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Ecalle's theory of multiple zeta values

The goal of this book is to understand Ecalle's proofs of some of the major results concerning the algebra FZ of formal (or symbolic) multiple zeta values and the associated Lie coalgebra nfz, which is the quotient of FZ modulo products. These results concern the freedom of the dual Lie algebra $nfz^{\vee}$, a canonical system of generators for nfz, the existence of a canonical rational Drinfeld associator, the relations between $\mathfrak{d}s$ and the associator relations ("gonal" relations: bigon, triangle, pentagon), period polynomials associated to modular forms, and finally, precise freeness/non-freeness results on algebras of formal Eulerian multiple zeta values and generalized values with roots of unity).

Contents

Chapter 1

Real and formal multiple zeta values

In this first chapter, we introduce some of the basic objects of study in the classical theory; the algebras of real and formal multiple zeta values, the real and formal Drinfel'd associators, the double shuffle Lie algebra, and the weight grading and depth filtrations.

1.1. Multiple zeta values and their regularizations

For every sequence $\mathbf{k} = (k_1, \ldots, k_r)$ of strictly positive integers with $k_1 \geq 2$, let $\zeta(k_1, \ldots, k_r)$ be the *multiple zeta value* defined by

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{1.1.1}$$

For every word in $\mathbb{Q}\langle x, y\rangle$, we define a multiple zeta value $\zeta(w)$ as follows. If w starts with x and ends with y, we write $w = x^{k_1-1}y \cdots x^{k_r-1}y$ with $k_1 \geq 2$, and set $\zeta(w) = \zeta(k_1, \ldots, k_r)$.

For general w, we write $w = y^r v x^s$ and set

$$\zeta(w) = \sum_{a=0}^{r} \sum_{b=0}^{s} (-1)^{a+b} \zeta\big(\pi(sh(y^a, y^{r-a}vx^{s-b}, x^b))\big), \tag{1.1.2}$$

where π is the projection of a polynomial onto the *convergent* words, i.e. those starting with x and ending with y, and ζ is considered to be additive. This way of extending the real zeta values of convergent words (called *convergent zeta values*) to all words is called the *shuffle regularization*, because of the following property that characterizes it.

Theorem 1.1. *For all words u, $v \in \mathbb{Q}\langle x, y\rangle$, the regularized ζ values defined in (1.1.2) satisfy the* shuffle[1] relations

$$\zeta\big(sh(u, v)\big) = \zeta(u)\zeta(v) \tag{1.1.3}$$

in the alphabet $\mathcal{X} = \{x, y\}$.

Multiple zeta values possess a second interesting multiplicative property. For all convergent words u, v, considered to be written in the variables $y_i = x^{i-1}y$, the convergent zeta

[1] The shuffle product of two words u and v in an alphabet $\mathcal{X}$ is defined recursively by $sh(u, 1) = sh(1, u) = u$ and $sh(Xu, Yv) = X\, sh(u, Yv) + Y\, sh(Xu, v)$ for any letters $X, Y \in \mathcal{X}$.

values satisfy the *stuffle*[2] *relations* $\zeta\big(st(u,v)\big)=\zeta(u)\zeta(v)$ in the alphabet $\mathcal{Y}=\{y_i|i\geq 0\}$, considered to be additive via the rule $y_i+y_j=y_{i+j}$. This result follows easily from the expression of $\zeta(k_1,\ldots,k_r)$ as a power series. But there is a second regularization of the zeta values, called the *stuffle regularization*, extending the stuffle relation to all words in the y_i. It is defined as follows.

Definition. The *Drinfel'd associator* Φ is given by

$$\Phi=1+\sum_{w\in\mathbb{Q}\langle x,y\rangle}\zeta(w)w. \tag{1.1.4}$$

Let π_y denote the projection of power series onto their words ending in y, rewritten in the y_i. Set

$$\Phi_*=exp\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\zeta(y_n)y_1^n\Big)\pi_y(\Phi), \tag{1.1.5}$$

and for every word v in the y_i, define $\zeta^*(v)$ to be the coefficient $\big(\Phi_*|v\big)$ of the word v in Φ_*. Since the exponential "correction" factor is a power series in y_1, it follows that for any convergent word v (i.e. any word in the y_i not starting with y_1), we have $\zeta^*(v)=\zeta(v)$. Inversely, the stuffle-regularized values $\zeta^*(1,\ldots,1)$ come entirely from the correction factor and are all polynomials in the single zeta values $\zeta(n)$; we see for instance that

$$\zeta^*(1)=\zeta(1)=0,\quad \zeta^*(1,1)=-\frac{1}{2}\zeta(2),\quad \zeta^*(1,1,1)=\frac{1}{3}\zeta(3),$$

$$\zeta^*(1,1,1,1)=-\frac{1}{4}\zeta(4)+\frac{1}{4}\zeta(2)^2=\frac{1}{2}\zeta(2,2)$$

given the stuffle identity $\zeta(2)^2=2\zeta(2,2)+\zeta(4)$. Thus, we can write the correction factor as

$$exp\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\zeta(y_n)y_1^n\Big)=\sum_{n\geq 1}\zeta^*(\underbrace{1,\ldots,1}_{n})y_1^n, \tag{1.1.6}$$

and for $w=y_1^iv$ with v a word in the y_i not starting with y_1, the stuffle regularized multizeta values are given by the formula

$$\zeta^*(w)=\Big(\Phi_*|v\Big)=\sum_{j=0}^{i}\zeta^*(\underbrace{1,\ldots,1}_{j})\big(\Phi|y_1^{i-j}v\big). \tag{1.1.7}$$

The values $\zeta^*(v)$ are called the *stuffle regularization* of the convergent multizeta values, because of the following theorem.

[2] Let $\mathcal{Y}$ be an additive alphabet, i.e. an alphabet equipped with a rule so that for every pair of letters $X,Y\in\mathcal{Y}$, $X+Y$ is also a letter in $\mathcal{Y}$. The stuffle product in the additive alphabet $\mathcal{Y}$ is defined recursively by $st(u,1)=st(1,u)=u$ and $st(Xu,Yv)=X\,st(u,YV)+Y\,st(Xu,v)+(X+Y)\,st(u,v)$ for all letters $X,Y\in\mathcal{Y}$.

Theorem 1.2. *For all words u, v in the variables y_i, the values $\zeta^*(v)$ satisfy the* stuffle relations

$$\zeta^*\big(st(u,v)\big) = \zeta^*(u)\zeta^*(v). \tag{1.1.8}$$

Definition. Let $\mathcal{Z}$ denote the $\mathbb{Q}$-algebra generated by the convergent multizeta values under the multiplication law (1.1.3). By (1.1.2) and (1.1.7), $\mathcal{Z}$ contains all the shuffle and stuffle regularized multizeta values. For every word $w \in \mathbb{Q}\langle x, y\rangle$ of length n containing d y's, the corresponding multiple zeta value $\zeta(w)$ is said to be of *weight* n and *depth* r. Let $\mathcal{Z}_n$ denote the $\mathbb{Q}$-vector space generated by the convergent multiple zeta values of weight n. We have $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = \langle 0\rangle$, $\mathcal{Z}_2 = \langle\zeta(2)\rangle$.

The algebra $\mathcal{Z}$ has a rich structure of which the shuffle and stuffle families of relations (known as the double shuffle relations) are only one aspect. There are many other known algebraic relations between elements of $\mathcal{Z}$, and also, of course, difficult problems of transcendence and irrationality. There are few results known on this aspect: irrationality of $\zeta(3)$, and the fact that an infinite number of the $\zeta(n)$ for odd n are transcendent. But the main transcendence conjectures, namely that all multiple zeta values are transcendent, seems still out of reach.

The transcendence conjecture can be stated as a structural conjecture on $\mathcal{Z}$ as follows.

Main transcendence conjecture. *The weight provides a grading of the $\mathbb{Q}$-algebra $\mathcal{Z}$; in other words, there are no linear relations between multizeta values of different weights.*

Indeed, if this is the case, then every multizeta value is transcendent, since otherwise, if some ζ of weight n were algebraic, there would be a minimal polynomial $P(x)$ such that $P(\zeta) = 0$. Each term of the polynomial would be a ζ^i, which when expanded out as a sum by the shuffle multiplication rule would yield a non-zero linear combination of multizetas of weight in, and the sum of all these terms of different weights would be zero, contradicting the main conjecture.

Because the conjectures concerning transcendence seem unprovable for the time being, and yet the combinatorial/algebraic structure of the multizeta algebra is still a rich subject of study – and indeed the double shuffle relations have been conjectured to generate *all* algebraic relations between multizeta values – it is useful to define a formal multiple zeta algebra of symbols satisfying the double shuffle relations, but which are taken automatically to be transcendent. This algebra, defined in the next section, is the main object of study in the theory of multiple zeta values.

1.2. Formal multiple zeta values

For every word w in x and y, let $\overline{Z}(w)$ denote a formal symbol associated to w, and let $\mathbb{Q}[\overline{Z}(w)]$ be the commutative $\mathbb{Q}$-algebra generated as a vector space by these symbols, equipped with the multiplication law

$$\overline{Z}(u)\overline{Z}(v) = \overline{Z}\big(sh(u,v)\big). \tag{1.2.1}$$

Let $\mathcal{SH}$ be the quotient of $\mathbb{Q}[\overline{Z}(w)]$ by the linear relations analogous to (1.1.2)

$$\overline{Z}(w)=\sum_{a=0}^{r}\sum_{b=0}^{s}(-1)^{a+b}\overline{Z}\big(\pi(sh(y^a,y^{r-a}vx^{s-b},x^b))\big) \tag{1.2.2}$$

for every non-convergent word w. As in theorem 1.1, this definition ensures that the multiplication law (1.2.1) passes to the quotient $\mathcal{SH}$. We write $\widetilde{Z}(w)$ for the image of $\overline{Z}(w)$ in $\mathcal{SH}$.

In analogy with (1.1.5), we define $\widetilde{Z}^*(\underbrace{1,\ldots,1}_{n})$ to be the coefficient of y_1^n in the formal power series with coefficients in $\mathcal{SH}$

$$exp\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\widetilde{Z}(y_n)y_1^n\Big),$$

so they are polynomials in the $\widetilde{Z}(y_i)$; note that all polynomials in the $\widetilde{Z}(w)$ can be expressed as linear combinations of convergent multizetas by using the multiplication rule (1.2.1) and then (1.2.2). In analogy with (1.1.7), we set

$$\widetilde{Z}^*(w)=\sum_{j=0}^{i}\widetilde{Z}^*(\underbrace{1,\ldots,1}_{j})\big(\Phi|y_1^{i-j}v\big)=\sum_{j=0}^{i}\widetilde{Z}^*(\underbrace{1,\ldots,1}_{j})\widetilde{Z}(y^{i-j}v), \tag{1.2.3}$$

for every word $w=y_1^iv$ where v is a word in the y_i not starting with y_1; thus these values can also be expressed as linear combinations of convergent $\widetilde{Z}(w)$. Therefore, $\mathcal{SH}$ is generated as a vector space by the $\widetilde{Z}(w)$ for convergent w.

Let $\mathcal{FZ}$, the *formal multizeta algebra*, be the vector space quotient of $\mathcal{SH}$ by the relations

$$\widetilde{Z}^*\big(st(u,v)\big)=\widetilde{Z}^*(u)\widetilde{Z}^*(v),$$

which although they appear algebraic, can be written as above as linear relations between the convergent $\widetilde{Z}(w)$. The multiplication (1.2.1) passes to $\mathcal{FZ}$, making it into a $\mathbb{Q}$-algebra. We write $Z(w)$ for the image of $\widetilde{Z}(w)$ in $\mathcal{FZ}$.

By definition, we have a surjection $\mathcal{FZ}\rightarrow\mathcal{Z}$. But the space $\mathcal{FZ}$ is easier to study than $\mathcal{Z}$ because the real multizeta values satisfy unknown numbers of other relations, including, as explained in 1.1, the fact that it is not even known whether they are transcendent, or whether there are any linear relations between real multizeta values of different weights. It is tempting to conjecture that $\mathcal{FZ}\simeq\mathcal{Z}$, but pending any kind of knowledge about the transcendence properties of real multizeta values, we adopt the strategy of replacing the real value algebra by the formal multizeta algebra $\mathcal{FZ}$ as the main object of study in the combinatorial/algebraic theory of multizetas.

By definition, $\mathcal{FZ}$ is a graded algebra, with $\mathcal{FZ}_0=\mathbb{Q}$, $\mathcal{FZ}_1=0$ and $\mathcal{FZ}_2$ a one-dimensional space generated by $Z(2)=Z(xy)$ (as for real multizetas, we use the notation

$Z(k_1,\ldots,k_r) = Z(x^{k_1-1}y\cdots x^{k_r-1}y))$. Let $\overline{\mathcal{FZ}}$ denote the quotient of $\mathcal{FZ}$ by the ideal generated by $Z(2)$.

The principal result we will need in the present chapter is that $\overline{\mathcal{FZ}}$ is a Hopf algebra, with a coproduct defined by Goncharov. Thus, the quotient nfz of $\mathcal{FZ}$ modulo the ideal generated by $\mathcal{FZ}_0$, $\mathcal{FZ}_2$ and products $\mathcal{FZ}^2_{>0}$, called the *new formal zeta space*, is not just a vector space but actually a Lie coalgebra, with a cobracket inherited from Goncharov's coproduct.

The dual of nfz is thus a Lie algebra, known as the double shuffle Lie algebra $\mathfrak{ds}$; in fact, the proof that $\overline{\mathcal{FZ}}$ is based on a direct proof given by Racinet [R] that $\mathfrak{ds}$ is a Lie algebra. The following section is devoted to a closer study of this Lie algebra, which is one of the main points of focus of the entire theory, thanks to the simplicity of its definition and the concrete nature of its elements, which make it into a valuable and attractive "way in" to the theory, accessible to explicit computation.

1.3. The double shuffle Lie algebra $\mathfrak{ds}$.

Definition. The Lie algebra $\mathfrak{ds}$ is the dual of the Lie coalgebra nfz of new formal multizeta values. It is the set of polynomials $f \in \mathbb{Q}\langle x,y\rangle$ having the two following properties:

(1) The coefficients of f satisfy the *shuffle relations*

$$\sum_{w\in sh(u,v)} (f|w) = 0,$$

where u,v are words in x,y and $sh(u,v)$ is the set of words obtained by shuffling them. This condition is equivalent to the assertion that $f \in \mathrm{Lie}[x,y]$.

(2) Let $f_* = \pi_y(f) + f_{\mathrm{corr}}$, where $\pi_y(f)$ is the projection of f onto just the words ending in y, and

$$f_{\mathrm{corr}} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(f|x^{n-1}y)y^n.$$

(When f is homogeneous of degree n, which we usually assume, then f_{corr} is just the monomial $\frac{(-1)^n}{n}(f|x^{n-1}y)y^n$.) The coefficients of f_* satisfy the *stuffle relations*:

$$\sum_{w\in st(u,v)} (f|w) = 0,$$

where now u, v and w are words ending in y, considered as rewritten in the variables $y_i = x^{i-1}y$, and $st(u,v)$ is the stuffle of two such words.

Definition. Defined as above, $\mathfrak{ds}$ is a vector space. However, Racinet [R] proved that $\mathfrak{ds}$ is actually a Lie algebra under the *Poisson bracket* defined as follows. We first define a derivation D_f of the Lie algebra $\mathrm{Lie}[x,y]$ associated to every $f \in \mathrm{Lie}[x,y]$, by setting

$$D_f(x) = 0, \quad D_f(y) = [y,f]$$

and extending as a derivation.

We then define the Poisson bracket by

$$\{f,g\} = [f,g] + D_f(g) - D_g(f).$$

This definition corresponds naturally to the Lie bracket on the space of derivations of Lie$[x,y]$; it is easy to check that

$$[D_f, D_g] = D_f \circ D_g - D_g \circ D_f = D_{\{f,g\}}.$$

The double shuffle Lie algebra is equipped with an increasing depth filtration

$$\mathfrak{ds}_1 \subset \mathfrak{ds}^2 \subset \cdots$$

where $f \in \mathfrak{ds}$ lies in $\mathfrak{ds}^d$ if the smallest number of y's appearing in any monomial of f is greater than or equal to d. The depth filtration is not a grading because there are known linear combinations of elements of depth d which are themselves in depth $> d$. This filtration is dual to the decreasing filtration on $\mathcal{Z}$ given by letting the depth of $\zeta(k_1, \ldots, k_r)$ be equal to r. It is a filtration rather than a grading since there can be linear relations mixing depths. For example, in weight 3, the shuffle relations are

$$\begin{cases} Z(x)Z(xx) = 3Z(xxx) \\ Z(x)Z(xy) = 2Z(xxy) + Z(xyx) \\ Z(x)Z(yx) = Z(xyx) + 2Z(yxx) \\ Z(x)Z(yy) = Z(xyy) + Z(yxy) + Z(yyx) \\ Z(y)Z(xx) = Z(yxx) + Z(xyx) + Z(xxy) \\ Z(y)Z(xy) = Z(yxy) + 2Z(xyy) \\ Z(y)Z(yx) = 2Z(yyx) + Z(yxy) \\ Z(y)Z(yy) = 3Z(yyy), \end{cases}$$

so in particular $Z(1,2) = Z(y,x,y) = -2Z(xyy) = -2Z(2,1)$ since $Z(x) = Z(y) = 0$. The stuffle relations are

$$Z^*(1)Z^*(2) = Z^*(1,2) + Z^*(2,1) + Z^*(3) = Z(1,2) + Z^*(1)Z(2) + Z(2,1) + Z(3) =$$

$$Z(1,2) + Z(2,1) + Z(3) = 0$$

and

$$Z^*(1)Z^*(1,1) = 3Z^*(1,1,1) + Z^*(2,1) + Z^*(1,2) = Z(3) + Z(2,1) + Z(1,2) = 0,$$

so the two stuffle relations are equivalent, and using the identity $Z(1,2) = -2Z(2,1)$ coming from shuffle, we obtain $Z(2,1) = Z(3)$. This relation, already known to Euler, is the first relation which mixes depths.

1.4. Properties of the double shuffle Lie algebra

There are several conjectures about the structure of the double shuffle Lie algebra $\mathfrak{ds}$, and a certain number of results have already been proven. Recall that the depth filtration is increasing, $\mathfrak{ds}^1 \subset \mathfrak{ds}^2 \subset \cdots$, and that it is not a grading since there can be linear combinations of depth d elements which are of depth greater than d. We will see examples of this in 3.3 below. The associated graded is $\mathrm{gr}\,\mathfrak{ds} = \oplus_{d\geq 1}\mathfrak{ds}^d/\mathfrak{ds}^{d+1}$. This is a doubly-graded vector space since it inherits the weight grading, so it is convenient to break it up by weight and study the finite-dimensional weight-graded pieces $(\mathrm{gr}\,\mathfrak{ds})_n^d = \mathfrak{ds}_n^d/\mathfrak{ds}_n^{d+1}$. The results on $\mathfrak{ds}$ and $\mathrm{gr}\,\mathfrak{ds}$ that we survey here, to approach them again later using Ecalle's theory, are the following.

(1) $\mathfrak{ds}$ is a Lie algebra.

(2) We have $\dim \mathfrak{ds}_n^d/\mathfrak{ds}_n^{d+1} = 0$ if $n \not\equiv d \bmod 2$.

(3) For $n \geq 3$, we have

$$\begin{aligned} \dim \mathfrak{ds}_n^1/\mathfrak{ds}_n^2 &= 1 && \text{if } n \text{ is odd} \\ \dim \mathfrak{ds}_n^2/\mathfrak{ds}_n^3 &= \Big[\frac{n-2}{6}\Big] && \text{if } n \text{ is even} \\ \dim \mathfrak{ds}_n^3/\mathfrak{ds}_n^4 &\geq \Big[\frac{(n-3)^2-1}{48}\Big] && \text{if } n \text{ is odd.} \end{aligned}$$

(4) For a homogeneous polynomial $f \in \mathfrak{ds}_n$, let d be the depth of f, i.e. the minimal number of y's occurring in any monomial of f, and let f^d denote the depth d part of f. Then f has the *cyclic permutation invariance property*, i.e. for every depth d word $x^{r_0}y\cdots yx^{r_d}$,

$$\big(f \,\big|\, x^{r_0}yx^{r_1}y\cdots yx^{r_d}\big) = \big(f \,\big|\, x^{r_d}yx^{r_0}y\cdots yx^{r_{d-1}}\big).$$

(5) Every element $f \in \mathfrak{ds}_n$ satisfies an analogous but more complicated property of cyclic invariance for the words in any depth: writing $f = f_x x + f_y y$, this property is best expressed by the statement

$$R_X(f_x + f_y) = (-1)^n(f_x + f_y),$$

where R_X denotes the "backwards writing" automorphism of $\mathbb{Q}\langle x, y\rangle$ that takes any word in x and y to its mirror image written from right to left.

COMMENT on the higher cyclic property

Remarks. The first statement implies that there exists a double shuffle element f_i of homogeneous weight n and depth 1 for each odd $n \geq 3$. For the second case, we actually have more precise knowledge of the situation. If we choose depth 1 elements $f_3, f_5, \ldots$ as allowed by the first statement, then it is known that $\mathfrak{ds}_n^2$ is generated by the $\big[\frac{n-4}{4}\big]$ Poisson brackets $\{f_i, f_{n-i}\}$. Furthermore, the exact relations between these generators in the quotient $\mathfrak{ds}_n^2/\mathfrak{ds}_n^3$ are known (period polynomial relations). For the third statement, the inequality is claimed by IKZ (but not proved there). They mention in a footnote that Goncharov has proved the actual equality in this case. This dimension corresponds to the more precise conjecture that $\mathfrak{ds}_n^3/\mathfrak{ds}_n^4$ is generated by all brackets $\{f_i,\{f_j,f_k\}\})$ with $i+j+k=n$, modulo the only relations $\{f_i,R\}$ where R denotes one of the explicitly known period polynomial relations in depth 2.

Further results on $\mathfrak{ds}$ concern its relation to other Lie algebras. The main structural conjecture on $\mathfrak{ds}$ is the following.

