Every acyclotomic element of the profinite Grothendieck-Teichmüller group is a twist

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Abstract

In this note we motivate, state and prove a profinite analog of the main result of [AET] (see also [AT]), to the effect that every acycotomic element of the profinite Grothendieck-Teichmüller group (i.e. every element of $\widehat{GT}_1$) is a twist in the sense first introduced by V.G.Drinfeld (in [D]).

1 Introduction

We will state and prove a profinite analog of the main result of [AET] (see also [AT]) concerning the connection between *associators* on the one hand and *solutions of the Kashiwara-Vergne problem* on the other. We will need to introduce comparatively little material and the text is (almost) self-contained. We start however with this brief introductory and hopefully motivational paragraph for which we do not (and possibly cannot) introduce the necessary background but which can be skipped without impairing the understanding of the more detailed technical parts.

Start from a solution F of the KV-problem, that is F is an automorphism of the (degree completion of) the free Lie algebra $\mathbb{L}(x,y)$ on two generators x and y, which maps each of these generators to a conjugate of itself (and satisfies certain additional properties). Then form the combination:

$$\Phi = (F^{12,3})^{-1}\ (F^{1,2})^{-1}\ F^{2,3}\ F^{1,23} \tag{1}$$

in which each factor, and so also Φ is an automorphism of the free Lie algebra $\mathbb{L}(x,y,z)$ on *three* generators x, y, z; multiplication is by composition, starting from the right. Here we use the (co)simplicial notation which has become common in terms of tensor categories. It is explained for instance in detail in [AT]; the reader may find it useful to recall that in terms of braids it corresponds to repeated 'strand doubling' and 'strand removing', as well as adjoining strands to the left or right of a given braid. We will need the multiplicative version and we thus give the explicit formulas in that case (see (9) to (12) below). Adapting these formulas to the present Lie algebra additive (and so partially commutative) setting simply requires replacing uv by $u+v$ and w^{-1} by $-w$ everywhere.

Expression (1) is easily, that is formally shown to satisfie the *pentagon equation* and to be an element of the appropriate version of the Kashiwara-Vergne group ([AT], Prop. 7.1). Also, Φ determines F up to an elementary exponential factor ([AT], Prop. 7.2). In the converse direction, if an automorphism F of (the degree completion of) $\mathbb{L}(x,y)$ gives rise to a Φ which belongs to the

KV-group, then F solves the KV-problem ([AT], Prop. 7.4). Finally and more deeply, start from any automorphism Φ of (the degree completion of) $\mathbb{L}(x,y,z)$ and assume it satisfies the pentagon equation; then (up to minor exponential factors), there *exists* a unique F such that Φ can be written in the form (1) and F solves the KV-problem ([AT], Theorem 7.5). It is worth noting that the proof of this last result proceeds in the way which is typical in a prounipotent (or pronilpotent) framework (see esp. [D], §5), namely constructively, by linearization and induction on the degree or weight. At each step one is confronted with a (linear) cohomological obstruction which has to vanish in order to carry on with the construction. In the case at hand, the relevant cohomology theory is constructed in [AT] (§§2,3) and the vanishing result is Theorem 3.17 there. Of course, this general strategy breaks completely in the profinite situation.

Now restrict attention to the Grothendieck-Teichmüller setting, still in a prounipotent framework. A Drinfeld associator is a Φ (not *a priori* of the form (1)) which can be expressed in terms of braids (see [AT], esp. Def. 9.4). Passing to the Lie algebra this amounts to considering an element $\psi \in \mathfrak{grt}$, the Grothendieck-Teichmüller Lie algebra (here we use H.Furusho's result which asserts that the pentagon is the only defining equation of $\mathfrak{grt}$). Then a little miracle happens: one finds an explicit, indeed almost tautological solution of the relevant cohomological equation ([AT], Prop. 4.7). This seed has been cultivated in [AET] in order to produce an *explicit* version of the connection between Φ and F as above, which represents the main result of that paper (see Introduction and Theorems 2.1 and 2.5 there). Here we will prove the profinite version of that same result. Because associators, not to mention Lie algebras, are typically prounipotent creatures, our statement will of necessity be group theoretical by nature. Finally the connection with Drinfeld's twists, which has been inspirational since the beginning of that story, can be gathered from [D] (see formula (1.11) there).

