Genus-0 K-theoretic FJRW invariants

Maxime Cazaux

February 29, 2024

Introduction

Quantum K-theory is a version of Gromov–Witten theory introduced by Y.P. Lee [Lee22], in which the usual intersection numbers are replaced by Euler characteristic of K^0-classes on the moduli space of stable maps.

The Grothendieck–Riemann–Roch theorem

$$\chi(X, F) = \int_X \mathrm{ch}(F)\mathrm{td}(T_X) \tag{1}$$

naturally connects quantum K-theory and quantum cohomology. Since the moduli space of stable maps is a stack, this formula must be generalized as in Kawasaki [Kaw79], or Toën [Toe98]. The integral now takes place over the *inertia stack* of X, rather than X itself. As a consequence, the computations in quantum K-theory are highly non-trivial: the moduli space of stable maps is already difficult to handle, and its inertia stack is worse.

Over the last decade, a parallel theory known as the Landau–Ginzburg model was built, in which the relevant spaces are much more tractable. This theory was built by Fan–Jarvis–Ruan [FJR12], and Polishchuk–Vaintrob [PV14] in K-theory, and depends on a homogeneous singularity W. The Landau–Ginzburg/Calabi–Yau correspondence relates the Gromov–Witten invariants of a projective hypersurface defined by a polynomial W, to the LG model of the singularity defined by W in the affine space, and has been shown to hold in many instances ([CR10], [CIR14], [PLS14],[Cla13], A FINIR).

In this paper, we fully compute the genus-0 quantum K-theory of the singularity $\sum_{i=1}^{r} X^r$, with symmetry group μ_r. This computation is inspired by [GT11], with some changes which are interesting to survey.

For the stack of stable maps $\overline{\mathcal{M}}(X)$, the inertia stack possesses a distinguished component isomorphic to $\overline{\mathcal{M}}(X)$ classifying points of $\overline{\mathcal{M}}(X)$ alongside with the identity morphism

$$I\overline{\mathcal{M}}(X) = \overline{\mathcal{M}}(X) \sqcup \text{ other components.}$$

Therefore, the formula (1) should be interpreted as one of the contributions of $I\overline{\mathcal{M}}(X)$ to the GRR formula. This contributions is usually called the "fake Euler characteristics", or the "fake invariant" [GT11]. Computing the fake invariants

is a crucial step in the general computation of the "true" K-theoretic invariants. This definition of the fake invariants must be adapted to our setting.

In the case of the LG model, the relevant moduli stack is the space of r-spin curves $\overline{\mathcal{M}}^r_{g,n}$, that is, curves with an rth root $\mathcal{L}$ of the log-canonical bundle. In genus 0, this stack is simply $\overline{\mathcal{M}}_{0,n}$, modified by iterated applications of the so-called rth root construction (see 3.1 below), which associates to a scheme X (or a stack) the universal stack where a given line bundle L admits an rth root, compatible with a section s of L (see [BC10] ET AUTRE).

This construction applies in two ways. If the section s vanishes globally, it transforms the initial scheme into a $\boldsymbol{\mu}_r$-gerbe over X. If the section vanishes along a Cartier divisor D, the construction yields a stack $X[D/r]$ over X, with an open substack isomorphic to $X \setminus D$.

This description of the moduli space of r-spin curves also provides a simple description of its inertia stack (prop. 3.5). As a result, we see that the moduli stack $\overline{\mathcal{M}}^r_{0,n}$ is simply a $\boldsymbol{\mu}_r$-gerbe over $\overline{\mathcal{M}}_{0,n}[\sum_D D/r]$ where the divisors are all boundary divisors. Thus, the inertia stack is the disjoint union of r *main components*, isomorphic to $\overline{\mathcal{M}}^r_{0,n}$, and boundary components

$$I\overline{\mathcal{M}}^r_{0,n} = \bigsqcup_{\xi \in \boldsymbol{\mu}_r} X_\xi \sqcup \text{ lower dimension strata.} \tag{2}$$

The K-theoretic FJRW invariants are defined as the Euler characteristics of the K^0 class

$$\left(\Lambda_{-1}(R^1\pi_*\mathcal{L}^\vee)\right)^{\otimes r} \bigotimes \mathrm{ev}_i^* E_i, \tag{3}$$

where

1) $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{M}}^r_{0,n}$ is the universal curve,

2) $\mathcal{L}$ is the universal rth root,

3) ev_i are evaluation maps to $IB\boldsymbol{\mu}_r$ defined on some gerbe over $\overline{\mathcal{M}}^r_{0,n}$ (see 1.2),

4) E_i are K-classes in $K^0(IB\boldsymbol{\mu}_r)$.

We define the *ξ-fake theories* as the contribution of the main component X_ξ to the GRR formula, which yields

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathrm{fake},\xi} = r\,\mathrm{ch}\left(\Lambda_{-\xi} R^1\pi_*\mathcal{L}^\vee\right)^r \prod_i \mathrm{ev}_i^*(\gamma_i)\mathrm{td}(T) \cap \left[\overline{\mathcal{M}}^r_{0,n}\right], \tag{4}$$

where γ_i are cohomology classes in $H^*(IB\boldsymbol{\mu}_r)$.

Notice that these theories coincide (up to the Todd class) with the Cohomological Field theories (CohFTs) constructed by Polishchuk and Vaintrob [PV14].

Each of these fake theories is a *twisted theory*, in the sense that it is obtained by capping the trivial CohFT with an obstruction class of the form $\exp\left(\sum s_k \,\mathrm{ch}_k(R\pi_*\mathcal{L})\right)$. Such theories have been studied by Chiodo–Zvonkine in

[CZ07], where it is shown that their Lagrangian cone is obtained from the trivial cone by applying an explicit symplectomorphism.

Finally, we reconstruct the true K-theoretic invariants from the fake invariants. The only non-trivial automorphisms of a genus-0 stable twisted curve are *ghost automorphisms* ([ACV01], Section 7), which only exist on reducible curves. These automorphisms fix the coarse curve, but rescale the spin structure $\mathcal{L}$ (see 3.12), just as the generic automorphisms of $\overline{\mathcal{M}}^r_{0,n}$ do. Thus, the contribution of the lower-dimensional strata are recovered by assembling different ξ-fake invariants on different components of the curve. The adelic characterization of Givental–Tonita is ideally suited to express this result.

Theorem 0.1. *The J-function of the K-theoretic FJRW invariants satisfies the properties*

(1) $J(t)$ has no poles outside of $0, \infty$ and the r-th roots of unity,

(2) the localization of $\underline{f} e_\xi$ at 1 lies in the fake cone $\mathcal{L}^{\xi,\mathrm{fake}}$,

(3) $f - \underline{f} \in \mathcal{K}_+$.

Conversely, these 3 conditions yield a recursive algorithm allowing to compute all the correlators of the J-function in terms of the fake invariants.

Interestingly, we obtain r^2 I-functions, one for each fake theory. $r(r-1)$ of them assemble in an I-function for the FJRW invariants (thm 3.24). We hope that these functions could shed new light on the 5^2 difference equation satisfied by the permutation-equivariant I-function of the quintic threefold (Givental [Giv]). More precicsely, we expect that the I-functions of the permutation-equivariant FJRW theory coincide with the solutions found in [Wen22].

Notations and conventions

All schemes and stacks are of finite type over $\mathbb{C}$.

A sheaf on a Deligne–Mumford stack $\mathcal{X}$, is a sheaf on the small étale site of $\mathcal{X}$. Note that all stacks considered in this paper have the resolution property, which allows us to consider classes of coherent sheaves in the K^0 ring.

Notations

- $\boldsymbol{\mu}_r$: the group of r-th roots of unity,
- BG : for a finite group G, $BG = [\mathrm{Spec}(\mathbb{C})/G]$,
- $\hat{G}$: the group of characters of a group G,
- $\overline{\mathcal{M}}_{g,n}$: the stack of stable curves of genus g with n marked points,
- $\overline{\mathcal{M}}_{g,n}(r)$: the stack of twisted stable curves of genus g with n marked gerbes,

- $\overline{\mathcal{M}}^r_{g,\underline{a}}$: for a multi-index $\underline{a} = (a_1, \ldots, a_n)$, the stack of rth roots of the line bundle $\omega_{\log}(-\sum a_i x_i)$,
- $I\mathcal{X}$: when $\mathcal{X}$ is a stack, $I\mathcal{X}$ is the inertia stack of $\mathcal{X}$,
- $\Lambda_t E$: for a vector bundle E on X, $\Lambda_t E = \sum_{k\geq 0} t^k \Lambda^k E \in K^0(X)[t]$,
- $T_X, \mathcal{N}$: Tangent sheaf and normal sheaf of a regular embedding,
- ch, td : Chern character and Todd class.

1 Quantum K-theory

In this section, we give a review of quantum K-theory as defined in [Lee22], and define the analog invariants for the FJRW theory.

1.1 Quantum K-theory

Let X be a smooth projective variety, and $X_{g,n,d}$ be the moduli space of stable maps into X with genus g, degree d, and n marked points. This space is equipped with the so-called "fundamental structure sheaf" $\mathcal{O}^{\mathrm{vir}}$ (see [BF97], [Lee22]), which is related to the virtual fundamental class by the formula $\mathrm{ch}(\mathcal{O}^{\mathrm{vir}})\mathrm{td}(\mathcal{T}^{\mathrm{vir}}) = [X]^{\mathrm{vir}}$.

Let L_i be the line bundle $\sigma_i^* \omega_{\mathcal{C}/X_{g,n,d}}$, where $\sigma_i \colon X_{g,n,d} \to \mathcal{C}_{g,n,d}$ is the section corresponding to the i-th marked point. Then, given a sequence $E_1, \ldots, E_n$ of elements of $K^0(X)$, we define the quantum K-invariants by

$$\left\langle E_1 L_1^{k_1}, \ldots, E_n L_n^{k_n} \right\rangle^X_{g,n,d} = \chi\left(X_{g,n,d}, \mathcal{O}^{\mathrm{vir}} \bigotimes ev_i^*(E_i) L_i^{k_i}\right). \tag{5}$$

The genus-0 quantum K-theory invariants are organized in a generating function called the J-function. In [GT11], the Grothendieck–Rimemann–Roch theorem was used to compute this generating function in terms of the quantum cohomology of X.

