot A)$ to be circ-neutral, the sum of the images of this expression under the five non-trivial powers of the six-cycle permutation $\sigma = (123456)$ must be zero.

In particular, the coefficient of every factor of B that occurs in that sum must sum to zero. Let us show this for the B-factor $B(v_2-v_5, v_3-v_5, v_4-v_5)$ that arises in the second term of (126). The terms in the complete sum containing this factor can only come from σ acting on the first term of (126), giving

$$A(v_5, v_6, v_1)B(v_2-v_5, v_3-v_5, v_4-v_5)$$

and from σ^5 acting on the third term of (126), giving

$$A(v_6, v_1, v_5)B(v_2-v_5, v_3-v_5, v_4-v_5).$$

Therefore the coefficient of $B(v_2-v_5, v_3-v_5, v_4-v_5)$ in the complete sum is equal to

$$A(v_1, v_5, v_6) + A(v_5, v_6, v_1) + A(v_6, v_1, v_5)$$

which is equal to zero by the circ-neutrality of A. The same holds for every B-factor that occurs in the sum; there will always be exactly three possible ways to obtain it by a unique permutation acting on each of the three terms of (126), and the coefficients will be a circ-sum of A's that add up to zero.

To conclude the proof of the lemma, we need to prove that the term $\overline{anit}(B)\cdot A$ is also circ-neutral, but the proof is analogous to the case of $\overline{amit}$. Finally, by exchanging A and B, this also shows that $\overline{amit}(A)\cdot B$ and $\overline{anit}(A)\cdot B$ are circ-neutral. This concludes the proof of the lemma. □

6. Appendix 2: Proof of Lemma 4.19

Let us recall the statement of the technical lemma 4.19. Recall that for $A \subset \{v_1,\ldots,v_r\}$, we let M_d^A denote the set of all monomials of degree d in the letters of A, and S_d^A the sum of all monomials in M_d^A. We will use the notation $\mathbf{W}$, $\mathbf{W^b}$ etc. given between the statement of Proposition 4.18 and its proof.

Lemma 4.19. *For $n > 1$ and any constant $c \neq 0$, let T_c^n be the homogeneous polynomial mould of degree n defined by*

$$T_c^n(v_1,\ldots,v_r) = \frac{c}{r}\, S_{n-r}^{\{v_1,\ldots,v_r\}}.$$

Then T_c^n is circ-constant and $\overline{ganit}(poc)\cdot pari(T_c^n)$ is circ-neutral.

Proof. The mould T_c^n is trivially circ-constant for the value c. For the rest of this proof we set $c=1$ and $T^n = T_1^n$; it suffices to multiply all identities in the proof below by the constant c to prove the general case.

Let $N = pari(T^n)$. In order to show that $\overline{ganit}(poc)\cdot N$ is circ-neutral, we start by recalling from the beginning of the proof of Proposition 4.18 that for each $r>1$, the cyclic sum

$$\overline{ganit}(poc)\cdot N(v_1,\ldots,v_r) + \cdots + \overline{ganit}(poc)\cdot N(v_r,v_1,\ldots,v_{r-1})$$

is equal to the expression (109)

$$\sum_{\mathbf{b}=\{\mathbf{b}_1,\ldots,\mathbf{b}_s\}} (-1)^{|\mathbf{a}|} poc(\lfloor\mathbf{b}_1)\cdots poc(\lfloor\mathbf{b}_s) S^{\mathbf{a}}_{n-|\mathbf{a}|}, \tag{127}$$

where the sum runs over the distinct $\mathbf{b}$-parts that can occur when decomposing the cyclic permutations $\sigma_r^i(\mathbf{v}) = (v_{i+1},\ldots,v_r,v_1,\ldots,v_i)$ into chunks $\mathbf{a}_1\mathbf{b}_1\cdots\mathbf{a}_s\mathbf{b}_s$ (in which only $\mathbf{b}_s$ can be empty). For each term of the sum, $\mathbf{a}$ denotes the subset of $\{v_1, \ldots, v_r\}$ which is the complement of the $\mathbf{b}$-part $\{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$.

To prove the Lemma, we will show that (127) is equal to zero for all $r > 1$, by breaking up the sum into simpler parts that can be expressed explicitly.

