LIAISON AND CASTELNUOVO-MUMFORD REGULARITY

By Marc Chardin and Bernd Ulrich*

Dedicated to Jürgen Herzog on his sixtieth birthday

Abstract. In this article we establish bounds for the Castelnuovo-Mumford regularity of projective schemes in terms of the degrees of their defining equations. The main new ingredient in our proofs is to show that generic residual intersections of complete intersection rational singularities again have rational singularities. When applied to the theory of residual intersections this circle of ideas also sheds new light on some known classes of free resolutions of residual ideals.

Introduction. In this article we prove bounds for the Castelnuovo-Mumford regularity of projective schemes in terms of the degrees of defining equations, very much in the spirit of Bertram, Ein, and Lazarsfeld ([BEL]). Our methods are based on liaison theory. They also lead to results in positive characteristic and provide information on defining ideals, even if these are not saturated or unmixed.

The following gives a flavor of one of our main results (Theorem 4.7(a)).

Theorem 0.1. *Let k be a field of characteristic zero and $\mathcal{S} \subset \mathbf{P}^n_k$ an equidimensional subscheme of codimension r with no embedded components, defined by forms of degrees $d_1 \geq \cdots \geq d_t \geq 1$. Assume $\mathcal{S}$ is locally a complete intersection except possibly at finitely many points, and has only rational singularities outside a subscheme of dimension one. If $\mathcal{S}$ is not a complete intersection, then*

$$\operatorname{reg}(\mathcal{S}) \leq d_1 + \cdots + d_r - r - 1.$$

To prove this result we pass to a sufficiently general link $\mathcal{S}'$ of $\mathcal{S}$ and apply an improved version of Kodaira vanishing to $\mathcal{S}'$. This gives a short proof if $\mathcal{S}$ is smooth of dimension at most three, since smoothness passes to $\mathcal{S}'$ ([Hi], [PS]). While this is no longer true in higher dimensions, we are able to show that a generic link of a complete intersection rational singularity again has rational singularities (Theorem 3.10). We deduce this from a general observation relating the Rees algebra of an ideal to various residual intersections. This result also has applications to the theory of residual intersections and sheds new light on some known classes of free resolutions for such ideals ([BKM], [BE], [KU1]) (Proposition 3.7, Corollary 3.8, and Remark 3.9).

Our main result in arbitrary characteristic (Theorem 4.7(b)) uses the notion of F-rationality that extends the concept of rational singularities to any characteristic ([HH1]).

---

* Supported in part by the National Science Foundation.

Manuscript received October 27, 2000.

American Journal of Mathematics.

THEOREM 0.2. *Let k be any field and $\mathcal{S} = \mathrm{Proj}(R/I) \subset \mathbf{P}^n_k = \mathrm{Proj}(R)$ a locally complete intersection scheme of codimension r with $1 < r < n$. Assume $\mathcal{S}$ has at most isolated irrational singularities if* $\mathrm{char}(k) = 0$ *or has at most F-rational singularities if* $\mathrm{char}(k) > 0$. *Further suppose I is generated by forms of degrees $d_1 \geq \cdots \geq d_t \geq 1$ and is not a complete intersection. Then*

$$\mathrm{reg}(R/I) \leq \frac{(\dim \mathcal{S} + 2)!}{2}(d_1 + \cdots + d_r - r - 1).$$

Notice that I is not required to be saturated or equidimensional. To prove the result we induct on the dimension of the scheme by passing to the intersection of $\mathcal{S}$ with a generic link $\mathcal{S}'$. Two ingredients are used in this reduction: first, a result of K. Smith ([S2]) that allows one to control the degrees of the equations defining $\mathcal{S}'$; second, a theorem we prove which says that F-rationality passes from $\mathcal{S}$ to $\mathcal{S} \cap \mathcal{S}'$ (Theorem 4.4). The latter theorem follows from a broader study of how F-rationality behaves under taking generic residual intersections (Theorem 3.13).

Acknowledgements. The first author is grateful to Michigan State University for its hospitality while this work was completed.

1. On the equations defining a link. To prove the above theorems we will need to control the degrees of the equations defining a direct link of a scheme. This is the content of Theorem 1.7. Its proof uses liaison theory and relies on a Kodaira vanishing theorem due to Ohsawa (in characteristic zero) as well as on work by K. Smith (in positive characteristic). To give a flavor of the results in this section, we start with the following observation:

PROPOSITION 1.1. *Let A be a standard graded algebra over a perfect field k such that $A_{\mathfrak{p}}$ is regular for every minimal prime $\mathfrak{p}$ of maximal dimension. Let $d := \dim A$, $\omega := \omega_A$ and let C be defined by the sequence*

$$0 \to (\omega_{\leq d}) \xrightarrow{\mathrm{can}} \omega \longrightarrow C \to 0.$$

Then $\mathrm{Supp}(C) \subset \mathrm{Sing}(A)$.

Proof. Let $\mathfrak{a} \subset A$ be the intersection of all primary components of 0 of maximal dimension. Without changing ω or $\dim A$, we may replace A by $A/\mathfrak{a}$ to assume that A is reduced and equidimensional. Consider the fundamental class

$$\mathbf{c}_{A/k} : \wedge^d \Omega_{A/k} \longrightarrow \omega_{A/k},$$

a natural A-linear map from the module of differential d-forms to the Dedekind complementary module, which is a particular graded canonical module of A sitting inside the module of meromorphic d-forms ([Elz, p. 34], [KW, 4.11 and 5.13], [Li, 3.1]). The homomorphism $\mathbf{c}_{A/k}$ is homogeneous and its cokernel is supported on the singular locus of A ([KW, 5.13]). Now the assertion follows since $\wedge^d \Omega_{A/k}$ is generated in degree d. $\square$

Let us recall the definition of the Castelnuovo-Mumford regularity of a graded module and of a sheaf of modules over an embedded projective scheme.

Definition. If S is a polynomial ring over a Noetherian ring B and M a finitely generated graded S-module, then

$$\begin{aligned}\mathrm{reg}(M) &:= \min\{\mu \mid H^i_{S_+}(M)_{>\mu-i} = 0,\ \forall i\} \\ &= \min\{\mu \mid \mathrm{Tor}^S_i(M, B)_{>\mu+i} = 0,\ \forall i\}.\end{aligned}$$

If $\mathcal{S}$ is a subscheme of $\mathbf{P}^n_B := \mathrm{Proj}(S)$, we set $\mathrm{reg}(\mathcal{S}) := \mathrm{reg}(S/I)$, where I is the unique saturated ideal such that $\mathcal{S} = \mathrm{Proj}(S/I)$.

If $\mathcal{M}$ is a coherent sheaf on $\mathbf{P}^n_B$ such that $\Gamma_\bullet(\mathcal{M}) := \bigoplus_{\mu\in\mathbf{Z}} H^0(\mathbf{P}^n_B, \mathcal{M}(\mu))$ is finitely generated over S, we set $\mathrm{reg}(\mathcal{M}) := \mathrm{reg}(\Gamma_\bullet(\mathcal{M}))$.

Corollary 1.2. *Let R be a standard graded Gorenstein algebra over a perfect field, and let $\mathcal{S}$ be a projective equidimensional reduced subscheme of $\mathcal{Z} := \mathrm{Proj}(R)$. Consider a direct link $\mathcal{S}'$ of $\mathcal{S}$ in $\mathcal{Z}$, given by forms of degrees $d_1, \dots, d_r$, and write $\sigma := \mathrm{reg}(R) + \sum_{i=1}^r (d_i - 1)$. If J is the defining ideal of $\mathcal{S}'$, then the scheme defined by $(J_{\leq\sigma})$ coincides with $\mathcal{S}'$ outside the singular locus of $\mathcal{S}$.*

Proof. Let I be the defining ideal of $\mathcal{S}$, $\omega := \omega_{R/I}$ and $d := \dim R/I$. We may assume that $\sigma \geq d_i$ for every i, since otherwise one reduces to the case where R is regular and $r = 1$. Let $\mathfrak{b}$ be the ideal generated by the forms linking $\mathcal{S}$ to $\mathcal{S}'$. Notice that $\mathfrak{b} = (\mathfrak{b}_{\leq\sigma})$. From the graded isomorphisms $J/\mathfrak{b} \simeq \mathrm{Hom}_{R/\mathfrak{b}}(R/I, R/\mathfrak{b}) \simeq \mathrm{Ext}^r_R(R/I, R)[-d_1 \cdots - d_r] \simeq \omega[d-\sigma]$ we obtain a commutative diagram

$$\begin{array}{ccc}
((J/\mathfrak{b})_{\leq\sigma}) & \xrightarrow{\sim} & (\omega[d-\sigma]_{\leq\sigma}) = (\omega_{\leq d})[d-\sigma] \\
\downarrow & & \downarrow \\
J/\mathfrak{b} & \xrightarrow{\sim} & \omega[d-\sigma] \\
\downarrow & & \downarrow \\
D & \longrightarrow & C[d-\sigma],
\end{array}$$

that identifies the cokernels D and $C[d-\sigma]$.

Now by Proposition 1.1, C is supported on $\mathrm{Sing}(R/I)$, therefore so is D. As $\mathfrak{b}$ is generated in degree $\leq \sigma$ the conclusion follows. $\square$

Definition. A scheme $\mathcal{S}$, essentially of finite type over a field of characteristic zero, has *rational singularities* if it is normal and, for $\mathcal{S}' \xrightarrow{\pi} \mathcal{S}$ a resolution of singularities of $\mathcal{S}$, one of the following equivalent properties holds,

(1) $\mathbf{R}^i\pi_*\mathcal{O}_{\mathcal{S}'} = 0$ for $i > 0$,

(2) $\mathcal{S}$ is Cohen-Macaulay and $\pi_*\omega_{\mathcal{S}'} = \omega_{\mathcal{S}}$.

Notice that it is implicit in the definition that these properties are independent of the desingularization. Notice also that $\pi_*\omega_{\mathcal{S}'}$ is always a subsheaf of $\omega_{\mathcal{S}}$ and that the normality of $\mathcal{S}$ implies that $\pi_*\mathcal{O}_{\mathcal{S}'} = \mathcal{O}_{\mathcal{S}}$.

Definition. A ring R of prime characteristic is *F-rational* if every parameter ideal in R is tightly closed. A scheme is F-rational if all its local rings are F-rational.

It has not been proved that the F-rational property is stable under localization in general. However this is known for rings that are homomorphic images of Cohen-Macaulay rings ([HH1, 4.2(f)]), a condition always satisfied in our context.

The notion of F-rationality extends to rings essentially of finite type over a field of characteristic zero (via reduction modulo $p \gg 0$, see [S1, 4.1] for a precise definition) in which case one talks about rings of *F-rational type*. Deviating slightly from standard terminology, we will say that a ring R is of *rational type* if either R is F-rational of prime characteristic, or R is essentially of finite type over a field of characteristic zero and has F-rational type. It was proved by K. Smith in [S1, 4.3] and by Hara and Mehta–Srinivas in [Har, 1.1] and [MS, 1.1], that in characteristic zero the notions of rational singularity and F-rational type coincide. We will denote by $\mathrm{Irr}(\mathcal{S})$ the locus of

the points of non rational type of a scheme $\mathcal{S}$. This locus is closed (at least under an excellence hypothesis satisfied in our context) by [Ve, 3.5].