Conjecture 1. *Let $\mathcal{F}$ be the free weight-graded Lie algebra $\mathcal{F}=\mathbb{Q}[s_3,s_5,\ldots]$ generated by one element in each odd weight ≥ 3. Then $\mathfrak{ds}$ is isomorphic to $\mathcal{F}$.*

This conjecture is related to the possibility of choosing depth 1 elements $f_3,f_5,\ldots$ in $\mathfrak{ds}$ as in (1) above. Each such choice gives a map $\mathcal{F}\rightarrow\mathfrak{ds}$ via $s_n\mapsto f_n$ and one conjectures that all such maps are isomorphisms. In the direction of conjecture 1, we do have the following fundamental inequality, whose proof is due to Goncharov.

(3) It is known that $\dim\mathcal{UF}_n=d_n$ where d_n is the sequence defined recursively by $d_0=1$, $d_1=0$, $d_2=1$ and $d_i=d_{i-2}+d_{i-3}$. Goncharov has proved using the theory of mixed Tate motives that $\dim\mathcal{U}\mathfrak{ds}\leq d_n$ for all n.

Remark. Goncharov's proof yields as a corollary that although a canonical choice of generators $f_3,f_5,\ldots$ is not known, there is a canonical image of $\mathcal{F}\rightarrow\mathfrak{ds}$. Ecalle, on the other hand, gives two ways of choosing systems $f_3,f_5,\ldots$ with good properties (kwa and kya). Neither of his systems appears canonical, and neither has the Bernoulli properties, but it is an interesting direction. Ecalle appears to have a third way of obtaining canonical generators f_i, independent of kwa and kya, which is the construction of *loma*. More strongly, *Ecalle claims to have proved conjecture 1.*

Conjecture 1 can be made more precise (see 3.3), namely one can equip $\mathcal{F}$ with a certain increasing depth filtration, the *special depth filtration* $\mathcal{F}^1\subset\mathcal{F}^2\subset\cdots$ which is conjectured to correspond to that of $\mathfrak{ds}$, and to yield the Broadhurst-Kreimer dimensions.

Refined conjecture 1. *Let $f_3,f_5,\ldots$ be any choice of depth 1 elements in $\mathfrak{ds}$. Then the Lie algebra map $s_n\mapsto f_n$ defines a Lie algebra isomorphism $\mathcal{F}\xrightarrow{\sim}\mathfrak{ds}$ such that $\mathcal{F}_n^d\xrightarrow{\sim}\mathfrak{ds}_n^d$ for each $n\geq 3$, $d\geq 1$.*

Proving statements analogous to (1) and (2) for $\mathcal{F}$ then means that (1) and (2) for $\mathfrak{ds}$ actually provide evidence for the refined version of conjecture 1. And indeed, we do have such statements, with equality in the case of depth 3 mod depth 4.

(4) For $n\not\equiv d$ mod 2, we have

$$\dim\mathcal{F}_n^d/\mathcal{F}_n^{d+1}=0,$$

and for $n \geq 3$, we have

$$\dim \mathcal{F}_n^1/\mathcal{F}_n^2 = 1 \qquad \text{if } n \text{ is odd}$$
$$\dim \mathcal{F}_n^2/\mathcal{F}_n^3 = [\frac{n-2}{6}] \qquad \text{if } n \text{ is even}$$
$$\dim \mathcal{F}_n^3/\mathcal{F}_n^4 = [\frac{(n-3)^2-1}{48}] \qquad \text{if } n \text{ is odd.}$$

The next set of known results on double shuffle concern its relationship with the Lie algebras $\mathfrak{grt}$ and kv (add definitions).

Conjecture 2. *$\mathfrak{ds} \simeq \mathfrak{grt}$.*

Conjecture 3. *$\mathfrak{ds} \simeq kv$.*

Partial results in the direction of these two conjectures are as follows.

(5) $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$ (Furusho);

(5) Double shuffle implies relation (I) of $\mathfrak{grt}$ (and maybe also relation (II)), Ecalle claim;

(7) Elements of $\mathfrak{ds}$ act on Lie$[x, y]$ as special automorphisms (in progress, [LS] following Ecalle);

(8) A property of double shuffle elements related to kv, namely the generalization of Ihara's theorem to TR, cf. [CS].

To conclude this brief survey of known results on $\mathfrak{ds}$, we add two symmetry properties of the elements of $\mathfrak{ds}$, which are useful in some of the proofs of the statements above.

(9) If f is a double shuffle element of depth d and f_d denotes the lowest depth part of f, then f_d is invariant under the $(d+1)$-cycle acting by cyclically permuting the power of x in the monomials $x^{d_0}y \ldots yx^{d_r}$ of f_d. It seems that this is the same (proof?) as Ecalle's "push-invariance of al/al", which is easy to prove. Ecalle generalizes this property to a symmetry property of double shuffle elements in all depth ("pushu-invariance").

(10) Ecalle proves the useful "cyclic property": let f be a polynomial in the y_i of homogeneous weight n satisfying stuffle, let $w = y_{i_1} \ldots y_{i_r}$ be a word, and let $\sigma^i(w)$ for $i = 1, \ldots, r$ be the r words obtained by applying the powers of the r-cycle permutation to the y_j in w. Then

$$\sum_{i=1}^{r} \big(f|\sigma^i(w)\big) = (-1)^{r-1}(f|y_n).$$

Chapter 2

The Lie algebra ARI

2.1. Moulds and bimoulds

Ecalle defines moulds on an arbitrary additive alphabet X, i.e. on a semigroup generated by the letters in a given alphabet X. In this book we will restrict ourselves to the case of moulds known as "bimoulds". These moulds are functions of two infinite sets of variables: $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$, which are considered to generate a ring within which all rational expressions in the variables make sense (and in whose completion, even power series in the variables make sense).

A *bimould* M is then a collection of functions

$$M\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix}$$

for each $r \geq 0$. These functions are arbitrary, but later, in the context of the study of multizeta values, we will restrict our attention to rational functions, polynomials, and constants.

Moulds on the same alphabet can be added, multiplied and, if $N^{\emptyset} = 0$, composed. Writing $w_i = \binom{u_i}{v_i}$ (or considering the variables w_i as belonging to an arbitrary alphabet), we have

$$\begin{aligned} (M+N)(w_1,\ldots,w_r) &= M(w_1,\ldots,w_r) + N(w_1,\ldots,w_r) \\ mu(M,N)(w_1,\ldots,w_r) &= \sum_{0\leq i\leq r} M(w_1,\ldots,w_i)N(w_{i+1},\ldots,w_r) \\ (M\circ N)(w_1,\ldots,w_r) &= \sum_{\substack{\mathbf{w}=\mathbf{w^1}\cdots\mathbf{w^s} \\ \mathbf{w}^i\neq\emptyset}} M(||\mathbf{w}^1||,\cdots,||\mathbf{w}^s||)N(\mathbf{w}^1)\cdots N(\mathbf{w}^s). \end{aligned} \tag{2.1.1}$$

Here, $||(w_1,\ldots,w_r)||$ denotes the single-letter word $w_1 + \cdots + w_r$, which is $\binom{u_1+\cdots+u_r}{v_1+\cdots+v_r}$ in the bimould case.

Remark. Moulds are generalizations of power series. If a mould M takes constant values on each word, then it can be identified with the power series

$$M = \sum_{(w_1,\ldots,w_r)} M(w_1,\ldots,w_r)w_1\cdots w_r.$$

Exercise. Check that in the power series case, the rules for addition, multiplication and composition are just the usual ones.

Examples. (1) The first examples are the Log and Exp moulds given by $Exp(\emptyset) = Log(\emptyset) = 0$,

$$\begin{cases} Log(w_1, \ldots, w_r) = \frac{(-1)^{r+1}}{r} \\ Exp(w_1, \ldots, w_r) = \frac{1}{r!}. \end{cases}$$

(2) The identity mould for multiplication $\mathbf{1}$ is given by $\mathbf{1}(\emptyset) = 1$ and all other values are 0. The identity mould $\mathbf{Id}$ for composition is given by

$$\mathbf{Id}(w_1, \ldots, w_r) = \begin{cases} 0 & \text{for } r = 0 \text{ and all } r > 1 \\ 1 & \text{for } r = 1. \end{cases}$$

Exercise. Show that Exp is the mould corresponding to the power series $e^t - t$ and Log to $\log(1+t)$. Show that as expected, $Exp \circ Log = \mathbf{Id}$.

2.2. The Lie algebra ARI

Let us define ARI as the full set of moulds on the fixed alphabet X satisfying $A(\emptyset) = 0$. In this book, as before, we restrict ourselves to the bimould situation with the alphabet made up of the double set of variables u_i and v_i, identifying w_i with $\binom{u_i}{v_i}$.

The vector space ARI is obviously a Lie algebra under the Lie bracket $limu$ defined by $limu(A, B) = mu(A, B) - mu(B, A)$. But Ecalle introduces an alternative bracket, the ari bracket, on ARI, making the same space into a different Lie algebra. In chapter 3, we will explore the precise analogy between what Ecalle does and the two types of Lie bracket on the free Lie algebra $\mathrm{Lie}[x, y]$.

Flexions. For every possible way of cutting the word $\mathbf{w}$ into three (possibly empty) subwords $\mathbf{w} = \mathbf{abc}$ with

$$\mathbf{a} = \binom{u_1^a, \ldots, u_k^a}{v_1^a, \ldots, v_k^a}, \quad ,\mathbf{b} = \binom{u_1^b, \ldots, u_l^b}{v_1^b, \ldots, v_l^b}, \quad ,\mathbf{c} = \binom{u_1^c, \ldots, u_m^c}{v_1^c, \ldots, v_m^c},$$

set

$$\begin{cases} \lceil \mathbf{c} = \mathbf{c} & \text{if } \mathbf{b} = \emptyset \\ \mathbf{a} \rceil = \mathbf{a} & \text{if } \mathbf{b} = \emptyset \\ \mathbf{b} \rfloor = \mathbf{b} & \text{if } \mathbf{c} = \emptyset \\ \lfloor \mathbf{b} = \mathbf{b} & \text{if } \mathbf{a} = \emptyset, \end{cases}$$

otherwise

$$\begin{cases} \lceil \mathbf{c} = \begin{pmatrix} u_1^b + \cdots + u_l^b + u_1^c & u_2^c & \cdots & u_m^c \\ v_1^c & v_2^c & \cdots & v_m^c \end{pmatrix} & \text{if } \mathbf{b} \neq \emptyset \\ \mathbf{b} \rfloor = \begin{pmatrix} u_1^b & u_2^b & \cdots & u_l^b \\ v_1^b - v_1^c & v_2^b - v_1^c & \cdots & v_l^b - v_1^c \end{pmatrix} & \text{if } \mathbf{c} \neq \emptyset \\ \mathbf{a} \rceil = \begin{pmatrix} u_1^a & u_2^a & \cdots & u_l^a + u_1^b + \cdots + u_l^b \\ v_1^a & v_2^a & \cdots & v_k^a \end{pmatrix} & \text{if } \mathbf{b} \neq \emptyset \\ \lfloor \mathbf{b} = \begin{pmatrix} u_1^b & u_2^b & \cdots & u_l^b \\ v_1^b - v_k^a & v_2^b - v_k^a & \cdots & v_l^b - v_k^a \end{pmatrix} & \text{if } \mathbf{a} \neq \emptyset. \end{cases}$$

To every bimould B, associate a derivation[1] $arit(B)$ of ARI (for the *limu* bracket) defined by

$$\big(arit(B)\cdot A\big)(w) = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}). \tag{2.2.1}$$

Define a "pre-Lie" operation on ARI by

$$preari(A,B)(w) = \big(arit(B)\cdot A + mu(A,B)\big)(w) = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}), \tag{2.2.2}$$

Then the ARI bracket is defined by the formula

$$ari(A,B) = preari(A,B) - preari(B,A),$$

so it is given explicitly by the formula

$$ari(A,B) = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b}\neq\emptyset}} \Big(A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor)\Big) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} \Big(A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}) - B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})\Big). \tag{2.2.3}$$

Notice that we then have the "Poisson bracket" type identity

$$ari(A,B) = arit(B)\cdot A - arit(A)\cdot B + limu(A,B) \tag{2.2.4}$$

(cf. *ARI/GARI et la décomposition des multizêtas en irréductibles*, p. 28 (75) and p. 29 (84)).

2.3. Symmetries alternal and alternil

For the study of multizeta values, Ecalle introduces two fundamental symmetries that bimoulds in ARI can have: alternal and alternil. The terms

The first one is based on the well-known definition of the shuffle product of two words $\mathbf{w}'$ and $\mathbf{w}''$ in an alphabet X, given by the recursive formula $sh(w,1) = sh(1,w) = w$ and $sh(xu,yv) = x\,sh(u,yv) + y\,sh(xu,v)$ for any letters x,y and words u,v in the alphabet X. Here we use the notation in which the shuffle or the stuffle of two words is written as a formal sum of words, for example on the alphabet $X = \{a,b,c,d\}$ we write

$$sh((ab),(cd)) = abcd + acbd + acdb + cabd + cadb + cdab,$$

and on the alphabet $X = \{x,y\}$, we write

$$sh((x,y),(x,y)) = 4xxyy + 2xyxy.$$

[1] The proof of this fact is given in the appendix.

Note that to verify whether a mould satisfies shuffle, i.e. satisfies the full set of shuffle relations, it is only necessary to check the relations

$$sh\big((w_1,\ldots,w_s),(w_{s+1},\ldots,w_r)\big) \quad 1\le s\le \Big[\frac{r}{2}\Big],$$

since all other shuffle relations can be obtained from these by substitution of variables.

Definition. A mould given by its coefficients $M(w_1,\ldots,w_r)$ (which can be a bimould if each $w_i = (\begin{smallmatrix} u_i \\ v_i \end{smallmatrix})$) is *alternal* if

$$M\big(sh(\mathbf{w}',\mathbf{w}'')\big)=0,$$

for all pairs of non-trivial words $\mathbf{w}',\mathbf{w}''$.

Example. Let $M\in ARI$ be the mould given by

$$\begin{cases} M(\emptyset)=0 \\ M\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = u_1^2 \\ M\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = -u_1+u_2 \\ M\begin{pmatrix} u_1 & \cdots & u_r \\ v_1 & \cdots & v_r \end{pmatrix} = 0 \qquad \text{if } r>2. \end{cases}$$

Then

$$M\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} + M\begin{pmatrix} u_2 & u_1 \\ v_2 & v_1 \end{pmatrix} = 0,$$

so M is alternal.

The second symmetry, alternil, is related to the stuffle product of words in an additive alphabet X. The stuffle product is given recursively by $st(w,1)=st(1,w)=w$, $st(y_iu,y_jv)=y_i\,st(u,y_jv)+y_j\,st(y_iu,v)+y_{i+j}\,st(u,v)$. Some examples for different alphabets (in the first one, we write $a+b$ for the sum of two letters, while in the third we write y_{i+j} for the sum of y_i and y_j; these are merely differences of notation for the sum):

$$\begin{aligned} st(a,b) &= (a,b)+(b,a)+(a+b) \\ st((a,b),(c)) &= abc+acb+cab+(a+b,c)+(a,b+c) \\ st((a,b),(b)) &= 2(a,b,b)+(b,a,b)+(a+b,b)+(a,2b) \\ st((y_1),(y_2,y_3)) &= (y_1,y_2,y_3)+(y_2,y_1,y_3)+(y_2,y_3,y_1)+(y_3,y_3)+(y_2,y_4) \\ st((2,1),(2)) &= 2(2,2,1)+(2,1,2)+(4,1)+(2,3). \end{aligned} \tag{2.3.1}$$

Note that as above, in order for a mould to satisfy the full set of stuffle relations, it is only necessary to check the relations

$$st\big((w_1,\ldots,w_s),(w_{s+1},\ldots,w_r)\big),\quad 1\le s\le \Big[\frac{r}{2}\Big],$$

since all other stuffle relations can be obtained from these by substitution.

The alternil symmetry is defined only for bimoulds*. The alternility relations are defined from the stuffle relations by replacing every term containing a contraction $M\binom{\dots,u_i+u_j,\dots}{\dots,v_i+v_j,\dots}$ with the following sum of two terms:

$$\frac{1}{v_i-v_j}M\begin{pmatrix}\dots,u_i+u_j,\dots\\ \dots,v_i,\dots\end{pmatrix}+\frac{1}{v_j-v_i}M\begin{pmatrix}\dots,u_i+u_j,\dots\\ \dots,v_j,\dots\end{pmatrix}. \tag{2.3.2}$$

Thus for example, the alternility condition in depth 2 is given by

$$0=M\begin{pmatrix}u_1&u_2\\ v_1&v_2\end{pmatrix}+M\begin{pmatrix}u_2&u_1\\ v_2&v_1\end{pmatrix}+\frac{1}{v_1-v_2}M\begin{pmatrix}u_1+u_2\\ v_1\end{pmatrix}+\frac{1}{v_2-v_1}M\begin{pmatrix}u_1+u_2\\ v_2\end{pmatrix}, \tag{2.3.3}$$

and one of the two depth 3 conditions is given by

$$\begin{aligned}0=M\begin{pmatrix}u_1&u_2&u_3\\ v_1&v_2&v_3\end{pmatrix}&+M\begin{pmatrix}u_2&u_1&u_3\\ v_2&v_1&v_3\end{pmatrix}+M\begin{pmatrix}u_2&u_3&u_1\\ v_2&v_3&v_1\end{pmatrix}\\ &+\frac{1}{v_1-v_2}M\begin{pmatrix}u_1+u_2&u_3\\ v_1&v_3\end{pmatrix}+\frac{1}{v_2-v_1}M\begin{pmatrix}u_1+u_2&u_3\\ v_2&v_3\end{pmatrix}\\ &+\frac{1}{v_1-v_3}M\begin{pmatrix}u_2&u_1+u_3\\ v_2&v_1\end{pmatrix}+\frac{1}{v_3-v_1}M\begin{pmatrix}u_2&u_1+u_3\\ v_2&v_3\end{pmatrix}\end{aligned} \tag{2.3.4}$$

(compare these with the first and fourth equations of (2.3.1)). Note that in fact, the second equation defining the alternility condition in depth 3 is given by taking the product

$$M\begin{pmatrix}u_1&u_2\\ v_1&v_2\end{pmatrix}M\begin{pmatrix}u_3\\ v_3\end{pmatrix},$$

but because these products are commutative (product of functions in u_i, v_i),

Exercise. Write the alternility conditions for $r=4$. There are two conditions, one coming from $st\big((w_1),(w_2,w_3,w_4)\big)$ and the other from $st\big((w_1,w_2),(w_3,w_4)\big)$.

Remark. (1) If M is a constant (i.e. constant-valued) mould, then the alternil conditions reduce to the shuffle together with the collection of equalities

$$M\begin{pmatrix}\dots&u_i+u_j&\dots\\ \dots&v_i&\dots\end{pmatrix}=M\begin{pmatrix}\dots&u_i+u_j&\dots\\ \dots&v_j&\dots\end{pmatrix}.$$

(2) If M is a polynomial-valued mould, then the left-hand sides of the alternil conditions are polynomials. To see this, it suffices to note that setting $v_i=v_j$ in the term

$$\frac{1}{v_i-v_j}M\begin{pmatrix}\dots&u_i+u_j&\dots\\ \dots&v_i&\dots\end{pmatrix}+\frac{1}{v_j-v_i}M\begin{pmatrix}\dots&u_i+u_j&\dots\\ \dots&v_j&\dots\end{pmatrix},$$

* See *ARI/GARI la dimorphie...*, p. 417-418.

it is clearly equal to zero.

2.4. Special subspaces of ARI

There are many interesting subspaces of ARI. In this section, we briefly look at three of the first ones that are relevant in the theory of multizeta values: the two most important ones for that theory, $ARI_{al/al}$ and $ARI_{al/il}$, are studied in the following sections.

• ARI^{ent}, the subspace of polynomial-valued bimoulds in variables u_i, v_i;

• ARI^{u-var} (resp. ARI^{v-var}), the subspace of bimoulds whose values are functions only in the variables u_i (v_i).

• ARI_{al}, the subspace of alternal bimoulds.

Proposition 2.4.1. *ARI^{ent} and ARI^{u-var} are Lie algebras under the ari bracket.*

This statement follows immediately from the fact that the ari bracket preserves polynomial expressions and expressions in the u_i only.

For our study of multizeta values in the context of mould calculus, it is useful to introduce the notion of homogeneous weight for polynomial-valued moulds that are functions of the u_i only.

Definition. For each $n \geq 1$, a polynomial-in-u_i-valued bimould M is said to be *homogeneous in weight* n if there exists an integer $n \geq 1$ such that $M\begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix}$ is a homogeneous polynomial in $u_1, \dots, u_r$ of degree $n - r$. In particular, such a mould is 0-valued for $r > n$. The homogeneous weight puts a grading on the Lie algebra $ARI^{ent,u-var}$.

