Some notes on Parker's conjecture

Definition. For any group G with two generators a, b, let *Parker's element* $P \in G \times G$ be defined by

$$P = \sum_{g \in G} (g^{-1}ag, g^{-1}bg).$$

Let M_P denote the matrix giving the action of P on the vector space $\mathbb{Q}[G \times G]$ by left multiplication, and let K_G denote the number field generated by the eigenvalues of M_P.

Parker's Conjecture: K_G is the moduli field, as a G-cover, of the dessin d'enfant determined by the data of G and its two generators a and b.

Let us prove this conjecture for all abelian groups $C_n \times C_m$ and dihedral groups D_n.

Abelian groups on at most two generators. Let a be a generator of C_n and b a generator of C_m, and consider the abelian group $G = C_n \times C_m$ equipped with these two generators. It corresponds to a dessin on a surface of genus $(nm-n-m-gcd(n,m)+2)/2$ with m n-petaled flowers over 0, n m-petaled flowers over 1, and $gcd(n,m)$ faces. In the case where $m=1$, the dessin is the n-petaled hedgehog with moduli field $\mathbb{Q}(\zeta_n)$.

Proposition 1. *$K_{C_n \times C_m} = \mathbb{Q}(\zeta_r)$ where $r = lcm(n,m)$.*

Proof. Parker's element is given by $P = nm\big((a,1),(1,b)\big)$. Set $P' = \big((a,1),(1,b)\big)$. Let us number the basis e_i of $\mathbb{Q}[G \times G]$, i.e. the elements of $G \times G$, as

$$e_r = \big((a^i, b^j), (a^k, b^l)\big), \quad 1 \le i, j \le n, \quad 1 \le k, l \le m, \quad 1 \le r \le n^2m^2$$

with $r = m^2n(i-1) + m^2(j-1) + m(k-1) + l$. The matrix $M_{P'}$ corresponding to the action of P' by left multiplication is given as an $nm \times nm$ matrix of $nm \times nm$ blocks

$$M_{P'} = \begin{pmatrix} 0 & 0 & \cdots & 0 & T & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & T & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & T \\ T & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & T & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where T is the $nm \times nm$ matrix given by

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and the T in the first column above is in the position of the $(m+1)$-st block in the column of nm blocks. The matrix M_P is just $nmM_{P'}$. The element P' is of order $r = lcm(n,m)$ in $G\times G$, and the matrix $M_{P'}$ is also of order r as P' does not act trivially on any element of $G\times G$. The action of $M_{P'}$ on the vector $(v_1,\ldots,v_{n^2m^2})$ is given explicitly, writing $k = jnm+km+i$ with $0\le j\le nm-1$, $0\le k\le n-1$ and $1\le i\le m$, by

$$M_{P'}(v_k) = M_{P'}(v_{jnm+km+i}) = v_{(j+m)nm+km+\widetilde{i+1}} \tag{1}$$

where $\widetilde{i+1}$ is the representative of $i+1$ between 0 and m, and indices are taken to be modulo n^2m^2. It is easy to check directly from this formula that the action of $M_{P'}$ is of order r.

To find the eigenvalues, we want to solve $M_{P'}(v_1,\ldots,v_{n^2m^2}) = \lambda(v_1,\ldots,v_{n^2m^2})$, so by (1), we must have

$$\begin{aligned} v_{jnm+km+i} &= \lambda^{-1}v_{(j+m)nm+km+\widetilde{i+1}} = \lambda^{-2}v_{(j+2m)nm+km+\widetilde{i+2}} \\ &= \lambda^{-r}v_{(j+rm)nm+km+\widetilde{i+r}} = \lambda^{-r}v_{jnm+km+i} = \lambda^{-r}v_k. \end{aligned}$$

Thus the eigenvalues of $M_{P'}$ are the r-th roots of unity, so the field generated by the eigenvalues of M_P (which are nm times these) is $\mathbb{Q}(\zeta_r)$. □

Dihedral groups. Let $G = D_n$ be the dihedral group of order $2n$. It corresponds to the genus 0 given by an n-gon with n black vertices (over 0) and a white vertex in the middle of each edge. (The dual is a pumpkin with n semi-meridians joining the north pole to the south pole.)

Proposition 2. $K_{D_n} = \mathbb{Q}(\{\zeta_n^i + \zeta_n^{-i} \mid 0\le i\le n-1\})$.