THEOREM 1.3. *Let $\mathcal{S}$ be a projective equidimensional scheme over a field of characteristic zero and $\mathcal{L}$ an ample invertible sheaf on $\mathcal{S}$, then $H^i(\mathcal{S}, \omega_{\mathcal{S}} \otimes \mathcal{L}) = 0$ for $i > \max\{0, \dim \operatorname{Irr}(\mathcal{S})\}$. In particular, if $\dim \operatorname{Irr}(\mathcal{S}) \leq 0$ then $\operatorname{reg}(\omega_{\mathcal{S}}) = \dim \mathcal{S} + 1$, for the embedding given by any very ample invertible sheaf $\mathcal{L}$.*

Proof. We may suppose that the field is algebraically closed and that $\mathcal{S}$ is reduced. Since the cohomology modules in question are epimorphic images of the direct sum of the corresponding cohomology modules of the irreducible components of $\mathcal{S}$, we may further assume that $\mathcal{S}$ is irreducible. Let $\mathcal{S}' \xrightarrow{\pi} \mathcal{S}$ be a desingularization of $\mathcal{S}$. One has an exact sequence of coherent sheaves (see e. g. [Elk, p. 141]),

$$(*) \quad 0 \to \pi_* \omega_{\mathcal{S}'} \xrightarrow{\tau} \omega_{\mathcal{S}} \to C \to 0,$$

with $\operatorname{Supp}(C) \subset \operatorname{Irr}(\mathcal{S})$, and from [Oh] (or [Ko, 2.1(iii)]), it follows that

$$H^i(\mathcal{S}, \pi_* \omega_{\mathcal{S}'} \otimes \mathcal{L}) = 0$$

for $i > 0$. The exact sequence in cohomology derived from $(*)$ proves our claim. For the equality $\operatorname{reg}(\omega_{\mathcal{S}}) = \dim \mathcal{S} + 1$ also notice that $H^{\dim \mathcal{S}}(\mathcal{S}, \omega_{\mathcal{S}}) \neq 0$ (see the proof of Proposition 4.2). $\square$

Remark 1.4 (i) If $\dim \mathcal{S} \geq 1$, $\mathcal{L}$ is very ample giving an embedding $\mathcal{S} \subset \mathbf{P}^n_k$, and $\mathcal{S} = \operatorname{Proj}(A)$ with $A = k[X_0, \ldots, X_n]/I$ for a homogeneous ideal I, then $\operatorname{reg}(\omega_{\mathcal{S}}) = \operatorname{reg}(\omega_A)$.

(ii) If $\mathcal{S} = \operatorname{Proj}(A)$, where A is a standard graded domain of dimension at least two over an algebraically closed field of characteristic zero, the above proof shows that $\Gamma_\bullet(\pi_* \omega_{\mathcal{S}'})$ is a submodule of ω_A which has Castelnuovo-Mumford regularity equal to $\dim A$, and which coincides with ω_A outside the irrational locus of A.

We will need the following partial generalization to positive characteristic, which again only needs to be proved for reduced and irreducible schemes:

THEOREM 1.5. ([S2, 3.2]) *Let $\mathcal{S}$ be a projective equidimensional scheme of rational type. If $\mathcal{L}$ is a globally generated ample invertible sheaf on $\mathcal{S}$, then $\omega_{\mathcal{S}} \otimes \mathcal{L}^{\dim \mathcal{S}+1}$ is generated by its global sections.*

Remark 1.6. Even without the rationality assumption, one can prove as in [S2, the proof of 3.2] using [Ve, 3.9] that $\omega_{\mathcal{S}} \otimes \mathcal{L}^{\dim \mathcal{S}+1}$ is generated by its global sections on the rational locus of $\mathcal{S}$. Furthermore according to [S3, Theorem 1], if in Theorem 1.5, $\mathcal{L}$ is very ample then $\omega_{\mathcal{S}} \otimes \mathcal{L}^{\dim \mathcal{S}}$ is globally generated, unless $\mathcal{S}$ is a projective space $\mathbf{P}$ and $\mathcal{L} = \mathcal{O}_{\mathbf{P}}(1)$.

THEOREM 1.7. *Let R be a standard graded Gorenstein algebra over a field k, and let $\mathcal{S}$ be a projective equidimensional reduced subscheme of $\mathcal{Z} := \operatorname{Proj}(R)$. Consider $\mathcal{S}'$, a direct link of $\mathcal{S}$ in $\mathcal{Z}$ given by forms of degrees $d_1, \ldots, d_r$ and write $\sigma := \operatorname{reg}(R) + \sum_{i=1}^r (d_i - 1)$. Denote by J the defining ideal of $\mathcal{S}'$ and set $J^\sharp := (J_{\leq \sigma})$.*

(i) *If $\dim \mathcal{S} = 1$, then $J = J^\sharp$.*

(ii) *If $\operatorname{char}(k) = 0$ and $\dim \operatorname{Irr}(\mathcal{S}) \leq 0$, then $J = J^\sharp$.*

(iii) *$J^\sharp = J \cap K$ where the scheme defined by K is supported on the irrational locus of $\mathcal{S}$. In particular, if $\mathcal{S}$ is of rational type then $\mathcal{S}' = \operatorname{Proj}(R/J^\sharp)$.*

Proof. First notice that in case (i), $\operatorname{reg}(\omega_{\mathcal{S}}) = 2$ as $H^1(\mathcal{S}, \omega_{\mathcal{S}}(j)) = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-j)) = 0$ for $j > 0$. The argument is then the same as in the proof of Corollary 1.2, replacing Proposition 1.1 by the above equality for (i), by Theorem 1.3 and Remark 1.4(i) for (ii), and by Remark 1.6 for (iii), each time with $\mathcal{L} = \mathcal{O}_{\mathcal{S}}(1)$. $\square$

2. Regularity results for schemes of dimension zero and one. If I is an ideal, I^{top} will denote the unmixed part of I, that is the intersection of the primary components of height equal to the height of I.

PROPOSITION 2.1. *Let R be a standard graded Gorenstein algebra of dimension $n+1$ over a field and I a homogeneous ideal of height n. Assume that $f_1, \ldots, f_{n+1}$ are forms of degrees $d_1, \ldots, d_{n+1}$ in I so that $\operatorname{ht}((f_1, \ldots, f_{n+1}) : I) \geq n+1$ and $f_1, \ldots, f_n$ form a regular sequence. Then either*

$$\operatorname{reg}(R/I) \leq \operatorname{reg}(R) + \sum_{i=1}^{n}(d_i - 1)$$

if $\operatorname{indeg}(I^{top}/(f_1, \ldots, f_n)) \geq d_{n+1} - 1$, *or* $\operatorname{reg}(R/I) \leq \operatorname{reg}(R) + \sum_{i=1}^{n+1}(d_i - 1) - \operatorname{indeg}(I^{top}/(f_1, \ldots, f_n))$ *otherwise.*

Proof. Write $\mathfrak{b} := (f_1, \ldots, f_n)$ and $J := (f_1, \ldots, f_{n+1})$. Notice that $I^{top} = J^{top}$ and $\operatorname{reg}(R/I) \leq \operatorname{reg}(R/J)$. Further, there is an exact sequence,

$$0 \to (R/(\mathfrak{b} : J))[-d_{n+1}] \xrightarrow{\times f_{n+1}} R/\mathfrak{b} \longrightarrow R/J \to 0,$$

which gives

$$\operatorname{reg}(R/J) \leq \max\{\operatorname{reg}(R/\mathfrak{b}), \operatorname{reg}(R/(\mathfrak{b} : J)) + d_{n+1} - 1\}.$$

Since $J^{top} = I^{top}$ and $\mathfrak{b} : J$ is Cohen-Macaulay,

$$\begin{aligned} \operatorname{reg}(R/(\mathfrak{b} : J)) &= \operatorname{reg}(R/\mathfrak{b}) - \operatorname{indeg}(\mathfrak{b} : (\mathfrak{b} : J)/\mathfrak{b}) \\ &= \operatorname{reg}(R/\mathfrak{b}) - \operatorname{indeg}(I^{top}/\mathfrak{b}), \end{aligned}$$

and this proves our claim. $\square$

PROPOSITION 2.2. *Let R be a standard graded Gorenstein algebra of dimension $n+1$ over a field k and I a homogeneous ideal of height $n-1 > 0$. Assume that I is a complete intersection locally in codimension n and that I^{top} is reduced. Let $f_1, \ldots, f_t$ be forms of degrees $d_1 \geq \cdots \geq d_t \geq 1$ in I such that $\operatorname{ht}((f_1, \ldots, f_t) : I) \geq n+1$. Writing $\sigma := \operatorname{reg}(R) + \sum_{i=1}^{n-1}(d_i - 1)$ one has*

$$\operatorname{reg}(R/I) \leq \max\{\sigma, 3\sigma - 3\},$$

unless $R = k[X, Y, Z]$ and $I = \ell K$ with ℓ a linear form and K a complete intersection of 3 forms of degree $d_1 - 1$ (in which case $\operatorname{reg}(R/I) = 3\sigma - 2$).

Proof. We may assume that the ground field is infinite, that $I = (f_1, \ldots, f_t)$ and that I is not a complete intersection. There exist forms $\alpha_1, \ldots, \alpha_{n-1}$ of degrees $d_1, \ldots, d_{n-1}$ so that $\alpha_1, \ldots, \alpha_{n-1}$ form a regular sequence and $J = (\alpha_1, \ldots, \alpha_{n-1}) : I$ is a geometric link of I^{top}. The exact sequence

$$0 \to R/(\alpha_1, \ldots, \alpha_{n-1}) \longrightarrow R/I \oplus R/J \longrightarrow R/I + J \to 0$$

shows that

$$\operatorname{reg}(R/I) \leq \max\{\operatorname{reg}(R/(\alpha_1, \ldots, \alpha_{n-1})), \operatorname{reg}(R/I+J)\}.$$

We first assume that R is not regular or that $n-1 \geq 2$. In this case $\sigma \geq d_n$. By Theorem 1.7(i), $J = (J_{\leq \sigma})$. Let $\mathfrak{P}$ be the finite set of primes of height n containing $I+J$. For $\mathfrak{p} \in \mathfrak{P}$ either $I_{\mathfrak{p}}$ is a complete intersection of height $n-1$ and $(I+J/I)_{\mathfrak{p}}$ is cyclic, or $I_{\mathfrak{p}}$ is a complete intersection of height n and $(I+J/I)_{\mathfrak{p}} = 0$. Since furthermore $\sigma \geq d_n$, we can choose forms $\beta_1, \ldots, \beta_{n-1}$ in I of degrees $d_1, \ldots, d_{n-1}$ and a form β_n in $I+J$ of degree σ so that $\beta_1, \ldots, \beta_n$ form a regular sequence, $(I+J)/(\beta_1, \ldots, \beta_n)$ is generated in degrees $\leq \sigma$, and $(I+J)_{\mathfrak{p}} = (\beta_1, \ldots, \beta_n)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathfrak{P}$. Now there exists a form β_{n+1} of degree σ in $I+J$ such that $\operatorname{ht}((\beta_1, \ldots, \beta_{n+1}) : (I+J)) \geq n+1$. Therefore

$$\operatorname{reg}(R/I+J) \leq \max\{\sigma + \sigma - 1, \sigma + 2(\sigma-1) - 1\}$$

by Proposition 2.1. Also notice that $2\sigma - 1 \leq \max\{\sigma, 3\sigma - 3\}$ for a non negative integer σ.