Proposition 2.4.2. *ARI_{al} is a Lie algebra under the ari bracket.*

Proof. We saw in (2.2.4) that

$$ari(A, B) = arit(B) \cdot A - arit(A) \cdot B + limu(A, B),$$

where $arit(B)$ and $arit(A)$ denote derivations of ARI viewed as a Lie algebra under $limu$. Let us show that if A and B are alternal, then $mu(A, B)$ is alternal. Since alternality is additive, we may assume that A is concentrated in depth i and B in depth $n-i$; for every fixed pair of non-trivial words (u,v) of lengths r and s with $n=r+s$, we have

$$\sum_{w\in sh(u,v)} mu(A,B)(w) = \sum_{w\in sh(u,v)} A(w_1,\ldots,w_i)B(w_{i+1},\ldots,w_n)$$

$$= \sum_{j=0}^{i} A\big(sh((u_1,\ldots,u_j),(v_1,\ldots,v_{i-j}))\big)B\big(sh((u_{j+1},\ldots,u_r),(v_{i-j+1},\ldots,v_s))\big)$$

$$= \sum_{j=1}^{i-1} A\big(sh((u_1,\ldots,u_j),(v_1,\ldots,v_{i-j}))\big)B\big(sh((u_{j+1},\ldots,u_r),(v_{i-j+1},\ldots,v_s))\big)$$

$$+ A\big(v_1,\ldots,v_i\big)B\big(sh((u_1,\ldots,u_r),(v_{i+1},\ldots,v_s))\big)$$

$$+ A\big(u_1,\ldots,u_i\big)B\big(sh((u_{i+1},\ldots,u_r),(v_1,\ldots,v_s))\big).$$

Now, because both A and B are alternal, every term in this expression will be equal to zero unless $i=r$ and $n-i=s$. In this case, the above yields

$$\sum_{w\in sh(u,v)} mu(A,B) = A\big(v_1,\ldots,v_r\big)B\big(sh((u_1,\ldots,u_r),(v_{r+1},\ldots,v_s))\big)$$

$$+ A\big(u_1,\ldots,u_r\big)B\big(v_1,\ldots,v_s\big)$$

$$= \begin{cases} A(u_1,\ldots,u_r)B(v_1,\ldots,v_s) & r\neq s \\ A(u_1,\ldots,u_r)B(v_1,\ldots,v_s)+A(v_1,\ldots,v_s)B(u_1,\ldots,u_r) & r=s. \end{cases}$$

In both cases, we obviously have $\sum_{w\in sh(u,v)} limu(A,B) = 0$, so we have $limu(A,B)\in ARI_{al}$. It remains to show that if $A,B\in ARI_{al}$, then $arit(B)\cdot A$ is alternal, which is done in the appendix. ♢

2.5. The Lie algebra $\mathbf{ARI}_{\underline{al}/\underline{al}}$

Let *push*, *neg*, *anti*, *swap* and *mantar* be the operators on moulds defined by

$$push(M)\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = M\begin{pmatrix} -u_1-\cdots-u_r & u_1 & \cdots & u_{r-1} \\ -v_r & v_1-v_r & \cdots & v_{r-1}-v_r \end{pmatrix}$$

$$neg(M)\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = M\begin{pmatrix} -u_1 & -u_2 & \cdots & -u_r \\ -v_1 & -v_2 & \cdots & -v_r \end{pmatrix}$$

$$anti(M)\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = M\begin{pmatrix} u_r & u_{r-1} & \cdots & u_1 \\ v_r & v_{r-1} & \cdots & v_1 \end{pmatrix}$$

$$swap(M)\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = M\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_2-v_3 & v_1-v_2 \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+u_2 & u_1 \end{pmatrix}$$

$$mantar(M)\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = (-1)^{r-1}M\begin{pmatrix} u_r & \cdots & u_1 \\ v_r & \cdots & v_1 \end{pmatrix}.$$

Definition. $ARI_{al/al}$ is the vector subspaces of bimoulds $A\in ARI$ such that A is alternal and $swap(A)$ is also alternal.

Lemma 2.5.1. *We have $neg \circ push = anti \circ swap \circ anti \circ swap$.*

We omit the proof of this lemma, which is obtained simply by direct composition of the variable changes (or multiplication of matrices, if the operators in depth r are written as $2r \times 2r$ matrices).

Lemma 2.5.2. *If $A \in ARI_{al}$, then*

$$anti(A)(w_1, \ldots, w_r) = (-1)^{r-1} A(w_1, \ldots, w_r),$$

in other words, A is mantar-invariant.

Proof. We first show that the sum of shuffle relations

$$sh\big((1), (2, \ldots, r)\big) - sh\big((2, 1), (3, \ldots, r)\big) + sh\big((3, 2, 1), (4, \ldots, r)\big) + \cdots$$
$$+(-1)^{r-1} sh\big((r-1, \ldots, 2, 1), (r)\big) = (1, \ldots, r) + (-1)^{r-1}(r, \ldots, 1).$$

Indeed, using the recursive formula for shuffle, we can write the above sum with two terms for each shuffle, as

$$\begin{aligned}
(1, \ldots, r) &+ 2 \cdot sh\big((1), (3, \ldots, r)\big) \\
&- 2 \cdot sh\big((1), (3, \ldots, r)\big) - 3 \cdot sh\big((2, 1), (4, \ldots, r)\big) \\
&+ 3 \cdot sh\big((2, 1), (4, \ldots, r)\big) + 4 \cdot sh\big((3, 2, 1), (5, \ldots, r)\big) \\
&+ \cdots + (-1)^{r-2}(r-1) \cdot sh\big((r-2, \ldots, 1), (r)\big) \\
&+ (-1)^{r-1}(r-1) \cdot sh\big((r-2, \ldots, 1), (r)\big) + (-1)^{r-1}(r, r-1, \ldots, 1) \\
&= (1, \ldots, r) + (-1)^{r-1}(r, \ldots, 1).
\end{aligned}$$

Using this, we conclude that if A satisfies the shuffle relations, then

$$A(w_1, \ldots, w_r) + (-1)^{r-1} A(w_r, \ldots, w_1),$$

which is the desired result. ♢

Lemma 2.5.3. *$ARI_{al/al}$ is $(neg \circ push)$-invariant.*

Proof. Using lemmas 2.5.1 and 2.5.2, we have

$$\begin{aligned}
neg \circ push(A)(w_1, \ldots, w_r) &= anti \circ swap \circ anti \circ swap(A)(w_1, \ldots, w_r) \\
&= (-1)^{r-1} anti \circ swap \circ swap(A)(w_1, \ldots, w_r) \\
&= (-1)^{r-1} anti(A)(w_1, \ldots, w_r) \\
&= A(w_1, \ldots, w_r),
\end{aligned}$$

which proves the result. ♢

Definition. Let $ARI_{\underline{al}/\underline{al}}$ denote the subspace of $ARI_{al/al}$ of moulds A such that $A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ is an even function, i.e. $A\begin{pmatrix} -u_1 \\ -v_1 \end{pmatrix} = A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$.

Proposition 2.5.4. *$ARI_{\underline{al}/\underline{al}}$ is neg-invariant and push-invariant.*

Proof. Let $A \in ARI_{al/al}$. By additivity, we may assume that A is concentrated in a fixed depth d, meaning that $A(w_1, \ldots, w_r) = 0$ for all $r \neq d$. Since $A = neg \circ push(A)$ by lemma 2.5.3, we have $neg(A) = push(A)$, so we only need to show that $neg(A) = A$, in other words that $A(w_1, \ldots, w_d)$ is an even function. For $d = 1$, this comes from the assumption on A. If $d = 2s$ is even, then since neg is of order 2 and commutes with $push$ and $push$ is of order $d + 1 = 2s + 1$, we have

$$A = (neg \circ push)^{2s+1}(A) = neg^{2s+1}(A) = neg(A).$$

If $d = 2s + 1$ is odd, we can write A as a sum of an even and an odd part

$$A = \frac{1}{2}\big(A(w_1, \ldots, w_d) + A(-w_1, \ldots, -w_d)\big) + \frac{1}{2}\big(A(w_1, \ldots, w_d) - A(-w_1, \ldots, -w_d)\big),$$

so we may assume that $A(w_1, \ldots, w_d)$ is odd, i.e. $neg(A) = -A$. Then, since A is alternal, using the shuffle $sh\big((w_1, \ldots, w_{2s})(w_{2s+1})\big)$, we have

$$\sum_{i=0}^{2s} A(w_1, \ldots, w_i, w_{2s+1}, w_{i+1}, \ldots, w_{2s}) = 0.$$

Making the variable change $w_0 \leftrightarrow w_{2s+1}$ gives

$$\sum_{i=0}^{2s} A(w_1, \ldots, w_i, w_0, w_{i+1}, \ldots, w_{2s}) = 0,$$

which we write out as

$$\sum_{i=0}^{2s} A\begin{pmatrix} u_1 & \ldots & u_i & u_0 & u_{i+1} & \ldots & u_{2s} \\ v_1 & \ldots & v_i & v_0 & v_{i+1} & \ldots & v_{2s} \end{pmatrix} = 0. \tag{2.5.1}$$

Now consider the shuffle relation $sh((w_1)(w_2, \ldots, w_{2s+1}))$, which gives

$$\sum_{i=1}^{2s+1} A(w_2, \ldots, w_i, w_1, w_{i+1}, \ldots, w_{2s+1}) = 0. \tag{2.5.2}$$

Set $u_0 = -u_1 - \cdots - u_{2s+1}$. Since $neg \circ push$ acts like the identity on A, we can apply it to each term of (2.5.2) to obtain

$$\sum_{i=1}^{2s} -A\begin{pmatrix} u_0 & u_2 & \ldots & u_i & u_1 & u_{i+1} & \ldots & u_{2s} \\ v_{2s+1} & v_2 - v_{2s+1} & \ldots & v_i - v_{2s+1} & v_1 - v_{2s+1} & v_{i+1} - v_{2s+1} & \ldots & v_{2s} - v_{2s+1} \end{pmatrix}$$

$$-A\begin{pmatrix} u_0 & u_2 & \dots & u_{2s} & u_{2s+1} \\ -v_1 & v_2-v_1 & \dots & v_{2s}-v_1 & v_{2s+1}-v_1 \end{pmatrix} = 0.$$

We apply $neg\circ push$ again to the final term of this sum in order to get the u_{2s+1} and v_{2s+1} to disappear, obtaining

$$\sum_{i=1}^{2s} -A\begin{pmatrix} u_0 & u_2 & \dots & u_i & u_1 & u_{i+1} & \dots & u_{2s} \\ -v_{2s+1} & v_2-v_{2s+1} & \dots & v_i-v_{2s+1} & v_1-v_{2s+1} & v_{i+1}-v_{2s+1} & \dots & v_{2s}-v_{2s+1} \end{pmatrix}$$

$$+A\begin{pmatrix} u_1 & u_0 & u_2 & \dots & u_{2s-1} & u_{2s} \\ v_1-v_{2s+1} & -v_{2s+1} & v_2-v_{2s+1} & \dots & v_{2s-2}-v_{2s-1} & v_{2s-1}-v_{2s} \end{pmatrix} = 0.$$

Making the variable changes $u_0 \leftrightarrow u_1$ and $v_1 \mapsto v_0 - v_1$, $v_i \mapsto v_i - v_1$ for $2 \le i \le 2s$, $v_{2s+1} \mapsto -v_1$ in this identity yields

$$\sum_{i=1}^{2s} -A\begin{pmatrix} u_1 & u_2 & \dots & u_i & u_0 & u_{i+1} & \dots & u_{2s} \\ v_1 & v_2 & \dots & v_i & v_0 & v_{i+1} & \dots & v_{2s} \end{pmatrix} + A\begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_{2s-1} & u_{2s} \\ v_0 & v_1 & v_2 & \dots & v_{2s-1} & v_{2s} \end{pmatrix} = 0. \tag{2.5.3}$$

Finally, adding (2.5.1) and (2.5.3) yields

$$2A\begin{pmatrix} u_0 & u_1 & \dots & u_{2s} \\ v_1 & v_2 & \dots & v_{2s} \end{pmatrix} = 0,$$

so $A = 0$. This concludes the proof that if $A \in ARI_{al/al}$, then $A(w_1, \dots, w_d)$ is an even function for all $d > 1$; thus if we assume in addition that A is even for $d = 1$, then $neg(A) = A$, and by lemma 2.5.3, we also have $push(A) = A$. ♢

In the appendix, we prove the following important identity, valid whenever $A, B \in ARI^{push}$, i.e. are both *push*-invariant bimoulds:

$$swap\big(ari(A,B)\big) = ari\big(swap(A), swap(B)\big), \tag{2.5.4}$$

This identity immediately yields the following result.

Proposition 2.5.5. *$ARI_{\underline{al}/\underline{al}}$ is a Lie algebra under the ari bracket.*

Proof. Let $A, B \in ARI_{\underline{al}/\underline{al}}$ and set $C = ari(A,B)$. The mould C is alternal by proposition 2.4.2, and by (2.5.4), we have $swap(C) = swap\big(ari(A,B)\big) = ari\big(swap(A), swap(B)\big)$, which is also alternal by proposition 2.4.2. It remains only to check that C is even in depth 1. But in fact, $C\binom{u_1}{v_1} = 0$, as the depth 1 part of an ari-bracket is always zero, which follows directly from the definition (cf. (2.2.3)). ♢

2.6. The space $\mathbf{ARI}_{\underline{al}/\underline{il}}$

Definition. Let $ARI_{al/il}$ denote the subspace of ARI consisting of bimoulds A which are alternal, and such that there exists a bimould $Z \in ARI$ such that

$$Z\begin{pmatrix} u_1 & \cdots & u_r \\ v_1 & \cdots & v_r \end{pmatrix} = \begin{cases} m_r \in \mathbb{C} & \text{all } v_j = 0 \\ 0 & \text{some } v_j \neq 0, \end{cases}$$

for which $A+Z$ is alternil. The subspace $ARI_{\underline{al}/\underline{il}} \subset ARI_{al/il}$ are the bimoulds which are even functions in depth 1.

Note the difference between this notation and the notation $ARI_{al/al}$, for which both the mould and its swap are alternal. Ecalle points out that using the strict definition for $ARI_{al/il}$ would lead to very small subspaces; for example restricting the strict definition to ARI^{u-var} would yield only zero.

Let us consider the definition of $ARI_{al/il}$ in the case of homogeneous polynomial-valued moulds of weight n. Here, the following statement holds.

Lemma 2.6.1. *Let A be a homogeneous alternal bimould of weight n in the variables u_i, and suppose that $swap(A)$ is alternil in depths $1 \leq d \leq n-1$. Then $swap(A)$ is not alternil in depth n, but if the value 0 in depth n is replaced by $\frac{(-1)^{n-1}}{n}$ (i.e. if this constant mould concentrated in depth n is added to A), the resulting mould is alternil in all depths.*

Proof. This is identical to the standard result on correction of double shuffle polynomials (cf. ref). In fact, if A is alternal, then $A=0$ in depth n, so $swap(A)=0$ in depth n. The mould $swap(A)$ does not satisfy the depth n alternility conditions, but there is a unique constant that can be put in depth n which will make it do so. ♢

Example. Let us use the homogeneous mould of degree 3 given by:

$$ma\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = u_1^2, \quad ma\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = -u_1 + u_2.$$

This mould is clearly alternal, and setting $mi = swap(ma)$, we have

$$mi\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = v_1^2, \quad mi\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = v_1 - 2v_2.$$

The alternility relation for depth 2 are given by

$$mi\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} + mi\begin{pmatrix} u_2 & u_1 \\ v_2 & v_1 \end{pmatrix} + \frac{1}{v_1 - v_2} mi\begin{pmatrix} u_1 + u_2 \\ v_1 \end{pmatrix} + \frac{1}{v_2 - v_1} mi\begin{pmatrix} u_1 + u_2 \\ v_2 \end{pmatrix},$$

i.e.

$$(v_1 - 2v_2) + (v_2 - 2v_1) + (v_1 + v_2) = 0.$$

But the alternility relation for depth 3 is given by

$$
\begin{aligned}
mi\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} + mi\begin{pmatrix} u_2 & u_1 & u_3 \\ v_2 & v_1 & v_3 \end{pmatrix} + mi\begin{pmatrix} u_2 & u_3 & u_1 \\ v_2 & v_3 & v_1 \end{pmatrix} \\
+\frac{1}{v_1 - v_2} mi\begin{pmatrix} u_1+u_2 & u_3 \\ v_1 & v_3 \end{pmatrix} + \frac{1}{v_2 - v_1} mi\begin{pmatrix} u_1+u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \\
+\frac{1}{v_2 - v_3} mi\begin{pmatrix} u_1 & u_2+u_3 \\ v_1 & v_2 \end{pmatrix} + \frac{1}{v_3 - v_2} mi\begin{pmatrix} u_1 & u_2+u_3 \\ v_1 & v_3 \end{pmatrix},
\end{aligned}
$$

i.e.

$$
0+0+0+\frac{(v_1-2v_3)-(v_2-2v_3)}{v_1-v_2}+\frac{(v_1-2v_2)-(v_1-2v_3)}{v_2-v_3} = 1-2 = -1,
$$

which is not zero. To make mi into a truly alternil mould, we need to set

$$
mi\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \frac{1}{3}.
$$

This lemma shows that for homogeneous polynomial moulds of weight n, we can use the following definition.

Definition. A polynomial-valued bimould A of homogeneous weight n lies in $ARI_{al/il}$ if it satisfies the alternility conditions in depths $0 \leq r < n$.

The proof that $ARI_{\underline{al}/\underline{il}}$ is also a sub Lie algebra of ARI for the ari bracket is difficult and lies at the heart of Écalle's theory. We explore this part of the theory in chapter 4. Before that, however, we will establish the connections between the standard theory of formal multiple zeta values and ARI.

Appendix to Chapter 2

For every mould $B \in ARI$, we define associated actions $amit(B)$ and $anit(B)$ on ARI as follows:

$$amit(B)\cdot A = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor),$$

$$anit(B)\cdot A = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}),$$

For every pair of moulds $B, C \in ARI$, we set

$$axit(B,C)\cdot A = amit(B)\cdot A + anit(C)\cdot A;$$

then *arit* arises as the special case of *axit* given by taking $C = -B$, so

$$arit(B)\cdot A = axit(B,-B)\cdot A = amit(B)\cdot A - anit(B)\cdot A.$$

We will use these notions in the proofs of results from the text, but in fact we will prove more: we will show that *amit*, *anit*, *axit* and *arit* are all derivations with respect to the usual mould multiplication *mu*, that *axit* thus gives rise to a Lie bracket on pairs of moulds

$$axi\big((A,B),(C,D)\big) =$$

$$\Big(axit(A,B)\cdot C - axit(C,D)\cdot A + limu(A,C)\,,\,axit(A,B)\cdot D - axit(C,D)\cdot B + limu(B,D)\Big),$$

and we will prove general identities between *swap*, *amit*, *anit*, *axit*, *arit*, *ari* and *axi*.

From §4.2.2: Proof that $arit(A)$ is a derivation with respect to mu.

We proceed by proving that $amit(A)$ and $anit(A)$ are derivations and then deduce the result for $arit(A)$.

For *amit*, we need to prove the identity

$$amit(A)\cdot mu(B,C) = mu\big(amit(A)\cdot B\,,\,C\big) + mu\big(B, amit(A)\cdot C\big).$$

Assuming that all moulds are in ARI and therefore 0-valued on the emptyset, we can remove $\mathbf{b} \neq \emptyset$ from the definition of *amit*; we have

$$
\begin{aligned}
amit(A)\cdot mu(B,C) &= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} mu(B,C)(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor)\\
&= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} \sum_{\mathbf{d}_1\mathbf{d}_2=\mathbf{a}\lceil\mathbf{c}} B(\mathbf{d}_1)C(\mathbf{d}_2)A(\mathbf{b}\rfloor)\\
&= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} \sum_{\mathbf{a}_1\mathbf{a}_2=\mathbf{a}} B(\mathbf{a}_1)C(\mathbf{a}_2\lceil\mathbf{c})A(\mathbf{b}\rfloor) + \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} \sum_{\substack{\mathbf{c}_1\mathbf{c}_2=\lceil\mathbf{c}\\ \mathbf{c}_1\neq\emptyset}} B(\mathbf{ac}_1)C(\mathbf{c}_2)A(\mathbf{b}\rfloor)\\
&= \sum_{\substack{\mathbf{w}=\mathbf{a}_1\mathbf{a}_2\mathbf{bc}\\ \mathbf{c}\neq\emptyset}} B(\mathbf{a}_1)C(\mathbf{a}_2\lceil\mathbf{c})A(\mathbf{b}\rfloor) + \sum_{\substack{\mathbf{w}=\mathbf{abc}_1\mathbf{c}_2\\ \mathbf{c}_1\neq\emptyset}} B(\mathbf{a}\lceil\mathbf{c}_1)C(\mathbf{c}_2)A(\mathbf{b}\rfloor)\\
&= \sum_{\substack{\mathbf{w}=\mathbf{a}_1\mathbf{d}\\ \mathbf{d}\neq\emptyset}} B(\mathbf{a}_1) \sum_{\substack{\mathbf{d}=\mathbf{a}_2\mathbf{bc}\\ \mathbf{c}\neq\emptyset}} C(\mathbf{a}_2\lceil\mathbf{c})A(\mathbf{b}\rfloor) + \sum_{\substack{\mathbf{w}=\mathbf{dc}_2\\ \mathbf{d}\neq\emptyset}} \sum_{\substack{\mathbf{d}=\mathbf{abc}_1\\ \mathbf{c}_1\neq\emptyset}} B(\mathbf{a}\lceil\mathbf{c}_1)A(\mathbf{b}\rfloor)C(\mathbf{c}_2)\\
&= \sum_{\substack{\mathbf{w}=\mathbf{a}_1\mathbf{d}\\ \mathbf{d}\neq\emptyset}} B(\mathbf{a}_1)\big(amit(A)\cdot C)(\mathbf{d}) + \sum_{\substack{\mathbf{w}=\mathbf{dc}_2\\ \mathbf{d}\neq\emptyset}} \big(amit(A)\cdot B\big)(\mathbf{d})C(\mathbf{c}_2).
\end{aligned}
$$

Noting that for $A, B, C \in ARI$ we always have $(amit(A)\cdot B)(\emptyset) = (amit(A)\cdot C)(\emptyset) = 0$, we can drop the requirement $\mathbf{d} \neq \emptyset$ under the sum, and therefore obtain exactly

$$mu\big(B, amit(A)\cdot C) + mu\big(amit(A)\cdot B, C).$$

Exercise. Show similarly that *anit* is a derivation.

We deduce immediately that $arit(B)\cdot A = amit(B)\cdot A - anit(B)\cdot A$ is a derivation with respect to *mu*. We have the explicit expression

$$arit(A)\cdot B = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a}\neq\emptyset}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b}).$$

From §4.2.4: Proof that if $A, B \in ARI_{al}$, then $arit(B)\cdot A \in ARI_{al}$.