1.2 K-theoretic FJRW invariants

Definition 1.1. Let r be a positive integer, and $n \geq 3$. $\overline{\mathcal{M}}^r_{g,n}$ is the moduli space of balanced twisted curves (with orbifold structure at the marked points), equipped with an r-th root of $\omega_{\log}$. Explicitly, objects over S are triples $(\mathcal{C}, (\Sigma_i)_{i\leq n}, \mathcal{L}, \phi)$ where $(\mathcal{C}, (\Sigma_i)_{i\leq n})$ is a balanced twisted curve over S [AV00], $\mathcal{L}$ is a line bundle over $\mathcal{C}$ and $\phi : \mathcal{L}^{\otimes r} \to \omega_{\log}$ is an isomorphism.

The universal curve $\pi \colon \overline{\mathcal{C}}^r_{g,n} \to \overline{\mathcal{M}}^r_{g,n}$ is equipped with the universal rth root $\mathcal{L}_{g,n}$ (or $\mathcal{L}$ if the context is clear). Let $\Sigma_i \subset \overline{\mathcal{C}}^r_{g,n}$ denote the i-th marked gerbe. The restriction $\mathcal{L}_{|\Sigma_i}$ defines a character of $\boldsymbol{\mu}_{r_i}$, which we denote $a_i \in \mathbb{Z}/r_i\mathbb{Z}$.

When r is prime, the band of a marked gerbe is either trivial, or $\boldsymbol{\mu}_r$. In that case, the moduli space $\overline{\mathcal{M}}^r_{g,n}$ is a disjoint union over multi-indexes $\underline{a} = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}/r\mathbb{Z}$

$$\overline{\mathcal{M}}^r_{g,n} = \bigsqcup_{\underline{a}} \overline{\mathcal{M}}^r_{g,\underline{a}}, \tag{6}$$

where $\overline{\mathcal{M}}^r_{g,\underline{a}}$ denotes the substack of $\overline{\mathcal{M}}^r_{g,n}$ made of the curves such that the character induced by $\mathcal{L}$ at Σ_i is given by a_i. The universal curve over $\overline{\mathcal{M}}^r_{g,\underline{a}}$ is denoted $\overline{\mathcal{C}}^r_{g,\underline{a}}$, and the universal root is $\mathcal{L}_{g,\underline{a}}$.

Alternatively, the stack $\overline{\mathcal{M}}^r_{g,a}$ may be defined as the moduli space of curves, with an r-th root of $\omega_{\log}(-\sum a_i x_i)$, without orbifold structure at the marked points ([Chi07]). We will use both definitions as convenient.

From now on, we assume that r is prime.

Definition 1.2. The restriction of the universal root $\mathcal{L}$ to the marked gerbe Σ_i is an r-th root of $\mathcal{O}_{\Sigma_i}$, which defines the i-th evaluation map

$$\mathrm{ev}_i \colon \Sigma_i \to I\left(B\boldsymbol{\mu}_r\right). \tag{7}$$

Let $\widetilde{\mathcal{M}}^r_{0,n}$ be the stack of r-spin curves with a section of each marked gerbe Σ_i. There is a natural isomorphism

$$\widetilde{\mathcal{M}}^r_{0,n} = \Sigma_1 \times_{\overline{\mathcal{M}}^r_{0,n}} \Sigma_2 \times \cdots \times_{\overline{\mathcal{M}}^r_{0,n}} \Sigma_n. \tag{8}$$

The universal curve $\pi : \widetilde{\mathcal{C}}^r_{0,n} \to \tilde{\mathcal{M}}^r_{0,n}$ has sections σ_i, and we define $\mathcal{L}_i = \sigma_i^* \omega_\pi$. There is a canonical projection $p \colon \widetilde{\mathcal{M}}^r_{0,n} \to \overline{\mathcal{M}}^r_{0,n}$. The evaluation maps extend to

$$\mathrm{ev}_i : \widetilde{\mathcal{M}}^r_{0,n} \to I\left(B\boldsymbol{\mu}_r\right).$$

Definition 1.3. A multi-index $\underline{a}$ is called concave, if for any closed point of $\overline{\mathcal{M}}^r_{0,a}$, the corresponding curve satisfies $H^0(\mathcal{C}, \mathcal{L}) = 0$. In that case, Grauert's theorem implies that $R^1\pi_*\mathcal{L}$ is a vector bundle.

Proposition 1.4 ([CR10])**.** *Let $b_1, \ldots, b_n \in \mathbb{N}$ be integers such that $b_i > 0$ for $i > 1$, and let $\mathcal{L}$ be the universal r-th root of $\omega_{\log}(-\sum b_i x_i)$. Then for any genus-0 curve C, we have $H^0(C, \mathcal{L}) = 0$.*

The ring $K^0\left(I(B\boldsymbol{\mu}_r)\right)$ is isomorphic to a direct sum of copies of $K^0\left(B\boldsymbol{\mu}_r\right)$ with basis ϕ_a:

$$K^0(I(B\boldsymbol{\mu}_r))_{\mathbb{C}} = \bigoplus_{a=0}^{r-1} \mathbb{C}[\hat{\boldsymbol{\mu}}_r]\phi_a. \tag{9}$$

We decompose this ring according to the band of the marked gerbe.

$$V_1 = \mathbb{C}[\hat{\boldsymbol{\mu}}_r]\phi_a$$
$$V_r = \bigoplus_{a=1}^{r-1} \mathbb{C}[\hat{\boldsymbol{\mu}}_r]\phi_a$$

Definition 1.5. The state space of the FJRW invariants is the vector space $V = K^0(I(B\boldsymbol{\mu}_r))$. It is equipped with the orbifold pairing, which is defined on characters η_1, η_2 by

$$(\eta_1\phi_i; \eta_2\phi_j) = \begin{cases} \chi\left(B\boldsymbol{\mu}_r; \eta_1\eta_2\right) & \text{if } i+j=0 \mod r, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Remark 1.6. The restriction of the scalar product to the subspace V spanned by $\{\phi_i\}_{i\neq 0}$ is non-degenerated.

Following [PV14] and [Gué21], we now define the genus-0 K-FJRW invariants in the concave case.

Definition 1.7. The fundamental class $\Lambda_{0,n} \in K^0(\overline{\mathcal{M}}^r_{0,n})$ is defined on each component of $\overline{\mathcal{M}}^r_{0,n}$ by

$$\Lambda_{0,\underline{a}} = \begin{cases} \Lambda_{-1}\left(\left(R^1\pi_*\mathcal{L}_a(-E)\right)^\vee\right)^{\otimes r} & \text{if } a_i \neq 0\ \forall i \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Let $E_1, \ldots, E_n$ be vector bundles on $IB\boldsymbol{\mu}_r$, and $k_1 \ldots, k_n \in \mathbb{Z}$ be integers. The K-FJRW invariants are

$$\left\langle E_1\mathcal{L}_1^{k_1}, \ldots, E_n\mathcal{L}_n^{k_n}\right\rangle^K_{0,n} = \chi\left(\widetilde{\mathcal{M}}^r_{0,n}; p^*\Lambda_{0,n} \bigotimes \mathrm{ev}_i^* E_i\mathcal{L}_i^{k_i}\right) \tag{12}$$

$$= \chi\left(\overline{\mathcal{M}}^r_{0,n}; \Lambda_{0,n} \bigotimes_i (\pi_i)_*\left(\mathrm{ev}_i^* E_i \otimes \mathcal{N}_i^{-k_i}\right)\right), \tag{13}$$

where $\mathcal{N}_i$ denotes the normal line bundle to the i-th gerbe, and $\pi_i : \Sigma_i \to \overline{\mathcal{M}}^r_{0,n}$ is the projection (there is a slight abuse of notation in using the same symbol ev_i to denote different applications).

Define $\mathcal{K}_+ := V_1[q, q^{-1}] \oplus V_r[q^{1/r}, q^{-1/r}]$. We extend the correlator notation to elements of $\mathcal{K}_+$ by linearity.

The genus-0 potential is the formal function of $t \in \mathcal{K}_+$ given by

$$\mathcal{F}^0(t) = \sum_{n\geq 3} \frac{1}{n!} \langle t, \ldots, t\rangle_{0,n} \tag{14}$$

Remark 1.8. One can further decompose the expression (13) by

$$\left\langle \eta_1\phi_{a_1}\mathcal{L}_1^{k_1}, \ldots, \eta_n\phi_{a_n}\mathcal{L}_n^{k_n}\right\rangle^K_{0,n} = \chi\left(\overline{\mathcal{M}}^r_{0,a}; \Lambda_{0,\underline{a}} \bigotimes_i \pi_{i*}\left(\mathrm{ev}_i^*\eta_i \otimes \mathcal{N}_i^{-k_i}\right)\right). \tag{15}$$

This invariant vanishes unless $\eta_i^{a_i}(\xi) = \xi^{k_i}$, $\forall \xi \in \boldsymbol{\mu}_r, \forall i$.

Proposition 1.9. *The K-theoretic FJRW invariants satisfy the dilaton equation.*

$$\langle(\mathcal{L}_1^r - 1)\phi_1[1], t, \ldots, t\rangle^K_{0,n+1} = (n-2)\langle t, \ldots, t\rangle^K_{0,n}. \tag{16}$$

The dilaton equation implies that $\mathcal{F}$ is quadratic with respect to the shifted origin $q-1$.