Next, assume that R is regular and $n-1 = 1$, hence $R = k[X, Y, Z]$. In this case $I = \ell K$ with ℓ a form of degree $s \geq 1$ and K a homogeneous ideal of height 2 or 3. If K is not a complete intersection of height 3, then $\operatorname{reg}(R/K) \leq \max\{d_1 - s - 1 + d_2 - s - 1, d_1 - s - 1 + d_2 - s - 1 + d_3 - s - 1 - 1\}$. This follows from Proposition 2.1 if K has height 2 and is obvious otherwise. Thus $\operatorname{reg}(R/I) = s + \operatorname{reg}(R/K) \leq \max\{2\sigma - 1, 3\sigma - 3\}$ as asserted. If on the other hand K is a complete intersection of 3 forms of degrees $s_i \leq d_i - s$, we obtain $\operatorname{reg}(R/I) = s + s_1 - 1 + s_2 - 1 + s_3 - 1$, which is $\leq 3\sigma - 3$ unless $s = 1$ and $s_1 = s_2 = s_3 = d_1 - 1$. $\square$

3. Controlling residual intersections via blowing up or deformation. This section contains the main technical results of the paper. We investigate the behavior of rational and F-rational singularities under generic residual intersections. Two methods are applied, the study of residual intersections via projecting free resolutions of blow-up algebras, and the use of deformation theory to reduce to the "generic" case. The first approach does not yield results about F-rationality, but gives insight into resolutions of residual intersections of ideals that are not necessarily complete intersections. The second method on the other hand, allows us to treat F-rationality, but only in the context of residual intersections of complete intersections.

We start with a very general lemma from homological algebra.

LEMMA 3.1. *Let R be a commutative ring, $S := R[T_1, \ldots, T_r]$, and $(D_\bullet, d_\bullet)$ a graded complex of S-modules with $D_i = 0$ for $i < 0$ and $D_i = \bigoplus_{j \in \mathbf{Z}} S[-j]^{\beta_{ij}}$. Set $b_i := \sup\{j \mid \beta_{ij} \neq 0\}$ and $\mathfrak{g} := (T_1, \ldots, T_r)$. Assume that $H_q((D_\bullet)_{T_\ell}) = 0$ for $q > 0$ and all ℓ, and set $M := H_0(D_\bullet)$. Then*

(i) *$H_p(D_\bullet) \simeq H_{r+p}(H^r_{\mathfrak{g}}(D_\bullet))$ for $p > 0$, and $H^q_{\mathfrak{g}}(M) \simeq H_{r-q}(H^r_{\mathfrak{g}}(D_\bullet))$ for $q \geq 0$,*

(ii) *$H_p(D_\bullet)_\nu = 0$ for $\nu > b_{r+p} - r$ and $p > 0$,*

(iii) *$H^q_{\mathfrak{g}}(M)_\nu = 0$ for $\nu > b_{r-q} - r$ and $q \geq 0$.*

(iv) *Let $k \geq 0$ and $t \geq 0$. If $\beta_{ij} = 0$ for $i < k+r$ and $j \geq k+r$ and for $k < i < k + \max\{2, t\}$ and $j \leq k$, one has an exact sequence,*

$$H^r_{\mathfrak{g}}(D_{k+r+t})_k \to \cdots \to H^r_{\mathfrak{g}}(D_{k+r})_k \xrightarrow{\tau} (D_k)_k \to \cdots \to (D_0)_k \xrightarrow{can} (M/H^0_{\mathfrak{g}}(M))_k \to 0.$$

Proof. Consider the two spectral sequences arising from the double complex $\mathcal{C}^\bullet_{\mathfrak{g}}(D_\bullet)$, where $\mathcal{C}^\bullet_{\mathfrak{g}}(N)$ denotes the Čech complex on N relatively to the ideal $\mathfrak{g}$. Note that as $\mathfrak{g}$ is generated by a

regular sequence, the Čech cohomology is the same as the local cohomology. From the hypotheses one immediately gets,

$$'(1) \quad {}'E_1^{pq} = \begin{cases} H_p(D_\bullet) \text{ if } q = 0, \\ \mathcal{C}_{\mathfrak{g}}^q(M) \text{ if } p = 0, \\ 0 \quad \text{else,} \end{cases}$$

$$'(2) \quad {}'E_2^{pq} = {}'E_\infty^{pq} = \begin{cases} H_p(D_\bullet) \text{ if } q = 0 \text{ and } p \neq 0, \\ H_{\mathfrak{g}}^q(M) \text{ if } p = 0, \\ 0 \quad \text{else,} \end{cases}$$

$$''(1) \quad {}''E_1^{pq} = \begin{cases} H_{\mathfrak{g}}^r(D_p) \text{ if } q = r, \\ 0 \quad \text{else,} \end{cases}$$

$$''(2) \quad {}''E_2^{pq} = {}''E_\infty^{pq} = \begin{cases} H_p(H_{\mathfrak{g}}^r(D_\bullet)) \text{ if } q = r, \\ 0 \quad \text{else.} \end{cases}$$

This immediately gives (i). Also (ii) and (iii) follow as the ${}''E_1^{r+p,r}$ and ${}''E_1^{r-q,r}$ terms are zero in the considered degrees.

For (iv), notice that our hypotheses imply that $({}''E_1^{ir})_k = 0$ for $i < k + r$ and $(D_{k+i})_k = 0$ for $0 < i < \max\{2, t\}$. This shows that the given complex is exact to the left of τ, and between τ and the map *can*. It also implies that $H_{k+r}(H_{\mathfrak{g}}^r(D_\bullet))_k = \operatorname{coker}(H_{\mathfrak{g}}^r(d_{k+r+1}))_k$. Furthermore $({}'E_2^{k0})_k = H_k(D_\bullet)_k = \ker(d_k)_k$ and $H_{\mathfrak{g}}^0(M)_k = 0$ if $k > 0$, whereas $({}'E_2^{00})_0 = H_{\mathfrak{g}}^0(M)_0$ and $(D_0)_0 = M_0$ if $k = 0$. Therefore, defining τ as composition of canonical maps

$$H_{\mathfrak{g}}^r(D_{k+r})_k \longrightarrow H_{k+r}(H_{\mathfrak{g}}^r(D_\bullet))_k \xrightarrow{\delta} ({}'E_2^{k0})_k \hookrightarrow (D_k)_k$$

proves our claim, as δ is an isomorphism. $\square$

In our situation, we will be interested in the more specific result below, that essentially follows from the one above. For a ring R, an R-module M and a sequence γ of elements of R, we will denote in the sequel by $K_\bullet(\gamma\,; M)$ the Koszul complex of γ on M and by $H_i(\gamma\,; M)$ the i-th homology module of this complex.

Proposition 3.2. *Let R be a commutative ring, $S := R[T_1, \ldots, T_r]$, $\mathfrak{g} := (T_1, \ldots, T_r)$, and M a graded S-module. Assume that M has a free S-resolution $F_\bullet$ such that $F_i = \bigoplus_{j=0}^{i} S[-j]^{\beta_{ij}}$. Let $\gamma = \gamma_1, \ldots, \gamma_s$ be a sequence of elements in S_1 such that $H_i(\gamma\,; M_{T_\ell}) = 0$ for $i > 0$ and all ℓ, let J be the ideal they generate in S and $D_\bullet := F_\bullet \otimes_S K_\bullet(\gamma\,; S)$. Then*

(i) *$H_i(\gamma\,; M) \simeq H_{r+i}(H_{\mathfrak{g}}^r(D_\bullet))$ for $i > 0$, and $H_{\mathfrak{g}}^i(M/JM) \simeq H_{r-i}(H_{\mathfrak{g}}^r(D_\bullet))$ for $i \geq 0$,*

(ii) *$H_{\mathfrak{g}}^i(M/JM)$ is concentrated in degrees $\leq -i$ for all i,*

(iii) *$H_i(\gamma\,; M)$ is concentrated in degree i for $i > 0$,*

(iv) *$H_{\mathfrak{g}}^0(M/JM)_0 = (0 :_{M/JM} \mathfrak{g})_0 = J_1 M_0 :_{M_0} \mathfrak{g}_1$,*

(v)

$$\Gamma(k) := \Gamma(\operatorname{Proj}(S), \widetilde{M/JM}(k)) = \begin{cases} M_0/(J_1 M_0 :_{M_0} \mathfrak{g}_1) \text{ if } k = 0, \\ (M/JM)_k \qquad \text{if } k > 0. \end{cases}$$

(vi) *Let $k \geq 0$. If $\beta_{ij} = 0$ for $i \neq j \leq k$, then one has an acyclic complex of free R-modules,*

$$\cdots \to H^r_{\mathfrak{g}}(D_{k+r+1})_k \to H^r_{\mathfrak{g}}(D_{k+r})_k \xrightarrow{\tau} (D_k)_k \to \cdots \to (D_0)_k \to 0$$

that resolves $\Gamma(k)$.

(vii) *Let $k \geq 0$. There exists a morphism of complexes $u : H^r_{\mathfrak{g}}(D_\bullet)_k\{r\} \longrightarrow (D_\bullet)_k$ such that $H_i(u)$ is an isomorphism for $i > 0$ and a monomorphism for $i = 0$, $H_i(u) = 0$ for $i \neq k$, and $H_k(u)$ is the canonical map $H_{k+r}(H^r_{\mathfrak{g}}(D_\bullet)_k) \longrightarrow H_k((D_\bullet)_k)$. Therefore the mapping cone of u is a free R-resolution of $\Gamma(k)$.*

Proof. For (i) and (ii), notice that $D_\bullet$ satisfies the hypotheses of Lemma 3.1, and, as $F_\bullet$ resolves M, $H_i(D_\bullet) \simeq H_i(\gamma\,; M)$. Lemma 3.1 also shows that $H_i(\gamma\,; M)_\nu = 0$ for $\nu > i$ if $i \neq 0$, and as M is generated in degree 0 and the γ_i's are of degree 1, $K_i(\gamma\,; M)_\nu = 0$ for $\nu < i$. Now (iii) follows.

To prove (iv) note that one always has $0 :_{M/JM} \mathfrak{g} \subset H^0_{\mathfrak{g}}(M/JM)$ and that $\mathfrak{g}[H^0_{\mathfrak{g}}(M/JM)]_0 \subset [H^0_{\mathfrak{g}}(M/JM)]_{\geq 1} = 0$ by (ii), thereby proving the other inclusion. Part (v) follows directly from (ii) and (iv), and part (vi) is a direct consequence of Lemma 3.1(iv).

Concerning (vii), set $L_\bullet := H^r_{\mathfrak{g}}(D_\bullet)_k\{r\}$ and $P_\bullet := (D_\bullet)_k$. By (i) and (iii), $L_\bullet$ and $P_\bullet$ have no homology in homological degrees $> k$ and there is a canonical map $H_k(L_\bullet) \xrightarrow{\delta} H_k(P_\bullet)$, which is an isomorphism for $k > 0$ and an embedding for $k = 0$. Now the truncated complexes $\cdots \to L_{k+1} \xrightarrow{\phi} L_k \to 0$ and $\cdots \to P_{k+1} \xrightarrow{\psi} P_k \to 0$ are free R-resolutions of $\mathrm{Coker}(\phi)$ and $\mathrm{Coker}(\psi)$ respectively.