Using (A2), we compute

$$\sum_{\mathbf{w}\in sh(\mathbf{x},\mathbf{y})} \big(arit(A)\cdot B\big)(\mathbf{w}) = \sum_{\mathbf{w}\in sh(\mathbf{x},\mathbf{y})} \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{c}\neq\emptyset}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a}\neq\emptyset}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})$$

$$= \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_3\mathbf{y}_3\neq\emptyset}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}\in sh(\mathbf{x}_2,\mathbf{y}_2)\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_3)}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_1\mathbf{y}_1\neq\emptyset}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}\in sh(\mathbf{x}_2,\mathbf{y}_2)\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_3)}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})$$

$$= \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_3\mathbf{y}_3\neq\emptyset\\ |\mathbf{x}_2\mathbf{y}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{x}_2 \text{ or } \mathbf{y}_2\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_3)}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_1\mathbf{y}_1\neq\emptyset\\ |\mathbf{x}_2\mathbf{y}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{x}_2 \text{ or } \mathbf{y}_2\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_3)}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})$$

$$= \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\\ \mathbf{x}_3\mathbf{y}_2\neq\emptyset\\ |\mathbf{x}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{x}_2\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_2)}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\\ \mathbf{x}_1\mathbf{y}_1\neq\emptyset\\ |\mathbf{x}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{x}_2\\ \mathbf{c}\in sh(\mathbf{x}_3,\mathbf{y}_2)}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})$$

$$+ \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_3\mathbf{y}_3\neq\emptyset\\ |\mathbf{y}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{y}_2\\ \mathbf{c}\in sh(\mathbf{x}_2,\mathbf{y}_3)}} B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{x}=\mathbf{x}_1\mathbf{x}_2\\ \mathbf{y}=\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\\ \mathbf{x}_1\mathbf{y}_1\neq\emptyset\\ |\mathbf{y}_2|=1}} \sum_{\substack{\mathbf{a}\in sh(\mathbf{x}_1,\mathbf{y}_1)\\ \mathbf{b}=\mathbf{y}_2\\ \mathbf{c}\in sh(\mathbf{x}_2,\mathbf{y}_3)}} B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})$$

From §4.2.5: Swap, push and the ari bracket

Here we prove the identity (2.5.4) given by

$$swap\big(ari(A,B)\big) = ari\big(swap(A), swap(B)\big)$$

when A and B are both push-invariant moulds. In fact, we introduce some general objects and prove some more general identities concerning the relation between *swap* and *amit*, *anit*, *axit* and *arit* on the way.

Precisely, we prove the desired identity by uncovering the commutation relations between swap and the derivations *amit* and *anit*: these are given by the following two identities:

$$swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big) = amit(B)\cdot A + mu(A,B) - swap\Big(mu\big(swap(A), swap(B)\big)\Big), \tag{A6}$$

$$swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big) = anit\big(push(B)\big)\cdot A. \tag{A7}$$

Using these identities, we can recover a more general version of the desired result (2.5.4).

Recall that

$$preari(A,B) = arit(B)\cdot A + mu(A,B) \quad \text{and} \quad ari(A,B) = preari(A,B) - preari(B,A). \tag{A10}$$

Thus in particular, we have

$$\begin{aligned} ari(A,B) &= arit(B)\cdot A - arit(A)\cdot B + limu(A,B)\\ &= axit(B,-B)\cdot A - axit(A,-A)\cdot B + limu(A,B). \end{aligned} \tag{A11}$$

The identities (A6) and (A7) immediately yield

$$\begin{aligned}
&swap\Big(arit\big(swap(B)\big)\cdot swap(A)\Big)\\
&\qquad = swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big) - swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big)\\
&\qquad = anit(B)\cdot A + mu(A,B) - swap\Big(mu\big(swap(A), swap(B)\big)\Big) - amit\big(push(B)\big)\cdot A\\
&\qquad = axit\big(B, -push(B)\big)\cdot A + mu(A,B) - swamu(A,B),
\end{aligned} \tag{A12}$$

$$\begin{aligned}
swap\Big(preari\big(swap(A), swap(B)\big)\Big) &= swap\Big(arit\big(swap(B)\big)\cdot A\Big) + swamu(A,B)\\
&= axit\big(B, -push(B)\big)\cdot A + mu(A,B),
\end{aligned} \tag{A13}$$

and finally from (A11) and (A13), we see that if A and B are push-invariant, then

$$\begin{aligned}
&swap\Big(ari\big(swap(A), swap(B)\big)\Big)\\
&\qquad = axit\big(B,-push(B)\big)\cdot A + mu(A,B) - axit\big(A,-push(A)\big)\cdot B - mu(B,A)\\
&\qquad = axit\big(B,-push(B)\big)\cdot A - axit\big(A,-push(A)\big)\cdot B + limu(A,B)\\
&\qquad = axi\Big(\big(A,-push(A)\big),\big(B,-push(B)\big)\Big)_L \quad \text{(left-hand part of } axi \text{ pair)}\\
&\qquad = axit\big(B,-B\big)\cdot A - axit\big(A,-A\big)\cdot B + limu(A,B) \quad \text{(since } A, B \text{ push-invariant)}\\
&\qquad = arit(B)\cdot A - arit(A)\cdot B + limu(A,B)\\
&\qquad = ari(A,B),
\end{aligned} \tag{A14}$$

which proves the desired equality (2.5.4).

It remains to prove the key identities (A6) and (A7).

To do this, we need the following explicit expressions for the flexions occurring in the definitions of the derivations, and the effect of *swap*:

$$\mathbf{a}\lceil\mathbf{c} = \begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix}\begin{pmatrix} u_{k+1}+\cdots+u_{k+l+1} & \cdots & u_r \\ v_{k+l+1} & \cdots & v_r \end{pmatrix},$$

$$\mathbf{b}\rfloor = \begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1} \end{pmatrix}$$

$$\mathbf{a}\rceil\mathbf{c} = \begin{pmatrix} u_1 & \cdots & u_{k-1} & u_k+\cdots+u_{k+l} \\ v_1 & \cdots & v_{k-1} & v_k \end{pmatrix}\begin{pmatrix} u_{k+l+1} & \cdots & u_r \\ v_{k+l+1} & \cdots & v_r \end{pmatrix}.$$

$$\lfloor\mathbf{b} = \begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_k & \cdots & v_{k+l}-v_k \end{pmatrix}.$$

Setting $SC = swap(C)$ for any mould C, we have

$$SC(\mathbf{a}\lceil\mathbf{c}) = SC\begin{pmatrix} u_1 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r \end{pmatrix}$$

$$= C\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k-v_{k+l+1} & v_{k-1}-v_k & \cdots \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_k & u_1+\cdots+u_{k-1} & \cdots \end{pmatrix}$$

$$SC(\mathbf{b}\rfloor) = SC\begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1} \end{pmatrix}$$

$$= C\begin{pmatrix} v_{k+l}-v_{k+l+1} & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}$$

$$SC(\mathbf{a}\rceil\mathbf{c}) = SC\begin{pmatrix} u_1 & \cdots & u_{k-1} & u_k+\cdots+u_{k+l} & u_{k+l+1} & \cdots & u_r \\ v_1 & \cdots & v_{k-1} & v_k & v_{k+l+1} & \cdots & v_r \end{pmatrix}$$

$$= C\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k-v_{k+l+1} & v_{k-1}-v_k & \cdot \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_{k+l} & u_1+\cdots+u_{k-1} & \cdot \end{pmatrix}$$

$$SC(\lfloor\mathbf{b}) = SC\begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_k & \cdots & v_{k+l}-v_k \end{pmatrix}$$

$$= C\begin{pmatrix} v_{k+l}-v_k & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}$$

Applying the swap

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} \mapsto \begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_1-v_2 \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1 \end{pmatrix},$$

i.e.

$$\begin{cases} u_1 \mapsto v_r \\ u_i \mapsto v_{r-i+1}-v_{r-i+2}, \text{ if } i>1 \\ u_1+\cdots+u_i \mapsto v_{r-i+1} \\ u_i+\cdots+u_j \mapsto -v_{r-i+2}+v_{r-j+1} \ \text{ if } i<j \\ v_i \mapsto u_1+\cdots+u_{r-i+1} \\ v_i-v_{i+1} \mapsto u_{r-i+1} \\ v_i-v_j \mapsto u_{r-j+2}+\cdots+u_{r-i+1} \ \text{ if } i<j \\ v_i-v_j \mapsto -u_{r-i+2}-\cdots-u_{r-j+1} \ \text{ if } i>j \end{cases}$$

to these four terms yields

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1}+\cdots+u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{r-k-l+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1}-v_{r-k+1} & v_{r-k-l+2}-v_{r-k} & \cdots & v_{r-k}-v_{r-k+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1}\cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k-l+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{r-k-l+2} - \cdots - u_{r-k+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1} - v_{r-k+1} & v_{r-k-l+2} - v_{r-k+1} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix}$$

Setting $m = r - k - l$, they can be written as

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1} - v_{r-k+1} & v_{m+2} - v_{r-k} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots + u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{m+2} - \cdots - u_{r-k+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1} - v_{r-k+1} & v_{m+2} - v_{r-k+1} & \cdots & v_{r-k} - v_{r-k+1} \end{pmatrix}$$

Now putting $r - k = m + l$ gives

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} + \cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+l+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{m+2} - \cdots - u_{m+l+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \end{pmatrix}$$

Using all these, we can now prove (A6) and (A7).

Proof of (A6). We have

$$swap\Big(amit\big(swap(B)\big) \cdot swap(A)\Big) = swap\Big(\sum_{\substack{\mathbf{w}=\mathbf{abc} \\ \mathbf{b},\mathbf{c}\neq\emptyset}} SA(\mathbf{a}\lceil\mathbf{c})SB(\mathbf{b}\rfloor)\Big)$$

$$= swap\Big[\sum_{l=1}^{r-1}\sum_{m=1}^{r-l}$$

$$A\begin{pmatrix} v_r & v_{r-1} - v_r & \cdots & v_{k+l+1} - v_{k+l+2} & v_k - v_{k+l+1} & v_{k-1} - v_k & \cdots & u \\ u_1 + \cdots + u_r & u_1 + \cdots + u_{r-1} & \cdots & u_1 + \cdots + u_{k+l+1} & u_1 + \cdots + u_k & u_1 + \cdots + u_{k-1} & \cdots & \end{pmatrix}$$

$$\cdot B\begin{pmatrix} v_{k+l}-v_{k+l+1} & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1}\end{pmatrix}\Bigg]$$

$$=\sum_{l=1}^{r-1}\sum_{m=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}+\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+l+1} & v_{m+l+2} & \cdots & v_r\end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_{m+l+1} & v_{m+2}-v_{m+l+1} & \cdots & v_{m+l}-v_{m+l+1}\end{pmatrix}$$

$$=\sum_{l=1}^{r-1}\sum_{k=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r\end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & v_{k+2}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1}\end{pmatrix}$$

$$=\sum_{l=1}^{r-1}\sum_{k=0}^{r-l-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r\end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & v_{k+2}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1}\end{pmatrix}$$

$$-\sum_{l=1}^{r-1} A\begin{pmatrix} u_1+\cdots+u_{l+1} & u_{l+2} & \cdots & u_r \\ v_{l+1} & v_{l+2} & \cdots & v_r\end{pmatrix}\cdot B\begin{pmatrix} u_1 & u_2 & \cdots & u_l \\ v_1-v_{l+1} & v_2-v_l & \cdots & v_l-v_{l+1}\end{pmatrix}$$

$$+\sum_{l=1}^{r-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-l} \\ v_1 & v_2 & \cdots & v_{r-l}\end{pmatrix}\cdot B\begin{pmatrix} u_{r-l+1} & u_{r-l+2} & \cdots & u_r \\ v_{r-l+1} & v_{r-l+2} & \cdots & v_r\end{pmatrix}$$

$$= amit(B)\cdot A - swap\Big(mu\big(swap(A), swap(B)\big)\Big) + mu(A,B).$$

Proof of (A7). We have

$$swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big) = swap\Big(\sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} SA(\mathbf{a}\rceil\mathbf{c})SB(\lfloor\mathbf{b})\Big)$$

$$= swap\Big[\sum_{l=1}^{r-1}\sum_{k=1}^{r-l}$$

$$A\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k-v_{k+l+1} & v_{k-1}-v_k & \cdots \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_{k+l} & u_1+\cdots+u_{k-1} & \cdots\end{pmatrix}$$

$$\cdot B\begin{pmatrix} v_{k+l}-v_k & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1}\end{pmatrix}\Bigg]$$

$$= \sum_{l=1}^{r-1} \sum_{m=0}^{r-l-1} A \begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} \cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot B \begin{pmatrix} -u_{m+2} - \cdots - u_{m+l+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_{m+l+1} & v_{m+2} - v_{m+l+1} & \cdots & v_{m+l} - v_{m+l+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1} \sum_{m=0}^{r-l-1} A \begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1} \cdots + u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B) \begin{pmatrix} u_{m+2} & u_{m+3} & \cdots & u_{m+l+1} \\ v_{m+2} - v_{m+1} & v_{m+3} - v_{m+1} & \cdots & v_{m+l+1} - v_{m+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1} \sum_{m=1}^{r-l} A \begin{pmatrix} u_1 & u_2 & \cdots & u_{m-1} & u_m \cdots + u_{m+l} & u_{m+l+1} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{m-1} & v_m & v_{m+l+1} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B) \begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1} - v_m & v_{m+2} - v_m & \cdots & v_{m+l} - v_m \end{pmatrix}$$

$$= \sum_{l=1}^{r-1} \sum_{k=1}^{r-l} A \begin{pmatrix} u_1 & u_2 & \cdots & u_{k-1} & u_k \cdots + u_{k+l} & u_{k+l+1} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{k-1} & v_k & v_{k+l+1} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B) \begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1} - v_k & v_{k+2} - v_k & \cdots & v_{k+l} - v_k \end{pmatrix}$$

$$= anit\big(push(B)\big) \cdot A.$$

Chapter 3

From multiple zeta values to ARI

In this chapter, we give a map from the double shuffle Lie algebra $\mathfrak{ds}$ introduced in chapter 1 to the Lie algebra ARI, and prove that it is a map of Lie algebras. In fact, the map can be defined on a larger space, $\mathbb{F}$, containing $\mathfrak{ds}$ (cf. §4.3.1). We identify the image of $\mathfrak{ds}$ in ARI, as well as the image of the associated graded for the depth filtration, then use the results of chapter 2 to show how Ecalle's methods can be used to prove one of the basic theorems on $\mathfrak{ds}$.

3.1. The ring $\mathbb{F}$

Consider the ring of polynomials $\mathbb{Q}\langle x,y\rangle$ in non-commutative variables x,y. Let ∂_x denote the differential operator with respect to x. Set $C_i = \mathrm{ad}(x)^{i-1}(y)$, $i \geq 1$, so $C_1 = y$, $C_2 = [x,y]$, $C_3 = [x,[x,y]], \ldots$.

Lemma 3.1.1. (Lazard elimination) *The set of polynomials $f \in \mathbb{Q}\langle x,y\rangle$ such that $\partial_x(f) = 0$ is a subring which is equal to the subring of polynomials f that can be written as polynomials in the C_i. The C_i are free generators of this ring, so if f can be written as such a polynomial, then it can be written so in a unique way.*

Let π_y be the projector onto polynomials ending in y (i.e. π_y forgets all the monomials ending in x). The usefulness of the ring $\mathbb{F}$ is that π_y has a section on $\mathbb{F}$. Indeed, for any polynomial g ending in y, define $\sec(g)$ by

$$\sec(g) = \sum_{i\geq 0} \frac{(-1)^i}{i!} \partial_x^i(g) x^i.$$

Lemma 3.1.2. (Racinet) *(1)* $\sec \circ \pi_y = \mathrm{id}$ *on* $\mathbb{F}$.

(2) $\pi_y \circ \sec = \mathrm{id}$ *on* $\mathbb{Q}\langle x,y\rangle y$.

Write $\mathbb{F}_n$ for the vector space of homogeneous polynomials in $\mathbb{F}$ of degree n.

3.2. Associating bimoulds to elements $f \in \mathbb{F}_n$

Let $f \in \mathbb{F}_n$. We need the following definitions.

Definitions. Write $f(x,y) = \sum_{r=0}^n f^r(x,y)$ where r denotes the depth r part of f (i.e. the part of f consisting of monomials containing exactly r y's).

Let $\pi_y(f)$ denote the projection of f onto the monomials ending in y as above, and let f_y denote $\pi_y(f)$ rewritten in the variables $y_i = x^{i-1}y$, $i \geq 1$, and similarly f_y^r the depth r part, i.e. $\pi_y(f^r)$ written in the y_i. Similarly, let $\pi_Y(f)$ denote the projection of f onto the monomials starting with y. Let $\mathrm{ret}_X : \mathbb{Q}\langle x,y\rangle \to \mathbb{Q}\langle x,y\rangle$ denote the "backwards writing" map

$$\mathrm{ret}_X(x^{a_0}y\cdots yx^{a_{r-1}}yx^{a_r}) = x^{a_r}yx^{a_{r-1}}y\cdots yx^{a_0}. \tag{3.2.1}$$

Note that $\mathrm{Lie}[x,y] \subset \mathbb{F}$. If $f \in \mathbb{F}_n$ is actually a Lie element, we have

$$\mathrm{ret}_X(f) = (-1)^{n-1} f. \tag{3.2.2}$$

Finally, let f_Y^r denote the polynomial $\mathrm{ret}_X\big(\pi_Y(f)\big)$ written in the variables y_i.

Recall from lemma 3.1.1 that $\mathbb{F}$ is exactly the set of polynomials that can be written as polynomials in the C_i, and such a writing is unique. Let f_C denote f written in this way.

Define three maps from monomials in non-commutative variables x, y (resp. $y_1, y_2, \ldots$ resp. $C_1, C_2, \ldots$) to monomials in commutative variables $z_0, z_1, \ldots$ (resp. $v_1, v_2, \ldots$ resp. $u_1, u_2, \ldots$) as follows:

$$\begin{aligned} \iota_X &: x^{a_0-1} y \cdots x^{a_{r-1}-1} y x^{a_r-1} \mapsto z_0^{a_0-1} \cdots z_r^{a_r-1} \\ \iota_C &: C_{a_1} \cdots C_{a_r} \mapsto u_1^{a_1-1} \cdots u_r^{a_r-1} \\ \iota_Y &: y_{a_1} \ldots y_{a_r} \mapsto v_1^{a_1-1} \cdots v_r^{a_r-1}. \end{aligned} \tag{3.2.3}$$

Then we define three moulds (a mould and two bimoulds) associated to $f \in \mathbb{F}_n$ as follows:

$$\begin{aligned} vimo(z_0, z_1, \ldots, z_r) &= \iota_X(f^r) \\ ma\binom{u_1, \ldots, u_r}{0, \ldots, 0} &= (-1)^{r+n} \iota_C(f_C^r) \\ mi\binom{0, \ldots, 0}{v_1, \ldots, v_r} &= \iota_Y(f_Y^r). \end{aligned} \tag{3.2.4}$$

All other values of these (bi)moulds are 0.

Note that by (2.2), if $f \in \mathrm{Lie}[x,y]$, we have

$$\pi_y(f) = (-1)^{n-1} \mathrm{ret}_X\big(\pi_Y(f)\big),$$

so $f_y^r = (-1)^{n-1} f_Y^r$. Thus, if $f \in \mathrm{Lie}[x,y]$, the mould mi can also be defined by

$$mi\binom{0, \ldots, 0}{v_1, \ldots, v_r} = (-1)^{n-1} \iota_Y(f_y^r). \tag{3.2.5}$$

When we turn our attention to the double shuffle algebra $\mathfrak{ds}$, we will be in this case.

Since the maps ι_X, ι_C and ι_Y are obviously invertible, we recover f from $vimo$, f_C from ma and f_Y from mi. But of course, we easily recover f from f_C by expanding out the C_i, and we also recover f from f_Y by setting $f = \mathrm{sec}(f_Y)$, as we have assumed that $f \in \mathbb{F}_n$. Thus, $f \in \mathbb{F}_n$, f_C, f_Y, $vimo$, ma and mi are all different encodings of the same information.

Example. Let $n = 3$ and

$$f = [x,[x,y]] + [[x,y],y] = x^2y - 2xyx + yx^2 + xy^2 - 2yxy + y^2x.$$

Then $\pi_y(f) = x^2y - 2yxy + xy^2$, $f_Y = y_3 - 2y_1y_2 + y_2y_1$ and $f_C = C_3 - C_1C_2 + C_2C_1$, and we have

$$\begin{cases} vimo(z_0) = 0 \\ vimo(z_0, z_1) = z_0^2 - 2z_0z_1 + z_1^2 \\ vimo(z_0, z_1, z_2) = z_0 - 2z_1 + z_2 \\ vimo(z_0, z_1, z_2, z_3) = 0, \end{cases} \quad \begin{cases} ma^{\emptyset} = 0 \\ ma\left(\begin{smallmatrix} u_1 \\ 0 \end{smallmatrix}\right) = u_1^2 \\ ma\left(\begin{smallmatrix} u_1,u_2 \\ 0,0 \end{smallmatrix}\right) = -u_1 + u_2 \\ ma\left(\begin{smallmatrix} u_1,u_2,u_3 \\ 0,0,0 \end{smallmatrix}\right) = 0, \end{cases} \quad \begin{cases} mi(\emptyset) = 0 \\ mi\left(\begin{smallmatrix} 0 \\ v_1 \end{smallmatrix}\right) = v_1^2 \\ mi\left(\begin{smallmatrix} 0,0 \\ v_1,v_2 \end{smallmatrix}\right) = -2v_2 + v_1 \\ mi\left(\begin{smallmatrix} 0,0,0 \\ v_1,v_2,v_3 \end{smallmatrix}\right) = 0. \end{cases}$$

3.3. Ecalle's presentation of these moulds

Ecalle introduces these moulds a little differently, but the definitions are equivalent. He first associates to $f \in \mathbb{F}_n$ the mould *vimo* by the formula:

$vimo(z_0, z_1, \ldots, z_r) =$

$$\begin{cases} \displaystyle\sum_{\substack{(d_0,\ldots,d_r) \\ d_i \geq 0,\ d_0+d_1+\cdots+d_r=n-r}} \big(f|x^{d_0}yx^{d_1}y\cdots x^{d_{r-1}}yx^{d_r}\big) z_0^{d_0}\cdots z_r^{d_r} & \text{if } 0 \leq r \leq n \\ 0 & \text{if } r > n, \end{cases}$$

and then recovers *ma* and *mi* from *vimo* by the formulae

$$ma\left(\begin{matrix} u_1, \ldots, u_r \\ 0, \ldots, 0 \end{matrix}\right) = vimo(0, u_1, u_1 + u_2, \ldots, u_1 + \cdots + u_r)$$

$$mi\left(\begin{matrix} 0, \ldots, 0 \\ v_1, \ldots, v_r \end{matrix}\right) = vimo(0, v_r, v_{r-1}, \ldots, v_1). \tag{3.3.1}$$

Cf. the Appendix for a complete proof that this definition is equivalent to the one in (3.2.4).