Proof. Let us fix a multi-index $\underline{a} = (a_1, \dots, a_n)$ and let $\underline{a}' = (1, a_1, \dots, a_n)$. There is a diagram

$$\begin{tikzcd} \overline{\mathcal{C}}^r_{0,\underline{a}'} \arrow[r, "\rho"] \arrow[d, "\pi'"'] & \overline{\mathcal{C}}^r_{0,\underline{a}} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}^r_{0,\underline{a}'} \arrow[r, "p"] & \overline{\mathcal{M}}^r_{0,\underline{a}} \end{tikzcd} \tag{17}$$

Such that $\rho^*\mathcal{L}_{\underline{a}} = \mathcal{L}_{\underline{a}'}$, and $R\pi'_*\mathcal{L}_{\underline{a}'} = p^* R\pi_*\mathcal{L}_{\underline{a}}$. The result then follows from the classical dilaton equation [Lee22]. □

2 Twisted cohomological field theories

In this section, we define the fake theories, which will be the building blocks for the K-theoretic invariants. The name "fake" is used in analogy with [GT11], and consists in some cohomological field theories twisted by the Todd class of the tangent space of $\overline{\mathcal{M}}^r_{0,n}$.

Definition 2.1. Let $\underline{a} = (a_1, \dots, a_n)$ be a multi-index, and let E be the divisor $E = \sum_{a_i=0} \Sigma_i$ of the universal curve. For any multiplicative transformation $\mathcal{A}\colon K^0 \to H^*$ (or to the Chow ring), we define the collection of classes

$$\mathcal{A}_{g,\underline{a}} = r^{1-g}\mathcal{A}(R\pi_*\mathcal{L}_{g,\underline{a}}(-E)) \in H^*(\overline{\mathcal{M}}^r_{g,\underline{a}}, \mathbb{Q}). \tag{18}$$

Let $q\colon \overline{\mathcal{M}}^r_{0,\underline{a}} \to \overline{\mathcal{M}}_{0,n}$ be the forgetful map. If $\mathcal{A}(\mathcal{O}) = 1$, then the classes

$$\Omega_{g,\underline{a}} = q_*\mathcal{A}_{g,\underline{a}} \tag{19}$$

form a CohFT over the state space V, with unit ϕ_1.

If $\mathcal{A}(\mathcal{O}) \neq 1$, then we make the following modification to the scalar product on V,

$$(\phi_a, \phi_0) := \delta_{a,0}\mathcal{A}(\mathcal{O})^{-1} \tag{20}$$

and the result is again a CohFT.

Following [Giv03], we associate to the genus-0 CohFT $\mathcal{A}_{0,n}$ an overruled Lagrangian cone as follows. Let $\mathcal{H} = V \otimes \mathbb{C}[\![z, z^{-1}]$. This is a symplectic vector space with symplectic form

$$\Omega(f(z), g(z)) = \mathrm{Res}_{z=0}\left[(f(-z); g(z))dz\right]. \tag{21}$$

This space has a natural polarization

$$\mathcal{H}_+ = V \otimes \mathbb{C}[\![z]\!] \qquad\qquad \mathcal{H}_- = V \otimes z^{-1}\mathbb{C}[z^{-1}].$$

This polarization identifies $\mathcal{H}$ with $T^*\mathcal{H}_+$. The genus-0 invariants are assembled in the genus-0 potential, which is a formal function on $\mathcal{H}_+$.

$$\mathcal{F}^0(t(z)) = \sum_n \frac{1}{n!} \langle t(\psi), \dots, t(\psi)\rangle^{\mathcal{A}}_{0,n} \tag{22}$$

Definition 2.2. The Lagrangian cone of $\mathcal{L}^{\mathcal{A}}$ the CohFT $\mathcal{A}_{0,n}$ is the graph of $d\mathcal{F}^0$ inside $\mathcal{H}$ shifted by $-z$. Explicitly, this is the image of the big J-function.

$$J(t,-z) = -z\phi_1 + t(z) + \sum_{n,a} \frac{\phi^a}{n!} \left\langle \frac{\phi_a}{-z-\psi}, t(\psi), \ldots, t(\psi) \right\rangle^{\mathcal{A}}_{0,n+1}. \tag{23}$$

We now recall recall how to compute the cone of the CohFT $\mathcal{A}_{0,n}$.

Theorem 2.3 (Chiodo - Zvonkine [CZ07, thm. 1.2.2])**.** *Let $(w_n)_{n\in\mathbb{N}}$ be formal variables, and $\mathcal{A}_w$ the multiplicative class given by $\mathcal{A}_w(E) = \exp\left(\sum_{i\geq 0} w_i \operatorname{ch}_i(E)\right)$. Let Ω be a genus-0 CohFT with unit. We consider the deformation $\Omega_w = \Omega\mathcal{A}_{w,0,n}$, which induces a family of cones $\mathcal{L}_w$. Then we have*

$$\mathcal{L}_w = \exp\left(\sum w_i L_i\right) \mathcal{L}_0 \tag{24}$$

with

$$L_i = \frac{z^i}{(i+1)!} \operatorname{diag}\left[B_{i+1}(1), B_{i+1}\left(\frac{1}{r}\right), \ldots, B_{i+1}\left(\frac{r-1}{r}\right)\right] \tag{25}$$

Lemma 2.4. *For $\mathcal{A}(E) = \operatorname{ch}\left(\Lambda_{-\lambda}(-E^\vee)\right)^r$, the coefficients are given by*

$$w_i = r \sum_{m>0} (-m)^i \frac{\lambda^m}{m}. \tag{26}$$

Definition 2.5. The untwisted cone $\mathcal{L}^{\mathrm{un}}$ is the overruled cone inside $\mathcal{H}$ of the CohFT obtained by choosing $\mathcal{A} = 1$.

Corollary 2.6. *Let w_i be defined as in lemma 2.4.*

The cone $\mathcal{L}^{H,\lambda}$ of the CohFT associated to the class $\mathcal{A}(E) = \operatorname{ch}(\Lambda_{-\lambda}E^\vee)^r$ is

$$\mathcal{L}^{H,\lambda} = \exp(\sum w_i L_i)\mathcal{L}^{un}$$

for w_i as above.

2.1 The fake theories

In view of applying the Grothendieck–Riemann–Roch theorem, one needs to modify the correlators to include the Todd class of the tangent space. The class of the tangent space in $K^0\left(\overline{\mathcal{M}}^r_{0,n}\right)$ is

$$T = \pi_*\left(\omega^{\otimes 2}(\sum x_i)\right)^\vee - \pi_*(\mathcal{O}_Z)^\vee, \tag{27}$$

where Z is the substack of singular points in $\overline{\mathcal{C}}^r_{0,n}$. These two terms are treated separately.

It was shown in [Ton14], that the first term induces a change in the dilaton shift (ie a translation of the potential), and that the second term induces a change of polarization of the ambient space.

Recall that if r is prime, $\overline{\mathcal{M}}^r_{0,n}$ has two kinds of nodes. If the multiplicity of $\mathcal{L}_{\underline{a}}$ at a node η is trivial, then η has trivial relative automorphism group. If the multiplicity is non trivial, then the relative automorphism group of η is isomorphic to $\boldsymbol{\mu}_r$. Thus the singular locus $\mathcal{Z} \subset \overline{\mathcal{M}}^r_{0,\underline{a}}$ has a decomposition

$$\mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{Z}_r, \tag{28}$$

where $\mathcal{Z}_i$ is the substack of nodes whose relative automorphism group has order i.

Definition 2.7. Let $\mathcal{B}$, $\mathcal{C}_1, \mathcal{C}_r$ be invertible multiplicative classes. Define

$$\begin{aligned}\mathcal{B}_{0,n} &= \mathcal{B}\left(\pi_*\left(\omega_{\log}^{-1} - 1\right)\right)\\ \mathcal{C}_{0,n} &= \mathcal{C}_1\left(\pi_*\mathcal{O}_{Z_1}\right)\mathcal{C}_r\left(\pi_*\mathcal{O}_{Z_r}\right)\end{aligned}$$

The twisted invariants are defined by

$$\left\langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n}\right\rangle_{0,n}^{\mathcal{A},\mathcal{B},\mathcal{C}} = \int_{\overline{\mathcal{M}}^r_{0,a}} \mathcal{A}_{0,\underline{a}}\mathcal{B}_{0,n}\mathcal{C}_{0,n}\prod \psi_i^{k_i}, \tag{29}$$

The twisted potential is the formal function

$$\mathcal{F}^{\mathcal{ABC}}(t) = \sum_n \frac{1}{n!}\left\langle t(\psi), \ldots, t(\psi)\right\rangle_{0,n}^{\mathcal{ABC}} \tag{30}$$

Theorem 2.8 (Tonita, [Ton14])**.** *The $\mathcal{B}$-type twist translates the potential.*

$$\mathcal{F}^{g,\mathcal{A},\mathcal{B},\mathcal{C}}(t) = \mathcal{F}^g_{\mathcal{A},\mathcal{C}}\left(t + z - z\mathcal{B}(L_z^{-1})\right) \tag{31}$$

When $\mathcal{B} = td^{-1}$, as in our example, the translation is responsible for the change in dilaton shift, which becomes $1 - e^z$, instead of $-z$.

We now describe the effect of twisting by the class $\mathcal{C}$ on the Lagrangian cone.

As follows from the dilaton equation, the potentials of CohFTs are quadratic with respect to the shifted origin z. Thus the twisted potential is quadratic with respect to the new origin $z\mathcal{B}(L_z^{-1})$, and the twisted J function should have the form $J^{\mathrm{tw}}(t, -z) = -z\mathcal{B}(L_z^{-1}) + t + d_t\mathcal{F}^{0,\mathrm{tw}}$. In order to define such a graph in $\mathcal{H}$, we need to choose a polarization. The content of the next theorem is that there exists a polarization adapted to our needs.