2 Statements of the results

Let us move to the profinite setting and start afresh. We will work with the profinite Grothendieck-Teichmüller group $\widehat{GT}$ and introduce only a bare minimum. The original and unavoidable reference is [D] but here all the necessary details can actually be found in [LS1]; first inputs and more recent references are also available in [LS2]. There is an embedding $\widehat{GT} \hookrightarrow Aut(\hat{F}_2)$ of $\widehat{GT}$ into the automorphism group of the profinite completion of the free group F_2 on two generators (call them x, y) and we identify and element $F \in \widehat{GT}$ with its image. Define $z \in F_2 \subset \hat{F}_2$ such that $xyz = 1$; then the automorphism which we identify with $F \in \widehat{GT}$ is given as a pair $(\lambda, f(x,y)) \in \hat{\mathbb{Z}}^* \times \hat{F}_2$ (actually f is fixed by requiring that it further belong to $\hat{F}_2'$, the derived subgroup of $\hat{F}_2$) and it acts according to:

$$F(x) = f(x,z)x^{\lambda}f(z,x) \quad ; \quad F(y) = f(y,z)y^{\lambda}f(z,y). \tag{2}$$

A few simple comments are in order. First $F \in \widehat{GT}$ satisfies the three defining equations of that group, of which the first two read (see (6) below for the third):

$$f(y,x)f(x,y) = 1; \tag{3}$$

$$f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1. \tag{4}$$

The first one is the duality (2-cycle) relation; the second one is the hexagonal (3-cycle) relation, in which one has set $\lambda = 2m+1$ with $m \in \hat{\mathbb{Z}}$, which is licit (i.e. λ is 'odd', that is $= 1 \bmod 2$). By (3), $F(x)$ and $F(y)$ are conjugate to x^λ and y^λ respectively; (4) implies that $F(z) = z^\lambda$. It is a little more customary to use a version where $F(x) = x^\lambda$ but the one above, which is tuned to the conventions of [AT] and [AET], differs by an inner automorphism (using again (4)). In more geometric terms, think of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and of x, y and z as describing loops around 0, 1 and ∞ respectively. Then the 'usual' convention corresponds to working with a tangential basepoint at 0, whereas we pick one at ∞. This amounts to a permutation of the points (or of the letters which label them).

We still need to explain, if only to fix notation (which is essentially standard), how an element $F \in \widehat{GT}$ acts on $\hat{\Gamma}_{0,5}$, the profinite completion of the Teichmüller (or mapping class) group for spheres with 5 labeled marked points ($\Gamma_{0,4} \simeq F_2$), referring to [LS2] for complete details. For $i,j \in \{1,2,3,4,5\}$, $i \neq j$, one denotes by x_{ij} the 'elementary colored braid' corresponding to point i circling around point j once (or vice versa; $x_{ij} = x_{ji}$). Then $\hat{\Gamma}_{0,5}$ can be given as the split extension:

$$1 \longrightarrow \hat{F}_3 \longrightarrow \hat{\Gamma}_{0,5} \longrightarrow \hat{F}_2 \longrightarrow 1 \tag{5}$$

in which $F_2 = \langle x_{12}, x_{23} \rangle$ and $F_3 = \langle x_{14}, x_{24}, x_{34} \rangle$ are free on two and three generators respectively. Geometrically speaking, this is the (completion of the) homotopy sequence derived from the fibration corresponding to erasing the 4-th point on the sphere. An $F \in \widehat{GT}$ now satisfies the third defining equation (after (3) and (4); in fact (6) below implies (3)), namely the pentagon or 5-cycle relation in $\hat{\Gamma}_{0,5}$, which says that:

$$f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51}) = 1 \tag{6}$$

Using these relations one shows that $F \in \widehat{GT}$ induces an automorphism of $\hat{\Gamma}_{0,5}$, which we denote again F. It is made explicit in (26) below for the case $\lambda = 1$; the formulas in the general case are identical, except that $F(x_{ij})$ is conjugate to x_{ij}^λ (by the same factor). Moreover that automorphism preserves the decomposition (5), acting on the normal subgroup $\hat{F}_3$ which is nothing but the fundamental group of the sphere with 4 points (labeled $1,2,3,5$) removed. The so-called sphere relation states that the 4 corresponding elementary loops are related by:

$$x_{14}\ x_{24}\ x_{34}\ x_{45} = 1. \tag{7}$$

Inside $\widehat{GT}$ we find the subgroup $\widehat{GT}_1$ of the elements with $\lambda = 1$, which we call *acyclotomic*. This terminology comes of course from the Galois case: recall that there is an essentially natural embedding $\mathrm{Gal}(\mathbb{Q}) \hookrightarrow \widehat{GT}$ of the absolute Galois group of $\mathbb{Q}$ into $\widehat{GT}$. It maps $\sigma \in \mathrm{Gal}(\mathbb{Q})$ to a pair $(\lambda_\sigma, f_\sigma)$ where $\lambda_\sigma = \chi(\sigma)$ and χ denotes the cyclotomic character. So for Galois elements, $\lambda_\sigma = 1$ corresponds to $\chi(\sigma) = 1$, i.e. to elements which act trivially on the roots of unity. Note also that the (topological) derived group $[\widehat{GT}, \widehat{GT}]$ is included in $\widehat{GT}_1$. In the Galois case, class field theory tells us that the derived subgroup of $\mathrm{Gal}(\mathbb{Q})$ *coincides* with the subgroup of the elements with $\chi(\sigma) = 1$ (the Kronecker-Weber theorem). It was asked a long time ago (and is recorded in

[LS2] as Question 1.2) whether this also holds true for the whole of $\widehat{GT}$. One can wonder whether the result of the present note may have some bearing on that matter.

We can now state the main result of this note. We write $\mathrm{Inn}(u)v = uvu^{-1}$ for the conjugacy in a group. So for instance (2) can be rewritten as: $F(x) = \mathrm{Inn}(f(x,z))(x^\lambda); F(y) = \mathrm{Inn}(f(y,z))(y^\lambda)$. Then we have:

Theorem 2.1. *Let $F = (1, f) \in \widehat{GT}_1$ and let $\hat{F}_3 \subset \hat{\Gamma}_{0,5}$ be the normal subgroup topologically generated by x_{14}, x_{24} and x_{34}. Slightly abusing notation we write $\mathrm{Inn}(f(x_{12}, x_{23}))$ for the restriction of this inner automorphism of $\hat{\Gamma}_{0,5}$ to the subgroup $\hat{F}_3$. Then we have the following equality of automorphisms of $\hat{F}_3$:*

$$\mathrm{Inn}(f(x_{12}, x_{23})) = F^{1,2}\ F^{12,3}\ (F^{1,23})^{-1}\ (F^{2,3})^{-1}. \tag{8}$$

This result thus states that every acyclotomic element of the profinite Grothendieck-Teichmüller group can be written as a twist in the sense of Drinfeld (see [D], (1.11)), and the twisting element corresponds to the same element of $\widehat{GT}_1$, but with a different kind of action – actually four different actions – on $\hat{F}_3$. Slightly different conventions explain the differences between (8) and (1); recall that (1) refers, as in [AT], [ATE] to the prounipotent graded version, that is to automorphisms of (degree completed) Lie algebras. We also remark that we believe that the statement is simply wrong when $\lambda \neq 1$. In fact, looking at the right-hand sides of formula (1) (or (8)) and abelianizing the situation implies that the resulting automorphism has parameter $\lambda = 1$.

