WEAK K.A.M. SOLUTIONS AND MINIMIZING ORBITS OF TWIST MAPS.

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Abstract. For exact symplectic twist maps of the annulus, we etablish a choice of weak K.A.M. solutions $u_c = u(\cdot, c)$ that depend in a Lipschitz-continuous way on the cohomology class c. This allows us to make a bridge between weak K.A.M. theory of Fathi, Aubry-Mather theory for semi-orbits as developped by Bangert and existence of backward invariant pseudo-foliations as seen by Katnelson & Ornstein. We deduce a very precise description of the pseudographs of the weak K.A.M. solutions and many interesting results as

- the Aubry-Mather sets are contained in pseudographs that are vertically ordered by their rotation numbers;
- on every image of a vertical of the annulus, there is at most two points whose negative orbit is minimizing with a given rotation number;
- all the corresponding pseudographs are filled by minimizing semi-orbits and we provide a description of a smaller selection of full pseudographs whose union contains all the minimizing orbits;
- there exists an exact symplectic twist map that has a minimizing negative semi-orbit that is not contained in the pseudograph of a weak K.A.M. solution.

1. Introduction and Main Results.

In the 80s, Aubry and Mather elaborated a deep theory describing the dynamics of an exact symplectic twist diffeomorphism (ESTwD) of the 2-dimensional annulus restricted to the union of its minimizing orbits [1, 40]. Twenty five years later, Katznelson and Ornstein introduced a notion of pseudograph that allowed them to reprove in a geometric way some part of Aubry-Mather theory as well as a theorem of Birkhoff, [35] .

Meanwhile, Fathi made a striking connection between Aubry-Mather theory for Hamiltonian dynamical systems and the PDE approach of Hamilton-Jacobi equation. His weak K.A.M. solutions also define pseudographs. But it seems that an in depth study of weak K.A.M. solutions in the context of exact symplectic twist diffeomorphism has little been done.

Here, we fill that gap and give a precise description of weak K.A.M. solutions for an ESTwD. Also, we revisit a theory for minimizing semi-orbits, developed by

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Bangert [10], in the spirit of Aubry-Mather results on minimizing full orbits. Our approach is based on a method of Lipschitz selection of weak K.A.M. solutions that we elaborate. Among other results, we prove that

- our selection of full pseudographs of weak K.A.M. solutions is a vertically ordered filling of the whole annulus; all the corresponding pseudographs are filled by minimizing semi-orbits and we provide a description of a smaller selection of full pseudographs whose union contains all the minimizing orbits;
- every minimizing semi-orbit has a rotation number[1]. These semi-orbits are vertically arranged by their rotation numbers;
- for a fixed rotation number, every twisted vertical[2] contains at most two minimizing semi-orbit having this rotation number and every vertical contains at least one minimizing semi-orbit having this rotation number;
- ultimately, we provide a detailed description of the pseudographs of the weak K.A.M. solutions, especially in case of a rational rotation number, see Proposition 4.7.

1.1. **Main results.** In this article we study weak K.A.M. solutions and infinite minimizing orbits of Exact Symplectic Twist Diffeomorphisms (ESTwDs in short). Along the way, we recover classical results of Aubry, Mather and Bangert with a more weak K.A.M. approach. The aim of this paper is to be as much self-contained as can be, only the most basic results of Aubry-Mather theory for twist maps are used.

We recall briefly that[3]

- there is natural variational setting for the ESTwDs: a generating function can be associated to a ESTwD as well as an action and minimizing orbits are minimizers of this action.
- a 1-parameter family $(T^c)_{c\in\mathbb{R}}$ of variational operators is defined on the set $C^0(\mathbb{T},\mathbb{R})$ of continuous functions on $\mathbb{T}$ whose fixed points are called weak K.A.M. solutions;
- then, if u is a fixed point of T^c, the associated pseudograph, which is the partial graph $\mathcal{G}(c+u')$ of $c+u'$, is backward invariant by the ESTwD; the corresponding parameter is called the cohomology class; the corresponding full pseudograph $\mathcal{PG}(c+u')$ is an essential curve[4] that is the union of $\mathcal{G}(c+u')$ and some vertical segments.

The following result is reminiscent of Aubry-Mather theory for two-sided minimizing orbits (see [9, 10]): on every twisted vertical, there are at most two points with a fixed rotation number and whose negative orbit is minimizing. If $\theta\in\mathbb{T}$, we set $V_\theta=\{\theta\}\times\mathbb{R}$. In all the article, $\mathbb{Z}_-$ will refer to the set of nonpositive integers. The beginning of the following already appears in Bangert [10]:

Theorem 1.1. *Let f be a C^1 ESTwD of $\mathbb{T}\times\mathbb{R}$. Then every negative minimizing orbit has a rotation number. Let $\theta\in\mathbb{T}$ and $\rho_0\in\mathbb{R}$, then*

- *if $\rho_0\notin\mathbb{Q}$, there exists at most one $(x,p)\in f(V_\theta)$ such that $\big(\pi_1\circ f^i(x,p)\big)_{i\in\mathbb{Z}_-}$ is minimizing with rotation number ρ_0;*

[1]This is already proved in [10].
[2] This refer tho the forward image of a vertical.
[3] Precise definitions will be given later.
[4] This refers to a simple loop that is not isotopic to a point.

- *if* $\rho_0 \in \mathbb{Q}$, *there exists at most two* $(x,p) \in f(V_\theta)$ *such that* $\big(\pi_1 \circ f^i(x,p)\big)_{i\in\mathbb{Z}_-}$ *is minimizing with rotation number* ρ_0.

Now we state the existence of a Lipschitz continuous choice of fixed point u_c of T^c, that generates a continuous and ordered choice of the associated pseudograph.

Theorem 1.2. *Let* f *be a* C^1 *ESTwD of* $\mathbb{T}\times\mathbb{R}$. *Then there exists a continuous map* $u : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ *such that*

(1) $u(0,c) = 0$;
(2) the map $(\theta, c) \mapsto \frac{\partial u_c}{\partial \theta}(\theta)$ *is continuous on its set of definition;*
(3) each $u_c = u(\cdot, c)$ *is a weak K.A.M. solution for the cohomology class* c, *this implies that:*
   - *each* $u_c = u(\cdot, c)$ *is semi-concave (hence derivable almost everywhere)*[5];
   - *each partial graph* $\mathcal{G}(c+u_c')$ *of* $c + \frac{\partial u_c}{\partial \theta}$ *is backward invariant by* f;
   - *the negative orbit* $(f^{-n}(\theta,r))_{n\geq 0}$ *of every point* $(\theta,r) \in \mathcal{G}(c+u_c')$ *is minimizing;*

(4) for all $c \leqslant c'$, *we have* $c + u_c'(\theta) \leqslant c' + u_{c'}'(\theta)$ *at all* $\theta \in \mathbb{T}$ *where both derivatives exist;*
(5) the function u *is locally Lipschitz continuous (and even* 1*–Lipschitz with respect to* c*).*

From Theorems 1.2 and 1.1, we deduce that the negative orbits of the points of $\mathcal{G}(c+u_c')$ have a unique rotation number that we denote by $\rho(c)$.

Next Theorem explains that the associated full pseudographs make a vertically ordered continuous filling of the whole annulus.

Theorem 1.3. *With the notations of Theorem 1.2, we have*

(1) the map $c \mapsto \mathcal{PG}(c+u_c')$ *is continuous for the Hausdorff topology;*
(2) $\displaystyle\bigcup_{c\in\mathbb{R}} \mathcal{PG}(c+u_c') = \mathbb{A}$;
(3) if $\rho(c) < \rho(c')$, *then for all* $(q,p) \in \mathcal{PG}(c+u_c')$ *and* $(q,p') \in \mathcal{PG}(c+u_{c'}')$, *we have* $p < p'$.

As a result of the proof, we will deduce (see Proposition 2.3) that the Aubry-Mather[6] sets are contained in pseudographs that are vertically ordered by their rotation numbers.

The next statement explains that the weak K.A.M. solutions reflect all the richness of negative minimizing semi-orbits.

Theorem 1.4. *With the notations of Theorem 1.2, let* $(\theta_i, r_i)_{i\in\mathbb{Z}_-} \in \mathbb{A}^{\mathbb{Z}_-}$ *be a minimizing negative orbit of* f, *then there exist* $c \in \mathbb{R}$ *and a weak K.A.M. solution* $u_c : \mathbb{T} \to \mathbb{R}$ *at cohomology* c *such that*

$$(\theta_i, r_i)_{i\in\mathbb{Z}_-} \subset \mathcal{PG}(c+u_c'),$$
$$(\theta_i, r_i)_{i<0} \subset \mathcal{G}(c+u_c').$$

Jean-Pierre Marco raised the following question.

QUESTION. If $(\theta_i, r_i)_{i\in\mathbb{Z}_-} \in \mathbb{A}^{\mathbb{Z}_-}$ is a minimizing negative semi-orbit of f, is it necessarily contained in $\overline{\mathcal{G}(c+u_c')}$?

[5] The definition of a semi-concave function is given in subsection 2.3.
[6] The definition of Aubry-Mather set is given in subsection 2.2.

In part A.3, we answer negatively to this question and provide an example where a minimizing negative semi-orbit is not contained in such a set.

Finally, we prove that we can use only a particular subset of $\{\mathcal{G}(c+u_c'); c \in \mathbb{R}\}$ to recover the union of all the pseudographs of weak K.A.M. solutions.

Theorem 1.5. *For every $\rho_0 \in \mathbb{R}$, $\rho^{-1}(\{\rho_0\})$ is a segment $[a,b]$. With the notations of Theorem 1.2, if $c \in [a,b]$ and u is a weak K.A.M. solution at cohomology c, then*

$$\mathcal{G}(c+u') \subset \mathcal{G}(a+u_a') \cup \mathcal{G}(b+u_b'),$$

and by taking closures:

$$\overline{\mathcal{G}(c+u')} \subset \overline{\mathcal{G}(a+u_a')} \cup \overline{\mathcal{G}(b+u_b')}.$$

More precisely,

- *when ρ_0 is irrational, $\rho^{-1}(\{\rho_0\})$ is a single point;*
- *when ρ_0 is rational, the union of two pseudographs contain all pseudographs with this rotation number. Moreover, those two pseudographs intersect along minimizing periodic orbits.*

Moreover, every minimizing semi-orbit $(\theta_i, r_i)_{i\in\mathbb{Z}_-}$ with rotation number ρ_0 is contained in

$$\mathcal{PG}(a+u_a') \cup \mathcal{PG}(b+u_b').$$

In fact, we will provide a more precise description of how the full pseudographs $\mathcal{PG}(a+u_a')$ and $\mathcal{PG}(b+u_b')$ are positioned and of the way $\mathcal{PG}(c+u')$ is built by taking pieces of $\mathcal{PG}(a+u_a')$ and $\mathcal{PG}(b+u_b')$ and gluing them with vertical segments.

Once we have proved that there always exists a continuous choice $u(\theta,c)$ of weak K.A.M. solutions, we wonder when u can be more regular. We recall that an ESTwD is said to be C^0-integrable if the annulus $\mathbb{T}\times\mathbb{R}$ is C^0-foliated by C^0 invariant graphs.

Theorem 1.6. *With the notations of Theorem 1.2, we have equivalence of*

1. *f is C^0-integrable;*
2. *the map u is C^1.*

Moreover, in this case, u is unique and we have[7]

- *the graph of $c+u_c'$ is a leaf of the invariant foliation;*
- *$h_c : \theta \mapsto \theta + \frac{\partial u}{\partial c}(\theta,c)$ is a semi-conjugation between the projected Dynamics $g_c : \theta \mapsto \pi_1 \circ f\big(\theta, c+\frac{\partial u}{\partial \theta}(\theta,c)\big)$ and a rotation R of $\mathbb{T}$, i.e. $h_c\circ g_c = R\circ h_c$.*

We will prove here the implication (2) $\Rightarrow$ (1). The reverse implication is addressed in the companion paper [8].

1.2. **A double pendulum.** Let us illustrate some of our results on a simple example. Let $H : \mathbb{T}\times\mathbb{R}$ be the Hamiltonian defined by $(\theta,p)\mapsto \frac{1}{2}|p|^2+\cos(4\pi\theta)$ and $f=\phi_H^{t_0} : \mathbb{A}\to\mathbb{A}$ be the Hamiltonian flow of H for a small time t_0. Then it is known that for small enough t_0, f is an ESTwD. Moreover, weak K.A.M. solutions for H and f can be proven to be the same.

With that in mind, we obtain that for $\rho_0=0$, then $\rho^{-1}(\{0\})=[-a,a]$, where $a=\int_0^1\sqrt{2-2\cos(4\pi\theta)}d\theta$. The integrated function $\theta\mapsto\sqrt{2-2\cos(4\pi\theta)}$, denoted f^+, corresponds to the upper part of the level set $H^{-1}(\{1\})$. The lower part is the graph of $-f_+$.

[7]See the notation π_1 at the beginning of subsection 2.1.

The unique (up to constants) weak K.A.M. solution u_a, at cohomology a is C^1 and such that $a + u_a'$ is the graph on f_+, in blue in figure 6. Similarly, the unique (up to constants) weak K.A.M. solution u_{-a}, at cohomology $-a$ is C^1 and such that $-a + u_{-a}'$ is the graph on $-f_+$, in red in figure 6. Note that those two graphs intersect at the only minimizing fixed points of f that are of coordinates $(0, 0)$ and $(\frac{1}{2}, 0)$. This fact will be generalized in Proposition 4.7.

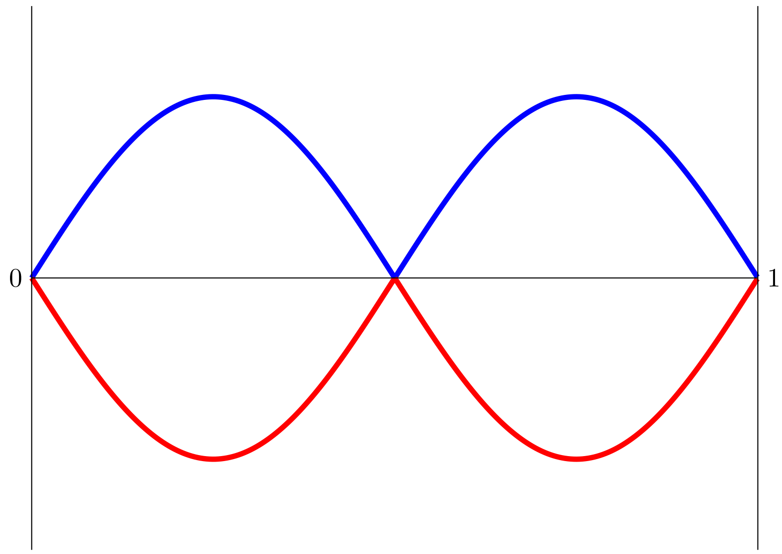

FIGURE 1. The level set $H^{-1}(\{1\})$ is the union of the graphs of $a + u_a'$ in blue and $-a + u_{-a}'$ in red.

Let us now focus at weak K.A.M. solutions at cohomology 0. Their derivative lie in $H^{-1}(\{1\})$. As weak K.A.M. solutions are semi–concave the derivative can only jump downwards and must have vanishing integral. So it looks like the red part in figure 2.