Let $\mathcal{H}_{\mathcal{A}}$ be the polarized symplectic space associated with the Cohft $\mathcal{A}_{g,n}$. We introduce a new polarized symplectic space $\mathcal{H}^{\mathcal{A},\mathcal{C}}$. As symplectic spaces we have $\mathcal{H}^{\mathcal{A},\mathcal{C}} = \mathcal{H}^{\mathcal{A}}$, and $\mathcal{H}^{\mathcal{A},\mathcal{C}}_+ = \mathcal{H}^{\mathcal{A}}_+$. Darboux coordinates on $\mathcal{H}^{\mathcal{A},\mathcal{C}}$ are constructed as follows. Let L be a line bundle with $c_1(L) = z$. Define

$$\frac{z}{u_i(z)} = \mathcal{C}_i(-L^\vee) \tag{32}$$

Then, one expands the following function

$$\frac{1}{u_i(-x-y)} = \sum_{k\geq 0}(u_i(x))^k v_{i,k}(y).$$

Darboux coordinates on $\mathcal{C}$ are given by

$$f = \sum_{\substack{k\geq 0\\ \alpha\neq 0}} q_k^\alpha(f) u_r(z)^k \phi_\alpha + \sum_{\substack{l\geq 0\\ \beta\neq 0}} p_l^\beta(f) v_{r,l}(z)\phi^\beta$$
$$+ \sum_{k\geq 0} q_k^0 u_0(z)^k \phi_0 + \sum_{l\geq 0} p_l^0(f) v_{0,l}(z)\phi^0.$$

Theorem 2.9 (Tonita [Ton14])**.** *Let $\mathcal{L}_\mathcal{A}$ be the cone of the Cohft $\mathcal{A}_{g,n}$ in $\mathcal{H}^\mathcal{A}$, and let $\mathcal{L}^{\mathcal{A},\mathcal{C}}$ be the cone of the twisted CohFT in $\mathcal{H}^{\mathcal{A},\mathcal{C}}$. Then $\mathcal{L}^\mathcal{A} = \mathcal{L}^{\mathcal{A},\mathcal{C}}$.*

In the remaining part of this paper, we fix

$$\begin{aligned}
&\mathcal{A}(E) = \mathrm{ch}\left(\Lambda_\lambda(-E^\vee)\right)^r \in H^*[\![\lambda]\!]\\
&\mathcal{B}(E) = \mathrm{td}\left(-E\right)\\
&\mathcal{C}_r(L) = \mathrm{td}\left(-L^{-1/r}\right)\\
&\mathcal{C}_0(L) = \mathrm{td}\left(-L^\vee\right)
\end{aligned}$$

Thus, a Darboux basis of $\mathcal{K}$ is given by $\left\{\phi_a(q^{1/r}-1)^k, \frac{1}{r}\phi^a \frac{q^{k/r}}{(1-q^{1/r})^{k+1}}\right\}_{a\neq 0}$ and $\left\{\phi_0(q-1)^k, \phi^0\frac{q^k}{(1-q)^{k+1}}\right\}$. Notice that $\mathcal{B}_{0,n}\mathcal{C}_{0,n} = \mathrm{td}\,(T)$.

Definition 2.10. Let V be the $\mathbb{C}$-vector space with basis $\phi_0,\dots,\phi_{r-1}$. V is the direct sum $V = V_0 \oplus V_r$, where V_0 is generated by ϕ_0, and V_r by $\{\phi_1,\dots,\phi_{r-1}\}$.

Let $\mathcal{K}^{\mathrm{fake},\lambda}$ be the free $\mathbb{C}[\![\lambda]\!]$ module of Laurent series in $(q-1)$

$$\mathcal{K}^{\mathrm{fake},\lambda} = V \otimes \mathbb{C}[\![(1-q),(1-q)^{-1}] \otimes \mathbb{C}[\![\lambda]\!]. \tag{33}$$

We endow $\mathcal{K}^{\mathrm{fake},\lambda}$ with the polarization prescribed by 2.9, that is,

$$\begin{aligned}
&\mathcal{K}_+^{\mathrm{fake},\lambda} = V_r \otimes \mathbb{C}[\![\lambda]\!][\![q^{1/r}-1]\!] \oplus V_1 \otimes \mathbb{C}[\![\lambda]\!][\![q-1]\!]\\
&\mathcal{K}_-^{\mathrm{fake},\lambda} = \mathrm{Span}\left(\left\{\phi_a \frac{q^{k/r}}{(1-q^{1/r})^{k+1}} | a\neq 0, k\in\mathbb{N}\right\} \cup \left\{\phi_0 \frac{q^k}{(1-q)^{k+1}} | k\in\mathbb{N}\right\}\right)
\end{aligned}$$

We define the fake correlators by

$$\begin{aligned}
\langle \tau_{j_1}(e_{a_1}),\dots,\tau_{j_n}(e_{a_n})\rangle_{0,n}^{\mathrm{fake}} &= \int_{\overline{\mathcal{M}}_{0,a}^r} \prod_i \mathrm{ch}(L_i)^{j_i} \mathcal{A}_{0,n}(\underline{a})\mathcal{B}_{0,n}\mathcal{C}_{0,n}\\
&= \int_{\overline{\mathcal{M}}_{0,a}^r} \prod_i \mathrm{ch}(L_i)^{j_i} \mathcal{A}_{0,n}(\underline{a})\mathrm{td}\left(T_{\overline{\mathcal{M}}_{0,a}^r}\right);
\end{aligned}$$

where L_i denotes the i-th cotangent line bundle to the coarse curve. We extend the corelators as n linear forms over $\mathcal{K}_+^{\mathrm{fake}}$, and define the genus-0 fake potential, by

$$\mathcal{F}^{\mathrm{fake},\lambda}(t(q)) = \sum_n \frac{1}{n!}\langle t(L),\dots,t(L)\rangle_{0,n}^{\mathrm{fake}}.$$

The fake J function is the graph of the (shifted) differential of the fake potential.

$$J^{\mathrm{fake},\lambda}(t) = 1 - q + t + \sum_{a\neq 0} \frac{1}{r}\frac{\phi^a}{n!}\left\langle \frac{\phi_a}{1-q^{1/r}L^{1/r}}, t(L),\dots,t(L)\right\rangle^{\mathrm{fake},\lambda}_{0,n+1}$$
$$+ \frac{\phi^0}{n!}\left\langle \frac{\phi_0}{1-qL}, t(L),\dots,t(L)\right\rangle^{\mathrm{fake},\lambda}_{0,n+1}$$

Proposition 2.11. *Let $\mathcal{L}^{\mathrm{fake},\lambda}$ be the range of the big fake J-function in $\mathcal{K}^{\mathrm{fake}}$, and $\mathcal{L}^{\mathcal{A}} \subset \mathcal{H}$ be the cone of the CohFT $\mathcal{A}$. Then*

$$\mathcal{L}^{\mathrm{fake}} = \mathrm{ch}^{-1}(\mathcal{L}^{\mathcal{A}}) \tag{34}$$

We summarize the situation in the following diagram.

$$\begin{array}{ccc} \mathcal{L}^{\mathrm{fake},\lambda} & \longrightarrow & \mathcal{L}^{\mathcal{A}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{K}^{\mathrm{fake},\lambda} & \xrightarrow{\mathrm{ch}} & \mathcal{H}^{\mathcal{A}} \end{array}$$

Since the Chern character map is an isomorphism, we will abuse notations and write $q = e^z$.

Evaluation In the next section we need to evaluate the class $\mathcal{A}(F) = \mathrm{ch}\left(\Lambda_\lambda(-F^\vee)\right)$ at specific values of $\lambda \in \mathbb{C}$. If $\lambda = -1$, then the result is well defined only if $-F$ is the isomorphism class of a vector bundle. It is shown in [CR10] that $R^1\pi_*\mathcal{L}(-E)$ is indeed a vector bundle. The evaluation of a correlator at $\lambda = -\xi \in \mathbb{C}$ will be denoted

$$\mathrm{ev}_{-\xi}\left\langle \phi_{a_1},\dots,\phi_{a_n}\right\rangle^{\mathrm{fake},\lambda} = \left\langle \phi_{a_1},\dots,\phi_{a_n}\right\rangle^{\mathrm{fake},\xi}. \tag{35}$$

3 The J-function

In this section we define a J-function for the FJRW-invariants, which has values in $K^0(IB\boldsymbol{\mu}_r)$, and we apply the Grothendieck–Riemann–Roch theorem to express this function in terms of the fake invariants. For notational reasons, we restrict to the case where r is prime. Thus, the marked gerbes are either isomorphic to $\overline{\mathcal{M}}^r_{0,\underline{a}}$, or $\boldsymbol{\mu}_r$-gerbes.

Recall that $K^0(IB\boldsymbol{\mu}_r)\otimes\mathbb{C} = \bigoplus_{a=0}^{r-1}\mathbb{C}[\hat{\boldsymbol{\mu}}_r]\phi_a \simeq V\otimes_{\mathbb{C}}\mathbb{C}[\hat{\boldsymbol{\mu}}_r]$.

Definition 3.1. Let $\mathcal{K}_1, \mathcal{K}_r$ be the spaces

$$\mathcal{K}_1 = V_1 \otimes \mathbb{C}[\hat{\boldsymbol{\mu}}_r]\otimes\mathbb{C}(q)$$
$$\mathcal{K}_r = V_r \otimes \mathbb{C}[\hat{\boldsymbol{\mu}}_r]\otimes\mathbb{C}(q^{1/r}).$$

The loop space of the FJRW theory is

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_r. \tag{36}$$

The scalar product on $V \otimes \mathbb{C}[\hat{\boldsymbol{\mu}}_r]$ is

$$(E\phi_a; F\phi_b) = \chi(B\boldsymbol{\mu}_r; E \otimes F)\delta_{a+b,r}, \tag{37}$$

and the symplectic form Ω is

$$\Omega(f(q)\phi_0, g(q)\phi_0) = [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}]\left((f(q); g(q^{-1}))\frac{dq}{q}\right),$$

$$\Omega(f(q^{1/r})\phi_a, g(q^{1/r})\phi_b) = \left[\mathrm{Res}_{q^{1/r}=0} + \mathrm{Res}_{q^{1/r}=\infty}\right]\left((f(q^{1/r})\phi_a; g(q^{-1/r})\phi_b)\frac{dq^{1/r}}{q^{1/r}}\right).$$

The polarization is given by

$$(\mathcal{K}_i)_+ = V_i[q^{1/i}, q^{-1/i}] \otimes \mathbb{C}[\hat{\boldsymbol{\mu}}_r] \text{ for } i = 1, r$$
$$\mathcal{K}_- = \{f \in \mathcal{K} \text{ such that } f(0) \neq \infty \text{ and } f(\infty) = 0\}$$