For each $0 \le i \le r$, let $\mathcal{B}_i$ denote the set of all $\mathbf{b}$-parts (occurring in the sum in (127)) that contain v_i but not $v_{i+1}, \ldots, v_r$; in other words a given $\mathbf{b}$-part $\mathbf{b} = \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$ lies in $\mathcal{B}_i$ if and only if i is the largest index such that v_i occurs in $\mathbf{b}$.

Examples for $i = 0, 1, 2$. We have $\mathcal{B}_0 = \emptyset$, corresponding to all the decompositions with empty $\mathbf{b}$-parts, namely the trivial decompositions

$$\sigma_r^j(\mathbf{v}) = (v_{j+1}, \ldots, v_r, v_1, \ldots, v_j) = \mathbf{a}_1$$

for $0 \le j \le r-1$. The set $\mathcal{B}_1$ contains only the single element $\mathbf{b} = (v_1)$, and corresponds to the decompositions

$$\sigma_r^j(\mathbf{v}) = (v_{j+1}, \ldots, v_r, v_1, \ldots, v_j) = (v_{j+1}, \ldots, v_r)(v_1)(v_2, \ldots, v_i) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2$$

for $1 \le j \le r-1$ (in fact just $\mathbf{a}_1\mathbf{b}_1$ for $j = 1$). The set $\mathcal{B}_2$ contains two different $\mathbf{b}$-parts, namely (v_2) and (v_1, v_2). The $\mathbf{b}$-part (v_2) occurs in the decompositions

$$\begin{cases} (v_1, \ldots, v_r) = (v_1)(v_2)(v_3, \ldots, v_r) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \\ (v_3, \ldots, v_r, v_1, v_2) = (v_3, \ldots, v_r, v_1)(v_2) = \mathbf{a}_1\mathbf{b}_1 \\ (v_j, \ldots, v_r, v_1, \ldots, v_{j-1}) = (v_{j+1}, \ldots, v_r, v_1)(v_2)(v_3, \ldots, v_j) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \ \text{ for } 4 \le j \le r. \end{cases}$$

The $\mathbf{b}$-part (v_1, v_2) occurs in the decompositions

$$\begin{cases} (v_3, \ldots, v_r)(v_1, v_2) = \mathbf{a}_1\mathbf{b}_1 \\ (v_j, \ldots, v_r)(v_1, v_2)(v_3, \ldots, v_{j-1}) = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2 \ \text{ for } 4 \le j \le r \end{cases}$$

Indeed, for $1 \le i \le r-1$, the set $\mathcal{B}_i$ is simply in bijection with the set of all subsets $B \subset \{1, \ldots, i-1\}$, by associating B to the $\mathbf{b}$-part $\{v_j | j \in B\} \cup \{v_i\}$); when $i = r$, $\mathcal{B}_r$ is in bijection with the set of all strict subsets of $\{1, \ldots, r-1\}$.

The $\mathbf{b}$-part of each decomposition of each cyclic permutation $\sigma_r^j(\mathbf{v})$ lies in a unique $\mathcal{B}_i$. Therefore setting

$$R_i^r := \sum_{\mathbf{b} \in \mathcal{B}_i} (-1)^{|\mathbf{a}|} poc(\lfloor \mathbf{b}_1) \cdots poc(\lfloor \mathbf{b}_s) S_{n-|\mathbf{a}|}^{\mathbf{a}} \tag{128}$$

for $0 \le i \le r$, we can write the sum (127) as

$$\sum_{i=0}^{r-1} \big(\overline{ganit}(poc) \cdot N\big)\big(\sigma_r^i(\mathbf{v})\big) = R_0^r + \cdots + R_r^r. \tag{129}$$