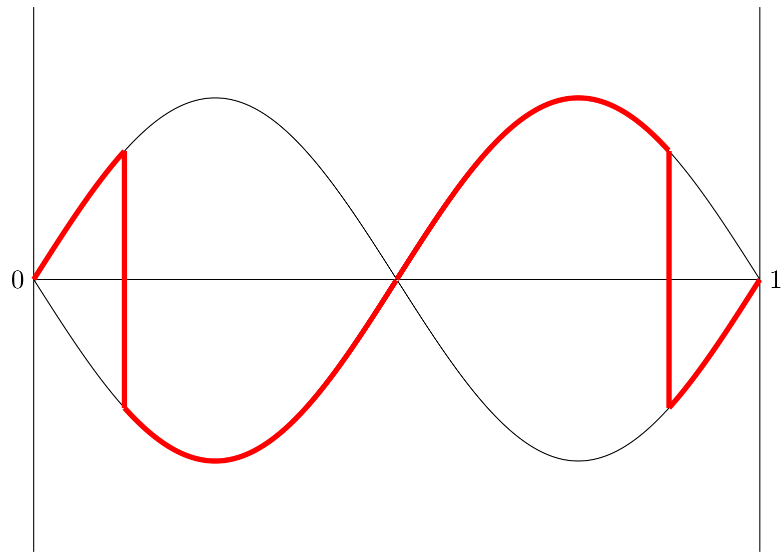

FIGURE 2. The full pseudograph of a weak K.A.M. solution at cohomology 0 in red.

The construction that we propose to prove Theorem 1.2 respects the $\frac{1}{2}$ periodicity of H, hence the weak K.A.M. solution obtained is itself $\frac{1}{2}$-periodic as shown in the next figure 3.

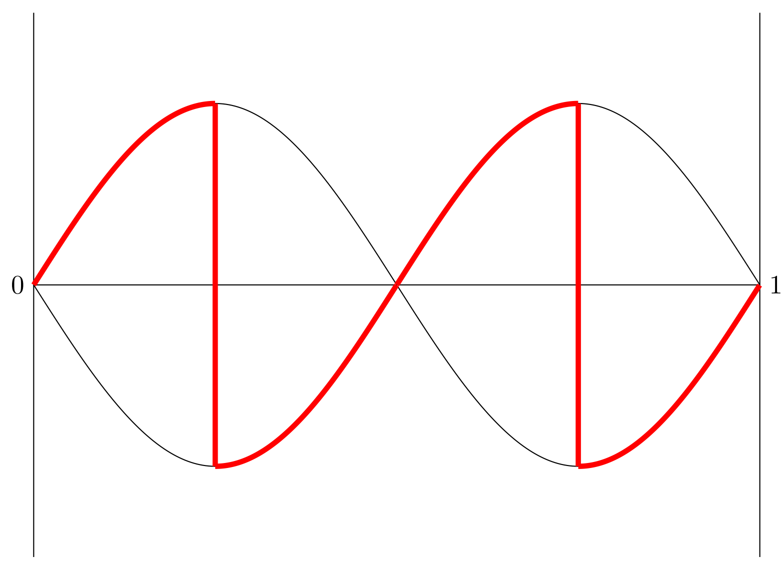

FIGURE 3. The full pseudograph of the weak K.A.M. solution at cohomology 0 selected by the construction in Theorem 1.2 in red.

1.3. **Further comments and related results.**

- The beginning of Theorem 1.1 is already present in Bangert's [10]. Actually, Bangert proves it for a more general setting that contains finite compositions of ESTwDs. However, restricting to an ESTwD allows us to obtain the two final items of Theorem 1.1 that would not hold in Bangert's setting. The same happens for other geometrical results as Proposition 4.8 that holds for ESTwDs but not in Bangert's more general setting.

  From a methodology point of view, Bangert uses Buseman functions, that are functions defined on $\mathbb{R}$. We rather focus on weak K.A.M. solutions that are defined on the circle $\mathbb{T}$ and have been widely studied in recent years. This allows for proofs that we hope more accessible to people familiar with weak K.A.M. theory. Moreover as already explained in the Introduction, this draws a parallel with a variational approach to Katznelson and Orstein's results on backward invariant pseudographs.
- **Theorem 1.2** selects in a continuous way a unique weak K.A.M. solution u_c for every cohomology class $c \in \mathbb{R}$. Let us mention two related results.
  - The recent works in [21] for the autonomous case and in [22] and [46] for the discrete case select a unique solution, called discounted solution, for every cohomology class. We give in Appendix A.2 an example of a C^∞ integrable ESTwD (coming from an autonomous Tonelli Hamiltonian) for which the discounted method doesn't select a transversally continuous weak K.A.M. solution. Hence our method is different from the discounted one.
  - If we have not a unique choice of a weak K.A.M. solution for every cohomology class $c \in H^1(M, \mathbb{R})$, we cannot speak of C^1 regularity with respect to c for the map $c \mapsto \{u_c\}$ that sends c to the whole set of weak K.A.M. solutions of cohomology class c. Observe nevertheless that a kind of local Lipschitz regularity was studied in [37] (for weak K.A.M. solutions for Tonelli Hamiltonians) with no uniqueness.
  - Around the same time this research was done, similar results were established in [49]. However our results are more precise with some respect (for instance the Lipschitz selection of weak K.A.M. solutions). Moreover, our study and description of weak K.A.M. solutions has not been obtained elsewhere.
- **Theorem 1.3** compares the cohomology classes of pseudographs that correspond to distinct rotation numbers. In the setting of Hamiltonian flows with two degrees of freedom, an analogous statement is proved in [19] concerning the cyclic order of rotation and cohomology vectors.
- Similarly to Theorem 1.3, Katznelson & Ornstein provide in [35] a continuous covering of the annulus by full pseudographs.

1.4. **Content of the different sections.** We chose to present our results in an order other than the order of the proofs.

To prove all these results, we will use together Aubry-Mather theory, weak K.A.M. theory in the discrete case. Let us detail what will be in the different sections

- Section 2 contains some reminders on ESTwDs, Aubry-Mather theory, on discrete weak K.A.M. theory, some new results on the weak K.A.M. solutions and the proof of Theorems 1.2 and 1.3;
- the second implication of Theorem 1.6 is proved in section 3;

- results on minimizing sequences and weak K.A.M. solutions are stated and proved in section 4, where we prove Theorems 1.1, 1.4 and 1.5;
- Appendices A contains some examples, Appendix B deals with full pseudographs, Appendix C explains a point that is useful to prove Theorem 1.2.

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2. Aubry-Mather and weak K.A.M. theories for ESTwDs and proof of Theorems 1.2 and 1.3

2.1. **The setting.** The definitions and results that we give here are very classical now. Good references are [29, 31, 42, 43, 12, 39].

Let us introduce some notations

Notations.

- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle and $\mathbb{A} = \mathbb{T}\times\mathbb{R}$ is the annulus ; $\pi : \mathbb{R} \to \mathbb{T}$ is the usual projection;
- the universal covering of the annulus is denoted by $p : \mathbb{R}^2 \to \mathbb{A}$;
- the corresponding projections are $\pi_1 : (\theta, r) \in \mathbb{A} \mapsto \theta \in \mathbb{T}$ and $\pi_2 : (\theta, r) \in \mathbb{A} \mapsto r \in \mathbb{R}$; we denote also the corresponding projections of the universal covering by $\pi_1,\ \pi_2\ : \mathbb{R}^2 \to \mathbb{R}$;
- the Liouville 1-form is defined on $\mathbb{A}$ as being $\lambda = \pi_2 d\pi_1 = rd\theta$; then $\mathbb{A}$ is endowed with the symplectic form $\omega = -d\lambda$.

Let us give the definition of an exact symplectic twist diffeomorphism.

Definition. An exact symplectic twist diffeomorphism (in short ESTwD) $f : \mathbb{A} \to \mathbb{A}$ is a C^1 diffeomorphism such that

- f is isotopic to identity;
- f is exact symplectic, i.e. if $f(\theta, r) = (\Theta, R)$, then the 1-form $Rd\Theta - rd\theta$ is exact;
- f has the twist property i.e. if $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f, for any $\tilde\theta \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde\theta, r) \in \mathbb{R}$ is an increasing C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

A C^2 generating function $S : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ that satisfies the following definition can be associated to any lift F of such an ESTwD f.

Definition. The C^2 function $S : \mathbb{R}^2 \to \mathbb{R}$ is a generating function of the lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of an ESTwD if

- $S(\tilde\theta + 1, \widetilde\Theta + 1) = S(\tilde\theta, \widetilde\Theta)$;
- $\displaystyle\lim_{|\widetilde\Theta-\tilde\theta|\to\infty} \frac{S(\tilde\theta, \widetilde\Theta)}{|\widetilde\Theta - \tilde\theta|} = +\infty$; we say that S is superlinear;
- for every $\tilde\theta_0, \widetilde\Theta_0 \in \mathbb{R}$, the maps $\tilde\theta \mapsto \frac{\partial S}{\partial \widetilde\Theta}(\tilde\theta, \widetilde\Theta_0)$ and $\widetilde\Theta \mapsto \frac{\partial S}{\partial \tilde\theta}(\tilde\theta_0, \widetilde\Theta)$ are decreasing diffeomorphisms of $\mathbb{R}$;

- for $(\tilde\theta, r), (\widetilde\Theta, R) \in \mathbb{R}^2$, we have the following equivalence

$$F(\tilde\theta, r) = (\widetilde\Theta, R) \Leftrightarrow r = -\frac{\partial S}{\partial\tilde\theta}(\tilde\theta, \widetilde\Theta) \quad \text{and} \quad R = \frac{\partial S}{\partial\widetilde\Theta}(\tilde\theta, \widetilde\Theta). \tag{1}$$

Remark. J. Moser proved in [43] that such an ESTwD is the time 1 map of a C^2 1-periodic in time Hamiltonian $H : \mathbb{T}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ that is C^2 convex in the fiber direction[8], i.e. such that

$$\frac{\partial^2 H}{\partial r^2}(\theta, r, t) > 0.$$

Then there exists a relation between the Hamiltonian that was built by J. Moser and the generating function. Indeed, if we denote by (Φ_t) the time t map of the Hamiltonian H that is defined on $\mathbb{R}^2$ and by L the associated Lagrangian that is defined by

$$L(\tilde\theta, v, t) = \max_{r\in\mathbb{R}} \big(rv - H(\tilde\theta, r, t)\big),$$

then we have

- for every $t \in (0, 1]$, Φ_t is an ESTwD and $\Phi_1 = F$;
- there exists a C^1 time-dependent family of C^2 generating functions S_t of Φ_t such $S_1 = S$ and for all $(\tilde\theta, r), (\widetilde\Theta, R) \in \mathbb{R}^2$,

$$\Phi_t(\tilde\theta, r) = (\widetilde\Theta, R) \Rightarrow S_t(\tilde\theta, \widetilde\Theta) = \int_0^t L\Big(\pi_1\circ\Phi_s(\tilde\theta, r), \frac{\partial}{\partial s}\big(\pi_1\circ\Phi_s(\tilde\theta, r)\big), s\Big) ds.$$

In other words, the generating function is also the Lagrangian action.

2.2. Aubry-Mather theory.

Good references for what is in this section are [9], [31] and [5]. Let us recall the definition of some particular invariant sets.

Definition. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of an ESTwD f.

- a subset E of $\mathbb{R}^2$ is well-ordered if it is invariant under the translation $(\tilde\theta, r) \mapsto (\tilde\theta + 1, r)$ and F and if for every $x_1, x_2 \in E$, we have

$$\big[\pi_1(x_1) < \pi_1(x_2)\big] \Rightarrow \big[\pi_1\circ F(x_1) < \pi_1\circ F(x_2)\big];$$

  this notion is independent from the lift of f we use;
- a subset E of $\mathbb{A}$ is well-ordered if $p^{-1}(E)$ is well-ordered;
- an Aubry-Mather set for f is a compact well-ordered set or the lift of such a set;
- a piece of orbit $(\tilde\theta_k, r_k)_{k\in[a,b]}$ for F is minimizing if for every sequence $(\tilde\theta'_k)_{k\in[a,b]}$ with $\tilde\theta_a = \tilde\theta'_a$ and $\tilde\theta_b = \tilde\theta'_b$, it holds

$$\sum_{j=a}^{b-1} S(\tilde\theta_j, \tilde\theta_{j+1}) \leqslant \sum_{j=a}^{b-1} S(\tilde\theta'_j, \tilde\theta'_{j+1});$$

  then we say that $(\tilde\theta_j)_{j\in[a,b]}$ is a minimizing sequence or segment;
- an infinite piece of orbit, or a full orbit for F is minimizing if all its finite subsegments are minimizing;
- an invariant set is said to be minimizing if all the orbits it contains are minimizing.

[8] In fact J. Moser assumed that f is smooth.

The following properties of the well-ordered sets are well-known

(1) a minimizing orbit and its translated orbits by $(\tilde\theta, r) \mapsto (\tilde\theta + 1, r)$ define a well-ordered set;
(2) the closure of a well-ordered set is a well-ordered set;
(3) any well-ordered set E is contained in the (non-invariant) graph of a Lipschitz map $\eta : \mathbb{T} \to \mathbb{R}$; it follows that the map $N = \big(\cdot, \eta(\cdot)\big) : \mathbb{T} \to \mathrm{Graph}(\eta)$ is Lipschitz and so are the maps $\pi_1 \circ f \circ N_{|\pi_1(E)}$ and $\pi_1 \circ f^{-1} \circ N_{|\pi_1(E)}$. This implies that the projected restricted Dynamics $\pi_1 \circ f\big(\cdot, \eta(\cdot)\big)_{|\pi_1(E)}$ to an Aubry-Mather set is the restriction of a biLipschitz orientation preserving circle homeomorphism;
(4) any well-ordered set E in $\mathbb{R}^2$ has a unique rotation number $\rho(E)$ $\big($the one of the circle homeomorphism we mentioned in Point (3)$\big)$, i.e.
$$\forall x \in E, \quad \lim_{k\to\pm\infty} \frac{1}{k}\big(\pi_1 \circ F^k(x) - \pi_1(x)\big) = \rho(E);$$
(5) for every $\alpha \in \mathbb{R}$, there exists a minimizing Aubry-Mather set E such that $\rho(E) = \alpha$;
(6) if α is irrational, there is a unique minimizing Aubry-Mather that is minimal (resp. maximal) for the inclusion; the minimal one is then a Cantor set or a complete graph and the maximal one $\mathcal{M}(\alpha)$ is the union of the minimal one and orbits that are homoclinic to the minimal one;
(7) if α is rational, any Aubry-Mather set that is minimal for the inclusion is a periodic orbit;
(8) any essential invariant curve by an ESTwD is in fact a Lipschitz graph (Birkhoff theorem, see [16], [25] and [32]) and a well-ordered minimizing set.

We will need more precise properties for minimizing orbits.

Definition. Let $a = (a_k)_{k\in I}$ and $b = (b_k)_{k\in I}$ be two finite or infinite sequences of real numbers. Then

- if $k \in I$, we say that a and b cross at k if $a_k = b_k$;
- if $k, k+1 \in I$, we say that a and b cross between k and $k+1$ if
$$(a_k - b_k)(a_{k+1} - b_{k+1}) < 0.$$

Note that concerning the first item, the traditional terminology also imposes that $(a_{k-1} - b_{k-1})(a_{k+1} - b_{k+1}) < 0$ when k is in the interior of I. However, due to the twist condition, this is automatic for projections of orbits of F as soon as $a_k = b_k$ if the two orbits are distinct.

Proposition 2.1. (Aubry fundamental lemma) *If $(a, b, a', b') \in \mathbb{R}^4$ verify $(a-b)(a'-b') < 0$ then*
$$S(a, a') + S(b, b') > S(a, b') + S(a', b).$$

As a consequence, two distinct minimizing sequences cross at most once except possibly at the two endpoints when the sequence is finite.

2.3. **Classical results on weak K.A.M. solutions.** Good references are [11], [12] or [30]. We assume that S is a generating function of a lift F of an ESTwD f.

We define on $C^0(\mathbb{T}, \mathbb{R})$ the so-called negative Lax-Oleinik maps T^c for $c \in \mathbb{R}$ as follows:

if $u\in C^0(\mathbb{T},\mathbb{R})$, we denote by $\tilde u:\mathbb{R}\to\mathbb{R}$ its lift and

$$\forall\tilde\theta\in\mathbb{R},\quad \widetilde{T}^c\tilde u(\tilde\theta)=\inf_{\tilde\theta'\in\mathbb{R}}\big(\tilde u(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c(\tilde\theta'-\tilde\theta)\big).\tag{2}$$

The function $\widetilde{T}^c\tilde u$ is then 1-periodic and the negative Lax-Oleinik operator is defined as the induced map $T^cu:\mathbb{T}\to\mathbb{R}$.