Proof. The elements of the dihedral group are given by $g_i = a^{i-1}$, $1\le i\le n$, and $g_{n+i} = a^{i-1}b$. Parker's element is given by

$$P = \sum_{i=0}^{n-1}\big((a,a^ib) + (a^{-1},a^ib)\big) \in \mathbb{Q}[D_n\times D_n].$$

Case 1: n odd. In order to write down a matrix action M_P of P on $\mathbb{Q}[D_n\times D_n]$, we choose a numbering for the elements of $D_n\times D_n$ by setting

$$\begin{cases} e_k = e_{n(i-1)+j} = (g_i,g_j) & \text{for } 1\le i\le 2n,\ 1\le j\le n, \text{ so } 1\le k\le 2n^2 \\ e_k = e_{2n^2+n(i-1)+j} = (g_i,g_j) & \text{for } 1\le i\le 2n,\ n+1\le j\le 2n, \text{ so } 2n^2+1\le k\le 4n^2. \end{cases}$$

Then the action of P on the basis $e_1,\ldots,e_{4n^2}$ by right (or left) multiplication is given by a nice symmetric matrix given as follows:

$$M_P = \begin{pmatrix} 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \\ T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \end{pmatrix} \tag{2}$$

where T is the $n^2 \times n^2$ matrix given by

$$T = \begin{pmatrix} 0 & 1_n & 0 & 0 & 0 & \cdots & 0 & 0 & 1_n \\ 1_n & 0 & 1_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1_n & 0 & 1_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 1_n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1_n & 0 & 1_n \\ 1_n & 0 & 0 & 0 & 0 & \cdots & 0 & 1_n & 0 \end{pmatrix},$$

where 1_n denotes the $n \times n$ block consisting only of 1's.

Now, set $\underline{v} = (v_1, \ldots, v_{4n^2})$ and

$$\underline{v}_i = (v_{(i-1)n^2+1}, v_{(i-1)n^2+2}, \ldots, v_{in^2}) \quad \text{for} \quad i = 1, 2, 3, 4.$$

Then using (2), we see that $M_P \cdot \underline{v} = \lambda v$ is equivalent to

$$T\underline{v}_3 = \lambda \underline{v}_1, T\underline{v}_4 = \lambda \underline{v}_2, T\underline{v}_1 = \lambda \underline{v}_3, T\underline{v}_2 = \lambda \underline{v}_4.$$

Putting the third and first equalities together yields

$$T^2 \underline{v}_1 = \lambda T \underline{v}_3 = \lambda^2 \underline{v}_1,$$

so the eigenvalues λ of M_P are equal to the square roots of those of T^2.

To compute the eigenvalues of T^2 we first square T, giving the matrix

$$T^2 = \begin{pmatrix} 2_n & 0 & 1_n & 0 & 0 & \cdots & 0 & 1_n & 0 \\ 0 & 2_n & 0 & 1_n & 0 & \cdots & 0 & 0 & 1_n \\ 1_n & 0 & 2_n & 0 & 1_n & \cdots & 0 & 0 & 0 \\ 0 & 1_n & 0 & 2_n & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2_n & 0 & 1_n \\ 1_n & 0 & 0 & 0 & 0 & \cdots & 0 & 2_n & 0 \\ 0 & 1_n & 0 & 0 & 0 & \cdots & 1_n & 0 & 2_n \end{pmatrix},$$

where 2_n denotes the $n \times n$ block consisting only of 2's. Let S be the $n \times n$ matrix given by replacing the blocks 1_n in T by 1 and 2_n by 2 (and the 0 blocks by 0), so

$$S = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 \end{pmatrix}.$$

Then the eigenvalues of S are the eigenvalues of T^2 divided by n, so it is enough to compute the eigenvalues of S. This matrix is given by cyclically permuting the entries in the first line, as is the matrix $S - xId$ whose determinant will yield the eigenvalues. The determinant of such a matrix is given by a classical formula; it turns out to be

$$\prod_{i=1}^{n}(2 - x + \zeta^{2i} + \zeta^{-2i}) = \prod_{i=1}^{n}((\zeta^i + \zeta^{-i})^2 - x).$$

Thus, the eigenvalues of T^2 are $\{(\zeta^i + \zeta^{-i})^2/n \mid 1 \leq i \leq n-1\}$ and those of M_P are the square roots $\{(\zeta^i + \zeta^{-i})/\sqrt{n} \mid 0 \leq i \leq n-1\}$. Noticing that for $i = 0$, the eigenvalue is $2/\sqrt{n}$, which shows that $\sqrt{n} \in K_{D_n}$, so that also $\zeta^i + \zeta^{-i} \in K_{D_n}$ for $0 \leq i \leq n-1$. But $\mathbb{Q}(\sqrt{n})$ is the discriminant field of $x^{n-1} + \cdots + x + 1$, and since it is real, it lies inside the maximal real subfield of $\mathbb{Q}(\zeta_n)$ which is generated by the $\zeta^i + \zeta^{-i}$. Thus K_{D_n} is itself generated by the $\zeta^i + \zeta^{-i}$, $0 \leq i \leq n-1$. This concludes the proof. $\square$