To express the fact that $f \in \mathbb{F}_n$ directly in terms of moulds, Ecalle has the following condition.

Lemma 2.3. (Ecalle, PAL1) *If $f \in \mathbb{Q}_n\langle x, y\rangle$ and vimo is defined as in (2.4), then $f \in \mathbb{F}_n$ if and only if*

$$vimo(z_0, \ldots, z_r) = vimo(0, z_1 - z_0, z_2 - z_0, \ldots, z_r - z_0).$$

3.4. The swap

Ecalle's presentation underlines the symmetry between *ma* and *mi*, and makes it trivial to prove that they are exchanged by an order 2 operation on bimoulds that Ecalle calls the *swap*. The swap is defined by

$$\mathrm{swap}(M)\left(\begin{matrix} u_1, \ldots, u_r \\ v_1, \ldots, v_r \end{matrix}\right) = M\left(\begin{matrix} v_r, & v_{r-1} - v_r, & \ldots, & v_1 - v_2 \\ u_1 + \cdots + u_r, & u_1 + \cdots + u_{r-1}, & \ldots, & u_1 \end{matrix}\right) \tag{3.4.1}$$

The swap is easily seen to be of order 2. It is immediate from applying the swap to (3.3.1) that

$$mi = swap(ma).$$

To understand the swap in terms of polynomials, one can express it as follows (which is what Racinet was intending in Appendix A, only the definition of swap and the statement and proof of his prop. 3.3 are all slightly wrong).

Lemma 2.4. *Let $f \in \mathbb{F}_n$ and let f_Y, f_C, ι_Y and ι_C be as in (2.3). For $0 \le r \le n$, define the map of polynomial rings $S_r : \mathbb{Q}[u_1, \ldots, u_r] \to \mathbb{Q}[v_1, \ldots, v_r]$ by*

$$S_r(u_1) = v_r, \quad S_r(u_i) = v_{r-i+1} - v_{r-i+2} \quad \text{for} \quad 2 \le i \le r.$$

Then

$$S_r\big((-1)^{r+n}\iota_C(f_C^r)\big) = \iota_Y(f_Y^r).$$

In other words, identifying S_r with the swap applied to variables u_i, $(-1)^{r+n}\iota_C(f_C^r)$ with $ma_f(u_1, \ldots, u_r)$ and $\iota(f_Y^r)$ with $mi_f(v_1, \ldots, v_r)$, this is $swap(ma_f) = mi_f$.

Remark. The linear map S_r and its inverse S_r^{-1} are given by the matrix

$$S_r = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad S_r^{-1} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

These matrices appear in IKZ and are directly related to Zagier's notation f, $f^{\#}$.

3.5. From ds to $\mathrm{ARI}_{al/il}$

Definition. The space $\mathrm{ARI}_{al/il}$ consists in moulds ma such that ma is alternal and $mi = swap(ma)$ is alternil up to a constant (cf. 2.6).

According to Ecalle, the passage from $\mathbb{F}$ to ARI defined in 2.2 maps $\mathfrak{d}s$ to $\mathrm{ARI}_{al/il}$. More precisely, Ecalle's properties translate the double shuffle properties on polynomials, but also generalize them to all bimoulds.

Lemma 3.5.1. *Let $f \in \mathbb{F}_n$. Then*

(1) f satisfies shuffle if and only if ma is alternal;

(2) f_Y satisfies stuffle in depth $1 \le r < n$ if and only if mi is alternil.

Corollary. *Let $f \in \mathbb{F}_n$. Then*

$$f \text{ satisfies double shuffle} \Rightarrow ma/mi \text{ is alternal/alternil}.$$

In other words, the association $f \mapsto ma$ maps $\mathfrak{ds}$ to $ARI_{al/il}$.

Proof. In the special case where f satisfies shuffle, we have $f_Y = (-1)^{n-1} f_y$, so under the shuffle assumption, (2) can be expressed as saying that mi is alternil if and only if f_y satisfies stuffle in depths $1 \le r < n$. But it is known ([CS]) that if this is the case, then $f_* = f_y + \frac{(-1)^{n-1}}{n}(f|x^{n-1}y)y_1^n$ satisfies stuffle in depths $1 \le r \le n$, so under shuffle, mi is alternil if and only f_* satisfies stuffle, i.e. f satisfies double shuffle. ♢

Cf. the Appendix for the proof of lemma 4.1 (sort of).

Example. We take the same example as in 2.2.2, and check that ma/mi is al/il. Recall that $ma\binom{u_1}{0} = u_1^2$, $ma\binom{u_1,u_2}{0,0} = -u_1 + u_2$. To show that ma is alternal, we have only to check that

$$ma\binom{u_1,u_2}{0,0} + ma\binom{u_2,u_1}{0,0} = 0,$$

which follows from $(-u_1+u_2)+(-u_2+u_1) = 0$.

To show that mi is alternil, we have only to check the stuffle relation for depth $r = 2$, i.e. $st((w_1),(w_2)) = (w_1,w_2)+(w_2,w_1)+(w_1,w_2)$. The corresponding alternility relation is

$$mi\binom{0,0}{v_1,v_2} + mi\binom{0,0}{v_2,v_1} + P(v_1-v_2)mi\binom{0}{v_1} + P(v_2-v_1)mi\binom{0}{v_2} = 0,$$

which follows since $mi\binom{0}{v_1} = v_1^2$ and $mi\binom{0,0}{v_1,v_2} = v_1 - 2v_2$, and

$$(v_1-2v_2)+(v_2-2v_1)+\frac{v_1^2}{v_1-v_2}+\frac{v_2^2}{v_2-v_1} = (-v_1-v_2)+(v_1+v_2) = 0.$$

3.6. The Poisson bracket and the ARI bracket

In this section we prove the relation between the Poisson bracket on $\mathfrak{ds}$ and the ARI bracket defined in (1.3). The main result of this section is the equality (2.4.9) relating the two brackets precisely.

Recall the definitions (1.2) and (1.3) from 2.1:

$$\big(S_B(A)\big)(\mathbf{w}) = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}) \tag{3.6.1}$$

and

$$[A,B]_{ari} = ari(A,B) = S_B(A) - S_A(B),$$

so

$$ari(A,B) = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b}\neq\emptyset}} \Big(A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - B(\mathbf{a}\lceil\mathbf{c})A(\mathbf{b}\rfloor)\Big) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} \Big(A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}) - B(\mathbf{a}\rceil\mathbf{c})A(\lfloor\mathbf{b})\Big). \tag{3.6.2}$$

We start by setting

$$D_B(A) = arit(B)\cdot A = S_B(A) - AB = \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) - \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}), \quad (3.6.3)$$

where $AB = mu(A,B)$ denotes the product of two moulds by the multiplication rule (2.1.1), so as to obtain the following expressions for the ARI bracket $[A,B]_{ari} = ari(A,B)$:

$$\begin{aligned} ari(A,B) &= S_B(A) - S_A(B) \\ &= arit(B)\cdot A + AB - arit(A)\cdot B - BA \\ &= AB - BA + arit(B)\cdot A - arit(A)\cdot B \\ &= limu(A,B) + arit(B)\cdot A - arit(A)\cdot B \end{aligned}$$

where $mu(A,B) = AB$ and $limu(A,B) = AB - BA$ (cf. *ARI/GARI et la décomposition des multizêtas en irréductibles*, p. 28 (75) and p. 29 (84)).

The notation $D_A(B) = arit(A)\cdot B$ given by Ecalle can be considered as an action of the mould A on the mould B analogous to $D_f(g)$. The use of the two notations $D_A(B) = arit(A)\cdot B$ underlines this similarity while moving towards the systematic use of Ecalle's notation (similarly, we will be replacing AB by $mu(A,B)$ and $[A,B]_{ari}$ by $ari(A,B)$). We will explain the analogy precisely below (see (3.6.6)), showing in particular that the action of $D_A = arit(A)$ is a derivation (proposition 3.6.2).

We first need a useful lemma. Observe that if $f \in \mathbb{F}_n$, then $\partial_x([x,f]) = 0$, so by lemma 3.1.1, $[x,f] \in \mathbb{F}_{n+1}$. By lemma 3.1.1, we can consider both f and $[x,f]$ as being polynomials in the C_i.

Lemma 3.6.1. (Racinet) *Let $f \in \mathbb{F}_n$. Then for $0 \le r \le n$, we have*

$$\iota_C([x,f^r]) = (u_1 + \cdots + u_r)\iota_C(f^r). \quad (3.6.4)$$

Proof. Note first that $a \mapsto [x,a]$ is a derivation, i.e. $[x,ab] = [x,a]b + a[x,b]$. Thus, writing $f^r = \sum_{\mathbf{a}} c_{\mathbf{a}} C_{a_1}\cdots C_{a_r}$, we have

$$\begin{aligned} [x,f^r] = \sum_{\mathbf{a}} c_{\mathbf{a}}[x, C_{a_1}\cdots C_{a_r}] &= \sum_{\mathbf{a}} c_{\mathbf{a}} \sum_{i=1}^r C_{a_1}\cdots C_{a_{i-1}}[x,C_{a_i}]C_{a_{i+1}}\cdots C_{a_r} \\ &= \sum_{\mathbf{a}}\sum_{i=1}^r c_{\mathbf{a}}\; C_{a_1}\cdots C_{a_{i-1}}C_{a_i+1}C_{a_{i+1}}\cdots C_{a_r}. \end{aligned}$$

So the left-hand side of (3.6.4) is equal to

$$\sum_{\mathbf{a}}\sum_{i=1}^r c_{\mathbf{a}}\; u_1^{a_1-1}\cdots u_i^{a_i}\cdots u_r^{a_r-1}. \quad (3.6.5)$$

But since $\iota_C(f^r) = \sum_{\mathbf{a}} c_{\mathbf{a}}\; u_1^{a_1-1} \cdots u_r^{a_r-1}$, (3.6.5) is equal to $\iota_C(f^r)$ multiplied by $(u_1 + \cdots + u_r)$, proving (3.6.4). $\diamondsuit$

Proposition 3.6.2. (Racinet) *For any mould A, the operator $arit(A) = D_A$ on moulds defined by $arit(A) \cdot B = D_A(B) = S_A(B) - BA$ is a derivation of ARI.*

Proof. Straightforward. Cf. Appendix for complete details.

Proposition 4.3. *Let $f \in \mathbb{F}_n$ be of homogeneous depth r and $g \in \mathbb{F}_m$ of homogeneous depth s. Let D_f be the derivation of $\mathbb{F}$ defined by $D_f(x) = 0$, $D_f(y) = [y, f]$. Then*

$$\iota_C\big(D_f(g)\big)\big) = D_{\iota_C(f)}\big(\iota_C(g)\big) = (-1)^{n+r+m+s-1} D_{ma_f}(ma_g). \tag{3.6.6}$$

Proof. We have $D_{f+g} = D_f + D_g$, so we may assume that $f = C_{a_1} \cdots C_{a_r}$ is a monomial in the C_i. Furthermore, a derivation of $\mathbb{F}$ is defined by its action on the generators C_i, so we may take $g = C_b = \mathrm{ad}(x)^{b-1}(y)$. Let $F_0 = [y, f]$, and for $i \geq 1$, let $F_i = \mathrm{ad}(x)^i([y, f])$. In particular, we have

$$D_f(g) = [x, [x, \cdots, [x, [y, f]] \cdots] = \mathrm{ad}(x)^{b-1}([y, f]) = F_{b-1}.$$

Then by lemma 5.1, since all the F_i are in depth $r + 1$, we have

$$\iota_C(F_i) = (u_1 + \ldots + u_{r+1})\iota_C(F_{i-1}) \;\text{ for }\; i > 0,$$

so

$$\iota_C(F_i) = (u_1 + \ldots + u_{r+1})^i \iota_C(F_0),$$

so the left-hand side of (5.4) is equal to

$$\begin{aligned}
\iota_C(D_f(g)) &= \iota_C(F_{b-1}) \\
&= (u_1 + \ldots + u_{r+1})^{b-1} \iota_C([y, f]) \\
&= (u_1 + \cdots + u_{r+1})^{b-1} \iota_C(C_1 C_{a_1} \cdots C_{a_r} - C_{a_1} \cdots C_{a_r} C_1) \\
&= (u_1 + \cdots + u_{r+1})^{b-1} \big(u_2^{a_1-1} \cdots u_{r+1}^{a_r-1} - u_1^{a_1-1} \cdots u_r^{a_r-1}\big).
\end{aligned} \tag{3.6.7}$$

Now consider the right-hand side of (3.6.6). By (3.2.4), we have

$$ma_f\binom{u_1, \ldots, u_r}{0, \ldots, 0} = (-1)^{r+n} \iota_C(f) = (-1)^{r+n} u_1^{a_1-1} \cdots u_r^{a_r-1},$$

where $n = a_1 + \cdots + a_r$, and

$$ma_g\binom{u_1}{0} = (-1)^{b-1} u_1^{b-1}.$$

Since $D_A(B) = S_A(B) - BA$, (3.6.1) yields

$$\big(D_A(B)\big)(\mathbf{w}) = \sum_{\mathbf{w}=\mathbf{abc}} B(\mathbf{ac}')A(\mathbf{b}) - \sum_{\substack{\mathbf{w}=\mathbf{abc} \\ \mathbf{a}\neq\emptyset}} B(\mathbf{a}''\mathbf{c})A(\mathbf{b}) - \sum_{\mathbf{w}=\mathbf{ab}} B(\mathbf{a})A(\mathbf{b}). \tag{3.6.8}$$

Since ma_g has value zero on any word of length greater than 1, this formula simplifies when $A = ma_f$, $B = ma_g$ to

$$\begin{aligned}\big(D_{ma_f}(ma_g)\big)(\mathbf{u}) &= ma_g(u_1)ma_f(u_2,\ldots,u_{r+1})\\ &\quad + ma_g(u_1+\cdots+u_{r+1})ma_f(u_1,\cdots,u_r)\\ &\quad - ma_g(u_1+\cdots+u_{r+1})ma_f(u_2,\ldots,u_{r+1})\\ &\quad - ma_g(u_1)ma_f(u_2,\ldots,u_{r+1})\end{aligned}$$

$$\begin{aligned}&= ma_g(u_1+\cdots+u_{r+1})\Big(ma_f(u_1,\cdots,u_r) - ma_f(u_2,\ldots,u_{r+1})\Big)\\ &= (-1)^{b+r+n}(u_1+\cdots+u_{r+1})^{b-1}\big(u_2^{a_1-1}\cdots u_{r+1}^{a_r-1} - u_1^{a_1-1}\cdots u_r^{a_r-1}\big).\end{aligned}$$

This proves (3.6.6) since $g = C_b$, so $m = b$ and $s = 1$. ♢

Corollary. *Let $f \in \mathbb{F}_n$ be of homogeneous depth r and $g \in \mathbb{F}_m$ of homogeneous depth s. Let D_f be the derivation of $\mathbb{F}$ defined by $D_f(x) = 0$, $D_f(y) = [y, f]$. Then*

$$\iota_C(\{f,g\}) = (-1)^{n+r+m+s}\ ari(ma_f, ma_g). \tag{2.4.9}$$

Proof. By (3.6.6), we have

$$\begin{aligned}\iota_C(\{f,g\}) &= \iota_C(fg - gf) + \iota_C\big(D_f(g)\big) - \iota_C\big(D_g(f)\big)\\ &= (-1)^{n+r+m+s}\Big[ma_{fg} - ma_{gf} - D_{ma_f}(ma_g) + D_{ma_g}(ma_f)\Big]\end{aligned}$$

while by definition, we have

$$\begin{aligned}ari(ma_f, ma_g) &= S_{ma_g}(ma_f) - S_{ma_f}(ma_g)\\ &= D_{ma_g}(ma_f) + mu(ma_f, ma_g) - D_{ma_f}(ma_g) - mu(ma_g, ma_f).\end{aligned}$$

It remains only to note that for f and g monomials in the C_i, we have $ma_{fg} = mu(ma_f, ma_g)$. Since mu is distributive, this extends to sums of monomials f and g as in the statement.♢

3.7. Parity property of the depth filtration on ds

We now show how to use the basic results on ARI obtained in chapter 2 to give an Ecalle-style proof of the following result.

Theorem 3.7.1. *Considering $\mathfrak{ds}$ as a weight-graded vector space equipped with the depth filtration, we have*

$$\dim \mathfrak{ds}_n^d/\mathfrak{ds}_n^{d+1} = 0 \ \text{ if } \ n \not\equiv d \mod 2. \tag{3.7.1}$$

Although this result is fairly familiar "folklore", the only written proof appears to be the proof of an equivalent but differently formulated result which appears in [IKZ]. However, there is some work involved in translating the authors' formulation of their theorem back to the language of $\mathfrak{ds}$. In fact, [IKZ] define spaces $DSh_r(s)$ of polynomials in commutative variables, which can be shown to be isomorphic to the graded filtered quotients $\mathfrak{ds}^r_{r+s}/\mathfrak{ds}^{r+1}_{r+s}$, and they prove that $DSh_r(s) = 0$ if s is odd. Their proof uses some astute tricks with permutations. Instead, the proof given here, which does not seem to appear anywhere in Ecalle's papers but is constructed by putting together basic facts from the ARI universe, is a perfect example of the real simplicity and magic of Ecalle's methods.

Theorem 3.7.2. *Let $A \in ARI^{\underline{al}/\underline{al}}$ be a homogeneous polynomial mould of weight n, concentrated in depth d. If $n \not\equiv d$ mod 2, then $A = 0$.*

Proof. Let A be as in the statement; then by proposition 2.5.4, A is neg-invariant. But by the homogeneity,

$$neg(A)\begin{pmatrix} u_1 & \cdots & u_d \\ v_1 & \cdots & v_d \end{pmatrix} = (-1)^{n-d} A \begin{pmatrix} u_1 & \cdots & u_d \\ v_1 & \cdots & v_d \end{pmatrix},$$

which gives the result. $\diamondsuit$

Ecalle-style proof of theorem 3.7.1. Let $f \in \mathfrak{ds}_n$, let d be the depth of f (the minimal number d such that some monomial of f contains d y's), let f^d be the part of f of depth d, and let ma_{f^d} the associated bimould via (3.2.4). Then ma_f is alternal since f^d is a Lie element. Furthermore, $(f^d)_Y$ satisfies shuffle in the y_i since the stuffle relations consist of the shuffle relations plus terms of lower depth, but f^d contains no terms of depth lower than d, so $(f^d)_Y$ is also alternal, which means that $ma_{f^d} \in ARI_{al/al}$. If $d > 1$, then $ma_f \in ARI_{\underline{al}/\underline{al}}$; thus by theorem 3.7.2, if $n \not\equiv d$ mod 2, $ma_{f^d} = 0$, so $f^d = 0$; in other words there exist no elements of weight n and depth d of $\mathfrak{ds}$ if $n \not\equiv d$ mod 2.

In the case $d = 1$, another argument is needed, because the existence of the mould $A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = u_1^{n-1}$, for instance, shows that a bialternal mould concentrated in depth 1 even when n is even. HOLE

Appendix to chapter 3

Proof that Ecalle's ma/mi in (3.3.1) are equivalent to the ones defined in (3.2.4).

By (3.2.4), we have $mi\binom{0,\dots,0}{v_1,\dots,v_r} = \iota_Y(f_Y^r)$. Since mi is additive on moulds, we may assume that f is a monomial, $f = x^{a_0-1}y\cdots yx^{a_r-1}$. Then

$$\pi_Y(f) = \begin{cases} f & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\mathrm{ret}_X(\pi_Y(f)) = \begin{cases} x^{a_r-1}y\cdots x^{a_1-1}y & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_Y = \begin{cases} y_{a_r}\cdots y_{a_1} & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$mi\binom{0,\dots,0}{v_1,\dots,v_r} = \iota_Y(f_Y) = \begin{cases} v_1^{a_r-1}\cdots v_r^{a_1-1} & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now using Ecalle's definition, we have

$$vimo(z_0,\dots,z_r) = z_0^{a_0-1}z_1^{a_1-1}\cdots z_r^{a_r-1},$$

so

$$mi\binom{0,\dots,0}{v_1,\dots,v_r} = vimo(0,v_r,\dots,v_1) = \begin{cases} v_r^{a_1-1}\cdots v_1^{a_r-1} & \text{if } a_0 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The case of ma seems a bit more complicated. We can assume that f is a monomial $C_{a_1}\cdots C_{a_r}$ in the C_i. We will prove it by induction on r (though there might be a better way). For the base case, $r = 1$, we have $n = a_1$ and

$$f = C_{a_1} = \sum_{i=0}^{a_1-1}(-1)^i C_{a_1-1}^i x^{a_1-1-i}yx^i,$$

$$vimo(z_0,z_1) = \sum_{i=0}^{a_1-1}(-1)^i C_{a_1-1}^i z_0^{a_1-1-i}z_1^i,$$

$$vimo(0,u_1) = (-1)^{a_1-1}u_1^{a_1-1} = (-1)^{r+n}u_1^{a_1-1} = ma\binom{u_1}{0}$$

using Ecalle's definition, and comparing with the (3.2.4), we have

$$ma\binom{u_1}{0} = (-1)^{r+n}\iota_C(C_{a_1}) = (-1)^{r+n}u_1^{a_1-1},$$

which is the same.