Let us now spell out the definitions of the terms on the right-hand side of (8). All the terms appearing there are automorphisms of $\hat{F}_3 = \langle x_{14}, x_{24}, x_{34}, x_{45}\rangle$ (see (7)). We identify this subgroup of $\hat{\Gamma}_{0,5}$ with (the completion of) a'standard' copy of $F_3 = \langle x, y, z, w\rangle$ $(xyzw = 1)$ by setting $x = x_{14}$, $y = x_{24}$, $z = x_{34}$, $w = x_{45}$. Finally, below we use of course (2) but replace there z by $(xy)^{-1}$, apologizing for the unfortunate notational clash. With this preparation (and with $\lambda = 1$) we have:

$$\begin{aligned} F^{1,2}(x) &= f(x,(xy)^{-1})xf((xy)^{-1},x) \\ F^{1,2}(y) &= f(y,(xy)^{-1})yf((xy)^{-1},y) \\ F^{1,2}(z) &= z \end{aligned} \tag{9}$$

$$\begin{aligned} F^{2,3}(x) &= x \\ F^{2,3}(y) &= f(y,(yz)^{-1})yf((yz)^{-1},y) \\ F^{2,3}(z) &= f(y,(yz)^{-1})zf((yz)^{-1},z) \end{aligned} \tag{10}$$

$$\begin{aligned} F^{12,3}(x) &= f(xy,w)xf((w,xy) \\ F^{12,3}(y) &= f(xy,w)yf((w,xy) \\ F^{12,3}(z) &= f(z,w)zf(w,z) \end{aligned} \tag{11}$$

$$\begin{aligned} F^{1,23}(x) &= f(x,w)xf((w,x) \\ F^{1,23}(y) &= f(yz,w)yf((w,yz) \\ F^{1,23}(z) &= f(yz,w)zf(w,yz) \end{aligned} \tag{12}$$

We break the statement of Theorem 2.1 into two, essentially for the sake of clarity. The first statement reads:

Proposition 2.2. *As automorphisms of $\hat{F}_3$ (topologically generated by x_{14}, x_{24}, x_{34}) the following equality holds true:*

$$F^{2,3}\ F^{1,23} = \mathrm{Inn}(f(x_{23}, x_{12}))\ F,$$

where $F \in \widehat{GT}$ is considered as an automorphism of $\hat{F}_3 \subset \hat{\Gamma}_{0,5}$ by restriction.

For the statement of the next proposition we need to introduce notation for the *noncolored* braids. We thus simply recall that σ_i $(i = 1, 2, 3, 4)$ denotes, as is usual, the elementary braid which intertwines points or strands i and $i+1$. In particular $\sigma_i^2 = x_{i,i+1}$. Although we will not really need it, we remark that in [LS1] the reader will find more generally the definition of elements σ_{ij} such that $\sigma_{ij}^2 = x_{ij}$ (any number of points or strands). Topologically speaking, imagine the points i (say $i \in \mathbb{Z}/n$ for some $n > 1$) placed on a circle. Then σ_{ij} will correspond to swapping i and j by letting these points travel along a chord. Returning to our problem we can now state:

Proposition 2.3. *As automorphisms of $\hat{F}_3$ (again topologically generated by x_{14}, x_{24}, x_{34}):*

$$F^{1,2}\ F^{12,3} = \mathrm{Inn}(\sigma_4)\ F\ \mathrm{Inn}(\sigma_4^{-1}),$$

where again $F \in \widehat{GT}$ is considered as an automorphism of $\hat{F}_3 \subset \hat{\Gamma}_{0,5}$ by restriction.

These two propositions will be proved in the next section. Together they imply the result. Indeed rewriting the first one we get:

$$F = \mathrm{Inn}(f(x_{12}, x_{23}))\ F^{2,3}\ F^{1,23}.$$

Next and using the fact that the standard action of $\widehat{GT}$ on $\hat{\Gamma}_{0,5}$ satisfies $F(\sigma_4) = \sigma_4$ (cf. e.g. [LS1]), we find that actually:

$$\mathrm{Inn}(\sigma_4)\ F\ \mathrm{Inn}(\sigma_4^{-1}) = \mathrm{Inn}(\sigma_4)\mathrm{Inn}(F(\sigma_4^{-1})\ F = F,$$

so that the statement of Proposition 2.3 simplifies to:

$$F = F^{1,2}\ F^{12,3}.$$

Comparing these two expressions of F yields the result.