We have

$$R_0^r = \sum_{j=0}^{r-1} N\big(\sigma_r^j(\mathbf{v})\big) = (-1)^r S_{n-r}^{v_1, \ldots, v_r} \tag{130}$$

since $N = pari(T^n)$ and T^n is circ-constant. For R_1^r, the only possible $\mathbf{b}$-part is (v_1) and we have

$$R_1^r = \frac{(-1)^{r-1}}{v_1 - v_r} S_{n-r+1}^{v_2, \ldots, v_r}. \tag{131}$$

For $i > 1$, note that if $\mathbf{b}_j = (v_{k+1}, v_{k+2}, \ldots, v_l)$ (with indices k and l considered mod r from 1 to r, for example $\mathbf{b}_j = (v_{r-1}, v_r, v_1)$ with $k = r-2$ and $l = 1$) is a chunk of any decomposition

$$\sigma_r^j(\mathbf{v}) = \mathbf{a}_1\mathbf{b_1}\cdots\mathbf{a_s}\mathbf{b}_s, \tag{132}$$

of any cyclic permutation of $\mathbf{v}$, then by the definition of *poc*, we have

$$\begin{aligned} poc(\lfloor \mathbf{b}_j) &= poc(v_{k+1} - v_k, v_{k+2} - v_k, \ldots, v_l - v_k) \\ &= \frac{1}{(v_{k+1} - v_k)(v_{k+1} - v_{k+2})\cdots(v_{l-1} - v_l)} \\ &= -\prod_{v_m \in \mathbf{b}_j} \frac{1}{(v_{m-1} - v_m)}, \end{aligned}$$

again with indices m mod r with values from 1 to r. Thus, writing as usual $\mathbf{a}$ for the $\mathbf{a}$-part of a decomposition as in (132), (128) can be written

$$R_i^r = \sum_{\mathbf{b'} \subseteq \{v_1, \ldots, v_{i-1}\}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1} - v_j)} \tag{133}$$

for $1 \leq i \leq r-1$, where $\mathbf{b'}$ runs over all subsets of $\{v_1, \ldots, v_{i-1}\}$ so $\mathbf{b} = \mathbf{b'} \cup \{v_i\}$ runs over the elements of $\mathcal{B}_i$, and for $i = r$ we have

$$R_r^r = \sum_{\mathbf{b'} \subsetneq \{v_1, \ldots, v_{r-1}\}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1} - v_j)}. \tag{134}$$

We will use these explicit expressions to show that the sum

$$R_0^r + \cdots + R_r^r = 0$$

for all $r \geq 2$. In order to prove this, we will give simple rational function expressions for $R_1^r, \ldots, R_r^r$ in Claim 1 below, generalizing the simple expression

$$R_0^r = (-1)^r S_{n-r}^{v_1, \ldots, v_r} \tag{135}$$

for R_0^r from (130). These will allow us to show in Claim 2 that $R_0^r + \cdots + R_r^r = 0$, thus completing the proof of Lemma 4.19.

Claim 1. *(i) For $i = 1$, we have*

$$R_1^r = \frac{(-1)^{r-1} S_{n-r+1}^{v_2, \ldots, v_r}}{(v_r - v_1)}. \tag{136}$$

(ii) For $2 \leq i \leq r-1$, we have

$$R_i^r = \frac{(-1)^{r-i} S_{n-r+i}^{\{v_{i-1}, v_{i+1}, \ldots, v_{r-1}\}}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{137}$$

(iii) For $i = r$, we have

$$R_r^r = \frac{v_{r-1}^{n-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})}. \tag{138}$$

The proof of this claim is long and we have moved it to the end of the Appendix. Our next claim uses Claim 1 to give a simple rational expression for the sum $R_0^r + \cdots + R_i^r$. We first note the following trivial but useful identity. Recall the notation $V_m = \{v_1, \ldots, v_m\}$ for $1 \le m \le r$. Let $A' \subsetneq V_r$, let $v_j \notin A$, and let $A = A' \cup \{v_j\}$: then we have the useful identity

$$S^{A'}_{d+1} + v_j S^A_d = S^A_{d+1}. \tag{139}$$

Indeed, the first term is the sum of all monomials of degree d in the elements of A', and the second is the sum of all monomials in the letters of A containing v_j, so their sum forms the sum of all monomials of degree d in the letters of A.