Now make the induction hypothesis that the two definitions of ma coincide up to depth $r-1$, and let $f = C_{a_1} \cdots C_{a_{r-1}} C_{a_r}$. Using the definition of (3.2.4), we find that

$$ma_f\binom{u_1,\ldots,u_r}{0,\ldots,0} = (-1)^{r+n}\iota_C(f) = (-1)^{r+n} u_1^{a_1-1}\cdots u_r^{a_r-1}.$$

Let us write $g = C_{a_1}\cdots C_{a_{r-1}}$. Then using the definition from (3.2.4), we have

$$ma_f\binom{u_1,\ldots,u_r}{0,\ldots,0} = ma_g\binom{u_1,\ldots,u_{r-1}}{0,\ldots,0} ma_{C_{a_r}}\binom{u_r}{0}.$$

By the induction hypothesis, we have

$$\begin{cases} ma_{C_{a_r}} = vimo_{C_{a_r}}(0,u_r) = (-1)^{a_r-1}u_r^{a_r-1} \\ ma_g\binom{u_1,\ldots,u_{r-1}}{0,\ldots,0} = vimo_g(0,u_1,\ldots,u_1+\cdots+u_{r-1}). \end{cases}$$

So to prove that Ecalle's definition coincides with (3.2.4), we just have to show that

$$\begin{aligned} vimo_f(0,u_1,\ldots,u_1+\cdots+u_r) &= vimo_g(0,u_1,\ldots,u_1+\cdots+u_{r-1})vimo_{C_{a_r}}(0,u_r) \\ &= (-1)^{a_r-1}vimo_g(0,u_1,\ldots,u_1+\cdots+u_{r-1})u_r^{a_r-1}. \end{aligned}$$

Write

$$g = \sum_{\mathbf{a}=(a_0,\ldots,a_{r-1})} c_{\mathbf{a}} x^{a_0-1}y\cdots yx^{a_{r-1}-1}.$$

Then

$$vimo_g(z_0,\ldots,z_{r-1}) = \sum_{\mathbf{a}=(a_0,\ldots,a_{r-1})} c_{\mathbf{a}} z_0^{a_0-1}z_1^{a_1-1}\cdots z_{r-1}^{a_{r-1}-1},$$

and

$$vimo_g(0,u_1,\ldots,u_1+\cdots+u_{r-1}) = \sum_{\mathbf{a}=(1,a_1,\ldots,a_{r-1})} c_{\mathbf{a}} u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\ldots+u_{r-1})^{a_{r-1}-1},$$

Then

$$\begin{aligned} &vimo_g(0,u_1,\ldots,u_1+\cdots+u_{r-1})vimo_{C_{a_r}}(0,u_r) \\ &= (-1)^{a_r-1}\sum_{\mathbf{a}=(1,a_1,\ldots,a_{r-1})} c_{\mathbf{a}} u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\ldots+u_{r-1})^{a_{r-1}-1}u_r^{a_r-1}. \quad (A1) \end{aligned}$$

But also

$$f = gC_{a_r} = \sum_{\mathbf{a}=(a_0,\ldots,a_{r-1})}\sum_{j=0}^{a_r-1}(-1)^j C_{a_r-1}^j c_{\mathbf{a}} x^{a_0-1}y\cdots yx^{a_{r-1}-1}x^{a_r-1-j}yx^j,$$

so

$$vimo_f(z_0,\ldots,z_r) = \sum_{\mathbf{a}=(a_0,\ldots,a_{r-1})}\sum_{j=0}^{a_r-1}(-1)^j C_{a_r-1}^j c_{\mathbf{a}} z_0^{a_0-1}z_1^{a_1-1}\cdots z_{r-1}^{a_{r-1}-2+a_r-j}z_r^j,$$

so

$$vimo_f(0,z_1,\ldots,z_r)=\sum_{\mathbf{a}=(1,a_1,\ldots,a_r)}\sum_{j=0}^{a_r-1}(-1)^jC^j_{a_r-1}c_{\mathbf{a}}z_1^{a_1-1}z_2^{a_2-1}\cdots z_{r-1}^{a_{r-1}-2+a_r-j}z_r^j,$$

so finally

$$vimo_f(0,u_1,\ldots,u_1+\cdots+u_r)=$$

$$\sum_{\mathbf{a}=(1,a_1,\ldots,a_r)}\sum_{j=0}^{a_r-1}(-1)^jC^j_{a_r-1}c_{\mathbf{a}}u_1^{a_1-1}(u_1{+}u_2)^{a_2-1}\cdots(u_1{+}\cdots{+}u_{r-1})^{a_{r-1}-2+a_r-j}(u_1{+}\cdots{+}u_r)^j$$

$$=\sum_{\mathbf{a}=(1,a_1,\ldots,a_{r-1})}c_{\mathbf{a}}u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\cdots+u_{r-1})^{a_{r-1}-1}\cdot$$

$$\left(\sum_{j=0}^{a_r-1}(-1)^jC^j_{a_r-1}(u_1+\cdots+u_{r-1})^{a_r-j}(u_1+\cdots+u_r)^j\right)$$

$$=(-1)^{a_r-1}\sum_{\mathbf{a}=(1,a_1,\ldots,a_{r-1})}c_{\mathbf{a}}u_1^{a_1-1}(u_1+u_2)^{a_2-1}\cdots(u_1+\cdots+u_{r-1})^{a_{r-1}-1}\cdot u_r^{a_r-1}$$

since the factor between large parenthesis is just the binomial expansion of

$$\big((u_1+\ldots+u_{r-1})-(u_1+\cdots+u_r)\big)^{a_r-1}=(-1)^{a_r-1}u_r^{a_r-1}.$$

But this is equal to (A1), which concludes the proof.

Proof of Lemma 3.1. (1) Let $f \in \mathbb{F}_n$. We show that f satisfies shuffle if and only if ma is alternal. Assume that $f \in \mathbb{F}_n$ satisfies shuffle, set $vimo(z_0, z_1, \ldots, z_r) = \iota_X(f^r)$ for $r \geq 0$ and $ma(\binom{u_1,\ldots,u_r}{0,\ldots,0}) = vimo(0, u_1, \ldots, u_1 + \cdots + u_r)$. We have to prove that

$$\sum_{\mathbf{w} \in sh(\mathbf{w}', \mathbf{w}'')} (f|\mathbf{w}) = 0 \Leftrightarrow \sum_{\mathbf{u} \in sh(\mathbf{u}', \mathbf{u}'')} ma(\mathbf{u}) = 0,$$

where $\mathbf{w}$, $\mathbf{w}'$ and $\mathbf{w}''$ are words in the non-commutative variables x and y, and $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ are words in the commutative (but ordered) variables $u_1, u_2, \ldots$. For ma to be alternal, it is enough to consider all pairs

$$(\mathbf{u}', \mathbf{u}'') = \big((u_1, \ldots, u_s), (u_{s+1}, \ldots, u_r)\big) \tag{A2}$$

for $1 \leq s \leq r-1$. Indeed if the shuffle relations for these variables are satisfied, then any other variables can be substituted for these and the relation will still hold (cf. IKZ). The notation used in IKZ is convenient here: let $SH_s \subset S_r$ be the set of permutations $\sigma \in S_r$ such that $\sigma(1) < \ldots \sigma(s)$ and $\sigma(s+1) < \ldots \sigma(r)$; then using (A2), the shuffle relations, i.e. the condition of being alternal, can be written

$$\sum_{\sigma \in SH_s} ma(u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(r)}) = 0, \quad 1 \leq s \leq [r/2]. \tag{A3}$$

We know that

$$\iota_C^{-1}\big(ma(\binom{u_1, \ldots, u_r}{0, \ldots, 0})\big) = f_C^r,$$

i.e. writing

$$ma(\binom{u_1, \ldots, u_r}{0, \ldots, 0}) = \sum_{\substack{\mathbf{a}=(a_1,\ldots,a_r) \\ a_1+\cdots+a_r=n}} c_{\mathbf{a}}\, u_1^{a_1-1} \cdots u_r^{a_r-1}$$

we have

$$f_C^r = \sum_{\mathbf{a}=(a_1,\ldots,a_r)} c_{\mathbf{a}}\, C_{a_1} \cdots C_{a_r}.$$

Applying ι_C^{-1} to (A3) for $1 \leq s \leq [r/2]$ then yields

$$\begin{aligned}
0 &= \iota_C\Big(\sum_{\sigma \in SH_s} ma(u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(r)})\Big) \\
&= \sum_{\sigma \in SH_s} \sum_{\mathbf{a}=(a_1,\ldots,a_r)} c_{\mathbf{a}}\, \iota_C\big(u_{\sigma^{-1}(1)}^{a_1-1} \cdots u_{\sigma^{-1}(r)}^{a_r-1}\big) \\
&= \sum_{\sigma \in SH_s} \sum_{\mathbf{a}=(a_1,\ldots,a_r)} c_{\mathbf{a}}\, C_{a_{\sigma(1)}} \cdots C_{a_{\sigma(r)}} \\
&= \sum_{\sigma \in SH_s} f_C(C_{\sigma(1)}, \ldots, C_{\sigma(r)}).
\end{aligned}$$

In other words, we have shown that

$$ma \text{ alternal} \quad \Leftrightarrow \quad f_C \text{ satisfies shuffle},$$

so we just have to prove that

$$f \text{ satisfies shuffle for } x, y \quad \Leftrightarrow \quad f_C \text{ satisfies shuffle for the } C_i.$$

But this follows from the fact that f satisfies shuffle if and only if f is a Lie element, and for any weight $n > 1$, the weight n part of the Lie algebra $\mathrm{Lie}[x, y]$ is generated by C_n and weight n Lie brackets of the lower C_i. Therefore f is necessarily a Lie element in the C_i, which proves (1) of lemma 3.1.

(2) "Proof by example". In fact, the coefficients of the left-hand sides of the alternility relations are exactly the coefficients of the stuffle relations. We do it on an example that is sufficiently big to see exactly what happens.

Assume $n > 5$, so we can work in depth 4. Recall that f_Y is the polynomial $\mathrm{ret}_X(\pi_Y(f))$ written in the variables y_i. Write

$$f_Y = \sum_{r=0}^{n} f_Y^r = \sum_{r=0}^{n} \sum_{\substack{\mathbf{a}=(a_1,\ldots,a_r) \\ a_1+\cdots+a_r=n}} c_{\mathbf{a}} y_{a_1} \cdots y_{a_r}$$

so by (3.2.4), for $0 \le r \le n$ we have

$$mi\binom{0,\ldots,0}{v_1,\ldots,v_r} = \sum_{\substack{\mathbf{a}=(a_1,\ldots,a_r) \\ a_1+\cdots+a_r=n}} c_{\mathbf{a}} v_1^{a_1-1} \cdots v_r^{a_r-1}.$$

We first assume that mi is alternil. As explained above for the shuffle, it is enough to assume that mi satisfies all of the stuffle relations

$$st((w_1,\ldots,w_s),(w_{s+1},\ldots,w_r)) \quad 2 \le r \le n, \quad 1 \le s \le [r/2],$$

where $w_i = \binom{0}{v_i}$, since all others can be obtained from these by substitutions.

We want to show that if mi satisfies the relation $st((w_1,\ldots,w_s),(w_{s+1},\ldots,w_r))$, then f_Y satisfies all stuffle relations of the form $st((a_1,\ldots,a_s),(a_{s+1},\ldots,a_r))$ for integers a_i such that $a_1+\ldots+a_r = n$.

Let us take $r = 4$ and the stuffle relation $st((w_1,w_2),(w_3,w_4))$ as an example. For mi to satisfy this stuffle relation means that

$$mi\binom{u_1,u_2,u_3,u_4}{v_1,v_2,v_3,v_4} + mi\binom{u_1,u_3,u_2,u_4}{v_1,v_3,v_2,v_4} + mi\binom{u_1,u_3,u_4,u_2}{v_1,v_3,v_4,v_2}$$

$$+mi\binom{u_3,u_1,u_2,u_4}{v_3,v_1,v_2,v_4} + mi\binom{u_3,u_1,u_4,u_2}{v_3,v_1,v_4,v_2} + mi\binom{u_3,u_4,u_1,u_2}{v_3,v_4,v_1,v_2}$$

$$+\frac{1}{v_2-v_3}mi(\begin{smallmatrix}u_1,u_2+u_3,u_4\\ v_1,v_2,v_4\end{smallmatrix})+\frac{1}{v_3-v_2}mi(\begin{smallmatrix}u_1,u_2+u_3,u_4\\ v_1,v_3,v_4\end{smallmatrix})$$

$$+$$

$$\frac{1}{v_1-v_3}mi(\begin{smallmatrix}u_1+u_3,u_2,u_4\\ v_1,v_2,v_4\end{smallmatrix})+\frac{1}{v_3-v_1}mi(\begin{smallmatrix}u_1+u_3,u_2,u_4\\ v_3,v_2,v_4\end{smallmatrix})$$

$$+\frac{1}{v_1-v_3}mi(\begin{smallmatrix}u_1+u_3,u_4,u_2\\ v_1,v_4,v_2\end{smallmatrix})+\frac{1}{v_3-v_1}mi(\begin{smallmatrix}u_1+u_3,u_4,u_2\\ v_3,v_4,v_2\end{smallmatrix})$$

$$+\frac{1}{v_2-v_4}mi(\begin{smallmatrix}u_1,u_3,u_2+u_4\\ v_1,v_3,v_2\end{smallmatrix})+\frac{1}{v_4-v_2}mi(\begin{smallmatrix}u_1,u_3,u_2+u_4\\ v_1,v_3,v_4\end{smallmatrix})$$

$$+\frac{1}{v_2-v_4}mi(\begin{smallmatrix}u_3,u_1,u_2+u_4\\ v_3,v_1,v_2\end{smallmatrix})+\frac{1}{v_4-v_2}mi(\begin{smallmatrix}u_3,u_1,u_2+u_4\\ v_3,v_1,v_4\end{smallmatrix})$$

$$+\frac{1}{v_1-v_4}mi(\begin{smallmatrix}u_3,u_1+u_4,u_2\\ v_3,v_1,v_2\end{smallmatrix})+\frac{1}{v_4-v_1}mi(\begin{smallmatrix}u_3,u_1+u_4,u_2\\ v_3,v_4,v_2\end{smallmatrix})$$

$$+\frac{1}{(v_1-v_3)(v_2-v_4)}mi(\begin{smallmatrix}u_1+u_3,u_2+u_4\\ v_1,v_2\end{smallmatrix})+\frac{1}{(v_1-v_3)(v_4-v_2)}mi(\begin{smallmatrix}u_1+u_3,u_2+u_4\\ v_1,v_4\end{smallmatrix})$$

$$+\frac{1}{(v_3-v_1)(v_2-v_4)}mi(\begin{smallmatrix}u_1+u_3,u_2+u_4\\ v_3,v_2\end{smallmatrix})+\frac{1}{(v_3-v_1)(v_4-v_2)}mi(\begin{smallmatrix}u_1+u_3,u_2+u_4\\ v_3,v_4\end{smallmatrix})=0.$$

Note that we have written in all the u_i here to make sure the alternil condition is correct, but in fact all the $u_i=0$ in our situation.

This is then equivalent to

$$\sum_{(a_1,a_2,a_3,a_4)} c_{\mathbf{a}}\big(v_1^{a_1-1}v_2^{a_2-1}v_3^{a_3-1}v_4^{a_4-1}+v_1^{a_1-1}v_3^{a_2-1}v_2^{a_3-1}v_4^{a_4-1}+v_1^{a_1-1}v_3^{a_2-1}v_4^{a_3-1}v_2^{a_4-1}$$

$$+v_3^{a_1-1}v_1^{a_2-1}v_2^{a_3-1}v_4^{a_4-1}+v_3^{a_1-1}v_1^{a_2-1}v_4^{a_3-1}v_2^{a_4-1}+v_3^{a_1-1}v_4^{a_2-1}v_1^{a_3-1}v_2^{a_4-1}\big)$$

$$\sum_{(b_1,b_2,b_3)} c_{\mathbf{b}}\Bigg[\frac{1}{v_2-v_3}\big(v_1^{b_1-1}v_2^{b_2-1}v_4^{b_3-1}-v_1^{b_1-1}v_3^{b_2-1}v_4^{b_3-1}\big)$$

$$+\frac{1}{v_1-v_3}\big(v_1^{b_1-1}v_2^{b_2-1}v_4^{b_3-1}-v_3^{b_1-1}v_2^{b_2-1}v_4^{b_3-1}\big)$$

$$+\frac{1}{v_1-v_3}\big(v_1^{b_1-1}v_4^{b_2-1}v_2^{b_3-1}-v_3^{b_1-1}v_4^{b_2-1}v_2^{b_3-1}\big)$$

$$+\frac{1}{v_2-v_4}\big(v_1^{b_1-1}v_3^{b_2-1}v_2^{b_3-1}-v_1^{b_1-1}v_3^{b_2-1}v_4^{b_3-1}\big)$$

$$+\frac{1}{v_2-v_4}\big(v_3^{b_1-1}v_1^{b_2-1}v_2^{b_3-1}-v_3^{b_1-1}v_1^{b_2-1}v_4^{b_3-1}\big)$$

$$+\frac{1}{v_1-v_4}\big(v_3^{b_1-1}v_1^{b_2-1}v_2^{b_3-1}-v_3^{b_1-1}v_4^{b_2-1}v_2^{b_3-1}\big)\Bigg]$$

$$+\sum_{(c_1,c_2)} c_{\mathbf{c}}\left[\frac{1}{(v_1-v_3)(v_2-v_4)}\big(v_1^{c_1-1}v_2^{c_2-1}-v_1^{c_1-1}v_4^{c_2-1}-v_3^{c_1-1}v_2^{c_2-1}+v_3^{c_1-1}v_4^{c_2-1}\big)\right]=0.$$

We write it as

$$\sum_{(a_1,a_2,a_3,a_4)} c_{\mathbf{a}}\big(v_1^{a_1-1}v_2^{a_2-1}v_3^{a_3-1}v_4^{a_4-1}+v_1^{a_1-1}v_3^{a_2-1}v_2^{a_3-1}v_4^{a_4-1}+v_1^{a_1-1}v_3^{a_2-1}v_4^{a_3-1}v_2^{a_4-1}$$

$$+v_3^{a_1-1}v_1^{a_2-1}v_2^{a_3-1}v_4^{a_4-1}+v_3^{a_1-1}v_1^{a_2-1}v_4^{a_3-1}v_2^{a_4-1}+v_3^{a_1-1}v_4^{a_2-1}v_1^{a_3-1}v_2^{a_4-1}\big)$$

$$\sum_{(b_1,b_2,b_3)} c_{\mathbf{b}}\Bigg[v_1^{b_1-1}v_4^{b_3-1}\Big(\frac{v_2^{b_2-1}-v_3^{b_2-1}}{v_2-v_3}\Big)+v_2^{b_2-1}v_4^{b_3-1}\Big(\frac{v_1^{b_1-1}-v_3^{b_1-1}}{v_1-v_3}\Big)$$

$$+v_4^{b_2-1}v_2^{b_3-1}\Big(\frac{v_1^{b_1-1}-v_3^{b_1-1}}{v_1-v_3}\Big)+v_1^{b_1-1}v_3^{b_2-1}\Big(\frac{v_2^{b_3-1}-v_4^{b_3-1}}{v_2-v_4}\Big)$$

$$+v_3^{b_1-1}v_1^{b_2-1}\Big(\frac{v_2^{b_3-1}-v_4^{b_3-1}}{v_2-v_4}\Big)+v_3^{b_1-1}v_2^{b_3-1}\Big(\frac{v_1^{b_2-1}-v_4^{b_2-1}}{v_1-v_4}\Big)\Bigg]$$

$$+\sum_{(c_1,c_2)} c_{\mathbf{c}}\left[\Big(\frac{v_1^{c_1-1}-v_3^{c_1-1}}{v_1-v_3}\Big)\Big(\frac{v_2^{c_2-1}-v_4^{c_2-1}}{v_2-v_4}\Big)\right]=0.$$

This is actually a polynomial expression which can be written as

$$\sum_{(a_1,a_2,a_3,a_4)} c_{\mathbf{a}}\big(v_1^{a_1-1}v_2^{a_2-1}v_3^{a_3-1}v_4^{a_4-1}+v_1^{a_1-1}v_2^{a_3-1}v_3^{a_2-1}v_4^{a_4-1}+v_1^{a_1-1}v_2^{a_4-1}v_3^{a_2-1}v_4^{a_3-1}$$

$$+v_1^{a_2-1}v_2^{a_3-1}v_3^{a_1-1}v_4^{a_4-1}+v_1^{a_2-1}v_2^{a_4-1}v_3^{a_1-1}v_4^{a_3-1}+v_1^{a_3-1}v_2^{a_4-1}v_3^{a_1-1}v_4^{a_2-1}\big)$$

$$\sum_{(b_1,b_2,b_3)} c_{\mathbf{b}}\Bigg[\sum_{i=0}^{b_2-2} v_1^{b_1-1}v_2^{b_2-2-i}v_3^{i}v_4^{b_3-1}+\sum_{i=0}^{b_1-2} v_1^{b_1-2-i}v_2^{b_2-1}v_3^{i}v_4^{b_3-1}$$

$$+\sum_{i=0}^{b_1-2} v_1^{b_1-2-i}v_2^{b_3-1}v_3^{i}v_4^{b_2-1}+\sum_{i=0}^{b_3-2} v_1^{b_1-1}v_2^{b_3-2-i}v_3^{b_2-1}v_4^{i}$$

$$+\sum_{i=0}^{b_3-2} v_1^{b_2-1}v_2^{b_3-2-i}v_3^{b_1-1}v_4^{i}+\sum_{i=0}^{b_2-2} v_1^{b_2-2-i}v_2^{b_3-1}v_3^{b_1-1}v_4^{i}\Bigg]$$

$$+\sum_{(c_1,c_2)} c_{\mathbf{c}}\sum_{i=0}^{c_1-2}\sum_{j=0}^{c_2-2}\left[v_1^{c_1-2-i}v_2^{c_2-2-j}v_3^{i}v_4^{j}\right]=0. \qquad (A4)$$

Now choose any four integers $a, b, c, d \geq 1$ with $a+b+c+d=n$. They are not all equal to 1 since by definition, the depth 4 conditions are not used in weight $n=4$, so we are assuming that $n > 5$.