As $L_{k-1} = 0$, the canonical monomorphism $H_k(L_\bullet) \xrightarrow{i} \mathrm{Coker}(\phi)$ is an isomorphism. Now the map

$$\overline{u} : \mathrm{Coker}(\phi) \xrightarrow{i^{-1}} H_k(L_\bullet) \xrightarrow{\delta} H_k(P_\bullet) \xrightarrow{can} \mathrm{Coker}(\psi)$$

can be lifted to a morphism between the truncated complexes, which, extended by 0 to a morphism from $L_\bullet$ to $P_\bullet$, satisfies our requirements. Notice that $L_\bullet$ is trivial in homological degrees $< k$ and that by (iii), $P_\bullet$ has only homology in homological degrees 0 and k. The zeroth homology of the mapping cone of u is given by (v) if $k > 0$ and by (i) and (v) if $k = 0$. $\square$

Remark 3.3. Assume R is an ℓ-algebra, graded by a finitely generated free abelian group G, each R_i is a finite free ℓ-module, M is bigraded and the elements γ_i are bihomogeneous of degree, say, $(e_i, 1)$. Then the complexes that appear in Proposition 3.2 are also bigraded. Writing $\chi_{S[-k]}(X, Y) := Y^k \sum_{i \in G, j \in \mathbf{Z}} (\mathrm{rank}_\ell R_i) X^i Y^j$, and $\chi_{C_\bullet} := \sum_p (-1)^p \chi(C_p)$ for any bigraded complex $C_\bullet$ of free S-modules, one has

$$\chi_{D_\bullet}(X, Y) = \chi_{F_\bullet}(X, Y) \prod_{i=1}^{s} (1 - X^{e_i} Y).$$

From Proposition 3.2(vii) one sees that the Hilbert-Poincaré series of $\bigoplus_k \Gamma(k)$ is $\chi_{D_\bullet}(X, Y) - (-Y)^{-r} \chi_{D_\bullet}(X, Y^{-1})$, which in turn can be expressed in terms of $\chi_{F_\bullet}(X, Y)$.

The following proposition will be a key to passing rational singularities from the blow-up to a generic residual intersection.

Proposition 3.4. *Let R be a local ring essentially of finite type over a field of characteristic zero, and let A be a standard graded R-algebra with Castelnuovo-Mumford regularity 0 such that*

$\mathrm{Proj}(A)$ *has rational singularities. If* $x_1, \dots, x_r$ *generate* A_1 *set* $g_i := \sum_{j=1}^r U_{ij} x_j \in A_1 \otimes_R R'$, *where* $R' := R[U_{ij}]$ *is a polynomial ring and* $1 \le i \le s$. *Then the ring*

$$R'/((g_1, \dots, g_s)R' :_{R'} A_1)$$

has rational singularities.

Proof. Let $A' := A \otimes_R R'$, $S' := R'[T_1, \dots, T_r]$ and $\mathfrak{g} := (T_1, \dots, T_r)$. The natural homogeneous map $S' \to A'$ sending T_i to x_i makes A' a graded S'-module. The forms $\gamma_i := \sum_{j=1}^r U_{ij} T_j \in S'_1$ satisfy the hypotheses of Proposition 3.2. By assumption A' has an S'-resolution $F_\bullet$ satisfying the hypotheses of the same proposition. We set $J := (\gamma_1, \dots, \gamma_s)S'$ and notice that $JA' = (g_1, \dots, g_s)A'$.

Note that $\mathcal{Z} := \mathrm{Proj}(A'/JA')$ has a natural structure of a vector bundle over $\mathrm{Proj}(A)$ so that $\mathcal{Z}$ has rational singularities as well. Now consider the projection $\pi : \mathcal{Z} \to \mathrm{Spec}(R')$. One has

(1) $\mathbf{R}^p \pi_* \mathcal{O}_{\mathcal{Z}} = (H_{\mathfrak{g}}^{p+1}(A'/JA')_0)^\sim = 0$ for $p > 0$ by Proposition 3.2(ii),

(2) $\pi_* \mathcal{O}_{\mathcal{Z}} = (R'/((g_1, \dots, g_s)R' :_{R'} A_1))^\sim$ by Proposition 3.2(v).

By the Leray spectral sequence $\mathbf{R}^p \pi_* (\mathbf{R}^q \xi_* \mathcal{O}_{\widetilde{\mathcal{Z}}}) \Rightarrow \mathbf{R}^{p+q} (\pi \circ \xi)_* \mathcal{O}_{\widetilde{\mathcal{Z}}}$ where $\widetilde{\mathcal{Z}} \xrightarrow{\xi} \mathcal{Z}$ is a resolution of singularities of $\mathcal{Z}$, we get the result. $\square$

Remark 3.5. In any characteristic, one can define a scheme $\mathcal{S}$ to have rational singularities if there exists a desingularization $\mathcal{S}' \xrightarrow{\pi} \mathcal{S}$ such that $\mathbf{R}^\bullet \pi_* \mathcal{O}_{\mathcal{S}'} = \mathcal{O}_{\mathcal{S}}$ and $\mathbf{R}^\bullet \pi_* \omega_{\mathcal{S}'} = \omega_{\mathcal{S}}$. Using this definition one can show, along the same lines as in the above proof, that Proposition 3.4 holds without the hypothesis that R is essentially of finite type over a field of characteristic zero.

In the sequel, if M is a finitely generated module, $\mu(M)$ will denote the minimal number of generators of M. Let us recall some definitions and properties related to blow-up algebras.

Definition. Let R be a Noetherian local ring, $z := z_1, \dots, z_s$ a sequence of elements of R, and J the R-ideal they generate.

(i) z is a *d-sequence* if $\mu(J) = s$ and $((z_1, \dots, z_j) : z_{j+1}) \cap J = (z_1, \dots, z_j)$ for $0 \le j < s$,

(ii) z is a *proper sequence* if $z_{j+1} H_i(z_1, \dots, z_j; R) = 0$ for $0 \le j < s$ and $i > 0$,

(iii) J has *sliding depth* if $\operatorname{depth} H_i(z; R) \ge \dim R - s + i$ for all i,

(iv) $J : I$ is an *s-residual intersection* of the R-ideal I properly containing J if $\mathrm{ht}(J : I) \ge s$. The residual intersection is called *geometric* if in addition $\mathrm{ht}(I + (J : I)) > s$.

Proposition 3.6. *Let* R *be a Noetherian local ring,* $I \subset R$ *a proper ideal and* $z_1, \dots, z_r$ *minimal generators of* I. *Set* $S := R[T_1, \dots, T_r]$, $\mathcal{S}_I := \mathrm{Sym}_R(I)$, *and* $\mathcal{R}_I := R \oplus I \oplus I^2 \oplus \cdots$. *The generators of* I *define a natural commutative diagram of homogeneous epimorphisms of degree 0,*

$$\begin{tikzcd} & S \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathcal{S}_I \arrow[rr, "\alpha"] & & \mathcal{R}_I \end{tikzcd}$$

and

(i) α *is an isomorphism if and only if* $\alpha \otimes_R R/I$ *is,*

(ii) $\ker(\alpha)$ *is nilpotent if and only* $\ker(\alpha \otimes_R R/I)$ *is,*

(iii) *If* $z_1, \dots, z_r$ *is a proper sequence then the Castelnuovo-Mumford regularity of* $\mathcal{S}_I$ *is* 0, *and the converse holds if* R *has infinite residue field and* $z_1, \dots, z_r$ *are general,*

(iv) *If $z_1, \dots, z_r$ is a d-sequence then the Castelnuovo-Mumford regularity of $\mathcal{R}_I \otimes_R R/I$ is 0, and the converse holds if R has infinite residue field and $z_1, \dots, z_r$ are general,*

(v) *If $z_1, \dots, z_r$ is a d-sequence then α is an isomorphism,*

(vi) *$z_1, \dots, z_r$ is a regular sequence if and only if $\psi \otimes_R R/I$ is an isomorphism.*

For part (i) see [HSV, 3.1], for (ii) [HSV, 3.2], for (iii) and (iv) [HSV, 12.7, 12.8, 12.10], and for (v) [Hu, 3.1] or [Va, 3.15].

PROPOSITION 3.7. *Let R be a Noetherian local ring, $I \subset R$ an ideal generated by a proper sequence of length r, and $\mathcal{S}_I := \mathrm{Sym}_R(I)$. For $S := R[T_1, \dots, T_r]$ the map $S \to \mathcal{S}_I$ sending T_i to the i-th generator of I makes $\mathcal{S}_I$ a graded S-module.*

Let $J \subset I$ be an ideal generated by a sequence $\gamma := \gamma_1, \dots, \gamma_s$.

(a) *Assume $\operatorname{ht} I > 0$, R is universally catenary, and $\operatorname{Proj}(\mathcal{S}_I)$ is Cohen-Macaulay and equidimensional. Then $\gamma \subset (\mathcal{S}_I)_1$ form a regular sequence locally on $\operatorname{Proj}(\mathcal{S}_I)$ if and only if $\mu((I/J)_\mathfrak{p}) \leq \dim R_\mathfrak{p} - s + 1$ for every $\mathfrak{p} \in V(J : I)$. In this case $\operatorname{ht}(J : I) \geq s$.*

(b) *Assume $\gamma \subset (\mathcal{S}_I)_1$ form a regular sequence locally on $\operatorname{Proj}(\mathcal{S}_I)$.*

(i) $\operatorname{pd}_R R/(J : I) \leq \operatorname{pd}_S \mathcal{S}_I + s - r + 1$. *If R is regular, then*

$$\operatorname{depth} R/(J : I) \geq \operatorname{depth} \mathcal{S}_I - s - 1.$$

(ii) *If $\operatorname{ht} I > 0$, $J \neq I$, R is Cohen-Macaulay, and $\mathcal{S}_I$ is a perfect S-module, then $J : I$ is a perfect ideal of grade s.*

(iii) *Let $k > 0$ and $t \geq 0$. Assume that $\operatorname{pd}_R \mathrm{S}_j(I) \leq t + j$ for $1 \leq j \leq k$. Then $\operatorname{pd}_R \mathrm{S}_k(I/J) \leq \max\{t + k, \operatorname{pd}_S \mathcal{S}_I + s - r + 1\}$. In particular, if R is regular then*

$$\operatorname{depth} \mathrm{S}_k(I/J) \geq \min\{\dim R - t - k, \operatorname{depth} \mathcal{S}_I - s - 1\}.$$

Proof. (a) First notice that if $\operatorname{ht} I > 0$ and $\operatorname{Proj}(\mathcal{S}_I)$ is equidimensional, then $\mathcal{S}_I$, $\mathcal{R}_I$, and hence R are equidimensional, and $\dim \mathcal{S}_I = \dim \mathcal{R}_I = \dim R + 1$. Since $\operatorname{Proj}(\mathcal{S}_I)$ is Cohen-Macaulay, γ form a regular sequence on $\operatorname{Proj}(\mathcal{S}_I)$ if and only if $\operatorname{ht}((\gamma)\mathcal{S}_I : (\mathcal{S}_{I+})^\infty) \geq s$ (here and in what follows, (γ) denotes the ideal generated by the γ_i's as elements of $(\mathcal{S}_I)_1$). One has $(J :_R I, (\gamma))\mathcal{S}_I \subset (\gamma)\mathcal{S}_I : (\mathcal{S}_{I+})^\infty$, and by Proposition 3.2(ii), (iv) and Proposition 3.6(iii) equality holds if γ is a regular sequence on $\operatorname{Proj}(\mathcal{S}_I)$. Therefore the above height condition is equivalent to $\operatorname{ht}(J :_R I, (\gamma))\mathcal{S}_I \geq s$. As $\mathcal{S}_I$ is an equidimensional and catenary positively graded ring over a local ring, the last inequality means that $\dim \mathcal{S}_I/(J :_R I, (\gamma)) \leq \dim \mathcal{S}_I - s = \dim R - s + 1$. But $\mathcal{S}_I/(J :_R I, (\gamma)) \simeq \mathrm{Sym}_{R/J:I}(I/J)$ and by [HR, 2.6],

$$\dim \mathrm{Sym}_{R/J:I}(I/J) = \max\{\dim R/\mathfrak{p} + \mu((I/J)_\mathfrak{p}) \mid \mathfrak{p} \in V(J : I)\}.$$

Now the asserted equivalence follows because R is an equidimensional and catenary local ring. Furthermore if either condition holds, then for every $\mathfrak{p} \in V(I : J)$, $\dim R_\mathfrak{p} \geq s + \mu((I/J)_\mathfrak{p}) - 1 \geq s$.