Claim 2. *For $0 \le i \le r-1$ we have*

$$R^r_0 + \cdots + R^r_i = \frac{(-1)^{r-i} S^{v_i,\ldots,v_{r-1}}_{n-r+i}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \tag{140}$$

Proof. We prove the result by induction on i. The base case $i = 0$ is given by (130). Now let $1 \le i \le r-1$ and assume (140) up to $i-1$. Then by the induction hypothesis and Claim 1, we have

$$\begin{aligned} R^r_0 + \cdots + R^r_i &= (R^r_0 + \cdots + R^r_{i-1}) + R^r_i \\ &= \frac{(-1)^{r-i+1} S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-2} - v_{i-1})} + \frac{(-1)^{r-i} S^{v_{i-1},v_{i+1},\ldots,v_{r-1}}_{n-r+i}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)} \\ &= \frac{(-1)^{r-i+1}\Big(v_{i-1} S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i-1} - v_i S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i-1} - S^{v_{i-1},v_{i+1},\ldots,v_{r-1}}_{n-r+i}\Big)}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)}. \end{aligned}$$

By (139), the second term and third terms in the numerator sum to $-S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i}$, so the numerator becomes $(-1)^{r-i+1}\Big(v_{i-1} S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i-1} - S^{v_{i-1},\ldots,v_{r-1}}_{n-r+i}\Big)$ which again by (139) sums to $(-1)^{r-i}\Big(S^{v_i,\ldots,v_{r-1}}_{n-r+i}\Big)$. Thus we have

$$R^r_0 + \cdots + R^r_i = \frac{(-1)^{r-i} S^{v_i,\ldots,v_{r-1}}_{n-r+i}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{i-1} - v_i)},$$

which proves Claim 2. $\square$

End of the proof of Lemma 4.19. It suffices to note that by (140) when $i = r-1$ we have

$$R^r_0 + \cdots + R^r_{r-1} = \frac{-v^{n-1}_{r-1}}{(v_r - v_1)(v_1 - v_2)\cdots(v_{r-2} - v_{r-1})}. \tag{141}$$

But this is precisely the negative of the expression for R^r_r given in Claim 1 (iii). Thus we have $R^r_0 + \cdots + R^r_r = 0$. By (129) this means that

$$\sum_{i=0}^{r-1} \big(\overline{ganit}(poc) \cdot N\big)\big(\sigma^i_r(\mathbf{v})\big) = 0, \tag{142}$$

i.e. $\overline{ganit}(poc) \cdot N$ is circ-neutral, completing the proof of Lemma 4.19. $\square$

It remains only to give the proof of Claim 1.

Proof of Claim 1. (i) When $i = 1$ we have $\mathcal{B}_1 = \{v_1\}$, so here there is only one term in the sum (133) corresponding to $\mathbf{b}' = \emptyset$, $\mathbf{b} = \{v_1\}$, $\mathbf{a} = \{v_2, \ldots, v_r\}$, $|\mathbf{a}| = r - 1$, so that (133) for $i = 1$ comes down to (136).

(ii) Let $V_m = \{v_1, \ldots, v_m\}$ for $1 \le m \le r$. Fix a value of i with $2 \le i \le r - 1$. Recall that R_i^r was defined in (133) as the sum

$$R_i^r = \sum_{\mathbf{b}' \subseteq V_{i-1}} \frac{(-1)^{|\mathbf{a}|} S_{n-|\mathbf{a}|}^{\mathbf{a}}}{\prod_{v_j \in \mathbf{b}} (v_{j-1} - v_j)}. \tag{143}$$

where $\mathbf{b} = \mathbf{b}' \cup \{v_i\}$ and $\mathbf{a}$ is the complement of $\mathbf{b}$ in V_r.

Multiplying R_i^r by the common denominator $(v_r - v_1) \cdots (v_{i-1} - v_i)$ and setting $v_0 = v_r$ as usual, we rewrite (143) as

$$\prod_{j=1}^{i} (v_{j-1} - v_j) R_i^r = \sum_{\mathbf{b}' \subseteq V_{i-1}} (-1)^{|\mathbf{a}|} \prod_{v_j \in V_{i-1} \setminus \mathbf{b}'} (v_{j-1} - v_j) S_{n-|\mathbf{a}|}^{\mathbf{a}}. \tag{144}$$

To conclude the proof, we need one more claim.

Claim 3. *For each pair i, k with $1 < i < r$ and $1 \le k \le i-1$, define the polynomial Q_k^i by*

$$\sum_{v_1, \ldots, v_k \notin B' \subset V_{i-1}} (-1)^{r-|B'|+k-1} \Bigg(\prod_{v_j \in V_{i-1} \setminus (B' \cup \{v_1, \ldots, v_k\})} (v_{j-1} - v_j) \Bigg) S_{n-r+|B'|+k+1}^{V_r \setminus (B' \cup \{v_1, \ldots, v_{k-1}\} \cup \{v_i, v_r\})}$$

Then $Q_1^i = Q_2^i = \cdots = Q_{i-1}^i$ and they are all equal to the right-hand side of (144).