There is a morphism of localization at $q^{1/r} = 1$

$$\mathcal{K} \to \mathcal{K}^{fake} = V_1 \otimes \mathbb{C}[\![q^{-1}, (q-1)^{-1}] \oplus V_r \otimes \mathbb{C}[\![q^{1/r} - 1, (q^{1/r} - 1)^{-1}].$$

Definition 3.2. The J-function is the formal function $\mathcal{K}_+ \to \mathcal{K}$ given by

$$J(t(q)) = 1 - q + t + \sum_n \frac{1}{n!}(\mathrm{ev}_1)_* \left(\frac{\pi_1^*\left(\Lambda_{0,n+1} \bigotimes_{i=2}^{n+1} t(\mathcal{L}_i)\right)}{1 - q^{1/r}\mathcal{L}_1}\right). \tag{38}$$

Remark 3.3. The Grothendieck ring $K^0(B\boldsymbol{\mu}_r)$ is the representation ring of $\boldsymbol{\mu}_r$, which is isomorphic to $\mathbb{C}[\hat{\boldsymbol{\mu}}_r]$. This ring has two natural $\mathbb{C}$-bases. The first is given by the characters $[\chi] \in \mathbb{C}[\hat{\boldsymbol{\mu}}_r]$, and the second is the basis of idempotents $e_\xi = \frac{1}{r}\sum_{\chi \in \hat{\boldsymbol{\mu}}_r} \chi^{-1}(\xi)[\chi]$. For $E = \sum_\chi \lambda_\chi[\chi]$, we have that $E = \sum_{\xi \in \boldsymbol{\mu}_r} E \otimes e_\xi$ and $E \otimes e_\xi = \sum_\chi \lambda_\chi \chi(\xi) e_\xi$.

On the other hand, let $p : IB\boldsymbol{\mu}_r \to B\boldsymbol{\mu}_r$ be the projection from the inertia stack. The cohomology of $IB\boldsymbol{\mu}_r$ is $H^*(IB\boldsymbol{\mu}_r) = \bigoplus_{\xi \in \boldsymbol{\mu}_r} \mathbb{C}.v_\xi$. Then $\mathrm{ch}(\rho(p^*E)) = \sum_{\xi \in \boldsymbol{\mu}_r} \sum_\chi \chi(\xi) v_\xi$. Thus the coefficients of $\mathrm{ch}(\rho(p^*E))$ coincide with the coefficients of E in the basis e_ξ, and are given by the Riemann-Roch theorem [Toe98].

3.1 $\overline{\mathcal{M}}^r_{0,\underline{a}}$ and its inertia stack

We describe the inertia stack of $\overline{\mathcal{M}}^r_{0,n}$, and how the stabilizers act on the universal root.

We fix a multi-index $\underline{a} = (a_1, \ldots, a_n)$, and let $\overline{\mathcal{M}}^r_{0,\underline{a}}$ be the stack of rth roots of $\omega_{\log}(-\sum_i a_i x_i)$. An object of this stack over a scheme S is a family of twisted curves $\mathcal{C} \to S$ equipped with a line bundle $\mathcal{L}$, and an isomorphism $\mathcal{L}^{\otimes r} \to \omega_{\log}(-\sum_i a_i x_i)$, such that each node is balanced. Moreover, for any geometric point x of $\mathcal{C}$, the representation $\mathrm{Aut}(x) \to \mathbb{C}^*$ induced by $\mathcal{L}_x$ must

be faithful. These conditions determine the order of the automorphism group of every point of $\mathcal{C}$.

Indeed, a boundary divisor D of $\overline{\mathcal{M}}_{0,n}$ is given by a partition $\{1,\dots,n\} = A \sqcup B$. The universal curve over D is the union of two components $\mathcal{C}_A \cup \mathcal{C}_B$, and the restriction of $\mathcal{L}$ to $\mathcal{C}_A$ has degree $d_A = \frac{1}{r}(-2+1-\sum_{i\in A} a_i) \in \mathbb{Z}[\frac{1}{r}]$. Thus the order of the automorphism group of the node is $\rho_D = \#\langle rd_A\rangle$, where $\langle rd_A\rangle \subset \mathbb{Z}_r$ is the subgroup of $\mathbb{Z}_r$ generated by rd_A.

Let $\overline{\mathcal{M}}_{0,n}(r,\underline{a})$ be the stack of twisted n-pointed stable curves of genus 0 such that each node defining a divisor D has order ρ_D. There is a map $\overline{\mathcal{M}}^r_{0,\underline{a}} \to \overline{\mathcal{M}}_{0,n}(r,\underline{a})$, which is a $\boldsymbol{\mu}_r$-gerbe.

Definition 3.4. Let $\underline{a}$ be a multi-index, and let ρ be the function on boundary divisors defined above. An inertial stable graph is a pair (Γ,ν), where

- Γ is a stable graph with n legs,
- $\nu : E(\Gamma) \to \boldsymbol{\mu}_\infty$ is a function with values in the roots of unity, such that for all $e \in E(\Gamma)$, we have $\nu(e) \in \boldsymbol{\mu}_{\rho(e)} \setminus \{1\}$,

where $\rho : E(\Gamma) \to \mathbb{N}_+$ is the function on edges of Γ induced by ρ.

The set of isomorphism classes of inertial stable graphs relatively to $\underline{a}$ is $G_{\underline{a}}$.

One associates an inertial stable graph to each connected component of $I\overline{\mathcal{M}}(r,\underline{a})$ as follows. A closed point of such a component is given by $(\mathcal{C},g)$, where g is an automorphism of $\mathcal{C}$. The only automorphisms of a twisted, genus-0 stable curve are the ghost automorphisms (see [ACV01, Section 7]). Thus a generic curve of a connected component of $I\overline{\mathcal{M}}(r,\underline{a})$ has a dual graph Γ such that $\rho(e) \neq 1$ for all edges $e \in E(\Gamma)$. We orient the edges of Γ in the direction going away from the first marked point x_1. The orientation of Γ yields an isomorphism $\mathrm{Aut}(\mathcal{C}) \simeq \prod_{e\in E(\Gamma)} \boldsymbol{\mu}_{\rho(e)}$. Explicitly, given a node $p \in \mathcal{C}$, $\zeta \in \boldsymbol{\mu}_{\rho(e)}$ corresponds to the ghost automorphism $(z_-,z_+) \mapsto (\zeta z_-, z_+)$

With this isomorphism fixed, one defines ν such that the isomorphism of $\mathcal{C}$ induced by g is $(\nu(e))_{e\in E(\Gamma)}$.

Proposition 3.5. *The correspondence described above induces a bijection between $G_{\underline{a}}$, and the set of connected components of $I\overline{\mathcal{M}}(r,\underline{a})$.*

The component corresponding to Γ is denoted $I\overline{\mathcal{M}}(r,\underline{a})_\Gamma$.

Proof. The moduli space $\overline{\mathcal{M}}_{0,n}(r,\underline{a})$ is obtained from $\overline{\mathcal{M}}_{0,n}$ by performing the r_D-th root construction at each boundary divisor D ([BC10]). Writing $\mathcal{M}$ for $\overline{\mathcal{M}}_{0,n}$, we have

$$\overline{\mathcal{M}}_{0,n}(r,\underline{a}) = \mathcal{M}[D_1/r_{D_1}] \times_{\mathcal{M}} \mathcal{M}[D_2/r_{D_2}] \times_{\mathcal{M}} \cdots \tag{39}$$

where D_i are the boundary divisors. The inertia stack of $\overline{\mathcal{M}}_{0,n}[D/r]$ is $I\overline{\mathcal{M}}_{0,n}[D/r] = \overline{\mathcal{M}}_{0,n} \bigsqcup_{\xi\in\boldsymbol{\mu}_{r_D}\setminus 1} \mathcal{D}_\xi$, where $\mathcal{D}_\xi$ is a stack whose coarse space is isomorphic to D. Finally, $\overline{\mathcal{M}}_{0,n}$ is a scheme, so the inertia stack of $\overline{\mathcal{M}}_{0,n}(r,\underline{a})$ is the fiber product over $\overline{\mathcal{M}}_{0,n}$ of all the $I\overline{\mathcal{M}}_{0,n}[D/r_D]$. $\square$

Let $p\colon \overline{\mathcal{M}}^r_{\underline{a}} \to \overline{\mathcal{M}}(r,\underline{a})$ be the forgetful map, and $Ip\colon I\overline{\mathcal{M}}^r_{\underline{a}} \to I\overline{\mathcal{M}}(r,\underline{a})$ be the induced map.

Definition 3.6. Define

$$I\overline{\mathcal{M}}^r_{\Gamma}\colon = (Ip)^{-1}\left(I\overline{\mathcal{M}}(r,\underline{a})_{\Gamma}\right). \tag{40}$$

We get a decomposition of $I\overline{\mathcal{M}}^r_{\underline{a}}$ into open and closed substacks

$$I\overline{\mathcal{M}}^r_{\underline{a}} = \bigsqcup_{\Gamma\in G_{\underline{a}}} I\overline{\mathcal{M}}^r_{\Gamma}. \tag{41}$$

The map p is a gerbe banded by $\boldsymbol{\mu}_r$, and for any closed point $(\mathcal{C},\mathcal{L})$ of $\overline{\mathcal{M}}^r$, it induces an exact sequence

$$1 \to \boldsymbol{\mu}_r \to \mathrm{Aut}((\mathcal{C},\mathcal{L})) \to \mathrm{Aut}(\mathcal{C}) \to 1. \tag{42}$$

Recall that an automorphism of the object $(\mathcal{C},\mathcal{L})$ is the data of an automorphism g of $\mathcal{C}$, and an isomorphism $\alpha : g^*\mathcal{L} \to \mathcal{L}$ compatible with the r-th root structure. Since the restriction of g to the component containing x_1 is the identity map, α acts on $\mathcal{L}_{|x_1}$ by multiplication by some r-th root of unity $\xi(g,\alpha) \in \boldsymbol{\mu}_r$. The map $(g,\alpha) \mapsto \xi(g,\alpha)$ splits the exact sequence 42.