(b) Proposition 3.6(iii) shows that Proposition 3.2 applies. By Proposition 3.2(v), parts (vi) and (vii) of that proposition imply the present assertions (i) and (iii), respectively. For (ii), notice that if $\operatorname{ht} I > 0$, and R and $\mathcal{S}_I$ are Cohen-Macaulay, then $\dim \mathcal{S}_I = \dim R + 1$ and $\operatorname{grade}(J : I) \geq s$ by (a). $\square$

Corollary 3.8. *Let R be a regular local ring and $I \subset R$ an ideal satisfying sliding depth such that $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ for every $\mathfrak{p} \in V(I)$. Then any geometric s-residual intersection of I is a perfect ideal of grade s.*

Proof. We may assume that R has infinite residue field and $\operatorname{ht} I > 0$. As I satisfies sliding depth and $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + 1$ for every $\mathfrak{p} \in V(I)$, $\mathcal{S}_I$ is Cohen-Macaulay and I is generated by a proper sequence $z_1, \ldots, z_r$ ([HSV, 5.3, 10.1, 12.9]). Let $J : I$ be a geometric s-residual intersection. Write $J = (\gamma_1, \ldots, \gamma_s)$ and $\gamma_i := \sum_{j=1}^r a_{ij} z_j$ with $a_{ij} \in R$. Set $g_i := \sum_{j=1}^r U_{ij} z_j$ where U_{ij} are variables, and let $\mathfrak{m}$ be the maximal ideal of R. Write $R' := R[U_{ij}]_{(\mathfrak{m}, U_{ij} - a_{ij})}$, $J' := (g_1, \ldots, g_s)R'$, and let $\pi : R' \to R$ be the specialization map sending U_{ij} to a_{ij}. Notice that $\pi(J') = J$. The generic elements $g_1, \ldots, g_s$ form a regular sequence on $\operatorname{Proj}(\mathcal{S}_{IR'})$. Hence by Proposition 3.7(b)(ii), $J' : IR'$ is a perfect ideal of grade s. Since $J : I = \pi(J') : \pi(IR')$ is a geometric residual intersection, [HU3, 4.2(ii)] shows that $J : I$ is perfect of grade s as well. $\square$

Remark 3.9. (i) In the setting of Proposition 3.7(b) and Corollary 3.8, the complexes of Proposition 3.2(vi) and (vii) provide free resolutions of $R/(J : I)$ and of $\mathrm{S}_k(I/J)$, $k > 0$, respectively.

(ii) If I is generated by a d-sequence, one can apply Proposition 3.2 to $\mathcal{G}_I := \mathcal{R}_I \otimes_R R/I$ instead of $\mathcal{S}_I$. This gives results for $R/((J + I^2) : I)$ and $I^k/(JI^{k-1} + I^{k+1})$, $k > 0$, that are similar to the ones of Proposition 3.7.

(iii) Let X be a Cohen-Macaulay scheme, let $S \subset Y \subset X$ be closed subschemes with $\operatorname{codim}_X S > 0$, and let $S' \subset X$ be the subscheme with $\mathcal{I}_{S'} = \mathcal{I}_Y : \mathcal{I}_S$. We can compare S' to the residual scheme proposed in [Fu, 9.2.2]. Indeed, let $\pi : \widetilde{X} := \operatorname{Bl}_S X \to X$, $E := \pi^{-1} S$, and $\mathcal{Z} \subset \widetilde{X}$ the subscheme with $\mathcal{I}_{\mathcal{Z}} = \mathcal{I}_{\pi^{-1} Y} \mathcal{I}_E^{-1}$. If $\mathcal{I}_S$ satisfies sliding depth, $\mu(\mathcal{I}_{S,x}) \leq \dim \mathcal{O}_{X,x}$ for every $x \in S$, $\mu(\mathcal{I}_{Y,x}) \leq s$ for every $x \in Y$, and $\mu(\mathcal{I}_{S,x}/\mathcal{I}_{Y,x}) \leq \dim \mathcal{O}_{X,x} - s + 1$ for every $x \in S'$, then Propositions 3.2(v), 3.6, 3.7(a) and [HSV, 5.3, 9.1, 12.9] show that $\mathcal{O}_{S'} = \pi_* \mathcal{O}_{\mathcal{Z}}$.

Theorem 3.10. *Let R be a local ring essentially of finite type over a field of characteristic zero, and $I = (z_1, \ldots, z_t) \subset R$ an ideal generated by a proper sequence. For $1 \leq i \leq s$, let $g_i := \sum_{j=1}^t U_{ij} z_j \in R[U_{ij}]$, where U_{ij} are variables. If $\operatorname{Proj}(\mathcal{S}_I)$ has rational singularities then $R[U_{ij}]/((g_1, \ldots, g_s) : I)$ has rational singularities. This is in particular the case when I is generated by a regular sequence and R/I has rational singularities.*

Proof. The first statement is a combination of Proposition 3.4 and Proposition 3.6(iii). For the second one, Proposition 3.6(v) and (vi) imply that $\mathcal{R}_I \simeq \mathcal{S}_I$ and that $\mathcal{G}_I = \mathcal{R}_I \otimes_R R/I \simeq S \otimes_R R/I$ has rational singularities. Now $\operatorname{Proj}(\mathcal{G}_I)$ is a Cartier divisor in $\operatorname{Proj}(\mathcal{R}_I)$, so that the result is a consequence of the following classical lemma. $\square$

Lemma 3.11. *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over a field and x an R-regular element. If R/xR is a ring of rational type, then so is R.*

This is proved in [HH1, 4.2(h)] in positive characteristic. In characteristic zero, it is proved in [Elk, proof of Théorème 2], and can also be reduced to the case of positive characteristic.

Remark 3.12. In Theorem 3.10, if $\mathcal{G}_I$ has rational singularities then so does R/I by [Bo]. Also if $z_1, \ldots, z_t$ is a regular sequence, $\operatorname{Proj}(\mathcal{G}_I)$ is regular if and only if R/I is regular.

In any characteristic, we have:

Theorem 3.13. *Let R be a ring essentially of finite type over a field, $I := (z_1, \ldots, z_t)$ an ideal, and $s \geq 0$ an integer. Let M be a t by s matrix of variables U_{ij}, $S := R[U_{ij}]$, $(\alpha_1 \ldots \alpha_s) := (z_1 \ldots z_t) \cdot M$, $J := (\alpha_1, \ldots, \alpha_s)S : IS$, and $\mathfrak{P} \in V(J)$.*

(i) *Assume that* $\mathfrak{P} \in V(IS)$, $I_{\mathfrak{P}\cap R}$ *is a complete intersection, and* $(R/I)_{\mathfrak{P}\cap R}$ *is of rational type. Then* $(S/IS+J)_{\mathfrak{P}}$ *is a ring of rational type.*

(ii) *If* $\mathfrak{P} \notin V(IS)$, *assume that* $R_{\mathfrak{P}\cap R}$ *is of rational type. If* $\mathfrak{P} \in V(IS)$, *suppose that* $I_{\mathfrak{P}\cap R}$ *is a complete intersection and* $(R/I)_{\mathfrak{P}\cap R}$ *is geometrically of rational type. In either case* $(S/J)_{\mathfrak{P}}$ *is a ring of rational type.*

Proof. Recall that (geometric) F-rationality and the rational singularity property descend and ascend under local ring extensions obtained by adjoining finitely many variables and localizing ([Ve, 3.1]). This takes care of the case where $\mathfrak{P} \notin V(IS)$, and we may assume from now on that $\mathfrak{P} \in V(IS)$.

Localizing at $\mathfrak{P} \cap R$, we may suppose that $(R, \mathfrak{m}, k)$ is a local ring with $\mathfrak{m} = \mathfrak{P} \cap R$, $I = (z_1, \ldots, z_t)$ is a complete intersection of height r, and R/I is of rational type (in (i)) or geometrically of rational type (in (ii)).

After a change of variables over R, we may suppose that $z_i = 0$ for $i > r$. We now replace R by $R[U_{ij} \mid i > r]_{\mathfrak{P}\cap R[U_{ij} \mid i>r]}$. Recall that this does not affect R/I being (geometrically) of rational type. We may then assume that the matrix M has size r by s. Let $n \geq 0$ be such that $I_n(M) \not\subset \mathfrak{P}$ and $I_{n+1}(M) \subset \mathfrak{P}$. Suppose for simplicity that the upper left n by n minor of M is not in $\mathfrak{P}$. Again we replace R by $R[U_{ij} \mid i \leq n \text{ or } j \leq n]_{\mathfrak{P}\cap R[U_{ij} \mid i\leq n \text{ or } j\leq n]}$, modify the generators of I, and perform a change of variables over the ring R to assume that $M = \begin{pmatrix} I_{n\times n} & 0 \\ 0 & N \end{pmatrix}$, where N is a matrix of variables contained in $\mathfrak{P}$. Finally, passing to the ring $R/(z_1, \ldots, z_n)$ we may suppose that $M = N$ is an r by s matrix of variables U_{ij} over R, $S = R[U_{ij}]$, and $\mathfrak{P} = (\mathfrak{m}, U_{ij})S$.

Notice that $J_{\mathfrak{P}} = (\alpha_1, \ldots, \alpha_s, I_r(M))S_{\mathfrak{P}}$ by [HU3, 3.4], and that the rings $(S/IS+J)_{\mathfrak{P}}$ and $(S/J)_{\mathfrak{P}}$ are domains by [HU3, 3.3]. Moreover, we may assume by induction on the dimension that $(S/IS+J)_{\mathfrak{P}}$ and $(S/J)_{\mathfrak{P}}$ are of rational type locally on the punctured spectrum.