We want to show that the coefficient of the monomial $V = v_1^{a-1}v_2^{b-1}v_3^{c-1}v_4^{d-1}$ in the polynomial (A4), being equal to 0, implies exactly the stuffle relation $st((a,b)(c,d))$ on the coefficients of f_Y. It is enough to simply calculate the contribution to this coefficient from each term from (A4). We obtain:

$$c_{a,b,c,d} + c_{a,c,b,d} + c_{a,c,d,b} + c_{c,a,b,d} + c_{c,a,d,b} + c_{c,d,a,b}$$

$$+c_{a,b+c,d} + c_{a+c,b,d} + c_{a+c,d,b} + c_{a,c,b+d} + c_{c,a,b+d} + c_{c,a+d,b}$$
$$+c_{a+c,b+d} = 0.$$

This essentially illustrates the general case. I will eventually write it down in general with proper notation, but I really think this calculation makes it clearer than that would.

Chapter 4

Commutation of *swap* with the *ari* operators

§4.1. The fundamental identities - ARI situation

Recall the definitions

$$amit(B)\cdot A=\sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor),$$

$$anit(B)\cdot A=\sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}),$$

$$axit(B,C)\cdot A=amit(B)\cdot A+anit(C)\cdot A$$

$$arit(B)\cdot A=axit(B,-B)\cdot A=amit(B)\cdot A-anit(B)\cdot A.$$

In this chapter we investigate the behavior of these operators with respect to the *swap*. The identities (4.1.1)-(4.1.2) and (4.1.7)-(4.1.9) are the main results of the section concerning operators on the Lie algebra ARI. The following section §4.2 will turn to similar identities in the GARI situation above all Ecalle's first fundamental identity, given in Theorem 4.2.1.

§4.1.1. The first two identities

In this section we prove the two identities:

$$swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big)=amit(B)\cdot A+mu(A,B)-swap\Big(mu\big(swap(A),swap(B)\big)\Big), \tag{4.1.1}$$

$$swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big)=anit\big(push(B)\big)\cdot A. \tag{4.1.2}$$

We have the following explicit expressions for the flexions occurring in the definition of *axit*:

$$\mathbf{a}\lceil\mathbf{c}=\begin{pmatrix}u_1&\cdots&u_k\\ v_1&\cdots&v_k\end{pmatrix}\begin{pmatrix}u_{k+1}+\cdots+u_{k+l+1}&\cdots&u_r\\ v_{k+l+1}&\cdots&v_r\end{pmatrix},$$

$$\mathbf{b}\rfloor=\begin{pmatrix}u_{k+1}&\cdots&u_{k+l}\\ v_{k+1}-v_{k+l+1}&\cdots&v_{k+l}-v_{k+l+1}\end{pmatrix}$$

$$\mathbf{a}\rceil\mathbf{c}=\begin{pmatrix}u_1&\cdots&u_{k-1}&u_k+\cdots+u_{k+l}\\ v_1&\cdots&v_{k-1}&v_k\end{pmatrix}\begin{pmatrix}u_{k+l+1}&\cdots&u_r\\ v_{k+l+1}&\cdots&v_r\end{pmatrix}.$$

$$\lfloor\mathbf{b}=\begin{pmatrix}u_{k+1}&\cdots&u_{k+l}\\ v_{k+1}-v_k&\cdots&v_{k+l}-v_k\end{pmatrix}.$$

Setting $SC = swap(C)$ for any mould C, we have

$$SC(\mathbf{a}\lceil\mathbf{c}) = SC\begin{pmatrix} u_1 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r \end{pmatrix}$$

$$= C\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k - v_{k+l+1} & v_{k-1}-v_k & \cdots \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_k & u_1+\cdots+u_{k-1} & \cdots \end{matrix}$$

$$SC(\mathbf{b}\rfloor) = SC\begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1} \end{pmatrix}$$

$$= C\begin{pmatrix} v_{k+l}-v_{k+l+1} & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}$$

$$SC(\mathbf{a}\rceil\mathbf{c}) = SC\begin{pmatrix} u_1 & \cdots & u_{k-1} & u_k+\cdots+u_{k+l} & u_{k+l+1} & \cdots & u_r \\ v_1 & \cdots & v_{k-1} & v_k & v_{k+l+1} & \cdots & v_r \end{pmatrix}$$

$$= C\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k - v_{k+l+1} & v_{k-1}-v_k & \cdots \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_{k+l} & u_1+\cdots+u_{k-1} & \cdots \end{matrix}$$

$$SC(\lfloor\mathbf{b}) = SC\begin{pmatrix} u_{k+1} & \cdots & u_{k+l} \\ v_{k+1}-v_k & \cdots & v_{k+l}-v_k \end{pmatrix}$$

$$= C\begin{pmatrix} v_{k+l}-v_k & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}$$

Applying the swap

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} \mapsto \begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_1-v_2 \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1 \end{pmatrix},$$

i.e.

$$\begin{cases} u_1 \mapsto v_r \\ u_i \mapsto v_{r-i+1}-v_{r-i+2}, \text{ if } i > cru_1+\cdots+u_i \mapsto v_{r-i+1} \\ u_i+\cdots+u_j \mapsto -v_{r-i+2}+v_{r-j+1} \text{ if } i<j \\ v_i \mapsto u_1+\cdots+u_{r-i+1} \\ v_i - v_{i+1} \mapsto u_{r-i+1} \\ v_i - v_j \mapsto u_{r-j+2}+\cdots+u_{r-i+1} \text{ if } i<j \\ v_i - v_j \mapsto -u_{r-i+2}-\cdots-u_{r-j+1} \text{ if } i>j \end{cases}$$

to these four terms yields

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1}+\cdots+u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{r-k-l+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1}-v_{r-k+1} & v_{r-k-l+2}-v_{r-k} & \cdots & v_{r-k}-v_{r-k+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-k-l} & u_{r-k-l+1}\cdots+u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{r-k-l} & v_{r-k-l+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{r-k-l+2}-\cdots-u_{r-k+1} & u_{r-k-l+2} & \cdots & u_{r-k} \\ v_{r-k-l+1}-v_{r-k+1} & v_{r-k-l+2}-v_{r-k+1} & \cdots & v_{r-k}-v_{r-k+1} \end{pmatrix}$$

Setting $m = r - k - l$, they can be written as

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}+\cdots+u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{r-k+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1}-v_{r-k+1} & v_{m+2}-v_{r-k} & \cdots & v_{r-k}-v_{r-k+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots+u_{r-k+1} & u_{r-k+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{r-k+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{m+2}-\cdots-u_{r-k+1} & u_{m+2} & \cdots & u_{r-k} \\ v_{m+1}-v_{r-k+1} & v_{m+2}-v_{r-k+1} & \cdots & v_{r-k}-v_{r-k+1} \end{pmatrix}$$

Now putting $r - k = m + l$ gives

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}+\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+l+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_{m+l+1} & v_{m+2}-v_{m+l+1} & \cdots & v_{m+l}-v_{m+l+1} \end{pmatrix}$$

$$C\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$C\begin{pmatrix} -u_{m+2}-\cdots-u_{m+l+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_{m+l+1} & v_{m+2}-v_{m+l+1} & \cdots & v_{m+l}-v_{m+l+1} \end{pmatrix}$$

We can now prove (4.1.1) and (4.1.2).

Proof of (4.1.1). We have

$$swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big) = swap\Big(\sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} SA(\mathbf{a}\lceil\mathbf{c})SB(\mathbf{b}\rfloor)\Big)$$

$$= swap\Big[\sum_{l=1}^{r-1}\sum_{m=1}^{r-l}$$

$$A\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k-v_{k+l+1} & v_{k-1}-v_k & \cdots & v \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_k & u_1+\cdots+u_{k-1} & \cdots & \end{matrix}$$

$$\cdot B\begin{pmatrix} v_{k+l}-v_{k+l+1} & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}\Big]$$

$$= \sum_{l=1}^{r-1}\sum_{m=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}+\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+l+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_{m+l+1} & v_{m+2}-v_{m+l+1} & \cdots & v_{m+l}-v_{m+l+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1}\sum_{k=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & v_{k+2}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1}\sum_{k=0}^{r-l-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_k & u_{k+1}+\cdots+u_{k+l+1} & u_{k+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_k & v_{k+l+1} & v_{k+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot B\begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1}-v_{k+l+1} & v_{k+2}-v_{k+l+1} & \cdots & v_{k+l}-v_{k+l+1} \end{pmatrix}$$

$$-\sum_{l=1}^{r-1} A\begin{pmatrix} u_1+\cdots+u_{l+1} & u_{l+2} & \cdots & u_r \\ v_{l+1} & v_{l+2} & \cdots & v_r \end{pmatrix} \cdot B\begin{pmatrix} u_1 & u_2 & \cdots & u_l \\ v_1-v_{l+1} & v_2-v_l & \cdots & v_l-v_{l+1} \end{pmatrix}$$

$$+\sum_{l=1}^{r-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_{r-l} \\ v_1 & v_2 & \cdots & v_{r-l} \end{pmatrix} \cdot B\begin{pmatrix} u_{r-l+1} & u_{r-l+2} & \cdots & u_r \\ v_{r-l+1} & v_{r-l+2} & \cdots & v_r \end{pmatrix}$$

$$= amit(B)\cdot A - swap\Big(mu\big(swap(A), swap(B)\big)\Big) + mu(A,B).$$

Proof of (4.1.2). We have

$$swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big) = swap\Big(\sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{a},\mathbf{b}\neq\emptyset}} SA(\mathbf{a}\rceil\mathbf{c})SB(\lfloor\mathbf{b})\Big)$$

$$= swap\Big[\sum_{l=1}^{r-1}\sum_{k=1}^{r-l}$$

$$A\begin{pmatrix} v_r & v_{r-1}-v_r & \cdots & v_{k+l+1}-v_{k+l+2} & v_k-v_{k+l+1} & v_{k-1}-v_k & \cdots \\ u_1+\cdots+u_r & u_1+\cdots+u_{r-1} & \cdots & u_1+\cdots+u_{k+l+1} & u_1+\cdots+u_{k+l} & u_1+\cdots+u_{k-1} & \cdots \end{pmatrix}$$

$$\cdot B\begin{pmatrix} v_{k+l}-v_k & v_{k+l-1}-v_{k+l} & \cdots & v_{k+1}-v_{k+2} \\ u_{k+1}+\cdots+u_{k+l} & u_{k+1}+\cdots+u_{k+l-1} & \cdots & u_{k+1} \end{pmatrix}\Big]$$

$$= \sum_{l=1}^{r-1}\sum_{m=0}^{r-l-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot B\begin{pmatrix} -u_{m+2}-\cdots-u_{m+l+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_{m+l+1} & v_{m+2}-v_{m+l+1} & \cdots & v_{m+l}-v_{m+l+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1}\sum_{m=0}^{r-l-1} A\begin{pmatrix} u_1 & u_2 & \cdots & u_m & u_{m+1}\cdots+u_{m+l+1} & u_{m+l+2} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+l+2} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B)\begin{pmatrix} u_{m+2} & u_{m+3} & \cdots & u_{m+l+1} \\ v_{m+2}-v_{m+1} & v_{m+3}-v_{m+1} & \cdots & v_{m+l+1}-v_{m+1} \end{pmatrix}$$

$$= \sum_{l=1}^{r-1}\sum_{m=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_{m-1} & u_m\cdots+u_{m+l} & u_{m+l+1} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{m-1} & v_m & v_{m+l+1} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B)\begin{pmatrix} u_{m+1} & u_{m+2} & \cdots & u_{m+l} \\ v_{m+1}-v_m & v_{m+2}-v_m & \cdots & v_{m+l}-v_m \end{pmatrix}$$

$$= \sum_{l=1}^{r-1}\sum_{k=1}^{r-l} A\begin{pmatrix} u_1 & u_2 & \cdots & u_{k-1} & u_k\cdots+u_{k+l} & u_{k+l+1} & \cdots & u_r \\ v_1 & v_2 & \cdots & v_{k-1} & v_k & v_{k+l+1} & \cdots & v_r \end{pmatrix}$$

$$\cdot push(B)\begin{pmatrix} u_{k+1} & u_{k+2} & \cdots & u_{k+l} \\ v_{k+1}-v_k & v_{k+2}-v_k & \cdots & v_{k+l}-v_k \end{pmatrix}$$

$$= anit\big(push(B)\big)\cdot A.$$

§4.1.2. Other linear identities

The identities (4.1.1) and (4.1.2) show that

$$\begin{aligned} swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big) &= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ a,b\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) \\ &= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ \mathbf{b},\mathbf{c}\neq\emptyset}} A(\mathbf{a}\lceil\mathbf{c})B(\mathbf{b}\rfloor) + \sum_{\mathbf{w}=\mathbf{ab}} A(\mathbf{a})B(\mathbf{b}) - \sum_{w=bc} A(\lceil\mathbf{c})B(\mathbf{b}\rfloor) \\ &= amit(B)\cdot A + mu(A,B) - swamu(A,B), \end{aligned}$$

proving (4.1.1), and

$$\begin{aligned} swap\Big(anit\big(swap(B)\big)\cdot swap(A)\Big) &= \sum_{\substack{\mathbf{w}=\mathbf{abc}\\ a,b\neq\emptyset}} A(\mathbf{a}\rceil\mathbf{c})B(\lfloor\mathbf{b}) \\ &= anit\big(push(B)\big)\cdot A, \end{aligned}$$

proving (4.1.2).

Since we have

$$axit(B,C)\cdot A = anit(B)\cdot A + amit(C)\cdot A, \tag{4.1.3}$$

$$arit(B)\cdot A = axit(B,-B)\cdot A = anit(B)\cdot A - amit(B)\cdot A, \tag{4.1.4}$$

$$preari(A,B)=arit(B)\cdot A+mu(A,B) \tag{4.1.5}$$

and

$$ari(A,B)=preari(A,B)-preari(B,A), \tag{4.1.6}$$

the identities (4.1.1) and (4.1.2) immediately yield

$$\begin{aligned} swap\Big(arit\big(swap(B)\big)\cdot swap(A)\Big) &= swap\Big(amit\big(swap(B)\big)\cdot swap(A)\Big)-swap\Big(anit\big(swap(B)\big)\cdot swa \\ &= amit(B)\cdot A+mu(A,B)-swap\Big(mu\big(swap(A),swap(B)\big)\Big)-anit\big(push(B)\big)\cdot A \\ &= axit\big(B,-push(B)\big)\cdot A+mu(A,B)-swamu(A,B) \end{aligned} \tag{4.1.7}$$

$$\begin{aligned} swap\Big(preari\big(swap(A),swap(B)\big)\Big) &= swap\Big(arit\big(swap(B)\big)\cdot A\Big)+swamu(A,B) \\ &= axit\big(B,-push(B)\big)\cdot A+mu(A,B) \\ &= amit\big(B\big)\cdot A+anit\big(-push(B)\big)\cdot A+mu(A,B) \\ &= arit\big(B\big)\cdot A+anit\big(B-push(B)\big)\cdot A+mu(A,B) \\ &= preari(A,B)+anit\big(B-push(B)\big)\cdot A, \end{aligned} \tag{4.1.8}$$

and finally from (4.1.6) and (4.1.8),

$$\begin{aligned} swap\Big(ari\big(swap(A),swap(B)\big)\Big) &= axit\big(B,-push(B)\big)\cdot A+mu(A,B)-axit\big(A,-push(A)\big)\cdot B-m \\ &= axi\Big(\big(A,-push(A)\big),\big(B,-push(B)\big)\Big)_L. \end{aligned} \tag{4.1.9}$$

We also have

$$preawi(A,B)=awit(B)\cdot A+mu(A,B)=amit(B)\cdot A+anit\big(anti\cdot neg(B)\big)\cdot A+mu(A,B),$$

and

$$\begin{aligned} swap\Big(preawi\big(swap(A),swap(B)\big)\Big) &= swap\Big(amit(swap(B))\cdot swap(A)\Big) \\ &\quad + swap\Big(anit\big(anti\cdot neg(swap(B))\big)\cdot swap(A)\Big)+swap\Big(mu(\\ &= amit(B)\cdot A+anit\big(push\cdot swap\cdot anti\cdot neg\cdot swap(B)\big)\cdot A+n \\ &= amit(B)\cdot A+anit\big(anti(B)\big)\cdot A+mu(A,B) \\ &= iwat(B)\cdot A+mu(A,B) \\ &= preiwa(A,B), \end{aligned}$$

with the definition

$$iwat(B)\cdot A=axit\big(B,anti(B)\big)\cdot A.$$

§4.2. The fundamental identity – GARI situation

The goal of this section is to prove the identity in Theorem 4.2.1. We need some work before we get there.

§4.2.1. A preliminary identity

$$r(\bullet)\cdot \ddot{O}_* = iwat(T\ddot{o})\cdot \ddot{O}_* + mu(\ddot{O}_*, T\ddot{o}) + mu(anti\cdot T\ddot{o}, \ddot{O}_*), \tag{4.2.1}$$

which is identity (11.97) of "Flexions", and use it to prove the fundamental separation identity

$$mu\big(anti(pal), pal\big)(u_1,\ldots,u_r) = \frac{1}{r!}\frac{1}{u_1\cdots u_r}. \tag{4.2.2}$$

The main point in the proof of (4.2.1) is the explicit computation of

$$iwat(T\ddot{o})\cdot \ddot{O}_*(u_1,\ldots,u_r).$$

Ecalle suggests first computing the individual terms

$$iwat(r\ddot{o}_p)\cdot \ddot{O}(u_1,\ldots,u_r)$$

with $r\geq p$, $r\ddot{o}_p = swap(re_p)$,

$$r\ddot{o}_p(u_1,\ldots,u_p) = \frac{ru_1+(r-1)u_2+\cdots+u_r}{u_1\cdots u_r(u_1+\cdots+u_r)},$$

$$T\ddot{o} = \sum_{r\geq 1}\frac{-1}{(r+1)!}r\ddot{o}_r, \tag{4.2.3}$$

$$\ddot{O}(u_1,\ldots,u_r) = \frac{1}{u_1\cdots u_r}$$

and

$$\ddot{O}_*(u_1,\ldots,u_r) = (-1)^{r-1}\frac{1}{r!}\ddot{O}(u_1,\ldots,u_r).$$

When you do this computation by computer, you appear to find the general expression:

$$\big(iwat(r\ddot{o}_p)\cdot\ddot{O}\big)(u_1,\ldots,u_{p+q}) = \frac{Q_p^r(u_1,\ldots,u_r)}{(u_1+\cdots+u_p)u_1\cdots u_{p+q}(u_{1+q}+\cdots+u_{p+q})} \tag{4.2.4}$$

where

$$Q_p^r(u_1,\ldots,u_r) = \sum_{i=1}^{p}\sum_{j=1}^{p}(r-p+j-i)u_iu_{r-p+j}. \tag{4.2.5}$$

Accepting this momentarily, let's prove (4.2.1). First, we find that

$$\big(iwat(r\ddot{o}_p)\cdot\ddot{O}_*\big)(u_1,\ldots,u_r) = (-1)^{r-p+1}\frac{1}{(r-p)!}\frac{Q_p^r(u_1,\ldots,u_r)}{(u_1+\cdots+u_p)u_1\cdots u_r(u_{r-p+1}+\cdots+u_r)}. \tag{4.2.6}$$

Now we add up these terms as in (4.2.3) to get

$$\begin{aligned}
iwat(T\ddot{o})\cdot\ddot{O}_*(u_1,\dots,u_r) &= -\sum_{p\geq 1}\frac{1}{(p+1)!}iwat(r\ddot{o}_p)\cdot\ddot{O}_*(u_1,\dots,u_r)\\
&= \sum_{p\geq 1}\frac{(-1)^{r-p}}{(p+1)!(r-p)!}\frac{Q_p^r(u_1,\dots,u_r)}{(u_1+\cdots+u_p)u_1\cdots u_r(u_{r-p+1}+\cdots+u_r)}\\
&= \sum_{p=1}^{r-1}\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{(-1)^{r-p}(r-p+j-i)}{(p+1)!(r-p)!}\frac{u_iu_{r-p+j}}{(u_1+\cdots+u_p)u_1\cdots u_r(u_{r-p+1}+\cdots
\end{aligned}$$

HOLE: Finish up this elementary proof.

Assuming (4.2.1), we will now prove (4.2.2). The first thing to notice is that in depth r, only terms of $\ddot{O}_*$ of depth up to $r-1$ occur, because $T\ddot{o}(\emptyset)=0$. So we will use (4.2.1) to prove (4.2.2) by induction.

We can't use $r=0$ for the base case, because $r(\bullet)\ddot{O}_*(\emptyset)=0$ whereas $gepar(pil)(\emptyset)=1$. So we'll use $r=1$ for the base case. Here we have

$$r(\bullet)\ddot{O}_*(u_1)=\ddot{O}_*(u_1)=-\frac{1}{u_1}$$

and since $iwat(T\ddot{o})\cdot\ddot{O}_*(u_1)=0$, we only need to check that

$$mu(\ddot{O}_*,T\ddot{o})(u_1)+mu(anti\cdot T\ddot{o},\ddot{O}_*)=\ddot{O}_*(\emptyset)T\ddot{o}(u_1)+anti\cdot T\ddot{o}(u_1)\ddot{O}_*(\emptyset)=-\frac{1}{u_1}.$$

We can do $r=2$ just for fun:

$$r(\bullet)\ddot{O}_*(u_1,u_2)=-\frac{1}{u_1u_2},$$

and

$$\begin{cases}
iwat(T\ddot{o})\cdot\ddot{O}_*(u_1,u_2)=\frac{1}{2u_1u_2}\\
mu(\ddot{O}_*,T\ddot{o})(u_1,u_2)=\frac{1}{6}\frac{u_1+2u_2}{u_1u_2(u_1+u_2)}\\
mu(anti\cdot T\ddot{o},\ddot{O}_*)(u_1,u_2)=\frac{1}{6}\frac{2u_1+u_2}{u_1u_2(u_1+u_2)},
\end{cases}$$

so the right-hand side of (4.2.1) in depth 2 is indeed also equal to $\frac{1}{u_1u_2}$.