We now decompose the inertia stack according to this splitting.

Definition 3.7. For $\xi \in \boldsymbol{\mu}_r$, and $\Gamma \in G_{\underline{a}}$, let $I\overline{\mathcal{M}}^{r,\xi}_{\Gamma}$ be the open and closed substack of $I\overline{\mathcal{M}}^r_{0,n}$ made of the curves $(\mathcal{C},\mathcal{L},g,\alpha) \in I\overline{\mathcal{M}}^r_{\Gamma}$ such that $\xi(g,\alpha) = \xi$. Then we have

$$I\overline{\mathcal{M}}^r_{0,n} = \bigsqcup_{\substack{\Gamma\in G_{\underline{a}}\\ \xi\in\boldsymbol{\mu}_r}} I\overline{\mathcal{M}}^{r,\xi}_{\Gamma}. \tag{43}$$

Let us denote the first marked gerbe by $\Sigma = \Sigma_1$, and let $\Sigma_{\underline{a}}$ be the first maked gerbe of $\overline{\mathcal{M}}^r_{0,\underline{a}}$. If $a_1 \neq 0$, the morphism, $\pi : \Sigma \to \overline{\mathcal{M}}^r_{0,n}$ is a gerbe banded by $\boldsymbol{\mu}_r$, so for any closed point x of Σ we obtain an exact sequence

$$1 \to \boldsymbol{\mu}_r \to \mathrm{Aut}(x) \to \mathrm{Aut}(\pi(x)) \to 1. \tag{44}$$

This sequence splits canonically, yielding an isomorphism $\mathrm{Aut}(x) \simeq \boldsymbol{\mu}_r \times \mathrm{Aut}(\pi(x))$. Thus $I\Sigma$ admits a first decomposition $I\Sigma = \bigsqcup_{\xi_1\in\boldsymbol{\mu}_r} I\Sigma^{\xi_1}$.

Let $I\pi_1 : I\Sigma \to I\overline{\mathcal{M}}^r_{0,n}$ be the morphism induced by π_1.

Definition 3.8. The inertia stack of $\Sigma_{\underline{a}}$ has a decomposition

$$I\Sigma_{\underline{a}} = \bigsqcup_{\Gamma\in G_{\underline{a}}} \bigsqcup_{\xi,\xi_1\in\boldsymbol{\mu}_r} I\Sigma^{\xi,\xi_1}_{\Gamma}, \tag{45}$$

where $I\Sigma^{\xi,\xi_1} = I\Sigma^{\xi_1} \times_{I\Sigma} (I\pi_1)^{-1}\left(I\overline{\mathcal{M}}^{r,\xi}_{\Gamma}\right)$.

Let us also define

$$I\Sigma_{\underline{a}}^{\xi,\xi_1} := \bigsqcup_{\Gamma\in G_{\underline{a}}} I\Sigma_{\Gamma}^{\xi,\xi_1}$$
$$I\Sigma_{0,n}^{\xi,\xi_1} := \bigsqcup_{\underline{a}} I\Sigma_{\underline{a}}^{\xi,\xi_1}$$

Lemma 3.9. *Let x be a $\mathbb{C}$-point of $I\Sigma_{0,\underline{a}}^{\xi,\xi_1}$, and let $I\mathrm{ev}_1 : I\Sigma \to IB\boldsymbol{\mu}_r$ be the morphism induced by $\mathrm{ev}_1 : \Sigma_{\underline{a}} \to B\boldsymbol{\mu}_r$. Recall that $I\boldsymbol{\mu}_r \simeq \bigsqcup_{\xi\in\boldsymbol{\mu}_r}(B\boldsymbol{\mu}_r)_\xi$.*

Then $I\mathrm{ev}_1(x) \in (B\boldsymbol{\mu}_r)_{\xi\xi_1^{a_1}}$.

Proof. The morphism $\Sigma \to B\boldsymbol{\mu}_r$ is given by the restriction of $\mathcal{L}$ to Σ_1, which is an r-th root of $\mathcal{O}_{\Sigma_1}$. This functor induces a morphism between the generic automorphism groups, which is given by

$$\begin{aligned}\boldsymbol{\mu}_r \times \boldsymbol{\mu}_r &\to \boldsymbol{\mu}_r\\ (\xi,\xi_1) &\mapsto \xi\xi_1^{a_1}.\end{aligned}$$

Ideed this corresponds to the action of $\boldsymbol{\mu}_r \times \boldsymbol{\mu}_r$ on $\mathcal{L}_{|\Sigma}$. □

We now describe the action of the automorphism groups of $\overline{\mathcal{M}}^r$ on $R^1\pi_*\mathcal{L}$ with the concavity assumption. We fix a connected component $I\overline{\mathcal{M}}_\Gamma^{r,\xi} \subset I\overline{\mathcal{M}}^{r,\xi}$. Then $R^1\pi_*\mathcal{L}$ decomposes as a direct sum $\bigoplus_{v\in V(\Gamma)} E_v$. Let v_0 be the component containing x_1. By hypothesis, the automorphism (g,α) acts on E_{v_0} by multiplication by ξ.

Lemma 3.10. *There exists a unique function $f : V(\Gamma) \to \boldsymbol{\mu}_r$ such that*

- *if v_1 and v_2 are two vertices connected by an edge e oriented from v_1 to v_2, then $\nu(e)^{\mathrm{mult}_{p_+}(\mathcal{L})} f(v_1) = f(v_2)$, where $\mathrm{mult}_{p_+}(\mathcal{L}) \in \mathbb{Z}_{\rho(e)}$ is the character of $\mathcal{L}$ at the node on the component given by v_2,*
- *$f(v_0) = \xi$.*

Recall that if $\mathcal{M}$ is a Deligne–Mumford stack, and E is a vector bundle on $I\mathcal{M}$, then E is locally decomposed as $E = \bigoplus_{l=0}^{k-1} E^l$, where k is the order of the isotropy subgroup generated by g. Then one defines the morphism $\rho : K^0(I\mathcal{M})_{\mathbb{C}} \to K^0(I\mathcal{M})_{\mathbb{C}}$ by $\rho(E) = \sum_l e^{\frac{2i\pi l}{k}} E^l$ (see [Toe98]).

Proposition 3.11. *Let $\underline{a}$ be a multi-index, and let $\Gamma \in G_{\underline{a}}$ be an inertial stable graph. Let f be the function defined by lemma 3.10. Then, over $I\overline{\mathcal{M}}_\Gamma^{r,\xi}$, one has*

$$\rho\left(R^1\pi_*\mathcal{L}\right) = \sum_{v\in V(\Gamma)} f(v)E_v. \tag{46}$$

Proof. For simplicity, let us assume that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a curve with one node p, and $x_1 \in \mathcal{C}_1$. Let g be an automorphism of $\mathcal{C}$. Note that $g_{|\mathcal{C}_i} \simeq \mathrm{id}_{\mathcal{C}_i}$. Let

$\alpha : g^*\mathcal{L} \to \mathcal{L}$ be an isomorphism such that $\alpha_{\mathcal{C}_1}$ is the multiplication by some root of unity ξ.

The local picture of g at p is given by $(z_1, z_2) \mapsto (\zeta z_1, z_2)$ (or any equivalent morphism). By [Chi07, prop. 2.5.3], we have

$$g^*\mathcal{L} \simeq \mathcal{L} \otimes T_{\mathcal{L}}, \tag{47}$$

where $T_{\mathcal{L}}$ is the line bundle whose sections over U are pairs of functions $(f_1, f_2) \in \mathcal{O}_{\mathcal{C}_1}(U) \oplus \mathcal{C}_{\mathcal{C}_2}(U)$ such that $\zeta^{\mathrm{mult}_{p_2}(\mathcal{L})} f_1(p) = f_2(p)$. Thus the morphism $\alpha_{|\mathcal{C}_2}$ is the multiplication by $\xi\zeta^{\mathrm{mult}_{p_2}(\mathcal{L})}$.

□

Corollary 3.12. *With the same setting as in the previous proposition, one has*

$$\rho\left(\Lambda_{-1}(R^1\pi_*\mathcal{L})^\vee\right) = \prod_{v \in V(\Gamma)} \Lambda_{-f(v)^{-1}}(E_v^\vee). \tag{48}$$

Proposition 3.13. *In the same setting as above, suppose a_i is prime to r, and let $T = \pi_{i*}\left(\mathrm{ev}_i^*[j/a_i] \otimes \mathcal{N}_i^{-j}\right)$. Let $v(i)$ be the vertex of Γ supporting the ith marked point. Then over $I\overline{\mathcal{M}}_\Gamma^{r,\xi}$ we have*

$$\rho(T) = f(v(i))^{j/a_i} \tag{49}$$

3.2 The J-function

We now apply the Grothendieck–Riemann–Roch theorem to compute the J-function of the K-theoretic invariants.