As for (i) we consider the natural map

$$\phi : R/I \longrightarrow B := (R[U_{ij}]/(I + I_r(M)))_{(\mathfrak{m},U_{ij})} = (S/IS+J)_{\mathfrak{P}}.$$

As $R[U_{ij}]/I_r(M)$ is R-flat, ϕ is flat as well. Furthermore ϕ is local with closed fiber $C := (k[U_{ij}]/I_r(M))_{(U_{ij})}$. The ring C is geometrically of rational type by [Ke1, Proposition 2] in characteristic 0 and by [HH2, 7.14] in any characteristic. Therefore B is of rational type according to [Elk, Théorème 5] in characteristic 0 and to [Has, 6.4] or [En, 2.27] in positive characteristic. To apply the latter, notice that C is geometrically F-injective, and that the generic fibre of ϕ is F-rational because B is F-rational on the punctured spectrum and C is F-rational.

To prove (ii) set $A' := k[z_1, \ldots, z_r, U_{ij}]/(\alpha_1, \ldots, \alpha_s, I_r(M))$ and let A be the localization of A' at the homogeneous maximal ideal. Notice that $z_1, \ldots, z_r, U_{ij}$ are indeterminates over k. In characteristic 0, A has a rational singularity by [Ke2, Ke3]. In positive characteristic, A' is F-split by [MT, (3), p. 362], hence F-injective. On the other hand $a(A') < 0$ (see for instance [KU2, 2.1]). Thus, since A is F-rational on the punctured spectrum, it follows from [FW] (see also [HW, 1.6]) that A is F-rational.

Now consider the natural map

$$\psi : A \longrightarrow B := (R[U_{ij}]/(\alpha_1, \ldots, \alpha_s, I_r(M)))_{(\mathfrak{m},U_{ij})} = (S/J)_{\mathfrak{P}}.$$

Since R is flat over $k[z_1, \ldots, z_r]_{(z_1,\ldots,z_r)}$, ψ is flat. In addition it is local with closed fiber R/I, which is geometrically of rational type by assumption. Again applying [Elk, Théorème 5] and [Has, 6.4] or [En, 2.27], we conclude that B is of rational type as well. $\square$

4. Castelnuovo-Mumford regularity. This last section contains our main result on Castelnuovo-Mumford regularity. We first prove several facts about the behavior of local cohomology under liaison. Then we investigate, in the context of ideals in a standard graded algebra, how several properties (such as reducedness, F-rationality, smoothness) are preserved, or improved, by passing to a generic link (Theorem 4.4). The main result (Theorem 4.7) will follow from these facts, combined with Kodaira vanishing theorems and Theorem 1.7.

PROPOSITION 4.1. *Let k be a field, R a positively graded Gorenstein k-algebra with homogeneous maximal ideal $\mathfrak{m}$, I, J homogeneous ideals linked by a homogeneous Gorenstein ideal $\mathfrak{b}$, set $d := \dim R/\mathfrak{b}$, $a := a(R/\mathfrak{b})$, $\mathcal{S} := \operatorname{Proj}(R/I)$, $\mathcal{S}' := \operatorname{Proj}(R/J)$ and write $-^*$ for the graded k-dual.*

(a) $H^d_{\mathfrak{m}}(R/I) \simeq (J/\mathfrak{b})^*[-a]$.

(b) $H^{d-1}_{\mathfrak{m}}(R/I) \simeq \operatorname{coker}(R/J \to \operatorname{End}(\omega_{R/J}))^*[-a]$.

Assume $d \geq 2$. Then $\operatorname{End}(\omega_{R/J}) \simeq \oplus_\mu H^0(\mathcal{S}'_2, \mathcal{O}_{\mathcal{S}'_2}(\mu))$ where $\mathcal{S}'_2$ is the S_2-ification of $\mathcal{S}'$. Furthermore, $H^{d-1}_{\mathfrak{m}}(R/I)$ has finite length if and only if $\mathcal{S}'$ is S_2, in which case $H^{d-1}_{\mathfrak{m}}(R/I) \simeq H^1_{\mathfrak{m}}(R/J)^*[-a]$. Finally, if $\mathcal{S}'$ is reduced over $\overline{k}$, then $\dim_k H^{d-1}_{\mathfrak{m}}(R/I)_a + 1$ is the number of components of $\mathcal{S}'$ over $\overline{k}$ that are connected in codimension 1.

(c) *$H^i_{\mathfrak{m}}(R/I) \simeq H^{d-i}_{\mathfrak{m}}(R/J)^*[-a]$ for $1 \leq i \leq d-1$, if $\mathcal{S}$ satisfies S_{i+1} or $\mathcal{S}'$ satisfies S_{d-i+1}.*

Proof. Part (a) is a consequence of local duality since $\omega_{R/I} \simeq (J/\mathfrak{b})[a]$.

To prove (b), we dualize the liaison sequence

$$0 \to \omega_{R/J}[-a] \to R/\mathfrak{b} \to R/I \to 0$$

into $\omega_{R/\mathfrak{b}}$. Using local duality, we obtain an exact sequence

$$(R/\mathfrak{b})[a] \longrightarrow \operatorname{End}(\omega_{R/J})[a] \longrightarrow \operatorname{Ext}^1_{R/\mathfrak{b}}(R/I, \omega_{R/\mathfrak{b}}) \simeq H^{d-1}_{\mathfrak{m}}(R/I)^* \longrightarrow 0,$$

which proves the first assertion. As for the last claim notice that if k is algebraically closed and $\mathcal{S}'$ is reduced, then $\dim_k H^0(\mathcal{S}'_2, \mathcal{O}_{\mathcal{S}'_2})$ is the number of connected components of $\mathcal{S}'_2$, which equals the number of components of $\mathcal{S}'$ connected in codimension 1.

(c) First assume that $\mathcal{S}'$ satisfies S_{d-i+1}. For $i = d-1$ our assertion follows from part (b). Hence we may suppose that $1 \leq i \leq d-2$. Now $\dim \operatorname{Ext}^j_{R/\mathfrak{b}}(R/J, \omega_{R/\mathfrak{b}}) \leq i-j-1$ for $1 \leq j \leq i-1$, and $\dim \operatorname{Ext}^i_{R/\mathfrak{b}}(R/J, \omega_{R/\mathfrak{b}}) \leq 0$. Thus we may use the $\omega_{R/\mathfrak{b}}$-dual of a free $R/\mathfrak{b}$-resolution of R/J to conclude that

$$H^{i+1}_{\mathfrak{m}}(\operatorname{Hom}_{R/\mathfrak{b}}(R/J, \omega_{R/\mathfrak{b}})) \simeq \operatorname{Ext}^i_{R/\mathfrak{b}}(R/J, \omega_{R/\mathfrak{b}}).$$

But the first module is $H^{i+1}_{\mathfrak{m}}(\omega_{R/J}) \simeq H^i_{\mathfrak{m}}(R/I)[a]$ by the liaison sequence, whereas the latter one is $H^{d-i}_{\mathfrak{m}}(R/J)^*$. Finally if $\mathcal{S}$ is S_{i+1}, our assertion follows by reversing the roles of $\mathcal{S}$ and $\mathcal{S}'$. $\square$

PROPOSITION 4.2. *If the assumptions of Proposition 4.1 are satisfied then*

$$\operatorname{reg}(R/I) \leq \operatorname{reg}(\omega_{R/J}) + a - 1.$$

Moreover, $\operatorname{reg}(R/I) > d + a - 1$ if and only if $\operatorname{reg}(\omega_{R/J}) \neq d$, and in this case, $\operatorname{reg}(R/I) = \operatorname{reg}(\omega_{R/J}) + a - 1$. If $d \geq 2$, $\mathcal{S}'$ is reduced over $\overline{k}$ and $H^i_{\mathfrak{m}}(\omega_{R/J})_{\geq d-i} = 0$ for $i \neq d$, then $\operatorname{reg}(R/I) \leq d + a - 2$ if and only if $J_1 = \mathfrak{b}_1$ and $\mathcal{S}'$ is connected in codimension 1 over $\overline{k}$.

Proof. The assertions are easily derived from Proposition 4.1 and the liaison sequence (see also [Ch, 5.2]). Also notice that $\operatorname{reg}(\omega_{R/J}) \geq d$ because $1 \in \operatorname{End}(\omega_{R/J})_0 \simeq H^d_{\mathfrak{m}}(\omega_{R/J})^*_0$. $\square$

Definition. If $\mathcal{S}$ is an embedded equidimensional projective scheme with $\dim \mathcal{S} \geq 1$, we will say that $\mathcal{S}$ satisfies *weak Kodaira vanishing* if $\operatorname{reg}(\omega_{\mathcal{S}}) = \dim \mathcal{S} + 1$. If $\mathcal{S}$ is reduced, this is equivalent to the condition $H^i(\mathcal{S}, \omega_{\mathcal{S}}(\dim \mathcal{S} + 1 - i)) = 0$ for $1 \leq i \leq \dim \mathcal{S} - 1$.

COROLLARY 4.3. *Let $\mathcal{Z}$ be an arithmetically Gorenstein closed subscheme of a projective space, and let $\mathcal{S}$ and $\mathcal{S}'$ be closed subschemes of $\mathcal{Z}$ of dimension ≥ 1 linked by forms of degrees $d_1, \ldots, d_r$. Then* $\operatorname{reg}(\mathcal{S}) < \operatorname{reg}(\mathcal{Z}) + \sum_{i=1}^{r}(d_i - 1)$ *if and only if $\mathcal{S}'$ satisfies weak Kodaira vanishing.*

Proof. Notice that if $\mathcal{Z} = \operatorname{Proj}(R)$ and $\mathfrak{b} \subset R$ is the ideal generated by the forms linking $\mathcal{S}$ to $\mathcal{S}'$, then $a(R/\mathfrak{b}) = a(R) + d_1 + \cdots + d_r = \operatorname{reg}(R) + \sum_{i=1}^{r}(d_i - 1) - \dim(R/\mathfrak{b})$, and therefore Proposition 4.2 gives the assertion. $\square$

THEOREM 4.4. *Let k be a field, R a standard graded Noetherian k-algebra, and $I \subset R$ a homogeneous ideal of height r generated by forms $f_1, \ldots, f_t$ of degrees $d_1 \geq \cdots \geq d_t \geq 1$. For an integer c with $r \leq c < \dim R$, assume that, locally in codimension c, R is Gorenstein and I is a complete intersection.*

Let $x_0, \ldots, x_n$ be linear forms spanning R_1, $a_{ij} := \sum_{|\mu|=d_j-d_i} U_{ij\mu}x^{\mu}$ for $r+1 \leq i \leq t$ and $1 \leq j \leq r$, where $U_{ij\mu}$ are variables, write $K := k(U_{ij\mu})$, consider the matrix $A := (a_{ij})$ and define

$$(\alpha_1 \cdots \alpha_r) := (f_1 \cdots f_t) \begin{pmatrix} \mathrm{I}_{r\times r} \\ A \end{pmatrix}.$$

Write $R' := R \otimes_k K$ and $J := (\alpha_1, \ldots, \alpha_r)R' : IR'$.

(a) *If R satisfies S_{r+1} and is reduced, then R'/J is reduced.*

(b) *$IR' + J$ has height at least $r+1$ and is a complete intersection locally in codimension c.*

(c) *J is a complete intersection locally in codimension* $\min\{r+3, c\}$.