Claim 3 allows us to compute the right-hand side of (144) by taking the sum Q_{i-1}^i, which is reduced to the single term corresponding to $B' = \emptyset$ and $B = \{v_i\}$, so is just $(-1)^{r-i} S_{n-r+i}^{v_{i-1}, v_{i+1}, \ldots, v_{r-1}}$. Thus by (144) we find that

$$\prod_{j=1}^{i} (v_{j-1} - v_j) R_i^r = (-1)^{r-i} S_{n-r+i}^{v_{i-1}, v_{i+1}, \ldots, v_{r-1}},$$

which proves part (ii) of Claim 1 as stated in (137).

Proof of Claim 3. We first show that the right-hand side of (144) is equal to Q_1^i, and subsequently that $Q_1^i = Q_2^i = \cdots = Q_k^i$.

For the first statement, we begin by breaking the right-hand side of (144) into $v_1 \in \mathbf{b}'$ and $v_1 \notin \mathbf{b}'$, which gives

$$\begin{aligned} \prod_{j=1}^{i} (v_{j-1} - v_j) R_i^r = & \sum_{v_1 \in \mathbf{b}' \subseteq V_{i-1}} (-1)^{|\mathbf{a}|} \prod_{v_j \in V_{i-1} \setminus \mathbf{b}'} (v_{j-1} - v_j) S_{n-|\mathbf{a}|}^{\mathbf{a}} \\ & + \sum_{v_1 \notin \mathbf{b}' \subseteq V_{i-1}} (-1)^{|\mathbf{a}|} \prod_{v_j \in V_{i-1} \setminus \mathbf{b}'} (v_{j-1} - v_j) S_{n-|\mathbf{a}|}^{\mathbf{a}}, \end{aligned}$$

where $\mathbf{b} = \mathbf{b}' \cup \{v_i\}$ and $\mathbf{a} = V_r \setminus \mathbf{b}$ as usual. Next we write $\mathbf{b}' = \mathbf{b}'' \cup \{v_1\}$ in the upper sum, simply rename $\mathbf{b}'$ to $\mathbf{b}''$ in the lower sum, and use the fact that

$$\mathbf{a} = V_r \setminus \mathbf{b} = \begin{cases} V_r \setminus (\mathbf{b}'' \cup \{v_1, v_i\}) & \text{in the upper sum} \\ V_r \setminus (\mathbf{b}'' \cup \{v_i\}) & \text{in the lower sum} \end{cases}$$

to rewrite this as

$$= \sum_{v_1 \notin \mathbf{b}'' \subseteq V_{i-1}} (-1)^{r-|\mathbf{b}''|-2} \prod_{v_j \in V_{i-1} \setminus (\mathbf{b}'' \cup \{v_1\})} (v_{j-1} - v_j) S_{n-r+|\mathbf{b}''|+2}^{V_r \setminus (\mathbf{b}'' \cup \{v_1, v_i\})}$$

$$+ \sum_{v_1 \notin \mathbf{b}'' \subseteq V_{i-1}} (-1)^{r-|\mathbf{b}''|-1} (v_r - v_1) \prod_{v_j \in V_{i-1} \setminus (\mathbf{b}'' \cup \{v_1\})} (v_{j-1} - v_j) S_{n-r+|\mathbf{b}''|+1}^{V_r \setminus (\mathbf{b}'' \cup \{v_i\})}.$$

This allows us to gather the terms in a single sum:

$$= \sum_{v_1 \notin \mathbf{b}'' \subseteq V_{i-1}} (-1)^{r-|\mathbf{b}''|} \prod_{v_j \in V_{i-1} \setminus (\mathbf{b}'' \cup \{v_1\})} (v_{j-1} - v_j) \times$$

$$\left(S_{n-r+|\mathbf{b}''|+2}^{V_r \setminus (\mathbf{b}'' \cup \{v_1, v_i\})} - (v_r - v_1) S_{n-r+|\mathbf{b}''|+1}^{V_r \setminus (\mathbf{b}'' \cup \{v_i\})} \right).$$