3 Proofs

Here we prove Propositions 2.2 and 2.3 together, thereby completing the proof of Theorem 2.1. The proofs appear as purely combinatorial, making good use of the braid relations in $\Gamma_{0,[5]}$, but in fact they rest on the *lego properties* of the action of $\widehat{GT}$, which were established in [HLS] in a much more general setting – to wit for Teichmüller groups of any genus and with any number of marked points.

We start with an auxiliary statement:

Lemma 3.1. *We have:*

$$F^{1,2}(xy) = xy,\ F^{2,3}(yz) = yz,$$
$$F^{1,2}(w) = F^{2,3}(w) = F^{12,3}(w) = F^{1,23}(w) = w.$$

Proof: It makes use of the hexagon identity (4) and the following variant:

$$f(x,y)x^{-m-1}f(z,x)z^{-m-1}f(y,z)y^{-m-1} = 1. \tag{13}$$

This identity holds true for any x, y, z such that $xyz = 1$, because if $F = (\lambda, f)$ is in $\widehat{GT}$, so is $(-\lambda, f)$, that is one can change λ into $-\lambda$, which maps m to $-m-1$ (recall that $\lambda = 2m+1$). Below, we will use only the case $\lambda = 1$ ($m = 0$) of (13).

We have

$$\begin{aligned} F^{1,2}(xy) &= f(x,z)xf(z,x)f(y,z)yf(z,y) \\ &= f(x,z)xf(y,x)yf(z,y) \\ &= \big(f(y,z)y^{-1}f(x,y)x^{-1}f(z,x)\big)^{-1} \\ &= z^{-1} = xy \end{aligned}$$

and then it follows that $F^{1,2}(w) = F^{1,2}(xyz)^{-1} = \big(F^{1,2}(xy)F^{1,2}(z)\big)^{-1} = (xyz)^{-1} = w$. The computation of $F^{2,3}(yz)$ is completely similar, leading to $F^{2,3}(yz) = yz$ and thus to $F^{2,3}(xyz) = xyz$, since $F^{2,3}(x) = x$. Next, applying the hexagon relation to the product $(xy)zw = 1$, we have

$$f(xy,z)f(w,xy)f(z,w) = f(xy,z)(xy)^{-1}f(w,xy)w^{-1}f(z,w)z^{-1} = 1,$$

so that

$$\begin{aligned} F^{12,3}(xyz) &= f(xy,w)xyf(w,xy)f(z,w)zf(w,z) \\ &= f(xy,w)xyf(z,xy)zf(w,z) \\ &= \big(f(z,w)z^{-1}f(xy,z)(xy)^{-1}f(w,xy)\big)^{-1} \\ &= w^{-1} = xyz. \end{aligned}$$

Finally, applying the hexagon relation to the product $x(yz)w = 1$, we get

$$f(x,yz)f(w,x)f(yz,w) = f(x,yz)x^{-1}f(w,x)w^{-1}f(yz,w)(yz)^{-1} = 1,$$

so we have

$$\begin{aligned} F^{1,23}(xyz) &= f(x,w)xf(w,x)f(yz,w)yzf(w,yz) \\ &= f(x,w)xf(yz,x)yzf(w,yz) \\ &= \big(f(yz,w)(yz)^{-1}f(x,yz)x^{-1}f(w,x)\big)^{-1} \\ &= w^{-1} = xyz \end{aligned}$$

which concludes the proof of the lemma. □

Before plunging into the computations proving Proposition 2.2 and 2.3, let us list some useful and well-known identities in $\Gamma_{0,5}$. For $i < j$ and the notation as above, we have:

$$x_{ij} = \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} \tag{14}$$

and we introduce the elements:

$$x'_{ij} = \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-1}. \tag{15}$$