(d) *If R/I is of rational type locally in codimension $c' < c$, and locally in codimension c at each prime that does not contain the unmixed part of I, then $R'/IR' + J$ is of rational type locally in codimension $c'+1$, and locally in codimension c at each prime that does not contain the unmixed part of I.*

(e) *Assume that R/I is geometrically of rational type locally in codimension $c' < c$. If, locally in codimension c at each prime that does not contain the unmixed part of I, R is of rational type and R/I is geometrically of rational type, then R'/J is of rational type locally in codimension $c'+1$, and locally in codimension c at each prime that does not contain the unmixed part of I.*

(f) *If R/I is smooth (or regular) locally in codimension $c' < c$, and R and R/I are smooth (or regular) locally in codimension c at each prime that does not contain the unmixed part of I, then R'/J is smooth (or regular, respectively) locally in codimension* $\min\{r+3, c'+1\}$ *at each prime that contains the unmixed part of I, and locally in codimension* $\min\{2\operatorname{ht} I_{\mathfrak{P}\cap R} - r, c\}$ *at each prime $\mathfrak{P}$ that does not contain the unmixed part of I.*

Proof. Let $\mathfrak{P} \in V(J)$ with $\dim R'_{\mathfrak{P}} \leq c$ and write $\mathfrak{p} := \mathfrak{P} \cap R$. As $R_1 \not\subset \mathfrak{p}$, one has $x_i \notin \mathfrak{p}$ for some i, and then the a_{ij}'s are algebraically independent over $R_{\mathfrak{p}}$ and $R'_{\mathfrak{p}}$ is a ring of fraction of a polynomial ring over $R_{\mathfrak{p}}[a_{ij}]$. Since the properties we are interested in are preserved by adjoining variables and localizing (see the proof of Theorem 3.13), we may replace R' by $R_a := R_{\mathfrak{p}}[a_{ij}]$ and $\mathfrak{P}$ by $\mathfrak{P}_a := \mathfrak{P} \cap R_a$. Notice that $\operatorname{ht}\mathfrak{P}_a \leq \operatorname{ht}\mathfrak{P}$. Let $B = (b_{ij})$ be an r by r matrix of variables. Replacing R_a by $R_b := R_a[b_{ij}]$ and $\mathfrak{P}_a$ by $\mathfrak{P}_b := \mathfrak{P}_a R_b$, and multiplying the matrix $\begin{pmatrix} \mathrm{I}_{r\times r} \\ A \end{pmatrix}$ by

B, we may assume that $(\alpha_1 \cdots \alpha_r) := (f_1 \cdots f_t) \cdot C$, where $C = \begin{pmatrix} B \\ AB \end{pmatrix}$ is a matrix of variables over $R_{\mathfrak{p}}$. Recall that our properties descend under flat local homomorphisms. Finally we replace R' by $S := R_{\mathfrak{p}}[c_{ij}]$, set $\mathfrak{P} := \mathfrak{P}_b \cap S$, and think of J as an ideal of S. Notice that $R_{\mathfrak{p}}$ is Gorenstein and $I_{\mathfrak{p}}$ is a complete intersection or the unit ideal.

As for (a) it suffices to consider the case $\dim S_{\mathfrak{P}} = c = r$. Now $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ (see for instance [HU1, the proof of 2.5]), and the reducedness passes from $R_{\mathfrak{p}}$ to $(S/J)_{\mathfrak{P}}$.

To prove the other assertions, first assume that $\operatorname{ht} I_{\mathfrak{p}} > r$. In this case $J_{\mathfrak{P}} = (\alpha_1, \ldots, \alpha_r)S_{\mathfrak{P}}$ since $R_{\mathfrak{p}}$ is Cohen-Macaulay. Now (b) and (c) are obvious. For (d)–(f) we may assume that $I_{\mathfrak{p}} \neq R_{\mathfrak{p}}$. Since $\mathfrak{p}$ cannot contain the unmixed part of I, our assumptions in (d) and (e) imply that $(R/I)_{\mathfrak{p}}$ is of rational type or geometrically of rational type, respectively. Now (d) and (e) follow from Theorem 3.13.

We now prove (f). After a linear change of variables and deleting indeterminates, we may assume that $t = \mu(I_{\mathfrak{p}}) = \operatorname{ht} I_{\mathfrak{p}}$. Since $\operatorname{ht} \mathfrak{P} \leq 2t - r$ and $I \subset \mathfrak{P}$, it follows that $I_r(C) \not\subset \mathfrak{P}$. Thus after inverting a maximal minor of C and changing generators of the ideal $IS_{\mathfrak{P}}$ we may suppose that $J_{\mathfrak{P}} = (f_1, \ldots, f_r)S_{\mathfrak{P}}$. Therefore $(S/J)_{\mathfrak{P}}$ is indeed smooth (resp. regular).

Finally suppose that $\operatorname{ht} I_{\mathfrak{p}} = r$. Then $J_{\mathfrak{p}} = L_1(I_{\mathfrak{p}})$ is a generic link of $I_{\mathfrak{p}}$. Now (c) follows from [HU1, 2.9(b)]. Also $\operatorname{ht}(IS + J)_{\mathfrak{P}} \geq r + 1$ ([HU1, 2.5]). This gives the second assertion of (b) according to [PS, 1.6] since $(R/I)_{\mathfrak{p}}$ is Gorenstein.

If $\dim R_{\mathfrak{p}} = \dim S_{\mathfrak{P}}$, then $S_{\mathfrak{P}} = S_{\mathfrak{p}S}$, and therefore $J_{\mathfrak{P}} = L_1(I_{\mathfrak{p}})_{\mathfrak{p}S} = L^1(I_{\mathfrak{p}})$ is a universal link of a complete intersection. But then $J_{\mathfrak{P}}$ is the unit ideal by [HU2, 2.13(f)], yielding a contradiction. Thus $\dim R_{\mathfrak{p}} < \dim S_{\mathfrak{P}}$.

Hence if $\dim S_{\mathfrak{P}} \leq c' + 1$, the assumptions of (d) and (e) give that $(R/I)_{\mathfrak{p}}$ is of rational type or geometrically of rational type, respectively. Again, an application of Theorem 3.13 leads to the conclusion. Finally as to (f), the ring $(R/I)_{\mathfrak{p}}$ is smooth (resp. regular). If moreover $\dim S_{\mathfrak{P}} \leq r+3$, then $(S/J)_{\mathfrak{P}}$ is Gorenstein by part (c). Therefore $(S/IS+J)_{\mathfrak{P}}$ is a complete intersection on $(S/J)_{\mathfrak{P}}$, which reduces us to showing that the former ring is smooth (resp. regular). However, after a change of variables we may again assume that $t = r$, and then $(S/IS + J)_{\mathfrak{P}} = S_{\mathfrak{P}}/(IS_{\mathfrak{P}} + (\det C)S_{\mathfrak{P}})$, which is smooth (resp. regular) as the determinant locus is smooth in codimension 3. $\square$

Remark 4.5. Assume k is an infinite field. In Theorem 4.4 parts (b) and (c), parts (a), (d) and (e) in characteristic zero, and part (f) for "smooth", one can replace the "generic" link J by a "general" link J_u defined by forms $\alpha_1(u), \ldots, \alpha_r(u)$, with u in a dense open subset of $\operatorname{Spec}(k[U_{ij\mu}])$. This is possible due to [Jo, 1.6 and 4.10] and the following lemma:

LEMMA 4.6. *Let k be a field of characteristic zero and $f : Y \to U$ a morphism of reduced k-schemes of finite type. If U is irreducible and Y has rational singularities, then there exists a non empty open subset Ω of U such that $Y_u = f^{-1}(u)$ has rational singularities for $u \in \Omega$.*

Proof. Let $Y' \xrightarrow{\pi} Y$ be a resolution of singularities of Y. There exists a non empty smooth open subscheme Ω of U over which f is flat, and such that π is a simultaneous resolution of singularities over Ω. If $u \in \Omega$, [Elk, Théorème 3] implies that Y_u has rational singularities. $\square$

If B is a standard graded Noetherian algebra over a field, we will set $B^{top} := B/\mathfrak{a}$, where $\mathfrak{a}$ is the intersection of primary components of 0 of maximal dimension, and $B^{low} := B/H^0_{\mathfrak{a}}(B)$. If $\mathcal{S} := \operatorname{Proj}(B)$, we will set accordingly $\mathcal{S}^{top} := \operatorname{Proj}(B^{top})$ and $\mathcal{S}^{low} := \operatorname{Proj}(B^{low})$. Notice that $\mathcal{S}^{low} = \emptyset$ does not imply $\mathcal{S} = \mathcal{S}^{top}$. Recall that $\operatorname{Irr}(\mathcal{S})$ denotes the locus where $\mathcal{S}$ is not of rational type.

THEOREM 4.7. *Let k be a field and let R be a standard graded Gorenstein k-algebra. Let $f_1, \dots, f_t$ be forms in R of degrees $d_1 \geq \cdots \geq d_t \geq 1$ and I the ideal they generate. Set $\mathcal{S} := \operatorname{Proj}(R/I)$, $\mathcal{Z} := \operatorname{Proj}(R)$ and $r := \operatorname{codim}_{\mathcal{Z}} \mathcal{S}$. Suppose $r > 0$ and I is not a complete intersection.*

(a) *Assume that one of the following four conditions holds,*

(i) $\dim \mathcal{S} = 0$,

(ii) $\dim \mathcal{S} = 1$, *$\mathcal{S}^{top}$ is generically a complete intersection and $\mathcal{Z}$ is reduced,*

(iii) $\dim \mathcal{S} \in \{2, 3\}$, $\operatorname{char}(k) \geq \dim \mathcal{S}$, *$\mathcal{S}$ is locally a complete intersection, $\mathcal{S}^{top}$ is smooth outside finitely many points and lifts to the ring of second Witt vectors $W_2(k)$, $\mathcal{Z}$ and $\mathcal{S}^{low}$ are smooth and* $\dim \mathcal{S}^{low} \leq 1$,

(iv) $\operatorname{char}(k) = 0$, $\dim \operatorname{Irr}(\mathcal{S}^{top}) \leq 1$ *and, outside finitely many points, $\mathcal{S}$ is locally a complete intersection in $\mathcal{Z}$ and $\mathcal{Z}$ and $\mathcal{S}^{low}$ have rational singularities.*

Then

$$\operatorname{reg}(\mathcal{S}^{top}) \leq \operatorname{reg}(R) + d_1 + \cdots + d_r - r - 1.$$

Moreover, if $d_r \geq 2$ and $\dim \mathcal{S} \geq 1$, assume that $\mathcal{S}''$ is a geometrically reduced link of $\mathcal{S}^{top}$ in some complete intersection defined by forms of degrees $d_1, \dots, d_r$. Then the inequality is strict if and only if $\mathcal{S}''$ is non degenerate and geometrically connected in codimension one.

(b) *Assume $d_1 \geq 2$, $\dim \mathcal{S} \geq 1$, $\mathcal{S}$ is locally a complete intersection in $\mathcal{Z}$ and one of the following two conditions holds,*

(i) *$\mathcal{S}$ is of rational type,*

(ii) $\operatorname{char}(k) = 0$, *and $\mathcal{S}$ has at most isolated irrational singularities.*

Then

$$\operatorname{reg}(R/I) \leq \frac{(\dim \mathcal{S} + 2)!}{2} (\operatorname{reg}(R) + d_1 + \cdots + d_r - r - 1),$$

unless R is a polynomial ring over k, $I = \ell K$ with ℓ a linear form and K a homogeneous ideal, and either $\dim R = 3$ and K is a complete intersection of 3 forms of degree $d_1 - 1$, or else K is generated by linear forms.