Setting $A' = V_r \setminus (\mathbf{b}'' \cup \{v_1, v_i\})$ and $A = A' \cup \{v_1\}$, the right-hand factor expands as the sum of three terms

$$S_{n-r+|\mathbf{b}''|+2}^{A'} - v_r S_{n-r+|\mathbf{b}''|+1}^{A} + v_1 S_{n-r+|\mathbf{b}''|+1}^{A}.$$

By applying (139) with $v_j = v_1$, this then simplifies as

$$S_{n-r+|\mathbf{b}''|+2}^{A} - v_r S_{n-r+|\mathbf{b}''|+1}^{A}.$$

Then applying (139) a second time with $v_j = v_r$, this simplifies to

$$S_{n-r+|\mathbf{b}''|+2}^{A'} = S_{n-r+|\mathbf{b}''|+2}^{V_r \setminus (\mathbf{b}'' \cup \{v_i, v_r\})}.$$

Thus we finally obtain the following expression for the right-hand side of (144):

$$= \sum_{v_1 \notin \mathbf{b}'' \subseteq V_{i-1}} (-1)^{r-|\mathbf{b}''|} \prod_{v_j \in V_{i-1} \setminus (\mathbf{b}'' \cup \{v_1\})} (v_{j-1} - v_j) \; S_{n-r+|\mathbf{b}''|+2}^{V_r \setminus (\mathbf{b}'' \cup \{v_i, v_r\})}.$$

From the definition of Q_k^i given in the statement of Claim 3, this is precisely equal to Q_1^i, so as claimed, we have

$$\prod_{j=1}^{i} (v_{j-1} - v_j) R_i^r = Q_1^i.$$

We can now proceed to the proof that $Q_1^i = Q_2^i = \cdots = Q_{i-1}^i$ by induction. Fix $1 \leq k < i-1$ and assume that $Q_1^i = \cdots = Q_k^i$. We will show by the same method that $Q_k^i = Q_{k+1}^i$. We break the expression for Q_k^i into the terms with $v_{k+1} \in B'$ and those with $v_{k+1} \notin B'$, writing Q_k^i as

$$\sum_{\substack{v_1, \ldots, v_k \notin B' \subseteq V_{i-1} \\ v_{k+1} \in B'}} (-1)^{r-|B'|+k-1} \Bigg(\prod_{v_j \in V_{i-1} \setminus (B' \cup \{v_1, \ldots, v_k\})} (v_{j-1} - v_j) \Bigg) S_{n-r+|B'|+k+1}^{V_r \setminus (B' \cup \{v_1, \ldots, v_{k-1}\} \cup \{v_i, v_r\}))}$$

$$+ \sum_{v_1, \ldots, v_{k+1} \notin B' \subseteq V_{i-1}} (-1)^{r-|B'|+k-1} \Bigg(\prod_{v_j \in V_{i-1} \setminus (B' \cup \{v_1, \ldots, v_k\})} (v_{j-1} - v_j) \Bigg) S_{n-r+|B'|+k+1}^{V_r \setminus (B' \cup \{v_1, \ldots, v_{k-1}\} \cup \{v_i, v_r\})}.$$

Next we write $B'' := B' \setminus \{v_{k+1}\}$ in the first line, and simply replace the notation B' by B'' in the second line, obtaining

$$\sum_{v_1, \ldots, v_{k+1} \notin B'' \subseteq V_{i-1}} (-1)^{r-|B''|+k-2} \Bigg(\prod_{v_j \in V_{i-1} \setminus (B'' \cup \{v_1, \ldots, v_{k+1}\})} (v_{j-1} - v_j) \Bigg) S_{n-r+|B''|+k+2}^{V_r \setminus (B'' \cup \{v_1, \ldots, v_{k-1}\} \cup \{v_{k+1}, v_i, v_r\}))}$$

$$+\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k-1}(v_k-v_{k+1})\Big(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Big)S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+1}.$$

Now we gather the terms as before, writing this as

$$\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k}\Big(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Big)\times$$

$$\left(S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_{k+1},v_i,v_r\}))}_{n-r+|B''|+k+2}-(v_k-v_{k+1})S^{V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+1}\right).$$