These formulas are actually general (not confined to $i, j \leq 5$); visually both x_{ij} and x'_{ij} correspond to the elementary intertwining of strands i and j, but with these strands passing in front of (resp. behind) the other strands of the braid (see e.g. [LS1]). Then the following identities hold in $\Gamma_{0,5}$ – and are easy to prove, either algebraically or visually, 'by inspection':

$$x_{12}x_{13}x_{23} = x_{45}, \tag{16}$$
$$x_{12}x_{13}x_{23} = x_{12}x_{23}x'_{13}, \tag{17}$$
$$x_{12}x_{14}x_{24} = x'_{35}, \tag{18}$$

together of course with all the identities obtained from these by *cyclic* permutation of the five indices $1, 2, 3, 4, 5$. Finally we will also make frequent use of the fact that if c commutes with a and b, and f is a commutator, we have

$$f(a, b) = f(ac, b) = f(a, bc). \tag{19}$$

Armed with these preliminaries we can proceed to the (brute?) computation of both sides of the identities in Propositions 2.2 and 2.3 as automorphisms of $\hat{F}_3$, that is their respective actions on $x = x_{14}$, $y = x_{24}$ and $z = x_{34}$ (as well as $w = x_{45}$; $xyzw = 1$). Let us begin with the left-hand sides, $F^{2,3} \circ F^{1,23}$ and $F^{1,2} \circ F^{12,3}$. We first have

$$F^{2,3}F^{1,23}(x_{14}) = F^{2,3}\big(f(x_{14}, x_{45})x_{14}f(x_{45}, x_{14}) = f(x_{14}, x_{45})x_{14}f(x_{45}, x_{14}). \tag{20}$$

where we used the fact that $F^{2,3}(x_{45}) = F^{2,3}(w) = w = x_{45}$ (see Lemma 3.1). Next we compute

$$\begin{aligned}
F^{2,3}F^{1,23}(x_{24}) &= F^{2,3}(f(x_{24}x_{34}, x_{45})x_{24}f(x_{45}, x_{24}x_{34})) \\
&= f\big(x_{24}x_{34}, x_{45}\big)f\big(x_{24}, (x_{24}x_{34})^{-1}\big)x_{24}f\big((x_{24}x_{34})^{-1}, x_{24}\big)f\big(x_{45}, x_{24}x_{34}\big) \\
&= f(x_{23}x_{24}x_{34}, x_{45})f(x_{24}, x_{45}x_{14})x_{24}f(x_{45}x_{14}, x_{24})f(x_{45}, x_{23}x_{24}x_{34}) \qquad (21) \\
&= f(x_{23}x_{24}x_{34}, x_{45})f(x_{24}, x_{45}x_{14}x_{51})x_{24}f(x_{45}x_{14}x_{51}, x_{24})f(x_{45}, x_{23}x_{24}x_{34}) \\
&= f(x_{51}, x_{45})f(x_{24}, x_{23})x_{24}f(x_{23}, x_{24})f(x_{45}, x_{51}).
\end{aligned}$$

Here, for the second equality we used the fact that $F^{2,3}(yz) = yz$ (Lemma 3.1), whereas in the third equality we inserted a factor of x_{23} inside the f, which is licit as it commutes with both $x_{24}x_{34}$ and x_{45} (see equation (19)); we also used $(x_{24}x_{34})^{-1} = x_{45}x_{14}$ from the sphere identity (see (7)). In the fourth equality we used again (19) to justify the insertion of a commuting factor of x_{51}, and in the final equality we used (a cyclic permutation of) identity (16).