An alternative but equivalent definition is as follows (see also [47] for similar constructions): define the function

$$\forall(\theta,\theta')\in\mathbb{T}\times\mathbb{T},\quad S^c(\theta,\theta')=\inf_{\substack{\pi(\tilde\theta)=\theta\\ \pi(\tilde\theta')=\theta'}}S(\tilde\theta,\tilde\theta')+c(\tilde\theta-\tilde\theta').\tag{3}$$

Then

$$\forall\theta\in\mathbb{T},\quad T^cu(\theta)=\inf_{\theta'\in\mathbb{T}}u(\theta')+S^c(\theta',\theta).$$

Then it can be proved that there exists a unique function $\alpha:\mathbb{R}\to\mathbb{R}$ such that the map $\widehat{T}^c=T^c+\alpha(c)$ that is defined by

$$\widehat{T}^c(u)=T^c(u)+\alpha(c)$$

has at least one fixed point in $C^0(\mathbb{T},\mathbb{R})$, i.e. if $u\in C^0(\mathbb{T},\mathbb{R})$ is such a fixed point, its lift verifies

$$\forall\tilde\theta\in\mathbb{R},\quad \tilde u(\tilde\theta)=\inf_{\tilde\theta'\in\mathbb{R}}\big(\tilde u(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c(\tilde\theta'-\tilde\theta)+\alpha(c)\big).\tag{4}$$

Such a fixed point is called a <u>weak K.A.M. solution</u>. It is not necessarily unique. For example, if u is a weak K.A.M. solution, so is $u+k$ for every $k\in\mathbb{R}$, but there can also be other solutions. We denote by $\mathcal{S}_c$ the set of these weak K.A.M. solutions. There is no link in general for solutions corresponding to distinct c's. We recall

Definition. Let $u:\mathbb{R}\to\mathbb{R}$ be a function and let $K>0$ be a constant. Then u is K-semi-concave if for every x in $\mathbb{R}$, there exists some $p\in\mathbb{R}$ so that:

$$\forall y\in\mathbb{R},\quad u(y)-u(x)-p(y-x)\leqslant\frac{K}{2}(y-x)^2.$$

A function $v:\mathbb{T}\to\mathbb{R}$ is K-semi-concave if its lift $\tilde v:\mathbb{R}\to\mathbb{R}$ is.

A good reference for semi-concave functions is the appendix A of [12] or [17].

Notation. If $u\in C^0(\mathbb{T},\mathbb{R})$ and $c\in\mathbb{R}$, we will denote by $\mathcal{G}(c+u')$ the partial graph of $c+u'$. This is a graph above the set of derivability of u.
When u is semi-concave, we sometimes say that $\mathcal{G}(c+u')$ is a <u>pseudograph</u>.

Let us end with definitions:

Definition. Let $g:\mathbb{T}\to\mathbb{R}$ be a Lipschitz function (hence derivable almost everywhere). We define

$$\forall x\in\mathbb{T},\quad \partial g(x)=\mathrm{co}\big\{(x,p)\in\mathbb{T}\times\mathbb{R},\quad (x,p)\in\overline{\mathcal{G}(g')}\big\}.$$

The notation co stands for the convex hull in the fiber direction. The sets $\partial g(x)$ are non empty, (obviously) convex and compact. They are particular instances of the Clarke subdifferential. This set is a good candidate for a generalized derivative because if g is derivable at x then $\big(x, g'(x)\big) \in \partial g(x)$. Moreover, if $\partial g(x)$ is a singleton, then g is derivable at x. The converse is in general not true, but it is however true for semi-concave functions.

Definition. If $g : \mathbb{T} \to \mathbb{R}$ is Lipschitz and $c \in \mathbb{R}$, we define $\mathcal{PG}(c + g') = \{(0, c) + \partial g(t), \quad t \in \mathbb{T}\}$. If g is semi-concave, we call it the full pseudograph of $c + g'$.

A proof of the following proposition is given in Appendix B.

Proposition 2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of equi-semi-concave functions from $\mathbb{T}$ to $\mathbb{R}$ that converges (uniformly) to a function f (that is hence also semi-concave). Then $\mathcal{PG}(f_n')$ converges to $\mathcal{PG}(f')$ for the Hausdorff distance.*

The following results can be found in the papers that we quoted

(a) the function α is convex and superlinear;
(b) if $u \in C^0(\mathbb{T}, \mathbb{R})$, then $\widehat{T}^c u$ is semi-concave and then differentiable Lebesgue almost everywhere;
(c) the function $\widehat{T}_c u$ is differentiable at x if and only if there is only one y where the minimum is attained in Equality (4); in this case, if u is semi-concave, then it is differentiable at y and we have
$$f\big(y, c + u'(y)\big) = \big(x, c + (\widehat{T}^c u)'(x)\big);$$
if u is a weak K.A.M. solution for $\widehat{T}^c$ that is differentiable at x then $\Big(f^k\big(x, c + u'(x)\big)\Big)_{k\in\mathbb{Z}_-}$ is a minimizing piece of orbit that is contained in $\mathcal{G}(c + u')$;
(d) moreover, for any compact subset K of $\mathbb{R}$, the weak K.A.M. solutions for T^c with $c \in K$ are uniformly semi-concave (i.e. for a fixed constant of semi-concavity) and then uniformly Lipschitz;
(e) if $u \in C^0(\mathbb{T}, \mathbb{R})$ is semi-concave, then
$$f^{-1}\big(\overline{\mathcal{G}(c + (\widehat{T}^c u)')}\big) \subset \mathcal{G}(c + u');$$
if $(x, p) \in \overline{\mathcal{G}(c + (\widetilde{T}^c \tilde{u})')}$ and $\big(y, c + \tilde{u}'(y)\big) = F^{-1}(x, p)$ then
$$\widetilde{T}^c \tilde{u}(x) = \tilde{u}(y) + S(y, x) + c(y - x);$$
if u is a weak K.A.M. solution for $\widehat{T}^c$, then $\mathcal{G}(c + u')$ satisfies
$$f^{-1}\big(\overline{\mathcal{G}(c + u')}\big) \subset \mathcal{G}(c + u')$$
and for every $(\tilde{\theta}_0, r) \in \overline{\mathcal{G}(c + \tilde{u}')}$, then $\big(\pi_1 \circ F^k(\tilde{\theta}_0, r)\big)_{k\in\mathbb{Z}_-} = (\tilde{\theta}_k)_{k\in\mathbb{Z}_-}$ is minimizing and satisfies
$$\forall k < 0, \quad \tilde{u}(\tilde{\theta}_0) - \tilde{u}(\tilde{\theta}_k) = \sum_{i=k}^{-1} S(\tilde{\theta}_i, \tilde{\theta}_{i+1}) + c(\tilde{\theta}_k - \tilde{\theta}_0) + |k|\alpha(c); \tag{5}$$
we will give in Appendix A.1 an example of a backward invariant pseudograph that doesn't correspond to any weak K.A.M. solution;

(f) moreover, if u is a weak K.A.M. solution for $\widehat{T}^c$, then the set

$$\bigcap_{n\in\mathbb{N}} f^{-n}\big(\mathcal{G}(c+u')\big)$$

is a f-invariant minimizing compact well-ordered set to which we can associate a unique rotation number. It results from Mather's theory that this rotation number only depends on c and is equal to $\rho(c)=\alpha'(c)$; because of the convexity of α, observe in particular that α is C^1 and ρ is continuous and non-decreasing;

(g) it then follows from the first (a) and the previous (d) and (f) points that, as in (d), for any compact subset K of $\mathbb{R}$, the weak K.A.M. solutions for T^c with $\rho(c)\in K$ are uniformly semi-concave (i.e. for a fixed constant of semi-concavity) and then uniformly Lipschitz;

(h) reciprocally, if u is a weak K.A.M. solution for $\widehat{T}^c$ and $(\tilde\theta_k)_{k\in\mathbb{Z}_-}$ verifies (5) (we say that $(\tilde\theta_k)_{k\in\mathbb{Z}_-}$ calibrates $\tilde u_c$), then the sequence $(\tilde\theta_k)_{k\in\mathbb{Z}_-}$ is minimizing. Setting for $k\in\mathbb{Z}_-$, $r_k=\frac{\partial S}{\partial\Theta}(\tilde\theta_{k-1},\tilde\theta_k)$, the sequence $(\tilde\theta_k,r_k)_{k\in\mathbb{Z}_-}$ is a piece of orbit of F such that $(\tilde\theta_0,r_0)\in\mathcal{PG}(c+\tilde u')$ and for all $k<0$, $(\tilde\theta_k,r_k)\in\mathcal{G}(c+\tilde u')$;

(i) in the setting of point (f), then for every weak K.A.M. solution for $\widehat{T}^c$, the graph $\mathcal{G}(c+\tilde u')$ contains any minimizing Aubry-Mather set with rotation number $\rho(c)$ that is minimal for the inclusion; we denote the union of these minimal Aubry sets by $\mathcal{M}^*\big(\rho(c)\big)$ (it is the Mather set). We denote $\mathcal{M}\big(\rho(c)\big)=\pi_1\Big(\mathcal{M}^*\big(\rho(c)\big)\Big)$. If $\rho(c)$ is irrational, then two possibilities may occur:
   - either $\mathcal{M}^*\big(\rho(c)\big)$ is an invariant Cantor set and $\mathcal{G}(c+\tilde u')$ is contained in the unstable set of the Cantor set $\mathcal{M}^*\big(\rho(c)\big)$;
   - or $\mathcal{M}^*\big(\rho(c)\big)=\mathcal{G}(c+\tilde u')$ and u is C^1.

   If $\rho(c)$ is rational, then $\mathcal{M}^*\big(\rho(c)\big)$ is the union of some periodic orbits and $\mathcal{G}(c+\tilde u')$ is contained in the union of the unstable sets of these periodic orbits.

We noticed that to any $c\in\mathbb{R}$ there corresponds a unique rotation number $\rho(c)$. But it can happen that distinct numbers c correspond to a same rotation number R. In this case, because $\rho(c)=\alpha'(c)$ is non decreasing (because of point (f)), $\rho^{-1}(R)=[c_1,c_2]$ is an interval. It can be proved that this may happen only for rational R's. This is a result of John Mather [41] (where he also attributes it to Aubry) and Victor Bangert [10]. A simple proof can be found in [13, Proposition 6.5]. We will recover this fact as a byproduct of our study.

Finally, when c corresponds to an irrational rotation number $\rho(c)$, then there exists only one weak K.A.M. solution up to constants. The argument comes from [13] and we will also provide a proof.

2.4. **More results on weak K.A.M. solutions.** We start by establishing that minimizing sequences that calibrate a weak K.A.M. solution admit a rotation number[9].

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[9] Actually, we will see in section 4 that all minimizing sequences calibrate a weak K.A.M. solution and have a rotation number.

Lemma 2.1. *Let $v : \mathbb{T} \to \mathbb{R}$ be a continuous function, $c \in \mathbb{R}$ and $\tilde\theta_1 < \tilde\theta_2$ two real numbers. Assume that $\tilde\theta_1'$ and $\tilde\theta_2'$ verify for $i \in \{1, 2\}$,*

$$\widetilde{T}^c \tilde v(\tilde\theta_i) = \min_{\tilde\theta' \in \mathbb{R}} \big(\tilde v(\tilde\theta') + S(\tilde\theta', \tilde\theta_i) + c(\tilde\theta' - \tilde\theta)\big) = \tilde v(\tilde\theta_i') + S(\tilde\theta_i', \tilde\theta_i) + c(\tilde\theta_i' - \tilde\theta_i),$$

then $\theta_1' \leqslant \theta_2'$.

Assume moreover that v is semi-concave, then the previous inequality is strict.

Proof. Let us argue by contradiction, then by Proposition 2.1 the following holds:

$$\tilde v(\tilde\theta_1') + S(\tilde\theta_1', \tilde\theta_1) + c(\tilde\theta_1' - \tilde\theta_1) + \tilde v(\tilde\theta_2') + S(\tilde\theta_2', \tilde\theta_2) + c(\tilde\theta_2' - \tilde\theta_2) >$$
$$> \tilde v(\tilde\theta_2') + S(\tilde\theta_2', \tilde\theta_1) + c(\tilde\theta_2' - \tilde\theta_1) + \tilde v(\tilde\theta_1') + S(\tilde\theta_1', \tilde\theta_2) + c(\tilde\theta_1' - \tilde\theta_2).$$

We infer that one of the two inequalities

$$\tilde v(\tilde\theta_1') + S(\tilde\theta_1', \tilde\theta_1) + c(\tilde\theta_1' - \tilde\theta_1) > \tilde v(\tilde\theta_2') + S(\tilde\theta_2', \tilde\theta_1) + c(\tilde\theta_2' - \tilde\theta_1),$$
$$\tilde v(\tilde\theta_2') + S(\tilde\theta_2', \tilde\theta_2) + c(\tilde\theta_2' - \tilde\theta_2) > \tilde v(\tilde\theta_1') + S(\tilde\theta_1', \tilde\theta_2) + c(\tilde\theta_1' - \tilde\theta_2),$$

is valid that is a contradiction.

Let us establish the last point. If v is semi-concave, by properties of the Lax-Oleinik semigroup (c), $\tilde v$ is derivable at $\tilde\theta_1'$ and $\tilde\theta_2'$ and $\tilde\theta_i = \pi_1 \circ F\big(\tilde\theta_i', c + \tilde v'(\tilde\theta_i')\big)$ for $i \in \{1, 2\}$ therefore $\tilde\theta_1' \neq \tilde\theta_2'$ as $\tilde\theta_1 \neq \tilde\theta_2$.

□

Lemma 2.2. *Let u be a weak K.A.M. solution for $\widehat{T}^c$. Let $(\theta_0, r) \in \overline{\mathcal{G}(c + u')}$, and let $\tilde\theta_0 \in \mathbb{R}$ be a lift of θ_0. Let $(\tilde\theta_k, r_k)_{k \in \mathbb{Z}_-} = \big(F^k(\tilde\theta_0, r)\big)_{k \in \mathbb{Z}_-}$. Then*

$$\lim_{k \to -\infty} \frac{\tilde\theta_k}{k} = \rho(c).$$

Proof. Let $x_0 \in \mathcal{M}\big(\rho(c)\big)$ such that $x_0 \leqslant \tilde\theta_0 \leqslant x_0 + 1$ and $(x_k)_{k \in \mathbb{Z}}$ the associated minimizing sequence. By successive applications of the previous lemma, it follows that $x_k \leqslant \tilde\theta_k \leqslant x_k + 1$ for all $k \leqslant 0$. The result follows as

$$\lim_{k \to -\infty} \frac{x_k}{k} = \rho(c).$$

□

Proposition 2.3. *Let u_1, u_2 be two weak K.A.M. solutions corresponding to T^{c_1}, T^{c_2}, such that $\rho(c_1) < \rho(c_2)$. Then we have*

- *$c_1 < c_2$;*
- *for any $t \in \mathbb{T}$, if $(t, p_1) \in \partial u_1(t)$ and $(t, p_2) \in \partial u_2(t)$ then $c_1 + p_1 < c_2 + p_2$;*
- *in particular, at every point of differentiability t of u_1 and u_2: $c_1 + u_1'(t) < c_2 + u_2'(t)$.*

Proof. Let $\tilde u_1$ and $\tilde u_2$ be the lifts of u_1 and u_2. We introduce the notation $v(t) = \tilde u_2(t) - \tilde u_1(t) + (c_2 - c_1)t$. Then v is Lipschitz and thus Lebesgue everywhere differentiable and equal to a primitive of its derivative. Let us assume by contradiction that there exist $(x, c_1 + p_1) \in \overline{\mathcal{G}(c_1 + u_1')}$ and $(x, c_2 + p_2) \in \overline{\mathcal{G}(c_2 + u_2')}$

$$c_2 + p_2 \leqslant c_1 + p_1. \tag{6}$$

As $\rho(c_1) \neq \rho(c_2)$, the two graphs correspond to distinct rotation numbers. Thanks to (e) their closures have no intersections. The inequality (6) is then strict.