Now we assume that $gepar(pil)=r(\bullet)\ddot{O}_*$ up through depth $r-1$. Then by (4.2.1) in depth r, using the facts

- $iwat(T\ddot{o})$ is a derivation for mu
- the identity $anti\cdot mu(anti\cdot A, anti\cdot B)=mu(B,A)$ and
- $anti\cdot iwat(T\ddot{o})\cdot anti=iwat(T\ddot{o})$, and finally
- $r(\bullet)pal=preiwa(pal,T\ddot{o})$, we have

$$
\begin{aligned}
r\ddot{O}_*(u_1,\ldots,u_r) &= \Bigg(\big(iwat(T\ddot{o})\cdot \ddot{O}_*\big) + mu(\ddot{O}_*, T\ddot{o}) + mu(anti\cdot T\ddot{o}, \ddot{O}_*)\Bigg)(u_1,\ldots,u_r) \\
&= \Bigg(\big(iwat(T\ddot{o})\cdot gepar(pil)\big) + mu\big(gepar(pil), T\ddot{o}\big) + mu\big(anti\cdot T\ddot{o}, gepar(pil)\big)\Bigg)(u_1, \ldots \\
&= \Bigg(\big(iwat(T\ddot{o})\cdot mu(anti\cdot pal, pal)\big) + mu(anti\cdot pal, pal, T\ddot{o}) + mu(anti\cdot T\ddot{o}, anti\cdot p \ldots \\
&= \Bigg(mu(iwat(T\ddot{o})\cdot anti\cdot pal, pal) + mu(anti\cdot pal, iwat(T\ddot{o})\cdot pal) \\
&\qquad + mu(anti\cdot pal, pal, T\ddot{o}) + mu(anti\cdot T\ddot{o}, anti\cdot pal, pal)\Bigg)(u_1,\ldots,u_r) \\
&= \Bigg(anti\cdot mu(anti\cdot pal, anti\cdot iwat(T\ddot{o})\cdot anti\cdot pal) + mu\big(anti\cdot pal, preiwa(pal, T\ddot{o})\big) \ldots \\
&\qquad + anti\cdot mu(anti\cdot pal, pal, T\ddot{o})\Bigg)(u_1,\ldots,u_r) \\
&= \Bigg(anti\cdot mu(anti\cdot pal, iwat(T\ddot{o})\cdot pal) + mu\big(anti\cdot pal, preiwa(pal, T\ddot{o})\big) \\
&\qquad + anti\cdot mu(anti\cdot pal, pal, T\ddot{o})\Bigg)(u_1,\ldots,u_r) \\
&= \Bigg(anti\cdot mu\big(anti\cdot pal, preiwa(pal, T\ddot{o})\big) + mu\big(anti\cdot pal, preiwa(pal, T\ddot{o})\big)\Bigg)(u_1,\ldots \\
&= \Bigg(anti\cdot mu\big(anti\cdot pal, r(\bullet)pal\big) + mu\big(anti\cdot pal, r(\bullet)pal\big)\Bigg)(u_1,\ldots,u_r)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{r}(r-i)antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + \sum_{i=0}^{r}(r-i)antipal(u_r,\dots,u_{r-i+1})pal(u_{r-i},\dots,u_1 \\
&= \sum_{i=0}^{r-1}(r-i)antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + \sum_{i=0}^{r-1}(r-i)antipal(u_r,\dots,u_{r-i+1})pal(u_{r-i},\dots,u_1 \\
&= \sum_{i=0}^{r-1}(r-i)antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + \sum_{i=0}^{r-1}(r-i)pal(u_{r-i+1},\dots,u_r)antipal(u_1,\dots,u_{r-i} \\
&= \sum_{i=0}^{r-1}(r-i)antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + \sum_{j=0}^{r-1}(r-j)antipal(u_1,\dots,u_{r-j})pal(u_{r-i+1},\dots,u_r \\
&= \sum_{i=0}^{r-1}(r-i)antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + \sum_{i=1}^{r} i\, antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) \quad (i = r \\
&= r\, pal(u_1,\dots,u_r) + \sum_{i=1}^{r-1} r\, antipal(u_1,\dots,u_i)pal(u_{i+1},\dots,u_r) + r\, antipal(u_1,\dots,u_r) \\
&= r\, mu\big(anti\cdot pal, pal\big) \\
&= r\, gepar(pil).
\end{aligned}
$$

This concludes the proof of (4.2.2).

§4.2.2. Proof of the fundamental identity

Here we prove the first main result of this chapter, Ecalle's fundamental identity for GARI.

Theorem 4.2.1. *We have*

$$
gira(A,B) = ganit_{rash(B)}\cdot gari(A, ras\cdot B). \tag{4.2.7}
$$

The proof will use the basic swap identity:

$$
gira(A,B) := swap\big(gari\big(swap\cdot A, swap\cdot B\big)\big) = gaxi\Big(\big(A,h(A)\big),\big(B,h(B)\big)\Big) \tag{4.2.8}
$$

with $h = push\cdot swap\cdot invmu\cdot swap$.

Let us first give a little explanation based on automorphisms of the group $\mathcal{A}$ consisting of elements in the power series ring ring $\mathbb{Q}\langle\langle x,y\rangle\rangle$ having constant term 1. Such a power series f gives rise to a polynomial-valued mould $ma(f)$.

For $f, g, g' \in \mathcal{A}$, we define endomorphisms $X_{(g,g')}$, R_f and N_f of $\mathcal{A}$ as follows: each one sends $x \mapsto x$, and

$$\begin{cases} X_{(g,g')}(y) = gyg' \\ R_f(y) = fyf^{-1} \\ N_f(y) = yf, \end{cases}$$

i.e. $R_f = X_{(f,f^{-1})}$ and $N_f = X_{(1,f)}$. In this restricted situation, these are to be viewed as the restrictions of the automorphisms $gaxit\big(ma(g), ma(g')\big)$, $garit\big(ma(f)\big)$ and $ganit\big(ma(f)\big)$.

We have

$$X_g \circ X_f = X_{X_{(g,g')}(f)},$$

so $f = invgaxi(g)$ if $X_{(g,g')}(f)g = 1$. The proof is based on the identity

$$X_{(g,g')} \circ R_f = N_{g'g}. \tag{4.2.9}$$

where $g^{-1} = X_{(g,g')}(f)$, i.e. $f = invgaxi_{(g,g')}(g)$. The identity (4.2.9) is easy to prove, since both automorphisms fix x, so we only need to compare their images on y. The RHS yields $N_{g'g}(y) = yg'g$. The LHS yields

$$\begin{aligned} X_{(g,g')}R_f(y) &= X_{(g,g')}(fyf^{-1}) \\ &= X_{(g,g')}(f)gyg'X_{(g,g')}(f^{-1}) \\ &= yg'X_{(g,g')}(f^{-1}) \\ &= yg'g. \end{aligned}$$

Thus (4.2.9) is easily proved. Its generalization is given as follows.

Lemma 4.2.2. *We have*

$$gaxit_{A,B} \cdot garit_{invgaxi_{A,B}(A)} = ganit_{BA}. \tag{4.2.10}$$

Proof. We have

$$garit_{invgaxi_{A,B}(A)} = gaxit_{invgaxi_{A,B}(A), invmu \cdot invgaxi_{A,B}(A)},$$

and the composition of two *gaxits* is given by

$$gaxit_{A,B} \cdot gaxit_{C,D} = gaxit_{gaxit_{A,B}(C)\, A, B\, gaxit_{A,B}(D)}, \tag{4.2.11}$$

so we can multiply the terms on the LHS of (4.2.10) to obtain

$$gaxit_{gaxit_{A,B}(invgaxi_{A,B}(A))A, B\, gaxit_{A,B}(invmu \cdot invgaxi_{A,B}(A))}. \tag{4.2.12}$$

But we have

$$gaxit_{A,B}(invgaxi_{A,B}(A)) = invmu\, A, \tag{4.2.13}$$

since by definition of the $gaxi$-multiplication, we have

$$mu(gaxit_{A,B}(invgaxi_{A,B}(A)),A) = gaxi(invgaxi_{A,B}(A),A) = 1.$$

Thus we can substitute (4.2.13) into (4.2.12) to obtain

$$gaxit_{1,B\,gaxit_{A,B}(invmu\cdot invgaxi_{A,B}(A))}. \tag{4.2.14}$$

Similarly, by (4.2.13) and because $gaxit$ is an automorphism for mu, we find that

$$gaxit_{A,B}\big(invmu\cdot invgaxi_{A,B}(A)\big) = invmu\Big(gaxit_{A,B}\big(invgaxi_{A,B}(A)\big)\Big) = invmu\cdot invmu\cdot A = A,$$

and replacing this into (4.2.14) yields the desired result $gaxit_{1,BA}$, which is equal to $ganit_{BA}$. This concludes the proof of Lemma 4.2.2. ♢

We now prove two useful identities.

Lemma 4.2.3. *We have*

$$\begin{cases} gaxit_{A,B}(invgaxi_{A,B}\cdot A) = invmu\cdot A \\ garit_C(invgari\cdot C) = invmu\cdot C \\ gaxit^h_C(invgaxi^h\cdot C) = invmu\cdot C. \end{cases} \tag{4.2.15}$$

Proof. Writing $garit_C = gaxit_{C,invmu\cdot C}$ and $gaxit^h_C = gaxit_{C,h(C)}$ shows that the first equality implies the second and third, so we only need to prove the first one. To prove the first one, we simply note that

$$1 = gaxi\big(invgaxi_{A,B}\cdot A, A\big) = mu\Big(gaxit_{A,B}\big(invgaxi_{A,B}\cdot A\big), A\Big),$$

which proves the result. ♢

Lemma 4.2.4. *We have*

$$ganit_{rash\cdot C}(ras\cdot C) = C. \tag{4.2.16}$$

Proof. Recall that $rash\cdot B = mu(h(B),B)$ and

$$ras\cdot B = invgari\cdot swap\cdot invgari\cdot swap\cdot B = invgari\cdot invgaxi_h\cdot B. \tag{4.2.17}$$

Let us apply (4.2.10) with $A = C$ and $B = h(C)$, so that

$$gaxit_{C,h(C)}\cdot garit_{invgaxi_{C,h(C)}(C)} = ganit_{rash\cdot C}. \tag{4.2.18}$$

The LHS of (4.2.16) is the RHS of (4.2.18) applied to $ras \cdot C$, so to compute it, we will study the LHS of (4.2.18) applied to $ras \cdot C$. We obtain

$$\begin{aligned} gaxit^h_C \cdot garit_{invgaxi^h_C(C)}\big(invgari \cdot invgaxi_h \cdot C\big) \\ &= gaxit^h_C \cdot invmu \cdot invgaxi^h_C(C) \quad \text{by(1.5.8)} \\ &= invmu \cdot gaxit^h_C \cdot invgaxi^h_C(C) \\ &= invmu \cdot invmu \cdot C \quad \text{by (4.2.15)} \\ &= C. \end{aligned}$$

This completes the proof. ♢

We can now prove Theorem 4.2.1. We start by using (4.2.8) to write

$$gira(A, B) = gaxi^h(A, B),$$

where

$$h = push \cdot swap \cdot invmu \cdot swap$$

and

$$gaxi^h(A, B) := gaxi\Big(\big(A, h(A)\big), \big(B, h(B)\big)\Big).$$

With this, the desired (4.2.7) becomes

$$gaxi_h(A, B) = ganit_{rash \cdot B} \cdot gari(A, ras \cdot B). \tag{4.2.19}$$

By (4.2.10), we have

$$gaxit_{A,B} \cdot garit_{invgaxi_{A,B}(A)} = ganit_{BA}.$$

Replacing the couple (A, B) by $(B, h(B))$, this gives

$$gaxit^h_B \cdot garit_{invgaxi^h_B(B)} = ganit_{rash \cdot B},$$

which we rewrite as

$$gaxit^h_B = ganit_{rash \cdot B} \cdot garit_{invgari \cdot invgaxi^h_B(B)}. \tag{4.2.20}$$

We will prove (4.2.7) by applying each side of this identity to A, then taking mu with B.

The LHS of (4.2.20) yields

$$mu\big(gaxit^h_B(A), B\big) = gaxi^h(A, B).$$

Recalling that

$$ras \cdot B = invgari \cdot swap \cdot invgari \cdot swap \cdot B = invgari \cdot invgaxi^h_B \cdot B,$$

the RHS yields

$$\begin{aligned} mu\big(ganit_{rash\cdot B}\cdot garit_{invgari\cdot invgaxi^h_B(B)}(A),B\big) &= mu\big(ganit_{rash\cdot B}\cdot garit_{ras\cdot B}(A),B\big)\\ &= mu\big(ganit_{rash\cdot B}\cdot garit_{ras\cdot B}(A), ganit_{rash\cdot B}(ras\cdot\\ &= ganit_{rash\cdot B}\cdot mu\big(garit_{ras\cdot B}(A), ras\cdot B)\big)\\ &= ganit_{rash\cdot B}\cdot gari\big(A, ras\cdot B)\big). \end{aligned}$$

This completes the proof of Theorem 4.2.1. ♢

Corollary 4.2.5. *We have*

$$swap\cdot fragari(swap\cdot A, swap\cdot B) = ganit_{crash\cdot B}\cdot fragari(A,B), \tag{4.2.21}$$

where $crash\cdot B = rash\cdot swap\cdot invgari\cdot swap\cdot B$.

Proof. By (4.2.7), we have

$$swap\cdot gari(swap\cdot A, swap\cdot C) = ganit_{rash\cdot C}\cdot gari(A, ras\cdot C).$$

Let $B = invgari\cdot ras\cdot C = swap\cdot invgari\cdot swap\cdot C$, so this translates to

$$swap\cdot gari(swap\cdot A, swap\cdot swap\cdot invgari\cdot swap\cdot B) ganit_{crash\cdot B}\cdot gari(A, invgari\cdot B),$$

which is exactly (4.2.21). ♢

§4.3. Investigation of the identities $crash(pal) = pac$ and $crash(pil) = pic$

First let's prove

Lemma 4.3.1. *We have*

$$crash(pal) := mu(push\cdot swap\cdot invmu\cdot invpil, swap\cdot invpil) = pac. \tag{4.3.1}$$

Proof. Since pil is symmetral, we have

$$mu\big(pari\cdot anti(pil), pil\big)) = 1, \tag{4.3.2}$$

and it's easy to see by the homogeneous degrees of pil that

$$anti\cdot neg(pil) = pari\cdot anti(pil), \tag{4.3.3}$$

so we find that

$$anti\cdot neg(pil) = invmu(pil). \tag{4.3.4}$$

Now, because of (4.3.1), we find that $pil \in GARI \cap GAWI$ (see p. 44), and thus the *gari* and *gawi* inverses are the same, so it makes sense to write $invpil \in GARI \cap GAWI$. This means that for *pil* and *invpil* we have

$$\begin{cases} push \cdot swap \cdot invmu \cdot swap \cdot swap(pil) = anti \cdot swap(pil) \\ push \cdot swap \cdot invmu \cdot swap \cdot swap(invpil) = anti \cdot swap(invpil). \end{cases} \tag{4.3.5}$$

Thus the LHS of (4.3.1) is equal to

$$crash(pal) = mu\big(anti \cdot swap(invpil), swap(invpil)\big),$$

which is nothing other than $gepar(invpil)$, so we can use §4.1.3 for $f(x) = -log(1-x)$ which shows that

$$gepar(invpil) = pic,$$

proving (4.3.1). ♢

Lemma 4.3.2. *We have*

$$ganit\big(pic\big) \cdot invpil = swap \cdot invpal. \tag{4.3.6}$$

Proof. From (4.2.21) applied to $A = 1$, $B = pal$, we have

$$swap \cdot invgari \cdot swap \cdot pal = swap \cdot invpil = ganit_{crash \cdot pal}(invpal). \tag{4.3.7}$$

Using (4.3.7), from (4.3.1) we also know that

$$ganit_{pac} \cdot invpal = swap \cdot invpil.$$

We need to use the elementary result

$$invgani(pac) = pari \cdot anti \cdot paj, \tag{4.3.8}$$

where

$$paj(u_1, \ldots, u_r) = \frac{1}{(u_1(u_1+u_2)(u_1+u_2+u_3)\cdots(u_1+\cdots+u_r)}.$$

This gives

$$invpal = ganit_{pari \cdot anti \cdot paj} \cdot swap \cdot invpil,$$

so

$$swap \cdot invpal = swap \cdot ganit_{pari \cdot anti \cdot paj} \cdot swap \cdot invpil.$$

It remains only to prove that the following two automorphisms of GARI are equal:

$$ganit_{pic} = swap \cdot ganit_{pari \cdot anti \cdot paj} \cdot swap. \tag{4.3.9}$$

Now, every mould C in the v_i such that $C(v_1,\dots,v_r)$ is actually a rational function B of the variables $v_2-v_1,\dots,v_r-v_1$ satisfies the identity $C = ganit_B(Y)$, by the calculation

$$\begin{aligned} ganit_B(Y)(v_1,\dots,v_r) &= \sum_{b^1c^c dots b^s c^s} Y(b^c dots b^s) B(\lfloor c^1)\cdots B(\lfloor c^2) \\ &= \sum_{b^1=(v_1),c^1=(v_2,\dots,v_r)} Y(v_1)B(v_2-v_1,\dots,v_r-v_1) \\ &= B(v_2-v_1,\dots,v_r-v_1) \\ &= C(v_1,\dots,v_r). \end{aligned} \tag{4.3.10}$$

Let us write $swap(Y) = Y$ a little abusively, since although the values in depths 0 and 1 are still 1, $swap(Y)$ is considered a mould in the u_i, we compute the right-hand side of (1.6.9) explicitly as

$$ganit_{pari\cdot anti\cdot paj}\cdot Y(u_1,\dots,u_r) = \frac{(-1)^{r-1}}{u_r(u_{r-1}+u_r)\cdots(u_2+\cdots u_r)}$$

(with $ganit_{pari\cdot anti\cdot paj}\cdot Y(\emptyset) = 1$, $ganit_{pari\cdot anti\cdot paj}\cdot Y(u_1) = 1$). Swapping this, we obtain

$$swap\cdot ganit_{pari\cdot anti\cdot paj}\cdot Y(u_1,\dots,u_r) = \frac{1}{(v_2-v_1)(v_3-v_1)\cdots(v_r-v_1)}.$$

Letting

$$C(v_1,\dots,v_r) = \frac{1}{(v_2-v_1)(v_3-v_1)\cdots(v_r-v_1)},$$

we see by (1.6.10) that $C = ganit_B(Y)$ where

$$B(v_1,\dots,v_r) = \frac{1}{v_c dots v_r}, \tag{4.3.11}$$

i.e. $B = pic$. ♢

We still can't prove that $crash(pil) = pic$, but the following result should be sufficient.

Proposition 4.3.3. We have

$$swap\cdot fragari(swap\cdot A, pal) = ganit_{pic}\cdot fragari(A, pil). \tag{4.3.12}$$

Proof. Applying the fundamental identity (1.5.1) to $A = swap\cdot M$ and $B = pal$ and using Lemma 4.3.1 yields

$$\begin{aligned} swap\cdot fragari(M, swap\cdot pal) &= ganit_{crash\cdot pal}\cdot fragari(swap\cdot M, pal) \\ &= ganit_{pac}\cdot fragari(swap\cdot M, pal). \end{aligned}$$

Thus by (4.3.8) we have

$$\begin{aligned} ganit_{invgani\cdot pac}\cdot swap\cdot fragari(M,pil) &= ganit_{pari\cdot anti\cdot paj}\cdot swap\cdot fragari(M,pil) \\ &= fragari(swap\cdot M,pal). \end{aligned}$$

Applying swap to both sides and (4.3.9), we have

$$\begin{aligned} swap\cdot ganit_{pari\cdot anti\cdot paj}\cdot swap\cdot fragari(M,pil) &= ganit_{pic}\cdot fragari(M,pil) \\ &= swap\cdot fragari(swap\cdot M,pal), \end{aligned}$$

which proves the desired (4.3.12). ◊

Proposition 4.3.4. *For every push-invariant mould M, we have*

$$swap\cdot adari(pal)\cdot M = ganit_{pic}\cdot adari(pil)\cdot swap\cdot M. \tag{4.3.13}$$

Proof. We use the defining identity

$$adari(A)\cdot B = fragari\big(preari(A,B),A\big) \tag{4.3.14}$$

and equation (1.8) given by

$$swap\big(preari(swap\cdot A, swap\cdot B)\big) = axit(B,-push(B))\cdot A + mu(A,B). \tag{4.3.15}$$

Using this for $A = pal$ and $B = M$, we find in particular that

$$\begin{aligned} preari(pil,swap\cdot M) &= swap\big(axit(M,-push(M))\cdot pal + mu(pal,M)\big) \\ &= swap\big(arit(M)\cdot pal + mu(pal,M)\big) \quad \text{because } M \text{ is push-inv} \\ &= swap\cdot preari(pal,M). \end{aligned} \tag{4.3.16}$$

Using (1.4.14) for $A = pal$, $B = M$, we have

$$\begin{aligned} swap\cdot adari(pal)\cdot M &= swap\cdot fragari\big(preari(pal,M),pal\big) \\ &= swap\cdot fragari\big(swap\big(swap\cdot preari(pal,M)\big),pal\big) \\ &= ganit_{pic}\cdot fragari(swap\cdot preari(pal,M),pil) \quad \text{by (1.4.12)} \\ &= ganit_{pic}\cdot fragari(preari(pil,swap\cdot M),pil) \quad \text{by (1.4.16)} \\ &= ganit_{pic}\cdot adari(pil)\cdot swap\cdot M, \end{aligned}$$

proving (4.3.13). ◊