The computation for x_{34} is completely similar, so we give only the result, namely

$$F^{2,3}F^{1,23}(x_{34}) = f(x_{51}, x_{45})f(x_{34}, x_{23})x_{34}f(x_{23}, x_{34})f(x_{45}, x_{51}). \tag{22}$$

We now pass to the computation of $F^{1,2} \circ F^{12,3}$ which is again quite similar to the above one. First

$$
\begin{aligned}
F^{1,2}F^{12,3}(x_{14}) &= F^{1,2}\big(f(x_{14}x_{24},x_{45})x_{14}f(x_{45},x_{14}x_{24})\big)\\
&= f(x_{14}x_{24},x_{45})f(x_{14},(x_{14}x_{24})^{-1})x_{14}f((x_{14}x_{24})^{-1},x_{14})f(x_{45},x_{14}x_{24})\\
&= f(x_{14}x_{24},x_{45})f(x_{14},x_{34}x_{45})x_{14}f(x_{34}x_{45},x_{14})f(x_{45},x_{14}x_{24})\\
&= f(x_{12}x_{14}x_{24},x_{45})f(x_{14},x_{34}x_{45}x'_{35})x_{14}f(x_{34}x_{45}x'_{35},x_{14})f(x_{45},x_{12}x_{14}x_{24})\\
&= f(x'_{35},x_{45})f(x_{14},x_{12})x_{14}f(x_{12},x_{14})f(x_{14},x'_{35}).
\end{aligned}
\tag{23}
$$

Here, and much as above, the second equality uses the fact that $F^{1,2}$ fixes xy and w (Lemma 3.1), the third equality uses the sphere relation, the fourth uses (cyclic permutations of) (16) and (17) whereas the last equality uses (18). Again the computation for x_{24} is very similar and we give only the result

$$
F^{1,2}F^{12,3}(x_{24}) = f(x'_{35},x_{45})f(x_{24},x_{12})x_{24}f(x_{12},x_{24})f(x_{45},x'_{35}). \tag{24}
$$

Finally the action on x_{34} is easier to compute, to wit

$$
F^{1,2}F^{12,3}(x_{34}) = F^{1,2}\big(f(x_{34},x_{45})x_{34}f(x_{45},x_{34})\big) = f(x_{34},x_{45})x_{34}f(x_{45},x_{34}). \tag{25}
$$

This completes the computation of the action on $\langle x_{14},x_{24},x_{34}\rangle$ of the automorphisms of (the profinite completion of) that group appearing on the left-hand sides of Proposition 2.2 and 2.3 respectively. There now remains to compare these results with the respective actions of the expressions appearing on the right-hand sides, to which end one uses the expression of the standard action of $F\in\widehat{GT}$ on the elements x_{14},x_{24},x_{34}. It is actually useful to list the 'standard' action of F on all ten (topological) generators x_{ij} of $\hat{\Gamma}_{0,5}$. The list below can be found in [LS1] but it can also be easily inferred as a special case of the lego developped in [HLS]. From that viewpoint, we are dealing here with the particular case of a sphere (genus 0) with 5 marked point (modular dimension 2), and we are writing the action for a particular 'pants decomposition', consisting of 2 (= the modular dimension) nonintersecting loops encircling the pairs of points labeled (1, 2) and (4, 5) respectively. This being said, one gets

$$
\begin{aligned}
F(x_{12}) &= x_{12}\\
F(x_{23}) &= f(x_{23},x_{12})x_{23}f(x_{12},x_{23})\\
F(x_{34}) &= f(x_{34},x_{45})x_{34}f(x_{45},x_{34})\\
F(x_{45}) &= x_{45}\\
F(x_{51}) &= f(x_{23},x_{12})f(x_{51},x_{45})x_{51}f(x_{45},x_{51})f(x_{12},x_{23})\\
F(x_{13}) &= f(x_{13},x_{12})x_{13}f(x_{12},x_{13})\\
F(x_{24}) &= f(x_{23},x_{12})f(x_{51},x_{45})f(x_{24},x_{23})x_{24}f(x_{23},x_{24})f(x_{45},x_{51})f(x_{12},x_{23})\\
F(x_{14}) &= f(x_{23},x_{12})f(x_{14},x_{45})x_{14}f(x_{45},x_{14})f(x_{12},x_{23})\\
F(x_{25}) &= f(x_{34},x_{45})f(x_{25},x_{12})x_{25}f(x_{12},x_{25})f(x_{45},x_{34})\\
F(x_{35}) &= f(x_{35},x_{45})x_{34}f(x_{45},x_{35}).
\end{aligned}
\tag{26}
$$