We introduce the notation $(x^1, y^1) = (x, c_1 + p_1)$ and $(x^2, y^2) = (x, c_2 + p_2)$. Then the orbit of (x^i, y^i) is denoted by $(x^i_k, y^i_k)_{k\in\mathbb{Z}}$. We know that the negative orbits $(x^i_k, y^i_k)_{k\in\mathbb{Z}_-}$, that are contained in the corresponding graphs, are minimizing. Hence the sequences $(x^i_k)_{k\in\mathbb{Z}_-}$ are minimizing. By Aubry's fundamental lemma, we know that they can cross at most once (hence only at x). But we have

- because of the twist condition, as $x^1_0 = x^2_0$ and $y^1_0 > y^2_0$, then $x^1_{-1} < x^2_{-1}$;
- as $\rho(c_1) < \rho(c_2)$, and thus for k close enough to $-\infty$, we have: $x^1_k > x^2_k$.

Finally we find two crossings for two minimizing sequences, a contradiction.

We have in particular for any point t of derivability of u_1 and u_2

$$c_1 + u_1'(t) < c_2 + u_2'(t).$$

Integrating this inequality, we deduce that $c_1 < c_2$.

Finally, for any $t \in \mathbb{T}$, as for all $(t, p_1) \in \overline{\mathcal{G}(c_1 + u_1')}$ and $(t, p_2) \in \overline{\mathcal{G}(c_2 + u_2')}$

$$c_2 + p_2 > c_1 + p_1, \tag{7}$$

taking convex hulls, we get the result.

□

In particular, we obtain the following consequence.

Corollary 2.1. *With the same notation as in Proposition 2.3, assume that $c_1 < c_2$ are such that at least one of $\rho^{-1}(\{\rho(c_1)\})$ and $\rho^{-1}(\{\rho(c_2)\})$ is a singleton. Then the function $t \in \mathbb{R} \mapsto \tilde{u}_{c_2}(t) - \tilde{u}_{c_1}(t) + t(c_2 - c_1)$ is strictly increasing.*

Remark. A consequence of Proposition 2.3 is that the pseudographs corresponding to the weak K.A.M. solutions having distinct rotation numbers are vertically ordered with the same order as the one between the rotation numbers.

Now we recall some results that are contained in [30] (see especially Theorem 9.3).

Notation. If $c \in \mathbb{R}$ and $n \geqslant 1$, we denote by $\mathcal{S}^c_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the function that is defined by

$$\mathcal{S}^c_n(\tilde{\theta}, \widetilde{\Theta}) = \inf_{\substack{\tilde{\theta}_0 = \tilde{\theta} \\ \tilde{\theta}_n - \widetilde{\Theta} \in \mathbb{Z}}} \left\{ \sum_{i=1}^{n} \big(S(\tilde{\theta}_{i-1}, \tilde{\theta}_i) + c(\tilde{\theta}_{i-1} - \tilde{\theta}_i) \big) \right\}.$$

Observe that $\mathcal{S}^c_n$ is $\mathbb{Z}^2$-periodic.

(1) If R is any rotation number, for any $c \in \rho^{-1}(R)$ and any weak K.A.M. solution u for $\widehat{T}^c$, the set of invariant Borel probability measures with support in $\mathcal{G}(c + u')$ is independent from $c \in \rho^{-1}(R)$ and u. Those measures are called Mather measures and the union of the supports of these measures is called the Mather set for R and corresponds to $\mathcal{M}^*(R)$; its projection on $\mathbb{T}$ is denoted $\mathcal{M}(R)$ whose lift to $\mathbb{R}$ is $\widetilde{\mathcal{M}}(R)$;

(2) We say that a function u defined on a part A of $\mathbb{T}$ is c-dominated if, denoting by $\widetilde{A}$ the lift of A to $\mathbb{R}$, and $\tilde{u}$ a lift of u, we have

$$\forall \tilde{\theta}, \tilde{\theta}' \in \widetilde{A}, \forall n \geqslant 1, \quad \tilde{u}(\tilde{\theta}) - \tilde{u}(\tilde{\theta}') \leqslant \mathcal{S}^c_n(\tilde{\theta}', \tilde{\theta}) + n\alpha(c);$$

a weak K.A.M. solution for $\widehat{T}^c$ is always c-dominated; if $A=\mathbb{T}$ a function $u:\mathbb{T}\to\mathbb{R}$ is c-dominated if and only if

$$\forall\tilde\theta,\tilde\theta'\in\mathbb{R},\quad \tilde u(\tilde\theta)-\tilde u(\tilde\theta')\leqslant S(\tilde\theta',\tilde\theta)+c(\tilde\theta'-\tilde\theta)+\alpha(c);$$

(3) If $u:\mathcal{M}\big(\rho(c)\big)\to\mathbb{R}$ is dominated, then there exists only one extension U of u to $\mathbb{T}$ that is a weak K.A.M. solution for $\widehat{T}^c$. This function is given by

$$\forall x\in\mathbb{T},\quad U(x)=\inf_{\substack{\pi(\tilde\theta)\in\mathcal{M}(\rho(c))\\ \pi(\tilde\theta')=x}}\tilde u(\tilde\theta)+\mathcal{S}^c(\tilde\theta,\tilde\theta'),$$

where

$$\mathcal{S}^c(\tilde\theta,\widetilde\Theta)=\inf_{n\in\mathbb{N}}\big(\mathcal{S}^c_n(\tilde\theta,\widetilde\Theta)+n\alpha(c)\big).$$

As we have not found it exactly stated in this way in the literature, we provide a sketch of proof for the reader's convenience in appendix C.

2.5. **Proof of Theorem 1.2.** When there is no ambiguity in the notations, we will put $\sim$ signs to signify that we consider lifts of functions defined on $\mathbb{T}$. We will need the following lemma.

Lemma 2.3. *Let (c_n) be a sequence of real numbers convergent to c and let (u_{c_n}) be a sequence of functions uniformly convergent to v such that u_{c_n} is a weak K.A.M. solution for $\widehat{T}^{c_n}$. Then v is a weak K.A.M. solution for $\widehat{T}^c$.*

Proof. We know from Equation (4) that

$$\tilde u_{c_n}(\tilde\theta)=\inf_{\tilde\theta'\in\mathbb{R}}\big(\tilde u_{c_n}(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c_n(\tilde\theta'-\tilde\theta)+\alpha(c_n)\big).$$

Because of the superlinearity of S and the fact that the u_{c_n} and c_n are uniformly bounded, there exists a fixed compact set I in $\mathbb{R}$ such that for every n, we have

$$\tilde u_{c_n}(\tilde\theta)=\inf_{\tilde\theta'\in I}\big(\tilde u_{c_n}(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c_n(\tilde\theta'-\tilde\theta)+\alpha(c_n)\big).$$

We deduce from the uniform convergence of (u_{c_n}) to v that

$$\tilde v(\tilde\theta)=\inf_{\tilde\theta'\in I}\big(\tilde v(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c(\tilde\theta'-\tilde\theta)+\alpha(c)\big).$$

As we could do the same proof for I as large as wanted, we have in fact

$$\tilde v(\tilde\theta)=\inf_{\tilde\theta'\in\mathbb{R}}\big(\tilde v(\tilde\theta')+S(\tilde\theta',\tilde\theta)+c(\tilde\theta'-\tilde\theta)+\alpha(c)\big).\tag{8}$$

□

Let us now prove Theorem 1.2. We will start with a fundamental uniqueness property of weak K.A.M. solutions for a wide class of cohomology classes.

Proposition 2.4. *Let $R\subset\mathbb{R}$ be a real number and set $[a_1,a_2]=\rho^{-1}(\{R\})$. Then, up to constants, there exists a unique weak K.A.M. solution for $\widehat{T}^{a_1}$ (resp. $\widehat{T}^{a_2}$).*

Proof. Let us prove the result for a_2, the proof being similar for a_1. Let $(c_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to a_2, such that $\big(\rho(c_n)\big)_{n\in\mathbb{N}}$ is decreasing (and converges to R). For all $n\in\mathbb{N}$, let $u_n:\mathbb{T}\to\mathbb{R}$ be a weak K.A.M. solution at cohomology c_n such that $u_n(0)=0$.

Then by Proposition 2.3, $(c_n+u'_n)_{n\in\mathbb{N}}$ is a decreasing sequence and then $\big(\tilde v_n : \tilde\theta \in [0,1] \mapsto c_n\tilde\theta+\tilde u_n(\tilde\theta)\big)_{n\in\mathbb{N}}$ is also a decreasing sequence, thus convergent and even uniformly convergent by the Ascoli Theorem. By Lemma 2.3, $\tilde u_{a_2}(\tilde\theta) = \lim_{n\to\infty} \tilde v_n(\tilde\theta) - c_n\tilde\theta$ defines a weak K.A.M. solution for T^{a_2} such that $u_{a_2}(0) = 0$ and $u'_{a_2} = \lim_{n\to\infty} u'_n$ almost everywhere.

Let us assume that v is another weak K.A.M. solution for $\widehat{T}^{a_2}$ that vanishes at 0. Because of Proposition 2.3, we have for all $n \in \mathbb{N}$,

$$c_n + u'_n > a_2 + v'.$$

Taking the limit in these inequalities and using the definition of u_{a_2}, we deduce that $v' \leqslant u'_{a_2}$. As $0 = \int_{\mathbb{T}} v' = \int_{\mathbb{T}} u'_{a_2}$, we deduce that $u'_{a_2} = v'$ Lebesgue almost everywhere and then $v = u_{a_2}$.

□

Notation. We use the notation $\mathcal{I} \subset \mathbb{R}$ is the set of $c \in \mathbb{R}$ such that $\rho^{-1}(\{\rho(c)\}) = \{c\}$. This is the set of cohomology classes where ρ is strictly increasing[10].

It is easily verified that the closure $\overline{\mathcal{I}}$ consists in the union of all the extremities $\{a_1, a_2\}$ of intervals $[a_1, a_2] = \rho^{-1}(\{R\})$ for $R \in \mathbb{R}$. This justifies the next:

Notation. When $c \in \overline{\mathcal{I}}$, we will denote by u_c the (unique) solution such that $u_c(0) = 0$.

Let us prove that any extension $c \mapsto u_c$ that maps c on a weak K.A.M. solution for $\widehat{T}^c$ that vanishes at 0 is continuous at every $c \in \overline{\mathcal{I}}$. Let us consider a sequence $(c_n)_{n\in\mathbb{N}}$ that converges to $c \in \overline{\mathcal{I}}$. Then the sequence $(u_{c_n})_{n\in\mathbb{N}}$ is made of equi semi-concave and then equiLipschitz functions. As all functions vanish at 0, the sequence is also equi-bounded. Because of the Ascoli Theorem it is relatively compact for the uniform convergence. Because of Lemma 2.3, all its accumulation points are weak K.A.M. solutions for $\widehat{T}^c$ that vanish at 0. It follows that the sequence uniformly converges to the unique such function u_c.
This gives the wanted continuity at every point of $\overline{\mathcal{I}}$.

Building a function u, the only problem of continuity we have now to consider is at the points of the set $\mathbb{R} \setminus \overline{\mathcal{I}}$.
Observe that if we find a continuous extension to $\mathbb{T}\times\mathbb{R}$ such that every u_c is a weak K.A.M. solution for $\widehat{T}^c$, replacing u_c by $u_c - u_c(0)$, we obtain an extension as wanted.

Let us now assume that R is a real number such that $\rho^{-1}(\{R\}) = [a_1, a_2]$, with $a_1 < a_2$[11]. Because u_{a_1} and u_{a_2} are weak K.A.M. solutions, they are dominated and we have

$$\forall x, y \in \mathbb{R}, \forall n \geqslant 1, \quad \tilde u_{a_i}(x) - \tilde u_{a_i}(y) \leqslant \mathcal{S}_n^{a_i}(x,y) + n\alpha(a_i).$$

Let $c = \lambda a_1 + (1-\lambda)a_2 \in [a_1, a_2]$. We use the notation $v_c = \lambda u_{a_1} + (1-\lambda)u_{a_2}$. Observe that $\alpha(c) = \lambda\alpha(a_1) + (1-\lambda)\alpha(a_2)$ because $\alpha' = R$ is constant on $[a_1, a_2]$. Then we have

$$\forall x, y \in \widetilde{\mathcal{M}}(R), \quad \tilde v_c(y) - \tilde v_c(x) \leqslant \mathcal{S}_n^c(x,y) + n\alpha(c);$$

[10]Let us remind the reader that $\mathcal{I}$ contains $\mathbb{R}\setminus\mathbb{Q}$. But we will not use this fact.

[11]Again, R is necessarily rational but we do not need this fact.

i.e. v_c is c-dominated on $\mathcal{M}\big(R\big)$. We deduce from Point (3) of subsection 2.4 that there exists only one extension u_c of v_c restricted to $\mathcal{M}\big(R\big)$ that is a weak K.A.M. solution for $\widehat{T}^c$.

Let us prove that $c \in [a_1, a_2] \mapsto u_c$ is continuous. By definition of u_c, the map $c \mapsto u_{c|\mathcal{M}(R)}$ is continuous. Let us now consider a sequence (c_n) in $[a_1, a_2]$ that converges to some $c \in [a_1, a_2]$. By the Ascoli Theorem the set $\{u_{c_n}, n \in \mathbb{N}\}$ is relatively compact for the uniform convergence distance. Let U be a limit point of the sequence (u_{c_n}). By Lemma 2.3, we know that U is a weak K.A.M. solution for $\widehat{T}^c$. Moreover, we have $U_{|\mathcal{M}(R)} = u_{c|\mathcal{M}(R)}$. Using Point (3) of subsection 2.4, we deduce that $U = u_c$ and the wanted continuity.

In appendix A of [12], it is proved that the uniform convergence of a sequence of equi-semi-concave functions implies their convergence C^1 in some sense. This implies for the function u given in Theorem that if $c_n \to c$, if $\theta_n \to \theta$ and if u_{c_n} is derivable at θ_n and u_c at θ, we have

$$\lim_{n\to\infty} \frac{\partial u}{\partial \theta}(\theta_n, c_n) = \frac{\partial u}{\partial \theta}(\theta, c),$$

i.e. that the map $(\theta, c) \mapsto \frac{\partial u_c}{\partial \theta}(\theta)$ is continuous.

We end this section with the proof of points (4) and (5) of Theorem 1.2. Let us state a lemma:

Lemma 2.4. *Let $c_1 < c_2$ be two real numbers. Let $v_1, v_2 : \mathbb{T} \to \mathbb{R}$ be continuous functions.*

If the function $\theta \mapsto (\tilde{v}_2 - \tilde{v}_1)(\tilde{\theta}) + (c_2 - c_1)\tilde{\theta}$ is non-decreasing, then so is the function $\tilde{\theta} \mapsto (\widetilde{T}^{c_2}\tilde{v}_2 - \widetilde{T}^{c_1}\tilde{v}_1)(\tilde{\theta}) + (c_2 - c_1)\tilde{\theta}$.