Proof. Let $J \subset R'$ and $\alpha_1, \dots, \alpha_r$ be as in Theorem 4.4, and write $\mathcal{S}' := \operatorname{Proj}(R'/J)$.

(a) By Proposition 4.2 it suffices to show that $H^i_{\mathfrak{m}}(\omega_{R'/J})_{>0} = 0$ for every i.

(i) The assertion is clear as R'/J is Cohen-Macaulay.

(ii) Since R'/J is reduced by Theorem 4.4(a), the claim follows from the proof of Theorem 1.7(i).

(iii) Applying Theorem 4.4(f) with $c := \dim R - 1$ and $c' := \dim R - 2$, we obtain that $\mathcal{S}'$ is smooth. Furthermore $\mathcal{S}'$ is liftable to $W_2(k)$, see [Bu, 6.3.10]. Thus indeed [DI, 2.8] implies the asserted vanishing.

(iv) Applying Theorem 4.4(e) with $c := \dim R - 2$ and $c' := \dim R - 3$, we deduce that $\dim \operatorname{Irr}(\mathcal{S}') \leq 0$. Now Theorem 1.3 yields the vanishing as claimed.

(b) Set $\sigma := \operatorname{reg}(R) + \sum_{i=1}^{r} (d_i - 1)$. First assume that R is not regular or $r \geq 2$. In this case $\sigma \geq d_{r+1}$ and also $\sigma \geq 2$, since $d_1 \geq 2$ and I is not a complete intersection. Inducting on $\dim \mathcal{S}$, we may assume that $\dim \mathcal{S} \geq 2$ according to Proposition 2.2. From Theorem 4.4(b) with $c := \dim R - 1$ it follows that $R'/IR' + J$ defines a locally complete intersection scheme $\mathcal{Y}$, necessarily of dimension $\dim \mathcal{S} - 1$. Also, Theorem 4.4(d) with $c := \dim R - 1$ in case (i), $c := \dim R - 2$ in case

(ii) and $c' := c-1$ implies that $\mathcal{Y}$ is of rational type in case (i) or has at most isolated irrational singularities in case (ii). Furthermore we have $\mathcal{Y} = \mathrm{Proj}(R'/IR' + (J)_{\leq\sigma})$ by Theorem 1.7(iii) in the setting of (i) and by Theorem 1.7(ii) in (ii). Since $\mathcal{Y}$ is a Cartier divisor in $\mathcal{S}$ there exist $d := \dim \mathcal{S} + 1$ forms $\beta_i \in J$ of degrees at most σ such that $\mathcal{Y} = \mathrm{Proj}(R'/(I, \beta_1, \ldots, \beta_d))$. Set $J' := (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_d)$. One has an exact sequence,

$$0 \to R'/IR' \cap J' \to R'/IR' \oplus R'/J' \to R'/IR' + J' \to 0.$$

Notice that

$$\mathrm{reg}(R/I) = \mathrm{reg}(R'/IR') \leq \max\{\mathrm{reg}(R'/IR' \cap J'), \mathrm{reg}(R'/IR' + J')\}.$$

Now, $IR' \cap J' = (\alpha_1, \ldots, \alpha_r)$ so that $\mathrm{reg}(R'/IR' \cap J) = \sigma$. Recall that $\sigma \geq d_{r+1}$. Hence if $IR' + J'$ is not a complete intersection then our induction hypothesis yields

$$\mathrm{reg}(R'/IR' + J') \leq \frac{d!}{2}(\sigma - 1 + d(\sigma - 1)) = \frac{(d+1)!}{2}(\sigma - 1).$$

If on the other hand $IR' + J'$ is a complete intersection then $\mathrm{reg}(R'/IR' + J') \leq \sigma + d(\sigma - 1)$. In either case our assertion follows.

Next, assume R is regular and $r = 1$. By Proposition 2.2 we may assume $d \geq 3$ or $\sigma = 1$. Now $I = \ell K$ where ℓ is a form of degree $s \geq 1$ and K a proper homogeneous ideal of dimension at most one. From Proposition 2.1, applied to K, we obtain $\mathrm{reg}(R/I) \leq s + (d+1)(\sigma - s)$. But the latter is at most $\frac{(d+1)!}{2}(\sigma - 1)$ unless $s = \sigma = 1$, in which case ℓ is a linear form and K is generated by linear forms. $\square$

Even with the assumptions of Theorem 4.7(b), the estimate of Theorem 4.7(a) does no longer hold if one replaces $\mathcal{S}^{top}$ by R/I, as one can see by taking $R := k[X, Y, Z]$ and $I := (X^3, XY, XZ)$.

Institut de Mathématiques, CNRS & Université Paris 6, 4, place Jussieu,
F–75252 Paris cedex 05, France
E-mail: chardin@math.jussieu.fr

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA
E-mail: ulrich@math.purdue.edu

REFERENCES

[BEL] A. Bertram, L. Ein and R. Lazarsfeld: Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.* **4** (1991), 587–602.

[Bo] J.-F. Boutot: Singularités rationnelles et quotients par les groupes réductifs, *Invent. Math.* **88** (1987), 65–68.

[BKM] W. Bruns, A. Kustin and M. Miller: The resolution of the generic residual intersection of a complete intersection, *J. Algebra* **128** (1990), 214–239.

[BE] D. Buchsbaum and D. Eisenbud: Remarks on ideals and resolutions, *Symp. Math.* **11** (1973), 193–204.

[Bu] R.-O. Buchweitz: *Contributions à la théorie des singularités,* Thèse de l'Université Paris VII (1981).

[Ch] M. Chardin: Applications of some properties of the canonical module in computational projective algebraic geometry, *J. Symbolic Comput.* **29** (2000), 527–544.

[DI] P. Deligne and L. Illusie: Relèvements modulo p^2 et décomposition du complexe de de Rham, *Invent. Math.* **89** (1987), 247–270.

[Elk] R. Elkik: Singularités rationelles et déformations, *Invent. Math.* **47** (1978), 139–147.

[Elz] F. Elzein: Complexe dualisant et applications à la classe fondamentale d'un cycle, *Bull. Soc. Math. France*, Mémoire **58** (1978).

[En] F. Enescu: On the behavior of F-rational rings under flat base change, *J. Algebra* **233** (2000), 543–566.

[FW] R. Fedder and K. Watanabe: A characterization of F-regularity in terms of F-purity, *Commutative Algebra (Berkeley, 1987)*, Math. Sci. Res. Inst. Publ. **15**, Springer, New York, 1989, 227–245.

[Fu] W. Fulton: *Intersection Theory*, Springer, Berlin, 1984.

[Har] N. Hara: A characterization of rational singularities in terms of injectivity of Frobenius maps, *Amer. J. Math.* **120** (1998), 981–996.

[HW] N. Hara and K. Watanabe: The injectivity of Frobenius acting on cohomology and local cohomology modules, *Manuscripta Math.* **90** (1996), 301–315.

[Has] M. Hashimoto: Cohen-Macaulay F-injective homomorphisms, *Geometric and Combinatorial Aspects of Commutative Algebra (Messina, 1999),* Lecture Notes in Pure and Appl. Math. **217**, Marcel Dekker, 2001, 231–244.

[HSV] J. Herzog, A. Simis and W. Vasconcelos: Koszul homology and blowing-up rings, *Commutative Algebra (Trento, 1981),* Lecture Notes in Pure and Applied Math. **84**, Marcel Dekker, New York, 1983, 79–169.

[Hi] H. Hironaka: Smoothing of algebraic cycles of small dimensions, *Amer. J. Math.* **90** (1968), 1–54.

[HH1] M. Hochster and C. Huneke: F-regularity, test elements and smooth base change, *Trans. Amer. Math. Soc.* **346** (1994), 1–62.

[HH2] M. Hochster and C. Huneke: Tight closure of parameter ideals and splitting in module-finite extensions, *J. Algebraic Geom.* **3** (1994), 599–670.

[Hu] C. Huneke: On the symmetric and Rees algebras of an ideal generated by a d-sequence, *J. Algebra* **62** (1980), 268–275.

[HR] C. Huneke and M. E. Rossi: The dimension and components of symmetric algebras, *J. Algebra* **98** (1986), 200–210.

[HU1] C. Huneke and B. Ulrich: Divisor class groups and deformations, *Amer. J. Math.* **107** (1985), 1265–1303.

[HU2] C. Huneke and B. Ulrich: The structure of linkage, *Ann. of Math.* **126** (1987), 277–334.

[HU3] C. Huneke and B. Ulrich: Residual intersections, *J. reine angew. Math.* **390** (1988), 1–20.

[Jo] J.-P. Jouanolou: *Théorèmes de Bertini et applications*, Progress in Math. **42**, Birkhäuser, Basel, 1983.

[Ke1] G. Kempf: On the geometry of a theorem of Riemann, *Ann. of Math.* **98** (1973), 178–185.

[Ke2] G. Kempf: Images of homogeneous vector bundles and varieties of complexes, *Bull. Amer. Math. Soc.* **81** (1975), 900–901.

[Ke3] G. Kempf: On the collapsing of homogeneous bundles, *Invent. Math.* **37** (1976), 229–239.

[Ko] J. Kollár: Higher direct images of dualizing sheaves I, *Ann. of Math.* **123** (1986), 11–42.

[KU1] A. Kustin and B. Ulrich: A family of complexes associated to an almost alternating map, with applications to residual intersections, *Mem. Amer. Math. Soc.* **461** (1992).

[KU2] A. Kustin and B. Ulrich: If the socle fits, *J. Algebra* **147** (1992), 63–80.

[KW] E. Kunz and R. Waldi: Regular differential forms, *Contemporary Math.* **79** (1988).

[Li] J. Lipman: Dualizing sheaves, differentials and residues on algebraic varieties, *Astérisque* **117** (1984).

[MS] V. B. Mehta and V. Srinivas: A characterization of rational singularities, *Asian J. Math.* **1** (1997), 249–271.

[MT] V. B. Mehta and V. Trivedi: Variety of complexes and F-splitting, *J. Algebra* **215** (1999), 352–365.

[Oh] T. Ohsawa: Vanishing theorems on complete Kähler manifolds, *Publ. Res. Inst. Math. Sci.* **20** (1984), 21–38.

[PS] C. Peskine and L. Szpiro: Liaison des variétés algébriques. I, *Invent. Math.* **26** (1974), 271–302.

[S1] K. Smith: F-rational rings have rational singularities, *Amer. J. Math.* **119** (1997), 159–180.

[S2] K. Smith: Fujita's freeness conjecture in terms of local cohomology, *J. Algebraic Geom.* **6** (1997), 417–429.

[S3] K. Smith: A tight closure proof of Fujita's freeness conjecture for very ample line bundles, *Math. Ann.* **317** (2000), 285–293.

[Va] G. Valla: On the symmetric and Rees algebras of an ideal, *Manuscripta Math.* **30** (1980), 239–255.

[Ve] J. Velez: Openness of the F-rational locus and smooth base change, *J. Algebra* **172** (1995), 425–453.