Proposition 3.14. *The J-function is the (shifted) differential of the genus-0 potential. Thus, its image is a Lagrangian cone in $\mathcal{K}$.*

Proof. For $h, t \in \mathcal{K}_+$, the differential of $\mathcal{F}$ at t is $d_t\mathcal{F}(h) = \sum \frac{1}{n!} \langle h, t, \ldots, t\rangle_{n+1}$. Let us suppose that $t \in V_r[q^{1/r}, q^{-1/r}] \otimes \mathbb{C}[\hat{\boldsymbol{\mu}}_r]$. Notice that $(\phi_a e_\xi; \phi_{r-a} e_{\xi'}) = \frac{1}{r}\delta_{\xi,\xi'}$, thus the dual of $\phi_a e_\xi$ is $r\phi^a e_\xi$. Then we have

$$d_t\mathcal{F} = \sum_{n, a\neq 0} \sum_{\xi \in \boldsymbol{\mu}_r} \frac{r\phi^a e_\xi}{n!} \left\langle \frac{\phi_a e_\xi}{(1-q^{1/r})\mathcal{L}_1}, t(\mathcal{L}), \ldots, t(\mathcal{L}) \right\rangle_{0,n+1}. \tag{50}$$

On the other hand, for any class $F \in K^0(\Sigma)$, we have

$$\begin{aligned}(\mathrm{ev}_1)_*(F) &= \sum_{a,\xi} r\chi\left(IB\boldsymbol{\mu}_r; (\mathrm{ev}_1)_*(F) \otimes \phi_a e_\xi\right) \phi_a e_\xi \\ &= \sum_{a,\xi} r\chi\left(\Sigma; F \otimes (\mathrm{ev}_1)^*\left(\phi_a e_\xi\right)\right) \phi_a e_\xi\end{aligned}$$

Taking F to be $\Lambda \otimes \bigotimes t(\mathcal{L}_i)$ yields the result. □

We define a linear action of $\boldsymbol{\mu}_r$ on $\mathcal{K}_r$ by

$$\zeta \cdot (q^{j/r}\phi_a e_\xi) = \zeta^j q^{j/r}\phi_a e_{\xi\zeta^a} \text{ over } \mathcal{K}_{r,+}$$
$$\zeta \cdot (q^{j/r}\phi^a e_\xi) = \zeta^j q^{j/r}\phi^a e_{\xi\zeta^a} \text{ over } \mathcal{K}_{r,-}$$

This action is related to the action of the relative automorphism group of Σ on $I\Sigma$.

Proposition 3.15. *The action of $\boldsymbol{\mu}_r$ on $\mathcal{K}_r$ is symplectic.*

Lemma 3.16. *Let $[d]$ be the character $\xi \mapsto \xi^d$. The invariant subspace of $\mathcal{K}_{r,+}$ is generated by the elements $q^{j/r}\phi^a[j/a]$ for $a \neq 0$.*

Proof. One computes that

$$\begin{aligned}\zeta \cdot \left(q^{j/r}\phi_a[d]\right) &= q^{j/r}\phi_a \sum_{\xi\in\boldsymbol{\mu}_r} \zeta^j \xi^d e_{\xi\zeta^a} \\ &= q^{j/r}\phi_a \zeta^{j-ad} \sum_{\xi\in\boldsymbol{\mu}_r} (\xi\zeta^a)^d e_{\xi\zeta^a} \\ &= q^{j/r}\phi_a \zeta^{j-ad}[d]\end{aligned}$$

Thus $\boldsymbol{\mu}_r$ acts diagonally on $\mathcal{K}_{r,+}$ with respect to the basis $\phi_a q^{j/r}[d]$, and the subspace of invariants has basis $\{q^{j/r}\phi^a[j/a]\}$ for $a \neq 0$. □

Let t be an element of $\mathcal{K}_{r,+}$, and denote $\underline{t} = \frac{1}{r}\sum_{\zeta\in\boldsymbol{\mu}_r} \zeta \cdot t(q^{1/r})$. The map $t \mapsto \underline{t}$ is a projection onto the subspace of invariant elements. Then by remark 1.8, we have that $\langle t, \ldots, t\rangle_{0,n} = \langle \underline{t}, \ldots, \underline{t}\rangle_{0,n}$ and

$$J(t) = 1 - q + t + \sum \frac{1}{n!}\mathrm{ev}_{1*}\left(\frac{\Lambda_{0,n+1}}{1-q^{1/r}\mathcal{L}_1}\bigotimes_{i=2}^{n+1}\underline{t}(\mathcal{L}_i)\right) \tag{51}$$

Proposition 3.17. *Suppose that $t = \underline{t} \in \mathcal{K}_+$. Then $J(t)$ is a $\boldsymbol{\mu}_r$-invariant point of $\mathcal{K}$.*

Proof. We only need to check that the $\mathcal{K}_-$ part of $J(t)$ is invariant. Recall that

$$J_-(t) = \sum_{a,n,\xi_0,\xi_1} \frac{r\phi^a e_{\xi_0}}{n!}\int_{I\Sigma_{0,n+1}^{\xi_0\xi_1^{-a},\xi_1}} \frac{\mathrm{ch}(\rho\Lambda_{0,n+1})}{\mathrm{ch}(1-\xi_1^{-1}q\mathcal{L}_1)}\bigotimes \mathrm{ch}\left(\rho t(\mathcal{L}_i)\right),$$

where we consider a_i as locally constant functions on $\overline{\mathcal{M}}_{0,n}^r$.

The stacks $I\Sigma_{0,n+1}^{\xi_0\xi_1^{-a},\xi_1}$ and $I\Sigma_{0,n+1}^{\xi_0\xi_1^{-a},\zeta\xi_1}$ are naturally isomorphic, since Σ is a $\boldsymbol{\mu}_r$-gerbe over $\overline{\mathcal{M}}_{0,n}^r$. Moreover, if t is $\boldsymbol{\mu}_r$-invariant, then proposition 3.13 implies that this isomorphism does not change the trace on $t(\mathcal{L}_i)$. Thus,

$$\begin{aligned}\zeta^{-1}\cdot J_-(t) &= \sum_{a,n,\xi_0,\xi_1} \frac{r\phi^a e_{\xi_0\zeta^{-a}}}{n!}\int_{I\Sigma_{0,n+1}^{\xi_0\xi_1^{-a},\xi_1}} \frac{\mathrm{ch}(\rho\Lambda_{0,n+1})}{\mathrm{ch}(1-(\zeta\xi_1)^{-1}q\mathcal{L}_1)}\prod \mathrm{ch}\left(\rho t(\mathcal{L}_i)\right) \\ &= \sum_{a,n,\xi_0,\xi_1} \frac{r\phi^a e_{\xi_0\zeta^{-a}}}{n!}\int_{I\Sigma_{0,n+1}^{\xi_0\xi_1^{-a},\xi_1\zeta}} \frac{\mathrm{ch}(\rho\Lambda_{0,n+1})}{\mathrm{ch}\left(1-q\rho\mathcal{L}_1\right)}\prod \mathrm{ch}\left(\rho t(\mathcal{L}_i)\right. \\ &= J_-(t)\end{aligned}$$

□

Proposition 3.18. *Suppose that $t = \underline{t} \in \mathcal{K}_+$. Then the localization at 1 of $J(t)$ is a point of the fake cone*

$$J(t)_{(1)} \in \mathcal{L}^{\mathrm{fake}} \tag{52}$$

Proof. The argument follows the adelic characterization of [GT11]. It consists in applying the Grothendieck–Riemann–Roch theorem for stacks (or GRR for short), and relating the different integrals with the fake theories introduced in section 2.

Let us decompose $J(t) = \sum_{\xi_0} J^{\xi_0} e_{\xi_0}$. We fix $\xi_0 \in \boldsymbol{\mu}_r$, and show that the function J^{ξ_0} lies on the fake cone $\mathcal{L}^{\xi_0,\mathrm{fake}}$. According to the GRR formula [Toe98], the component J^{ξ_0} is given by

$$1-q+t+\sum_{\xi_1\in\boldsymbol{\mu}_r, a\neq 0} \frac{r\phi^a}{n!}\int_{I\Sigma^{\xi_0\xi_1^{-a},\xi_1}} \frac{\mathrm{ch}\left(\rho\left(\Lambda_{0,n}\bigotimes t(\mathcal{L}_i)\right)\right)}{\left(1-\xi_1^{-1}q^{1/r}e^{\psi_1/r}\right)\mathrm{ch}\left(\rho\Lambda_{-1}\mathcal{N}^\vee\right)}\mathrm{td}(T). \tag{53}$$

Let us denote $\tilde{t}$ the terms of the sum with $\xi_1 \neq 1$. Explicitly, $\tilde{t}$ is the sum of the contributions of $I\Sigma^{\xi_0\xi_1^{-a},\xi_1}$ for $\xi_1 \neq 1$. Notice that $\tilde{t} \in \mathcal{K}_+^{\xi,\mathrm{fake}}$, thus we want to show that $J^{\xi_0}(t)_{(1)} = J^{\xi,\mathrm{fake}}(t+\tilde{t})$.

Let (C, g) be a geometric point of $I\Sigma^{\xi_0,1}$. For such a curve, the head of C is the maximal connected subcurve containing Σ_1 (the first marked gerbe), and such that the restriction of g to the head is the identity. The head has marked gerbes which are either nodes of C with non-trivial action of the ghost automorphism, or marked gerbes of C. In the first case, the connected component attached to the node is called an arm. An arm comes with a distinguished marked gerbe, called its horn.

The moduli space of heads is the union of all the first marked gerbes of the moduli space of spin curves,

$$\overline{\mathcal{M}}_n^{\mathrm{head},\xi_0} = \bigsqcup_{n\geq 3}\bigsqcup_{b_1,\ldots,b_n\neq 0}\Sigma_{\underline{b}}, \tag{54}$$

while the moduli space of arms is

$$\overline{\mathcal{M}}^{\mathrm{arm},\xi_0} = \bigsqcup_{n\geq 3}\bigsqcup_{c_1\neq 0,\xi_1\neq 1}\overline{\mathcal{M}}^r_{0,\underline{c}}. \tag{55}$$

Let us fix $\underline{a} = (a_1, \ldots, a_n)$, and let $\Gamma \in G_{\underline{a}}$ be an inertial graph. Let m be the number of legs of the vertex carrying the first marked point in Γ. The decomposition of curves into head and arms induces a morphism

$$I\Sigma_\Gamma^{\xi_0,1} \to \overline{\mathcal{M}}^{\mathrm{head},\xi_0} \times \left(\overline{\mathcal{M}}^{\mathrm{arm},\xi_0}\right)^m. \tag{56}$$

This morphism allows us to express the content of the GRR formula as a product of integrals over the moduli spaces of heads and arms, except for the normal bundle at nodes joining the head to an arm. Such a node of C contributes to

the conormal bundle $\mathcal{N}^\vee$ by $\mathcal{L}_+ \otimes \mathcal{L}_-$. Thus the contribution of $I\Sigma^{\xi_0,1}$ to the J-function is exactly

$$\sum_{m,n} \frac{1}{r} \frac{\phi^a}{m!n!} \left\langle \frac{\phi_a}{1-q^{1/r}\mathcal{L}^{1/r}}, t(\mathcal{L}^{1/r}), \ldots, t(\mathcal{L}^{1/r}), \tilde{t}(\mathcal{L}^{1/r}), \ldots, \tilde{t}(\mathcal{L}^{1/r}) \right\rangle^{\xi_0,\mathrm{fake}}$$
$$= \sum \frac{1}{r} \frac{\phi^a}{n!} \left\langle \frac{\phi_a}{1-q^{1/r}\mathcal{L}^{1/r}}, t(\mathcal{L}^{1/r}) + \tilde{t}(\mathcal{L}^{1/r}), \ldots, t(\mathcal{L}^{1/r}) + \tilde{t}(\mathcal{L}^{1/r}) \right\rangle^{\xi_0,\mathrm{fake}}_{0,n+1}$$

□

Theorem 3.19. *For all $t \in \mathcal{K}_+$, we have that,*

(1) $J(t)$ has no poles outside of $0, \infty$ and the fifth roots of unity,

(2) the localization of $\underline{f} e_\xi$ at 1 lies in the fake cone $\mathcal{L}^{\xi,\mathrm{fake}}$,

(3) $f - \underline{f} \in \mathcal{K}_+$.