Proof. Let $\tilde{\theta} < \tilde{\theta}'$ be two real numbers. By definition of the operators $\widetilde{T}_{c_i}$ there exist $\tilde{\theta}'_2$ and $\tilde{\theta}_1$ such that

$$\widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}') = \tilde{v}_2(\tilde{\theta}'_2) + S(\tilde{\theta}'_2, \tilde{\theta}') + c_2(\tilde{\theta}'_2 - \tilde{\theta}'),$$
$$\widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}) = \tilde{v}_1(\tilde{\theta}_1) + S(\tilde{\theta}_1, \tilde{\theta}) + c_1(\tilde{\theta}_1 - \tilde{\theta}).$$

There are two cases to consider:

- if $\tilde{\theta}'_2 < \tilde{\theta}_1$ we use Aubry's fundamental lemma to obtain

$$\begin{aligned}\widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}') + \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}) &= \tilde{v}_2(\tilde{\theta}'_2) + S(\tilde{\theta}'_2, \tilde{\theta}') + c_2(\tilde{\theta}'_2 - \tilde{\theta}') + \tilde{v}_1(\tilde{\theta}_1) + S(\tilde{\theta}_1, \tilde{\theta}) + c_1(\tilde{\theta}_1 - \tilde{\theta})\\ &> \tilde{v}_2(\tilde{\theta}'_2) + S(\tilde{\theta}'_2, \tilde{\theta}) + c_2(\tilde{\theta}'_2 - \tilde{\theta}') + \tilde{v}_1(\tilde{\theta}_1) + S(\tilde{\theta}_1, \tilde{\theta}') + c_1(\tilde{\theta}_1 - \tilde{\theta})\\ &\geqslant \widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}) + \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}') + (c_2 - c_1)(\tilde{\theta} - \tilde{\theta}').\end{aligned}$$

  After rearranging the terms, this reads

$$\widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}') - \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}') + (c_2 - c_1)\tilde{\theta}' > \widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}) - \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}) + (c_2 - c_1)\tilde{\theta}.$$

- if $\tilde{\theta}'_2 \geqslant \tilde{\theta}_1$ we use the hypothesis on $\tilde{\theta} \mapsto (\tilde{v}_2 - \tilde{v}_1)(\tilde{\theta}) + (c_2 - c_1)\tilde{\theta}$ to show that $\tilde{v}_2(\tilde{\theta}'_2) + \tilde{v}_1(\tilde{\theta}_1) \geqslant \tilde{v}_2(\tilde{\theta}_1) + \tilde{v}_1(\tilde{\theta}'_2) + (c_2 - c_1)(\tilde{\theta}_1 - \tilde{\theta}'_2)$ and then

$$\begin{aligned}\widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}') + \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}) &= \tilde{v}_2(\tilde{\theta}'_2) + S(\tilde{\theta}'_2, \tilde{\theta}') + c_2(\tilde{\theta}'_2 - \tilde{\theta}') + \tilde{v}_1(\tilde{\theta}_1) + S(\tilde{\theta}_1, \tilde{\theta}) + c_1(\tilde{\theta}_1 - \tilde{\theta})\\ &\geqslant \tilde{v}_2(\tilde{\theta}_1) + S(\tilde{\theta}'_2, \tilde{\theta}') + c_2(\tilde{\theta}_1 - \tilde{\theta}') + \tilde{v}_1(\tilde{\theta}'_2) + S(\tilde{\theta}_1, \tilde{\theta}) + c_1(\tilde{\theta}'_2 - \tilde{\theta})\\ &\geqslant \widetilde{T}^{c_2}\tilde{v}_2(\tilde{\theta}) + \widetilde{T}^{c_1}\tilde{v}_1(\tilde{\theta}') + (c_2 - c_1)(\tilde{\theta} - \tilde{\theta}').\end{aligned}$$

As before, this gives the result after rearranging terms.

□

Let us now conclude that the function u constructed verifies the requirements of (4) and (5). Let R be a real number and let us, as previously, introduce the notations $\rho^{-1}(R) = [a_1, a_2]$. As seen before, we denote by u_{a_1} and u_{a_2} the unique weak K.A.M. solutions for T^{a_1} and T^{a_2} vanishing at 0. We have proven that $\tilde\theta \mapsto (\tilde u_{a_2} - \tilde u_{a_1})(\tilde\theta) + (a_2 - a_1)\tilde\theta$ is non-decreasing.

Let $c = \lambda a_1 + (1-\lambda)a_2 \in [a_1, a_2]$. We use again the notation $v_c = \lambda u_{a_1} + (1-\lambda)u_{a_2}$ and recall that $\alpha(c) = \lambda\alpha(a_1) + (1-\lambda)\alpha(a_2)$ because $\alpha' = R$ is constant on $[a_1, a_2]$. It follows that $\tilde v_c$ is c dominated and that if $a_1 \leqslant c < c' \leqslant a_2$, the function $\tilde\theta \mapsto (\tilde v_{c'} - \tilde v_c)(\tilde\theta) + (c' - c)\tilde\theta$ is non decreasing.

Finally, as v_c is c-dominated, it can be proved that the function u_c constructed verifies

$$\forall \tilde\theta \in \mathbb{R}, \quad \tilde u_c(\tilde\theta) = \lim_{n\to+\infty} (\widetilde{T}^c)^n \tilde v_c(\tilde\theta) + n\alpha(c),$$

the limit being that of an increasing sequence. Hence the fact that $\tilde\theta \mapsto (\tilde u_{c'} - \tilde u_c)(\tilde\theta) + (c'-c)\tilde\theta$ is non decreasing follows from successive applications of the previous lemma.

To prove (5), if $c' \leqslant c$ and $\tilde\theta \in [0,1]$ then

$$0 = (\tilde u_{c'} - \tilde u_c)(0) \leqslant (\tilde u_{c'} - \tilde u_c)(\tilde\theta) + (c' - c)\tilde\theta \leqslant (\tilde u_{c'} - \tilde u_c)(1) + (c' - c) = c' - c.$$

It follows that

$$(c - c')\tilde\theta \leqslant (\tilde u_{c'} - \tilde u_c)(\tilde\theta) \leqslant (c' - c)(1 - \tilde\theta).$$

Hence $\tilde u$ is uniformly 1-Lipschitz in c and the result follows.

2.6. **More on the constructed function: proof of Theorem 1.3.** In this paragraph, $u : \mathbb{A} \to \mathbb{R}$ is any function given by Theorem 1.2 meaning that

- u is continuous;
- $u(0, c) = 0$;
- each $u_c = u(\cdot, c)$ is a weak K.A.M. solution for the cohomology class c.

We aim to give the range of the map $(\theta, c) \mapsto \big(\theta, c + \frac{\partial u}{\partial\theta}(\theta, c)\big)$. The following proposition asserts that any ESTwD is weakly integrable in the sense that $\mathbb{A}$ is covered by Lipschitz circles arising from weak K.A.M. solutions.

Recall that $\mathcal{PG}(c + u_c') = \{(0, c) + \partial u_c(t), \quad t \in \mathbb{T}\}$ is the full pseudograph of $c + u_c'$.

Proposition 2.5. *The following holds:*

$$\bigcup_{c\in\mathbb{R}} \mathcal{PG}(c + u_c') = \bigcup_{\substack{t\in\mathbb{T}\\ c\in\mathbb{R}}} (0, c) + \partial u_c(t) = \mathbb{A}. \tag{9}$$

Let us define two auxiliary functions with values in $\mathbb{R} \cup \{+\infty, -\infty\}$:

$$\forall \theta \in \mathbb{T}, \quad \eta_+(\theta) = \sup\big\{p \in \mathbb{R}; \quad \exists c \in \mathbb{R}; \quad (\theta, p) \in \overline{\mathcal{G}(c + u_c')}\big\},$$

and

$$\forall \theta \in \mathbb{T}, \quad \eta_-(\theta) = \inf\big\{p \in \mathbb{R}; \quad \exists c \in \mathbb{R}; \quad (\theta, p) \in \overline{\mathcal{G}(c + u_c')}\big\}.$$

Finally define $\mathbb{A}_0 = \big\{(\theta, c) \in \mathbb{A}, \quad \eta_-(\theta) < c < \eta_+(\theta)\big\}$.

The following lemma is proved in Appendix B.2.

Lemma 2.5. *For all $c \in \mathbb{R}$, $\mathcal{PG}(c+u_c')$ is a Lipschitz one dimensional compact manifold, hence it is an essential circle.*

It follows that the set $\mathbb{A}_0$ is open and connected (we will see at the end that it is in fact $\mathbb{A}$). Indeed, by Jordan's theorem and Proposition 2.3, for $c < c'$ such that $\rho(c) < \rho(c')$, the set $\big\{(t,p) \in \mathbb{A}, \quad c + \partial u_c(t) < p < c' + \partial u_{c'}(t)\big\}$ is open and connected. Now $\mathbb{A}_0$ is an increasing union of such sets.

Proposition 2.6. *The following equality holds:*

$$\mathbb{A}_0 = \bigcup_{c\in\mathbb{R}} \mathcal{PG}(c+u_c').$$

Proof. We denote by $\mathcal{B} = \bigcup_{c\in\mathbb{R}} \mathcal{PG}(c+u_c')$. Observe that $\mathcal{B} \subset \mathbb{A}_0$.

First we prove that $\mathcal{B}$ is closed in $\mathbb{A}_0$. Let $(t_n,p_n) \in \mathcal{PG}(c_n+u_{c_n}')$ be a sequence converging to $(t,p) \in \mathbb{A}_0$. By definition of $\mathbb{A}_0$, there are $C_0 < C_1$ and $(t,P_0) \in \mathcal{PG}(C_0+u_{C_0}')$, $(t,P_1) \in \mathcal{PG}(C_1+u_{C_1}')$ such that such that $P_0 < p < P_1$. Now let $c_- < C_0 < C_1 < c_+$ be such that $\rho(c_-)$, $\rho(c_+)$ are irrational and

$$\rho(c_-) < \rho(C_0) < \rho(C_1) < \rho(c_+).$$

As the pseudographs are vertically ordered (Proposition 2.3), (t,p) is trapped in the open sub-annulus between $\mathcal{PG}(c_-+u_{c_-}')$ and $\mathcal{PG}(c_++u_{c_+}')$. It follows that for n large enough, so is (t_n,p_n). Hence $\mathcal{PG}(c_n+u_{c_n}')$ is a full pseudograph that contains a point strictly between $\mathcal{PG}(c_-+u_{c_-}')$and $\mathcal{PG}(c_++u_{c_+}')$. Proposition 2.3 implies that $\rho(c_-) \leqslant \rho(c_n) \leqslant \rho(c_+)$. As $\rho(c_-)$, $\rho(c_+)$ are irrational, there is a unique weak K.A.M. solution for these rotation numbers and then $\rho(c_n) \notin \{\rho(c_-), \rho(c_+)\}$. We deduce that $\rho(c_-) < \rho(c_n) < \rho(c_+)$ and then that $c_- < c_n < c_+$.

Up to extracting, we may assume that $c_n \to c_\infty$ and by continuity of the pseudographs with respect to c (for the Hausdorff distance, see Proposition 2.2), it follows that $(t,p) \in \mathcal{PG}(c_\infty+u_{c_\infty}') \subset \mathbb{A}_0$.

Next we prove that $\mathcal{B} = \mathbb{A}_0$. We argue by contradiction, by the first part, if this is not the case, there is an open ball $B = (\theta_0,\theta_1)\times(r_0,r_1)$ such that $\overline{B} \subset \mathbb{A}_0 \setminus \mathcal{B}$.

We say that a topological essential circle $\mathcal{C}$ is above B if B is included in the lower connected component of $\mathbb{A}\setminus\mathcal{C}$[12] and $\mathcal{C}$ is under B if B is included in the upper connected component of $\mathbb{A}\setminus\mathcal{C}$. Therefore, if we set EC_B the set of essential circles $\mathcal{C} \subset \mathbb{A}\setminus B$, EC_B is the union of circles above B: EC_B^+ and those under B: EC_B^-.

We will prove that

Lemma 2.6. *Both EC_B^+ and EC_B^- are open subsets of EC_B for the Hausdorff distance.*

Proof. We prove it for EC_B^+. Let $\mathcal{C}^+$ be a circle above B. As the lower connected component of $\mathbb{A}\setminus\mathcal{C}^+$ is path connected, there is a continuous path $\gamma : [0,+\infty) \to \mathbb{A}\setminus\mathcal{C}^+$ such that $\gamma(0) \in B$ and $\gamma(t) = (0,-t)$ for all the t large enough. Let $\varepsilon > 0$ be such that $\mathcal{C}^+$ is at distance greater than ε from γ. If $\mathcal{C}^-$ is any circle under B, then it must intersect γ. Hence $d_H(\mathcal{C}^-,\mathcal{C}^+) > \varepsilon$ where d_H stands for the Hausdorff distance. This proves the lemma. $\square$

[12]Recall that by Jordan's theorem, $\mathbb{A}\setminus\mathcal{C}$ has two open connected components, one we call upper that contains $\mathbb{T}\times(k,+\infty)$ and one we call lower, that contains $\mathbb{T}\times(-\infty,-k)$ for k large enough.

We will obtain a contradiction as $\mathbb{R}$ is connected and the map $c \mapsto \mathcal{PG}(c+u_c')$ is continuous for the Hausdorff distance, provided we prove that for c large, $\mathcal{PG}(c+u_c')$ is above B while for c small $\mathcal{PG}(c+u_c')$ is under B.

Lemma 2.7. *For c large, $\mathcal{PG}(c+u_c')$ is above B while for c small $\mathcal{PG}(c+u_c')$ is under B.*

Proof. We establish only the first fact. Let $\theta_* \in (\theta_0, \theta_1)$. By definition of η_+, there exists C such that

$$\forall c > C,\ \forall (\theta_*, p) \in \mathcal{PG}(c+u_c'), \quad p > r_1.$$

It follows that for $t > 0$ small enough, we have

$$\forall (\theta_*, p) \in \varphi_{-t}\big(\mathcal{PG}(c+u_c')\big), \quad p > r_1$$

where φ denotes here the flow of the pendulum and

$$\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big) \cap B = \varnothing.$$

But it is proved in [4] that $\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$ is the Lipschitz graph of a function $\alpha_t : \mathbb{T} \to \mathbb{R}$ for small $t > 0$. Hence it follows from the intermediate value theorem that $\alpha(\theta) > r_1$ for $\theta \in (\theta_0, \theta_1)$ and it becomes obvious that $B = (\theta_0, \theta_1) \times (r_0, r_1)$ is under $\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$. Letting $t \to 0$ and passing to the limit, we obtain that B is under $\mathcal{PG}(c+u_c')$. □

□

In order to conclude, we have to prove that $\mathbb{A} = \mathbb{A}_0$ which is equivalent to proving that η_+ is identically $+\infty$ and η_- is identically $-\infty$. We will establish the result for u_+.

Lemma 2.8. *Let $[a,b]$ be a segment, there exists $C > 0$ depending on $[a,b]$ such that if $|c| > C$ then*

$$\forall \theta \in [0,1], \theta' \in [a,b], \quad S(\theta,\theta') + c(\theta - \theta') > \min_{n\in\mathbb{Z}} S(\theta, \theta'+n) + c(\theta - \theta' - n).$$

Proof. Let us set $\Delta = \max\left\{\left|\frac{\partial S}{\partial \theta'}(\theta,\theta')\right|, \quad \theta \in [0,1], \theta' \in [a-1, b+1]\right\}$ and $C = \Delta + 1$.