We will now use (139) twice as above to simplify the right-hand factor. First we take $A'=V_r\setminus(B''\cup\{v_1,\dots,v_{k-1}\}\cup\{v_{k+1},v_i,v_r\})$ and $A=A'\cup\{v_{k+1}\}$, and write the factor as

$$S^{A'}_{n-r+|B''|+k+2}-v_kS^{A}_{n-r+|B''|+k+1}+v_{k+1}S^{A}_{n-r+|B''|+k+1}.$$

Applying (139), this simplifies to

$$S^{A}_{n-r+|B''|+k+2}-v_kS^{A}_{n-r+|B''|+k+1}.$$

Next, since $v_k\notin B''$ we see that $v_k\in A$, so by applying (139) again we see that this simplifies to

$$S^{A\setminus\{v_k\}}_{n-r+|B''|+k+2}=S^{V_r\setminus(B''\cup\{v_1,\dots,v_k\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+2}.$$

Thus we rewrite Q^i_k as

$$\sum_{v_1,\dots,v_{k+1}\notin B''\subseteq V_{i-1}}(-1)^{r-|B''|+k}\Big(\prod_{v_j\in V_{i-1}\setminus(B''\cup\{v_1,\dots,v_{k+1}\})}(v_{j-1}-v_j)\Big)S^{V_r\setminus(B''\cup\{v_1,\dots,v_k\}\cup\{v_i,v_r\})}_{n-r+|B''|+k+2}.$$

But according to the definition of the polynomials Q^i_k, this is exactly equal to Q^i_{k+1}. This shows that $Q^i_1=Q^i_2=\cdots=Q^i_{i-1}$, and completes the proof of Claim 3. □

It remains only to prove part (iii) of Claim 1.

(iii) In this final part we have to prove that

$$\prod_{j=1}^{r-1}(v_{j-1}-v_j)R^r_r=v^{n-1}_{r-1}. \tag{145}$$

Recall from (134) that R^r_r is given by the formula

$$R^r_r=\sum_{\mathbf{b}'\subsetneq V_{r-1}}\frac{(-1)^{|\mathbf{a}|}S^{\mathbf{a}}_{n-|\mathbf{a}|}}{\prod_{v_j\in\mathbf{b}}(v_{j-1}-v_j)},$$

where $\mathbf{b}=\mathbf{b}'\cup\{v_r\}$. Thus the common denominator of all the terms in the sum is $(v_r-v_1)(v_1-v_2)\cdots(v_{r-1}-v_r)$, and we have

$$\prod_{j=1}^{r}(v_{j-1}-v_j)R^r_r=\sum_{\mathbf{b}'\subsetneq V_{r-1}}(-1)^{r-|\mathbf{b}'|-1}\prod_{v_j\in V_{r-1}\setminus\mathbf{b}'}(v_{j-1}-v_j)\ S^{V_{r-1}\setminus\mathbf{b}'}_{n-r+|\mathbf{b}'|+1}. \tag{146}$$

Let us write $\mathbf{c}=V_{r-1}\setminus\mathbf{b}'$, so this equality can be expressed as

$$\prod_{j=1}^{r}(v_{j-1}-v_j)R^r_r=\sum_{\emptyset\neq\mathbf{c}\subseteq V_{r-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|}. \tag{147}$$

For $1 \le i \le r-1$ and $n \ge 1$, define the sum T_i^n by

$$T_i^n := \sum_{\emptyset \neq \mathbf{c} \subseteq V_i} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S^{\mathbf{c}}_{n-|\mathbf{c}|},$$

where we set $S^{\mathbf{c}}_0 = 1$ and $S^{\mathbf{c}}_m = 0$ if $m < 0$. By this definition, the term T^n_{r-1} is equal to the right-hand side of (147). We will prove that

$$T_i^n = (v_i - v_r) v_i^{n-1}. \tag{148}$$

The equality (148) suffices to prove the desired result (145). Indeed, since T^n_{r-1} is equal to the right-hand side of (147), the left-hand side of (147) is equal to the right-hand side of (148) with $i = r-1$, i.e.

$$\prod_{j=1}^{r} (v_{j-1} - v_j) R^r_r = T^n_{r-1} = (v_{r-1} - v_r) v^{n-1}_{r-1}.$$

Canceling out the factor $(v_r - v_{r-1})$ from both sides yields the desired identity (145).