Conversely, the J function is uniquely determined by these 3 conditions, and can be recursively computed from the correlators of the fake theories.

Proof. It is clear that the J-function satisfies the three conditions above. Conversely, let us show that these conditions determine the J-function. We will construct all the correlators of J_-, and $\tilde{t}$ (the arm contribution) by recursion on the number of marked points. Let us first remark, that if we write $J_- = \sum J_-^{a,\xi_0} \phi^a e_{\xi_0}$ then the first condition implies that the partial fraction decomposition of J_- has the form $J_-^{a,\xi_0} = \sum_{\xi_1 \in \boldsymbol{\mu}_r} J_-^{a,\xi_0,\xi_1}$, where J_-^{a,ξ_0,ξ_1} is the part with pole at ξ_1. Then, the third condition implies that for all $\zeta \in \boldsymbol{\mu}_r$ we have

$$J_-^{a,\xi_0,\xi_1}(\zeta^{-1}q^{1/r}) = J_-^{a,\xi_0\zeta^{-a},\xi_1\zeta}(q^{1/r}) \tag{57}$$

We define $\tilde{t}$ to be

$$\tilde{t} = \sum_{a \neq 0, \xi_0 \in \boldsymbol{\mu}_r, \xi_1 \neq 1} \sum J_-^{a,\xi_0} \phi^a e_{\xi_0} \tag{58}$$

The previous equation shows that $\tilde{t}$ is determined by the functions $J_-^{a,\xi_0,1}$.

By definition, J_- has no terms of degree 0 and 1 in t, and the terms of degree 2 coincide with the fake invariants.

Suppose now that all correlators involving at most n marked points have been computed, and let us compute the correlators with $n+1$ marked points. The correlator $\left\langle \frac{\phi_a}{1-q^{1/r}\mathcal{L}^{1/r}}, t, \ldots, t \right\rangle_{0,n+1}$ is made of integrals over $I\Sigma_{0,n+1}$. Because of condition (3), we need only compute the contribution of $I\Sigma^{\xi,1}_{0,n+1}$, where the stability imposes that the arms have at most n marked points. By the recursion hypothesis, the arm contribution $\tilde{t}$ has been computed up to degree $n-1$ in t, which corresponds to n marked points. □

Corollary 3.20. *Let W be a vector space, and I be a formal function from W to $\mathcal{K}$ such that $I(0) = 1 - q$. If*

- *$I(t)$ has poles only at $0, \infty$ and the fifth roots of 1,*
- *Ie_ξ is a function on $\mathcal{L}^{\xi,\text{fake}}$,*
- *I is $\boldsymbol{\mu}_r$-invariant,*

then I lies on $\mathcal{L}$.

3.3 A function on the Lagrangian cone

3.4 I-functions for the fake theories

We now find a particular one-dimensional subvariety on $\mathcal{L}^{\lambda,\text{fake}}$.

We recall the following computation from [CCIT09] and [CR10]. Given a sequence of scalars w_d, we define the following formal function :

$$G_y(x,z) = \sum_{m,l\geq 0} w_{l+m-1} \frac{B_m(y)x^l}{m!l!} z^{m-1} \tag{59}$$

It satisfies the equations

$$G_y(x,z) = G_0(x,+yz,z)$$
$$G_0(x+z,z) = G_0(x,z) + s(x)$$

Let ∇ be a vector field on V.

Proposition 3.21. *Let $\tilde{J}$ be any formal function from V to $\mathcal{H}$. Suppose that $\tilde{J}(t,-z)$ lives on the untwisted cone. Then, for any sequence (w_i) and for any $y \in \mathbb{C}$, the function*

$$\tilde{J}_{w,y}(t,-z) = \exp(-G_y(z\nabla,z))\tilde{J}(t,-z) \tag{60}$$

also lives on the untwisted cone.

Recall that the small J-function of a point is $J(\tau) = (1-q)\exp\left(\frac{\tau}{1-q}\right)$. We extend the notation ϕ_n to all $n \in \mathbb{N}$ by $\phi_n = \phi_{n \mod r}$, and define an "untwisted" I-function by

$$I^{\text{un}}(\tau) = (1-q)\sum_{n\geq 0} \frac{\tau^n}{(1-q)^n}\phi_{n+1} \tag{61}$$

Lemma 3.22. *The function I^{un} lies on the untwisted cone $\mathcal{L}^{\text{un}} \subset \mathcal{K}^{\text{fake}}$.*

Proof. This is a consequence of 2.9. □

Thus we can apply 3.21 to I^{un} with the vector field $\nabla = \frac{\tau}{r}\partial_\tau$.

Corollary 3.23. *The following function lies on the cone $\mathcal{L}^{\mathrm{fake},\lambda}$.*

$$I^{\lambda}(\tau) = (1-q)\sum_{n\geq 0} \frac{\prod_{0\leq m<\lfloor n/r\rfloor}\left(1+\lambda q^{\frac{1}{r}+\{\frac{n}{r}\}+m}\right)^{r}\tau^{n}}{(1-q)^{n}}\phi_{n+1} \tag{62}$$

Evaluating at $\lambda = -\xi$, we obtain the function

$$I^{\xi}(\tau) = (1-q)\sum_{n\geq 0}\prod_{0\leq m<\lfloor \frac{n}{r}\rfloor}\left(1-\xi q^{\frac{1}{r}+\{\frac{n}{r}\}+m}\right)^{r}\frac{\tau^{n}}{(1-q)^{n}}\phi_{n+1} \tag{63}$$

Proof. See [CCIT09] and [CR10]. □

Theorem 3.24. *Let $\tilde{I}^{\xi}$ be the projection of I^{ξ} on V_r parallel to V_1. The function*

$$I = \bigoplus_{\xi\in\boldsymbol{\mu}_r}\tilde{I}^{\xi}e_{\xi} \tag{64}$$

has values in $\mathcal{L}$.

Proof. We apply 3.20. □

References

[ACV01] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers, June 2001.

[AV00] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. *arXiv:math/9908167*, March 2000.

[BC10] Arend Bayer and Charles Cadman. Quantum cohomology of [C^N/\mu_r]. *Compositio Mathematica*, 146(5):1291–1322, September 2010.

[BF97] K. Behrend and B. Fantechi. The intrinsic normal cone. *Inventiones Mathematicae*, 128(1):45–88, March 1997.

[CCIT09] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng. Computing Genus-Zero Twisted Gromov-Witten Invariants. *Duke Mathematical Journal*, 147(3), April 2009.

[Chi07] Alessandro Chiodo. Stable twisted curves and their r-spin structures. *arXiv:math/0603687*, August 2007.

[CIR14] Alessandro Chiodo, Hiroshi Iritani, and Yongbin Ruan. Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence. *Publications mathématiques de l'IHÉS*, 119(1):127–216, June 2014.

[Cla13] Emily Clader. Landau-Ginzburg/Calabi-Yau correspondence for the complete intersections X_{3,3} and X_{2,2,2,2}, April 2013.

[CR10] Alessandro Chiodo and Yongbin Ruan. Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations. *Inventiones mathematicae*, 182(1):117–165, October 2010.

[CZ07] Alessandro Chiodo and Dimitri Zvonkine. Twisted Gromov-Witten r-spin potential and Givental's quantization, November 2007.

[FJR12] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan. The Witten equation, mirror symmetry and quantum singularity theory. *arXiv:0712.4021 [hep-th, physics:math-ph]*, July 2012.

[Giv] Alexander Givental. Permutation-Equivariant Quantum K-theory V. Toric q-Hypergeometric Functions.

[Giv03] Alexander Givental. Symplectic geometry of Frobenius structures, May 2003.

[GT11] Alexander Givental and Valentin Tonita. The Hirzebruch–Riemann–Roch theorem in true genus-0 quantum K-theory, June 2011.

[Gué21] Jérémy Guéré. Congruences on K-theoretic Gromov–Witten invariants, February 2021.

[Kaw79] Tetsuro Kawasaki. The Riemann-Roch theorem for complex V-manifolds. *Osaka Journal of Mathematics*, 16(1):151–159, January 1979.

[Lee22] Y.-P. Lee. Quantum K-Theory I: Foundations, January 2022.

[PLS14] Nathan Priddis, Y.-P. Lee, and Mark Shoemaker. A proof of the Landau-Ginzburg/Calabi-Yau correspondence via the crepant transformation conjecture, October 2014.

[PV14] Alexander Polishchuk and Arkady Vaintrob. Matrix factorizations and Cohomological Field Theories, April 2014.

[Toe98] B. Toen. Riemann-Roch Theorems for Deligne-Mumford Stacks. *arXiv:math/9803076*, March 1998.

[Ton14] Valentin Tonita. Twisted orbifold Gromov-Witten invariants. *Nagoya Mathematical Journal*, 213:141–187, March 2014.

[Wen22] Yaoxiong Wen. Difference Equation for Quintic 3-Fold. *Symmetry, Integrability and Geometry: Methods and Applications*, June 2022.