If $|c| > C$ two cases may occur:

- either $c > \Delta + 1$. In this case, if $(\theta,\theta') \in [0,1] \times [a,b]$, by Taylor's inequality we find

$$S(\theta,\theta') + c(\theta-\theta') > S(\theta,\theta') + c\big(\theta - (\theta'+1)\big) + \Delta \geqslant S(\theta,\theta'+1) + c\big(\theta - (\theta'+1)\big);$$

- or $c < -\Delta - 1$, in which case

$$S(\theta,\theta') + c(\theta-\theta') > S(\theta,\theta') + c\big(\theta - (\theta'-1)\big) + \Delta \geqslant S(\theta,\theta'-1) + c\big(\theta - (\theta'-1)\big).$$

□

Corollary 2.2. *The function η_+ is identically $+\infty$.*

Proof. Let us fix $A > 0$. We assume that for all $(\theta, p) \in \mathcal{PG}(u_0')$, then $|p| \leqslant A$ (or in other words, u_0 is A-Lipschitz). As every map $\theta \mapsto \frac{\partial S}{\partial \Theta}(\theta, \Theta_0)$ is a decreasing diffeomorphism of $\mathbb{R}$, there exists a constant $B > 0$ such that for every $\Theta_0 \in [0,1]$, we have

$$\theta > B \Rightarrow \frac{\partial S}{\partial \Theta}(\theta, \Theta_0) < -(A+1) \quad \text{and} \quad \theta < -B \Rightarrow \frac{\partial S}{\partial \Theta}(\theta, \Theta_0) > A+1.$$

Let C be the constant given by Lemma 2.8 for the segment $[-B, B]$ and let us choose $c > \sup\{B, C\}$. Let $\tilde\theta_0 \in [0,1]$ be any derivability point of u_c. Because of Lemma 2.8, if $\tilde u_c$ is a lift of u_c and if $\tilde\theta_{-1}$ verifies

$$\tilde u_c(\tilde\theta_0) = \inf_{\tilde\theta\in\mathbb{R}} \tilde u_c(\tilde\theta) + S(\tilde\theta, \tilde\theta_0) + c(\tilde\theta - \tilde\theta_0) = \tilde u_c(\tilde\theta_{-1}) + S(\tilde\theta_{-1}, \tilde\theta_0) + c(\tilde\theta - \tilde\theta_0),$$

then $\tilde\theta_{-1} \notin [-B, B]$ and then $\left|\frac{\partial S}{\partial \Theta}(\tilde\theta_{-1}, \tilde\theta_0)\right| > A + 1$.

We deduce from point (c) of section 2.3 that $F\big(\tilde\theta_{-1}, c+\tilde u_c'(\tilde\theta_{-1})\big) = \big(\tilde\theta_0, c+\tilde u_c'(\tilde\theta_0)\big)$ and then

$$c + \tilde u_c'(\tilde\theta_0) = \frac{\partial S}{\partial \Theta}(\tilde\theta_{-1}, \tilde\theta_0),$$

and then $|c + \tilde u_c'(\theta_0)| > A + 1$.
As $\int_0^1 \big(c + \tilde u_c'(s)\big)ds = c > 0$, we can choose θ_0 such that $c + \tilde u_c'(\theta_0) > 0$ and so $c + \tilde u_c'(\theta_0) > A + 1$.

As the pseudographs are vertically ordered (Proposition 2.3), $\mathcal{PG}(c+u_c')$ is above $\mathcal{PG}(u_0')$. We conclude that for all derivability point θ of u_c then $c + \tilde u_c'(\tilde\theta) > A + 1$. Finally, the whole full pseudograph $\mathcal{PG}(c+u_c')$ lies above the circle $\{(t, A),\ \ t \in \mathbb{T}\}$.

We have just established that if $c > B$, then $\mathcal{PG}(c + u_c')$ lies above the circle $\{(t, A),\ \ t \in \mathbb{T}\}$, that concludes the proof.

□

Using technics given in [2], we will prove in Proposition B.2 of Appendix B.3 that the map that maps c on the full pseudograph[13] $\mathcal{PG}(c+u_c') = \{(0, c) + \partial u_c(t),\ \ t \in \mathbb{T}\}$ of $c + u_c'$ is continuous for the Hausdorff distance.

Point (3) of Theorem 1.3 is a result of Proposition 2.3.

3. Proof of the implication (2) ⇒ (1) in Theorem 1.6

We use the same notations as in Theorem 1.2. We assume that the map u is C^1. Then the graph of every $\eta_c = c + \frac{\partial u_c}{\partial\theta}$ is a continuous graph that is backward invariant, hence invariant. If for $c_1 \neq c_2$ the two graphs of η_{c_1} and η_{c_2} have a non-empty intersection, then their common rotation number is rational because an ESTwD has at most one invariant curve with a fixed irrational rotation number (see [32] or Theorem 4.2 below). Moreover, for every $c \in [c_1, c_2]$, we have $\rho(c) = \rho(c_1)$.
Using results of [9] (see section 5), we know that above any $\theta \in \mathbb{T}$, there are at most two $r_1, r_2 \in \mathbb{R}$ such that the orbit of (θ, r_i) is minimizing with rotation number $\rho(c_1)$. As $c_1 \neq c_2$, there exists then $\theta \in \mathbb{T}$ such that $r_1 = \eta_{c_1}(\theta) \neq \eta_{c_2}(\theta) = r_2$. But for $c \in [c_1, c_2]$, the orbit of $\big(\theta, \eta_c(\theta)\big)$ is minimizing with rotation number equal to $\rho(c_1)$ and then $\eta_c(\theta) \in \{r_1, r_2\}$. As $c \mapsto \eta_c(\theta)$ is continuous with values in $\{\eta_{c_1}(\theta), \eta_{c_2}(\theta)\}$ and satisfies $\eta_{c_1}(\theta) \neq \eta_{c_2}(\theta)$, we obtain a contradiction.

So finally the graphs of the η_c define a lamination of $\mathbb{A}$ and then f is C^0-integrable.

[13] see the definition in subsection 2.3

4. Properties of infinite minimizing sequences and weak K.A.M. solutions

In this section we study properties of sequences $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ that are minimizing. By symmetry, similar statements hold for sequences $(\tilde\theta_i)_{i\in\mathbb{Z}_+}$. We start with the following improvement of Lemma 2.2 that proves the beginning of Theorem 1.1.

Proposition 4.1. *Let $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ be a minimizing sequence, then the limit $\lim\limits_{k\to-\infty}\frac{\tilde\theta_k}{k}$ exists.*

Proof. Let us set for $i\in\mathbb{Z}_-$, $r_i=\frac{\partial S}{\partial\tilde\Theta}(\tilde\theta_{i-1},\tilde\theta_i)$ so that $(\tilde\theta_i,r_i)_{i\in\mathbb{Z}_-}$ is a piece of orbit of F. By Theorem 1.3, there exists $c\in\mathbb{R}$ and a weak K.A.M. solution $\tilde u_c$, for $\widetilde{T}^c$, such that $(\tilde\theta_0,r_0)\in\mathcal{PG}(c+\tilde u_c')$. Let $\{\tilde\theta_0\}\times[p_0,p_0']=\partial\tilde u_c(\tilde\theta_0)$ and for $k\in\mathbb{Z}_-$, let us define $x_k=\pi_1\circ F^k(\tilde\theta_0,c+p_0)$ and $x_k'=\pi_1\circ F^k(\tilde\theta_0,c+p_0')$. As $(\tilde\theta_0,c+p_0),(\tilde\theta_0,c+p_0')\in\overline{\mathcal{G}(c+\tilde u_c')}$, the sequences $(x_k)_{k\in\mathbb{Z}_-}$ and $(x_k')_{k\in\mathbb{Z}_-}$ are minimizing and

$$\lim_{k\to-\infty}\frac{x_k}{k}=\lim_{k\to-\infty}\frac{x_k'}{k}=\rho(c)$$

thanks to Lemma 2.2. Note that if $c+p_0=r_0$ or if $c+p_0'=r_0$ the result holds.

In the other cases, as $c+p_0<r_0<c+p_0'$, it follows from the twist condition that $x_{-1}>\tilde\theta_{-1}>x_{-1}'$. By Aubry's fundamental lemma (Proposition 2.1), we infer that for all $k<0$, $x_k>\tilde\theta_k>x_k'$ and the result follows. □

Before continuing our study, we need to introduce some notations. In the rest of this section, $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ will be a minimizing sequence and associated to it, $r_i=\frac{\partial S}{\partial\tilde\Theta}(\tilde\theta_{i-1},\tilde\theta_i)$ so that $(\tilde\theta_i,r_i)_{i\in\mathbb{Z}_-}$ is a piece of orbit of F. We will set ρ_0 the limit given by Proposition 4.1.

We anticipate on a Theorem (that implies Theorem 1.4) that will be proved later in two steps by distinguishing wether ρ_0 is rational or not:

Theorem 4.1. *Let $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ be a minimizing sequence. There exists a cohomology class $c\in\mathbb{R}$ and a weak K.A.M. solution $\tilde u_c$ for $\widetilde{T}^c$ such that $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ calibrates $\tilde u_c$.*

We can already deduce some properties of minimizing sequences reminiscent of orbits of circle homeomorphisms.

Corollary 4.1. *Let $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ be a minimizing sequence and $\rho_0\in\mathbb{R}$ its rotation number given by Proposition 4.1. If there exist p and q integers with $q<0$ and $\tilde\theta_q=\tilde\theta_0+p$ then $(\tilde\theta_i+p)_{i\in\mathbb{Z}_-}=(\tilde\theta_{i+q})_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$. In this case, $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\subset\widetilde{\mathcal{M}}(p/q)$.*

Proof. Let $c\in\mathbb{R}$ and $\tilde u_c$ be a weak K.A.M. solution given by Theorem 4.1 such that $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ calibrates $\tilde u_c$. We know from (h), recalling properties of weak K.A.M. solutions, that $(\tilde\theta_0,r_0)\in\mathcal{PG}(c+\tilde u_c')$ and $(\tilde\theta_k,r_k)\in\mathcal{G}(c+\tilde u_c')$ for all $k<0$. In particular, by periodicity, $\tilde u_c$ is derivable at $\tilde\theta_0$ and $r_0=c+\tilde u_c'(\tilde\theta_0)=c+\tilde u_c'(\tilde\theta_q)=r_q$. Then we obtain that

$$\tilde\theta_{k+q}=\pi_1\circ F^k(\tilde\theta_q,r_q)=\pi_1\circ F^k(\tilde\theta_0,r_0)+p=\tilde\theta_k+p,$$

for all $k\leqslant 0$. The last assertion follows from Aubry-Mather theory [9, Theorem 5.1]. □

The following result though not stated this way is present in [10]:

Corollary 4.2. *Let $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ be a minimizing sequence and $\rho_0\in\mathbb{R}$ its rotation number given by Proposition 4.1. Let p and q be integers with $q<0$.*

- *if $p/q<\rho_0$, then $\tilde\theta_q-\tilde\theta_0<p$;*
- *if $p/q>\rho_0$, then $\tilde\theta_q-\tilde\theta_0>p$.*

In particular $|\tilde\theta_k-\tilde\theta_0-k\rho_0|<1$ for all $k\leqslant 0$.

Proof. Let us prove the first item, the second being similar. Equality $\tilde\theta_q-\tilde\theta_0=p$ is excluded by the previous result. Let $c\in\mathbb{R}$ and $\tilde u_c$ be a weak K.A.M. solution given by Theorem 4.1 such that $(\tilde\theta_i)_{i\in\mathbb{Z}_-}\in\mathbb{R}^{\mathbb{Z}_-}$ calibrates $\tilde u_c$. And let p and q be integers with $q<0$ such that $p/q<\rho_0$.

Let us now assume that $\tilde\theta_q-\tilde\theta_0>p>q\rho_0$. As the sequence $(\tilde\theta_{k+q}-p)_{k\leqslant 0}$ also calibrates $\tilde u_c$, it follows from Lemma 2.1 and an induction that $\tilde\theta_{k+q}-p>\tilde\theta_k$ for all $k\leqslant 0$. Another induction then yields that the sequences $(\tilde\theta_{k+nq}-np)_{n\geqslant 0}$ are increasing. Applying for $k=0$ and dividing by nq, we deduce that (recall $q<0$)

$$\frac{\tilde\theta_{nq}}{nq}-\frac{p}{q}<\frac{\tilde\theta_0}{nq}.$$

Letting $n\to+\infty$ the inequality $\rho_0\leqslant\frac{p}{q}$ follows that is a contradiction.

To establish the last assertion, notice that if for some $q\leqslant 0$, $|\tilde\theta_q-\tilde\theta_0-q\rho_0|\geqslant 1$ then one of the following holds

$$\exists p\in\mathbb{Z},\quad \tilde\theta_q-\tilde\theta_0\leqslant p<q\rho_0,$$
$$\exists p\in\mathbb{Z},\quad \tilde\theta_q-\tilde\theta_0\geqslant p>q\rho_0.$$

This is not possible by the beginning of the Corollary that was just proved. □

Notation. If $\rho_0\in\mathbb{R}$ and $x\in\mathbb{R}$, we define two numbers:

- $r^+_{\rho_0}(x)$ is the largest $r\in\mathbb{R}$ such that $(x,r)\in\mathcal{PG}(c+\tilde u_c')$ for some weak K.A.M. solution $\tilde u_c$, associated some $c\in\rho^{-1}(\{\rho_0\})$.
- $r^-_{\rho_0}(x)$ is the smallest $r\in\mathbb{R}$ such that $(x,r)\in\mathcal{PG}(c+\tilde u_c')$ for some weak K.A.M. solution $\tilde u_c$, associated some $c\in\rho^{-1}(\{\rho_0\})$.

If $\rho_0\in\mathbb{R}$ then if $\rho^{-1}(\{\rho_0\})=[a,b]$, we established that u_a and u_b are unique up to constants. Then $r^+_{\rho_0}(x)-b$ is the left derivative of $\tilde u_b$ at x and $r^-_{\rho_0}(x)-a$ is the right derivative of $\tilde u_a$ at x.

Notation. If $\rho_0\in\mathbb{R}$ and $x\in\mathbb{R}$, we define two numbers $y^+_{\rho_0}(x)$ and $y^-_{\rho_0}(x)$ such that $y^+_{\rho_0}(x)=\min\{y\in\widetilde{\mathcal{M}}(\rho_0),\quad y\geqslant x\}$ and $y^-_{\rho_0}(x)=\max\{y\in\widetilde{\mathcal{M}}(\rho_0),\quad y\leqslant x\}$ where $\widetilde{\mathcal{M}}(\rho_0)$ is the lift of the projected Mather set of rotation number ρ_0. Obviously, $x\in[y^-_{\rho_0}(x),y^+_{\rho_0}(x)]$.

We denote by $\widetilde{\mathcal{M}^*(\rho_0)}$ the lift to $\mathbb{R}^2$ of the Mather set $\mathcal{M}^*(\rho_0)$. Then for every $c\in\rho^{-1}(\{\rho_0\})$ and u_c weak K.A.M. solution for T^c,

$$(y^\pm_{\rho_0}(x),p^\pm_{\rho_0}(x))=(y^\pm_{\rho_0}(x),c+\tilde u_c'(y^\pm_{\rho_0}(x)))$$

is the unique point of $\widetilde{\mathcal{M}}(\rho_0)$ that is above $y^\pm_{\rho_0}(x)$. Moreover, the sequences $\big(y^{\pm n}_{\rho_0}(x)\big)_{n\in\mathbb{Z}}$ where

$$y^{\pm n}_{\rho_0}(x)=\pi_1\circ F^n\big(y^\pm_{\rho_0}(x),p^\pm_{\rho_0}(x)\big)=\pi_1\circ F^n\big(y^\pm_{\rho_0}(x),c+\tilde u_c'\big(y^{\pm n}_{\rho_0}(x)\big)\big)$$

are contained in $\widetilde{\mathcal{M}}(\rho_0)$ and minimizing.

When not necessary, the subscripts ρ_0 will be omitted.

Proposition 4.2. *For all $n \in \mathbb{Z}_-$, $y^{-n}(\tilde\theta_0) \leqslant \tilde\theta_n \leqslant y^{+n}(\tilde\theta_0)$. In particular, $y^{\pm}(\tilde\theta_n) = y^{\pm n}(\tilde\theta_0)$.*

Proof. We use the notations of the proof of the previous Proposition 4.1 and recall that $x'_k \leqslant \tilde\theta_k \leqslant x_k$ for all $k \leqslant 0$. Using Lemma 2.1, a straightforward induction applied to $\tilde u_c$ yields that for all $n \in \mathbb{Z}_-$, $y^{-n}(\tilde\theta_0) \leqslant x'_n$ and that $x_n \leqslant y^{+n}(\tilde\theta_0)$. The result follows.