It remains only to prove (148). We proceed by induction on i. When $i = 1$, we have $\mathbf{c} = \{v_1\}$ and for all $n \ge 1$, we have

$$T_1^n = -(v_r - v_1) S^{v_1}_{n-1} = (v_1 - v_r) v_1^{n-1},$$

proving the base case.

Fix $n \ge 1$ and assume (148) holds for $i-1$. We break T_i^n into the sum over $\mathbf{c}$ containing v_i and $\mathbf{c}$ not containing v_i, as follows:

$$\begin{aligned}
T_i^n &= \sum_{\emptyset \neq \mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S^{\mathbf{c}}_{n-|\mathbf{c}|} \\
&\qquad + \sum_{\substack{\mathbf{c} \subseteq V_{i-1} \\ \mathbf{c}' = \mathbf{c} \cup \{v_i\}}} (-1)^{|\mathbf{c}'|} (v_{i-1} - v_i) \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S^{\mathbf{c}'}_{n-|\mathbf{c}'|} \\
&= T^n_{i-1} + (v_{i-1} - v_i) \sum_{\mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|+1} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S^{\mathbf{c}, v_i}_{n-|\mathbf{c}|-1} \\
&= T^n_{i-1} - (v_{i-1} - v_i) v_i^{n-1} - (v_{i-1} - v_i) \sum_{\emptyset \neq \mathbf{c} \subseteq V_{i-1}} (-1)^{|\mathbf{c}|} \prod_{v_j \in \mathbf{c}} (v_{j-1} - v_j) S^{\mathbf{c}, v_i}_{n-|\mathbf{c}|-1},
\end{aligned}$$

where the last line comes from separating the sum over $\mathbf{c} \subseteq V_{i-1}$ into $\mathbf{c} = \emptyset$ (giving the extra term $(v_{i-1} - v_i) v_i^{n-1}$) and the sum over $\mathbf{c} \neq \emptyset$. Since $\mathbf{c}$ does not contain v_i, we can write

$$S^{\mathbf{c}, v_i}_{n-|\mathbf{c}|-1} = S^{\mathbf{c}}_{n-|\mathbf{c}|-1} + v_i S^{\mathbf{c}}_{n-|\mathbf{c}|-2} + v_i^2 S^{\mathbf{c}}_{n-|\mathbf{c}|-3} + \cdots + v_i^{n-|\mathbf{c}|-2} S^{\mathbf{c}}_1 + v_i^{n-|\mathbf{c}|-1}.$$

Using this, the above equality becomes

$$= T_{i-1}^n - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)\times$$

$$\Big(S^{\mathbf{c}}_{n-|\mathbf{c}|-1} + v_i S^{\mathbf{c}}_{n-|\mathbf{c}|-2} + v_i^2 S^{\mathbf{c}}_{n-|\mathbf{c}|-3} + \cdots + v_i^{n-|\mathbf{c}|-2}S^{\mathbf{c}}_1 + v_i^{n-|\mathbf{c}|-1}\Big)$$

$$= T_{i-1}^n - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\sum_{k=0}^{n-|\mathbf{c}|-1}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|-1-k}v_i^k$$

$$= T_{i-1}^n - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k\sum_{\emptyset\neq\mathbf{c}\subseteq V_{i-1}}(-1)^{|\mathbf{c}|}\prod_{v_j\in\mathbf{c}}(v_{j-1}-v_j)S^{\mathbf{c}}_{n-|\mathbf{c}|-1-k}$$

$$= T_{i-1}^n - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k T_{i-1}^{n-k-1}$$

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)\sum_{k=0}^{n-2}v_i^k(v_{i-1}-v_r)v_{i-1}^{n-k-2} \quad \text{by induction}$$

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_i)(v_{i-1}-v_r)\sum_{k=0}^{n-2}v_i^k v_{i-1}^{n-k-2}$$

$$= (v_{i-1}-v_r)v_{i-1}^{n-1} - (v_{i-1}-v_i)v_i^{n-1} - (v_{i-1}-v_r)(v_{i-1}^{n-1}-v_i^{n-1})$$

$$= (v_i - v_r)v_i^{n-1}.$$

This proves (148) and thus completes the proof of Claim 1 (iii). □

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