Using this, we now first compute

$$
\mathrm{Inn}(f(x_{23},x_{12}))\circ F(x_{14}) = f(x_{14},x_{45})x_{14}f(x_{45},x_{14}),
$$

as well as

$$\mathrm{Inn}(f(x_{23},x_{12}))\circ F(x_{24}) = f(x_{51},x_{45})f(x_{24},x_{23})x_{24}f(x_{23},x_{24})f(x_{45},x_{51}),$$

and finally

$$\begin{aligned}\mathrm{Inn}(f(x_{23},x_{12}))\circ F(x_{34}) &= f(x_{12},x_{23})f(x_{34},x_{45})x_{34})f(x_{45},x_{34})f(x_{23},x_{12})\\ &= f(x_{51},x_{45})f(x_{34},x_{23})f(x_{12},x_{51})x_{34}f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})\\ &= f(x_{51},x_{45})f(x_{34},x_{23})x_{34}f(x_{23},x_{34})f(x_{45},x_{51})\end{aligned}$$

(The second equality uses the pentagon relation and the third uses commutation of x_{12} and x_{51} with x_{34}.) Checking that these three expressions are respectively equal to the quantities computed in (20), (21) and (22) completes the proof of Proposition 2.2. The proof of Proposition 2.3 is completely analogous, computing the expressions appearing there on the right-hand side; to start with

$$\begin{aligned}\mathrm{Inn}(\sigma_4)\circ F\circ\mathrm{Inn}(\sigma_4^{-1})(x_{14}) &= \mathrm{Inn}(\sigma_4)\circ F\big(\sigma_4 x_{14}\sigma_4^{-1}\big)\\ &= \mathrm{Inn}(\sigma_4)\circ F\big(\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\big)\\ &= \mathrm{Inn}(\sigma_4)\circ F(x_{51})\\ &= \mathrm{Inn}(\sigma_4)\big(f(x_{34},x_{45})f(x_{51},x_{12})x_{51}f(x_{12},x_{51})f(x_{45},x_{34})\big)\\ &= (f(x'_{35},x_{45})f(x_{14},x_{12})x_{14}f(x_{12},x_{14})f(x_{45},x'_{35}),\end{aligned}$$

then

$$\begin{aligned}\mathrm{Inn}(\sigma_4)\circ F\circ\mathrm{Inn}(\sigma_4^{-1})(x_{24}) &= \mathrm{Inn}(\sigma_4)\circ F\big(\sigma_4 x_{24}\sigma_4^{-1}\big)\\ &= \mathrm{Inn}(\sigma_4)\circ F\big(x_{25}\big)\\ &= \mathrm{Inn}(\sigma_4)\big(f(x_{34},x_{45})f(x_{25},x_{12})x_{25}f(x_{12},x_{25})f(x_{45},x_{34})\big)\\ &= f(x'_{35},x_{45})f(x_{24},x_{12})x_{24}f(x_{12},x_{24})f(x_{45},x'_{35})\end{aligned}$$

and last

$$\begin{aligned}\mathrm{Inn}(\sigma_4)\circ F\circ\mathrm{inn}(\sigma_4^{-1})(x_{34}) &= \mathrm{inn}(\sigma_4)\circ F\big(\sigma_4 x_{34}\sigma_4^{-1}\big)\\ &= \mathrm{inn}(\sigma_4)\circ F\big(x_{35}\big)\\ &= \mathrm{inn}(\sigma_4)\big(f(x_{35},x_{45})x_{35}f(x_{45},x_{35})\big)\\ &= f(x_{34},x_{45})x_{34}f(x_{45},x_{34}).\end{aligned}$$

One then confirms by inspection that these expressions coincide with those computed in (23), (24) and (25) respectively, concluding the proof of Proposition 2.3 and thus of Theorem 2.1. □

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