□

Next we give a property on minimizing sequences that almost cross twice:

Proposition 4.3. *Let $(\tilde\theta_i)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ and $(\tilde\theta'_i)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ be two minimizing sequences. Assume that*

- $\tilde\theta_0 = \tilde\theta'_0$,
- $\lim\limits_{n\to-\infty} \tilde\theta_n - \tilde\theta'_n = 0$,
- *there exists $c \in \mathbb{R}$ and $u_c : \mathbb{T} \to \mathbb{R}$, weak K.A.M. solution for T^c such that $(\tilde\theta'_i)_{i\in\mathbb{Z}_-}$ calibrates $\tilde u_c$, meaning that*

$$\forall n < 0, \quad \tilde u_c(\tilde\theta'_0) - \tilde u_c(\tilde\theta'_n) = \sum_{k=n}^{-1} S(\tilde\theta'_i, \tilde\theta'_{i+1}) + c(\tilde\theta'_n - \tilde\theta'_0) - n\alpha(c).$$

Then $(\tilde\theta_i, r_i)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ calibrates $\tilde u_c$.

Proof. Let us argue by contradiction and assume that

$$\tilde u_c(\tilde\theta_0) - \tilde u_c(\tilde\theta_{n_0}) < \sum_{k=n_0}^{-1} S(\tilde\theta_i, \tilde\theta_{i+1}) + c(\tilde\theta_{n_0} - \tilde\theta_0) - n_0\alpha(c) - 2\varepsilon,$$

for some $n_0 < 0$ and $\varepsilon > 0$. Recall now that if $n < n_0$ then

$$\tilde u_c(\tilde\theta_{n_0}) - \tilde u_c(\tilde\theta_n) \leqslant \sum_{k=n}^{n_0-1} S(\tilde\theta_i, \tilde\theta_{i+1}) - (n-n_0)\alpha(c) + c(\tilde\theta_n - \tilde\theta_{n_0}),$$

and summing we find that

$$\forall n < n_0, \quad \tilde u_c(\tilde\theta_0) - \tilde u_c(\tilde\theta_n) < \sum_{k=n}^{-1} S(\tilde\theta_i, \tilde\theta_{i+1}) - n\alpha(c) + c(\tilde\theta_n - \tilde\theta_0) - \varepsilon.$$

Pick $n < n_0$ such that $|S(\tilde\theta'_n, \tilde\theta'_{n+1}) - S(\tilde\theta_n, \tilde\theta'_{n+1})| < \varepsilon$ and $|\tilde u_c(\tilde\theta'_n) - \tilde u_c(\tilde\theta_n) + c(\tilde\theta'_n - \tilde\theta_n)| < \varepsilon$. We obtain that

$$\begin{aligned} S(\tilde\theta_n, \tilde\theta'_{n+1}) + \sum_{k=n+1}^{-1} S(\tilde\theta'_i, \tilde\theta'_{i+1}) &< \sum_{k=n}^{-1} S(\tilde\theta'_i, \tilde\theta'_{i+1}) + \varepsilon \\ &= \tilde u_c(\tilde\theta'_0) - \tilde u_c(\tilde\theta'_n) + n\alpha(c) - c(\tilde\theta'_n - \tilde\theta'_0) + \varepsilon \\ &< \tilde u_c(\tilde\theta'_0) - \tilde u_c(\tilde\theta_n) + n\alpha(c) - c(\tilde\theta_n - \tilde\theta'_0) + 2\varepsilon < \sum_{k=n}^{-1} S(\tilde\theta_i, \tilde\theta_{i+1}). \end{aligned}$$

As $\tilde\theta_0 = \tilde\theta'_0$, this contradicts the fact that $(\tilde\theta_i)_{i\in\mathbb{Z}_-}$ is minimizing.

□

In order to finish our study, we now discuss in function of the rationality of ρ_0.

4.1. **Case where $\rho_0 \notin \mathbb{Q}$.** In this case we recover that $\{c_0\} = \rho^{-1}(\{\rho_0\})$ is a singleton and that u_c is unique up to constants.

For all $x \in \mathbb{R}$, Aubry-Mather theory says that $\big(y_{\rho_0}^{\pm n}(x)\big)_{n\in\mathbb{Z}}$ are orbits of the lift of a circle homeomorphism of rotation number ρ_0. Looking back at their definition we see that if they do not coincide, the circle homeomorphism is a Denjoy counterexample and $\big(y^-(x), y^+(x)\big)$ projects to a wandering interval of this homeomorphism. In all cases, $\lim\limits_{n\to\pm\infty} |y^{-n}(x) - y^{+n}(x)| = 0$.

We deduce the following Theorem that contains Mather's and Bangert's result[14] and proves Theorem 1.4 for irrational rotation numbers:

Theorem 4.2. *There exists a unique $c_0 \in \mathbb{R}$ such that $\rho(c_0) = \rho_0$. Moreover, there exists a unique weak K.A.M. solution $\tilde u_{c_0}$ at cohomology c_0 such that $\tilde u_{c_0}(0) = 0$. Any minimizing sequence $(\tilde\theta_i)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ with rotation number ρ_0 calibrates $\tilde u_{c_0}$. In particular, $(\tilde\theta_i, r_i)_{i<0} \subset \mathcal{G}(c_0 + \tilde u'_{c_0})$ and $(\tilde\theta_0, r_0) \in \mathcal{PG}(c_0 + \tilde u'_{c_0})$.*

Proof. Let $[a,b] = \rho^{-1}(\{\rho_0\})$. We establish first that if $(\tilde\theta_i)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ is a minimizing sequence with rotation number ρ_0, then it calibrates the weak K.A.M. solution $\tilde u_b$. Let us define for all $n \leqslant 0$, $x_n = \pi_1 \circ F^n\big(\tilde\theta_0, r^+(\tilde\theta_0)\big)$[15] . It follows that $(x_n)_{n\leqslant 0}$ calibrates $\tilde u_b$ and is therefore minimizing. Arguing as in Proposition 4.1 and by Proposition 4.2 we discover that

$$\forall n \in \mathbb{Z}_-, \quad y^{-n}(\tilde\theta_0) \leqslant x_n \leqslant \tilde\theta_n \leqslant y^{+n}(\tilde\theta_0).$$

Using the previous discussion and applying Proposition 4.3 we obtain the result.

Let now $c \in [a,b]$ and $\tilde v : \mathbb{R} \to \mathbb{R}$ be the lift of a weak K.A.M. solution at cohomology c. Let $\tilde\theta_0 \in \mathbb{R}$ be a point of derivability of both $\tilde v$ and $\tilde u_b$. It follows there are respectively a unique sequence $(\tilde\theta_i^c)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ calibrating $\tilde v$ and a unique sequence $(\tilde\theta_i^b)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ calibrating $\tilde u_b$ such that $\tilde\theta_0 = \tilde\theta_0^c = \tilde\theta_0^b$. It follows from the beginning of the proof that $(\tilde\theta_i^c)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$ calibrates $\tilde u_b$ and by uniqueness that $(\tilde\theta_i^c)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-} = (\tilde\theta_i^b)_{i\in\mathbb{Z}_-} \in \mathbb{R}^{\mathbb{Z}_-}$. As a consequence, the equality

$$b + \tilde u_b'(\tilde\theta_0) = c + \tilde v'(\tilde\theta_0) = \frac{\partial S}{\partial\Theta}(\tilde\theta_{-1}^c, \tilde\theta_0)$$

is obtained. As this equality holds almost everywhere, we conclude that

$$b = \int_0^1 \big(b + \tilde u_b'(\tilde\theta)\big)d\tilde\theta = \int_0^1 \big(c + \tilde v'(\tilde\theta)\big)d\tilde\theta = c,$$

and then that $\tilde u_b - \tilde v$ is constant. □

4.2. **Case where $\rho_0 \in \mathbb{Q}$.** In this case we denote $[a,b] = \rho^{-1}(\{\rho_0\})$ and we write $\rho_0 = p/q$ in irreducible form with $q > 0$. It follows that for all $x \in \mathbb{R}$, $\big(y_{\rho_0}^{\pm n}(x)\big)_{n\in\mathbb{Z}} = \big(y_{\rho_0}^{\pm n+q}(x)+p\big)_{n\in\mathbb{Z}}$. Moreover, we have seen previously that u_a and u_b are unique up to constants. Hereafter, unless specified otherwise, $(\tilde\theta_i)_{i\leqslant 0}$ is a minimizing sequence with rotation number p/q.

In the spirit of Aubry-Mather theory, we start by a property of non-crossing of a minimizing sequence with its translates, in the spirit of Corollary 4.2:

[14] Recall that for all $i \leqslant 0$ we set $r_i = \frac{\partial S}{\partial\Theta}(\tilde\theta_{i-1}, \tilde\theta_i)$.

[15] See the notations after Corollary 4.2 for r^+.

Proposition 4.4. *Assume $(\tilde\theta_i)_{i\leqslant 0}$ does not verify $(\tilde\theta_i)_{i\leqslant 0} = (\tilde\theta_{i-q}-p)_{i\leqslant 0}$. One of the two holds:*

- $\tilde\theta_0 \leqslant \tilde\theta_{-q}-p$, $\tilde\theta_i < \tilde\theta_{i-q}-p$ *for all* $i<0$ *and* $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{+i}(\tilde\theta_0)| = 0$.
- $\tilde\theta_0 \geqslant \tilde\theta_{-q}-p$, $\tilde\theta_i > \tilde\theta_{i-q}-p$ *for all* $i<0$ *and* $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{-i}(\tilde\theta_0)| = 0$.

Proof. If $(\tilde\theta_i)_{i\leqslant 0}$ and $(\tilde\theta_{i-q}-p)_{i\leqslant 0}$ do not cross, we set $i_0=0$. Otherwise, there exists $i_0\leqslant 0$ such that $(\tilde\theta_i)_{i\leqslant 0}$ and $(\tilde\theta_{i-q}-p)_{i\leqslant 0}$ cross either at i_0 or between i_0 and i_0+1 for some $i_0<0$. As two minimizing sequences cross at most once, it follows that in all the previous cases, either $\tilde\theta_i < \tilde\theta_{i-q}-p$ for all $i<i_0$ or $\tilde\theta_i > \tilde\theta_{i-q}-p$ for all $i<i_0$. Let us assume the first holds, the second case is treated similarly. Then for $i<i_0$ the sequence $(\tilde\theta_{i-kq}-kp)_{k\geqslant 0}$ is increasing. Moreover, by Proposition 4.2, $\tilde\theta_{i-kq}-kp \leqslant y^{+i-kq}(\tilde\theta_0)-kp = y^{+i}(\tilde\theta_0)$. Hence the limit $z_i = \lim\limits_{k\to+\infty} \tilde\theta_{i-kq}-kp$ exists and verifies that $z_{i-q}-p=z_i$. As this sequence is minimizing (as a limit of minimizing sequences) and projects to a periodic sequence on the circle, we deduce that $(z_i)_{i<i_0} \subset \widetilde{\mathcal{M}}(\rho_0)$ and then $(z_i)_{i<i_0} = y^{+i}(\tilde\theta_0)$ by definition of $y^{+i}(\tilde\theta_0)$.

If now $(\tilde\theta_i)_{i\leqslant 0}$ and $(\tilde\theta_{i-q}-p)_{i\leqslant 0}$ cross either at i_0 or between i_0 and i_0+1 for some $i_0<0$, as we have also proven they are α-asymptotic, we obtain a contradiction with [9, Lemma 3.9].

□

The following result ends the proof of Theorem 4.1 and also gives precisions to Proposition 4.4 by excluding the possibility of an equality $\tilde\theta_0 = \tilde\theta_{-q}-p$ under its hypotheses.

Proposition 4.5. *Using the previous notations, if* $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{+i}(\tilde\theta_0)| = 0$ *then* $(\tilde\theta_i)_{i\leqslant 0}$ *calibrates* $\tilde u_b$.

If $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{-i}(\tilde\theta_0)| = 0$ *then* $(\tilde\theta_i)_{i\leqslant 0}$ *calibrates* $\tilde u_a$.

In particular, $\tilde\theta_0 = \tilde\theta_{-q}-p$ *if and only if* $(\tilde\theta_i)_{i\leqslant 0} = (\tilde\theta_{i-q}-p)_{i\leqslant 0}$.

In all cases, $(\tilde\theta_i, r_i)_{i<0} \subset \mathcal{G}(a+\tilde u_a') \cup \mathcal{G}(b+\tilde u_b')$ *and* $(\tilde\theta_i, r_i)_{i\leqslant 0} \subset \mathcal{PG}(a+\tilde u_a') \cup \mathcal{PG}(b+\tilde u_b')$.

Proof. Let us prove the first assertion, the rest is done in a similar way. If $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{+i}(\tilde\theta_0)| = 0$, let us define for all $n\leqslant 0$, $x_n = \pi_1\circ F^n\big(\tilde\theta_0, r^-(\tilde\theta_0)\big)$. It follows that $(x_n)_{n\leqslant 0}$ calibrates $\tilde u_a$ and is therefore minimizing. Arguing as in Proposition 4.1 and by Proposition 4.2 we discover that for all $n\in\mathbb{Z}_-$, $y^{-n}(\tilde\theta_0) \leqslant \tilde\theta_n \leqslant x_n \leqslant y^{+n}(\tilde\theta_0)$. Applying Proposition 4.3 we obtain the result.

The next assertion in the Theorem is a direct consequence of Corollary 4.1. The rest follows from general properties of weak K.A.M. solutions.

□

In particular we have finished proving Theorem 1.4 establishing more precisely that the cohomology can be taken to be a or b.

Next, the reciprocal question of existence of minimizing sequences verifying certain conditions is addressed:

Proposition 4.6. *For all* $\tilde\theta_0\in\mathbb{R}$ *there exist two minimizing sequences* $(\tilde\theta_i^{\pm})_{i\leqslant 0}$ *with rotation number* p/q *such that* $\tilde\theta_0^+ = \tilde\theta_0^- = \tilde\theta_0$ *and*

- $\lim\limits_{i\to-\infty} |\tilde\theta_i^+ - y^{+i}(\tilde\theta_0)| = 0$ *(and* $(\tilde\theta_i^+)_{i\leqslant 0}$ *calibrates* $\tilde u_a$*);*
- $\lim\limits_{i\to-\infty} |\tilde\theta_i^- - y^{-i}(\tilde\theta_0)| = 0$ *(and* $(\tilde\theta_i^-)_{i\leqslant 0}$ *calibrates* $\tilde u_b$*);*

Proof. If $\tilde\theta_0 \in \widetilde{\mathcal{M}}(p/q)$ then by Aubry-Mather theory, $y^{+i}(\tilde\theta_0) = y^{-i}(\tilde\theta_0)$ for all $i \leqslant 0$ and $\big(y^{\pm i}(\tilde\theta_0)\big)_{i\leqslant 0}$ is the only minimizing orbit starting at $\tilde\theta_0$ with rotation number p/q.

We handle now the other and more interesting case where $y^{-0}(\tilde\theta_0) < \tilde\theta_0 < y^{+0}(\tilde\theta_0)$ and prove the existence of $(\tilde\theta_i^-)_{i\leqslant 0}$ as the existence of $(\tilde\theta_i^+)_{i\leqslant 0}$ is established in a very similar way.

Let $(c_n)_{n>0}$ be a decreasing sequence of real numbers converging to b. Setting for $n > 0$, $\rho_n = \rho(c_n)$ it follows that $(\rho_n)_{n>0}$ is nonincreasing, converges to p/q and that $\rho_n > p/q$ for all $n > 0$. For all $n > 0$, let $(\tilde\theta_k^n)_{k\leqslant 0}$ be a minimizing sequence with rotation number ρ_n and such that $\tilde\theta_0^n = \tilde\theta_0$ (take any calibrating sequence for a weak K.A.M. solution at cohomology c_n starting at $\tilde\theta_0$). Up to extracting, we may assume that for all $k \leqslant 0$, the sequence $(\tilde\theta_k^n)_{n>0}$ converges to a $\tilde\theta_k^-$. It follows that $(\tilde\theta_k^-)_{k\leqslant 0}$ is minimizing, has rotation number p/q and verifies $\tilde\theta_0^- = \tilde\theta_0$.

Applying Corollary 4.2 yields the inequalities

$$\forall n > 0, \quad \tilde\theta_{-q}^n > \tilde\theta_0 + p.$$

Passing to the limit we get $\tilde\theta_{-q}^- > \tilde\theta_0 + p$, as equality is prohibited by Corollary 4.1. The rest now follows from Proposition 4.4. $\square$

This leads to a reciprocal to Proposition 4.5:

Theorem 4.3. *Let* $\tilde\theta_0 \notin \widetilde{\mathcal{M}}(p/q)$ *and* $(\tilde\theta_i)_{i\leqslant 0}$ *a minimizing sequence with rotation number* p/q*. The following assertions are equivalent:*

(1) $\tilde\theta_0 < \tilde\theta_{-q} - p$ *(resp.* $\tilde\theta_0 > \tilde\theta_{-q} - p$*);*
(2) $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{+i}(\tilde\theta_0)| = 0$ *(resp.* $\lim\limits_{i\to-\infty} |\tilde\theta_i - y^{-i}(\tilde\theta_0)| = 0$*);*
(3) $(\tilde\theta_i)_{i\leqslant 0}$ *calibrates* $\tilde u_a$ *(resp.* $(\tilde\theta_i)_{i\leqslant 0}$ *calibrates* $\tilde u_b$*).*

Proof. The only thing left to prove is that (3) implies (1).

Let us assume in a first step that $\tilde\theta_0$ is a point of derivability of $\tilde u_a$. Let $(\tilde\theta_i^\pm)_{i\leqslant 0}$ be the sequences given by Proposition 4.6 and let $(r_i^\pm)_{i\leqslant 0}$ be the associated sequences such that $(\tilde\theta_i^\pm, r_i^\pm)_{i\leqslant 0}$ are orbits of F. As $(\tilde\theta_i^+)_{i\leqslant 0}$ calibrates $\tilde u_a$, we discover that $r_0^+ = a + \tilde u_a'(\tilde\theta_0)$.

Assume now $\tilde\theta_0$ is arbitrary and $\{\tilde\theta_0\} \times [R_0, R_0'] = \partial\tilde u_a(\tilde\theta_0)$. Let $(\tilde\theta_0^n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to $\tilde\theta_0$, of derivability points of $\tilde u_a$. It follows that $\tilde u_a'(\tilde\theta_0^n) \to R_0'$. For all $n \geqslant 0$, let $(\tilde\theta_i^n)_{i\leqslant 0}$ be the unique sequence calibrating $\tilde u_a$, starting at $\tilde\theta_0^n$. Finally, let $(\tilde\theta_i')_{i\leqslant 0}$ be the limit of the sequences $(\tilde\theta_i^n)_{i\leqslant 0}$ so that for all $i \leqslant 0$, $\tilde\theta_i = \pi_1 \circ F^i(\tilde\theta_0, a + R_0')$. Thanks to the beginning of the proof, for all $n \geqslant 0$, the inequality $\tilde\theta_0^n \leqslant \tilde\theta_{-q}^n - p$. Passing to the limit we discover that $\tilde\theta_0 < \tilde\theta_{-q}' - p$, the inequality being strict as $\tilde\theta_0 \notin \widetilde{\mathcal{M}}(p/q)$.

If now $(\tilde\theta_i)_{i\leqslant 0}$ is any minimizing sequence starting at $\tilde\theta_0$ that calibrates $\tilde u_a$ and $(\tilde\theta_i, r_i)_{i\leqslant 0}$ the associated orbit of F we know that $(\tilde\theta_i)_{i\leqslant 0}$ and $(\tilde\theta_i')_{i\leqslant 0}$ can only cross at 0. As $r_0 \in [R_0, R_0']$, the twist condition implies that $\tilde\theta_{-1} \geqslant \tilde\theta_{-1}'$. We therefore conclude that $\tilde\theta_0 < \tilde\theta_{-q}' - p \leqslant \tilde\theta_{-q} - p$ that was to be proven.

$\square$

We deduce a further property concerning the pseudographs of u_a and u_b:

Proposition 4.7. *The full pseudographs $\mathcal{PG}(a+u_a)$ and $\mathcal{PG}(b+u_b)$ only intersect above $\mathcal{M}(p/q)$.*

Proof. Let $\tilde\theta_0 \notin \widetilde{\mathcal{M}}(p/q)$. We denote $\{\tilde\theta_0\}\times[r_0^a, R_0^a] = \partial\tilde u_a(\tilde\theta_0)$ and $\{\tilde\theta_0\}\times[r_0^b, R_0^b] = \partial\tilde u_b(\tilde\theta_0)$. We will prove that $b + r_0^b > a + R_0^a$ thus establishing that $\mathcal{PG}(b+\tilde u_b)$ is strictly above $\mathcal{PG}(a+\tilde u_a)$ on the interval $\big(y^-(\tilde\theta_0), y^+(\tilde\theta_0)\big)$. We set $\tilde\theta_0^a = \tilde\theta_0^b = \tilde\theta_0$ and for $i<0$, $\tilde\theta_i^a = \pi_1\circ F^i(\tilde\theta_0, a+R_0^a)$ and $\tilde\theta_i^b = \pi_1\circ F^i(\tilde\theta_0, b+r_0^b)$. From the previous Theorem 4.3 we know that $\lim\limits_{i\to-\infty}|\tilde\theta_i^a - y^{+i}(\tilde\theta_0)| = 0$ and that $\lim\limits_{i\to-\infty}|\tilde\theta_i^b - y^{-i}(\tilde\theta_0)| = 0$. Therefore for i small enough, $\tilde\theta_i^a > \tilde\theta_i^b$. As both minimizing sequences only can cross at 0, it follows that $\tilde\theta_{-1}^a > \tilde\theta_{-1}^b$. We conclude from the twist condition that $b + r_0^b > a + R_0^a$. □

As a Corollary we recover a famous result of Mather and Bangert:

Corollary 4.3. *The set $\rho^{-1}(p/q)$ is a singleton if and only if $\widetilde{\mathcal{M}}(p/q) = \mathbb{T}$.*

Proof. If $\widetilde{\mathcal{M}}(p/q) = \mathbb{T}$ then there is an invariant, 1–periodic Lipschitz graph $\tilde\theta \mapsto r_{\tilde\theta}$ for all $c\in\rho^{-1}(p/q)$, if $\tilde u:\mathbb{R}\to\mathbb{R}$ is a corresponding weak K.A.M. solution, it is of class C^1 and $c + \tilde u'(\tilde\theta) = r_{\tilde\theta}$ for all $\tilde\theta\in\mathbb{R}$. It follows that $c = \int_0^1 r_x dx$ is unique.

Reciprocally, if $\widetilde{\mathcal{M}}(p/q) \neq \mathbb{T}$, thanks to the preceding Proposition 4.7, there is $\tilde\theta_0 \notin \widetilde{\mathcal{M}}(p/q)$ such that $\tilde u_a$ and $\tilde u_b$ both are derivable at $\tilde\theta_0$ with $a + \tilde u_a'(\tilde\theta_0) < b + \tilde u_b'(\tilde\theta_0)$ and by property of semi–concave functions, this inequality is strict in a neighborhood of $\tilde\theta_0$. As $a + \tilde u_a'(\tilde\theta) \leqslant b + \tilde u_b'(\tilde\theta)$ holds almost everywhere, integrating on $[0,1]$ we find $a<b$. □

As a conclusion we obtain the following property on weak K.A.M. solutions:

Theorem 4.4. *Let $c\in[a,b]$ and u_c be a weak K.A.M. solution for T^c. Let $\big(\tilde\theta_0, c+\tilde u_c'(\tilde\theta_0)\big)\in\mathcal{G}(c+\tilde u_c')$ and $(\tilde\theta_i)_{i\leqslant 0}$ the associated minimizing sequence that calibrates $\tilde u_c$.*

- *If $\lim\limits_{i\to-\infty}\tilde\theta_i - y^{-i}(\tilde\theta_0) = 0$ then $\big(\tilde\theta_0, c+\tilde u_c'(\tilde\theta_0)\big)\in\mathcal{G}(b+\tilde u_b')$.*
- *If $\lim\limits_{i\to-\infty}\tilde\theta_i - y^{+i}(\tilde\theta_0) = 0$ then $\big(\tilde\theta_0, c+\tilde u_c'(\tilde\theta_0)\big)\in\mathcal{G}(a+\tilde u_a')$.*

Proof. Let us recall that for a semi-concave function $v:\mathbb{R}\to\mathbb{R}$, if $v'(x_0)$ exists then x_0 is a continuity point of $x\mapsto\partial v(x)$. On the other side, if $N\subset\mathbb{R}$ has Lebesgue measure 0 and contains the nonderivable points of v, and if $\lim\limits_{\substack{x\to x_0\\ x\notin N}} v'(x) = p$ exists, then $v'(x_0) = p$ exists.

Coming back to the Theorem, the result obviously holds is $\tilde\theta_0\in\widetilde{\mathcal{M}}(\rho_0)$. We now assume otherwise.

Let us introduce $N\subset\mathbb{R}$ to be the countable set of points where either $\tilde u_a$, $\tilde u_b$ or $\tilde u_c$ is not derivable.

Let us prove the first point. In this case, by Proposition 4.4, $\tilde\theta_{-q} - p < \tilde\theta_0$. There exists $\varepsilon>0$ such that if $\tilde\theta_0'\in\mathbb{R}\setminus N$ verifies $|\tilde\theta_0 - \tilde\theta_0'| < \varepsilon$, then $\tilde\theta_{-q}' - p < \tilde\theta_0'$ where $\tilde\theta_i' = \pi_1\circ F^i\big(\tilde\theta_0', c+\tilde u_c'(\tilde\theta_0')\big)$. Up to taking ε smaller, then $\tilde\theta_0'\in\big(y^-(\tilde\theta_0), y^+(\tilde\theta_0)\big)$, it then follows from Proposition 4.4 and Theorem 4.5 that $(\tilde\theta_i')_{i\leqslant 0}$ calibrates $\tilde u_b$ and then that

$$\forall i\leqslant 0,\quad \tilde\theta_i' = \pi_1\circ F^i\big(\tilde\theta_0', b+\tilde u_b'(\tilde\theta_0')\big).$$

Finally, we have established that

$$\lim_{\substack{\tilde\theta_0'\to\tilde\theta_0\\ \tilde\theta_0'\notin N}} b+\tilde u_b'(\tilde\theta_0') = \lim_{\substack{\tilde\theta_0'\to\tilde\theta_0\\ \tilde\theta_0'\notin N}} c+\tilde u_c'(\tilde\theta_0') = c+\tilde u_c'(\tilde\theta_0),$$

that proves our result. □

The following corollary ends the proof of Theorem 1.5.

Corollary 4.4. *Let $c\in[a,b]$ and u_c be a weak K.A.M. solution for T^c then*

$$\mathcal{G}(c+\tilde u_c') \subset \mathcal{G}(a+\tilde u_a') \cup \mathcal{G}(b+\tilde u_b'),$$

$$\overline{\mathcal{G}(c+\tilde u_c')} \subset \overline{\mathcal{G}(a+\tilde u_a')} \cup \overline{\mathcal{G}(b+\tilde u_b')}.$$

We may also provide a description of what $\mathcal{PG}(c+\tilde u_c')$ looks like, similar to the classical example of the pendulum. The open region between $\mathcal{PG}(a+\tilde u_a')$ and $\mathcal{PG}(b+\tilde u_b')$ has a connected component between two consecutive points of the projected Mather set that projects on an interval (y^-,y^+). Then, either $\mathcal{PG}(c+\tilde u_c')$ coincides with $\mathcal{PG}(a+\tilde u_a')$ on (y^-,y^+), either $\mathcal{PG}(c+\tilde u_c')$ coincides with $\mathcal{PG}(b+\tilde u_b')$ on (y^-,y^+), either there exists $z\in(y^-,y^+)$ such that $\mathcal{PG}(c+\tilde u_c')$ coincides with $\mathcal{PG}(b+\tilde u_b')$ on (y^-,z) and $\mathcal{PG}(c+\tilde u_c')$ coincides with $\mathcal{PG}(a+\tilde u_a')$ on (z,y^+).

4.3. **Pseudographs of weak K.A.M. solutions.** We end by a crucial property of pseudo-graphs associated to weak K.A.M. solutions that we believe is of independent interest:

Proposition 4.8. *Let $c\in\mathbb{R}$ and $u_c:\mathbb{T}\to\mathbb{R}$ be a weak K.A.M. solution for T^c. Then $f^{-1}\big(\mathcal{PG}(c+u_c')\big)$ is the graph of a continuous function.*

Proof. Recall that by [4], $\mathcal{PG}(c+u_c')$ is a Lipschitz embedded circle (see Lemma 2.5) that we can parametrize by a map $\gamma:\mathbb{T}\to\mathbb{A}$. Is $\tilde\gamma:\mathbb{R}\to\mathbb{R}^2$ is a lift of γ we assume without loss of generality that $\pi_1\circ\tilde\gamma$ is nondecreasing.

If $t<t'$ are real numbers we consider two cases: if $\pi_1\circ\tilde\gamma(t)=\pi_1\circ\tilde\gamma(t')$ then $\pi_2\circ\tilde\gamma(t)>\pi_2\circ\tilde\gamma(t')$ (because $\tilde u_c$ is semi-concave) and $\pi_1\circ F^{-1}\big(\tilde\gamma(t)\big)<\pi_1\circ F^{-1}\big(\tilde\gamma(t')\big)$ because of the twist condition.

If now $\pi_1\circ\tilde\gamma(t)<\pi_1\circ\tilde\gamma(t')$ then we consider $t\leqslant t_1<t_2\leqslant t'$ such that $\pi_1\circ\tilde\gamma(t)=\pi_1\circ\tilde\gamma(t_1)$, $\pi_1\circ\tilde\gamma(t')=\pi_1\circ\tilde\gamma(t_2)$ and $\{\tilde\gamma(t_1),\tilde\gamma(t_2)\}\in\overline{\mathcal{G}(c+\tilde u_c')}$. It follows that $\pi_2\circ\tilde\gamma(t)>\pi_2\circ\tilde\gamma(t_1)$ and $\pi_2\circ\tilde\gamma(t_2)>\pi_2\circ\tilde\gamma(t')$. Moreover, we deduce from Lemma 2.1 that $\pi_1\circ F^{-1}\big(\tilde\gamma(t_1)\big)\leqslant\pi_1\circ F^{-1}\big(\tilde\gamma(t_2)\big)$. Moreover, as $F^{-1}\big(\overline{\mathcal{G}(c+\tilde u_c')}\big)\subset\mathcal{G}(c+\tilde u_c')$ the previous inequality is strict. We conclude that

$$\pi_1\circ F^{-1}\big(\tilde\gamma(t)\big)\leqslant\pi_1\circ F^{-1}\big(\tilde\gamma(t_1)\big)<\pi_1\circ F^{-1}\big(\tilde\gamma(t_2)\big)\leqslant\pi_1\circ F^{-1}\big(\tilde\gamma(t')\big).$$

We have established that the function $\pi_1\circ F^{-1}\circ\tilde\gamma$ is increasing and that proves the Proposition. □

Remark. The preceding result may be interpreted in terms of positive Lax-Oleinik maps. Indeed if one defines $T_+^c\tilde u_c(x)=\max\limits_{x'\in\mathbb{R}}\tilde u_c(x')-S(x,x')+c(x'-x)$ then one may deduce that $T_+^c\tilde u_c$ is a C^1 function and that $F^{-1}\big(\mathcal{PG}(c+\tilde u_c')\big)=\mathcal{G}(c+T_+^c\tilde u_c')$. This is related to Lasry-Lyons type results, see [14, 15, 47, 28].

In Aubry-Mather Theory, it is known that given a rotation number ρ, on each vertical V_θ there is

- at most one bi-infinite minimizing orbit of rotation number ρ intersecting V_θ if $\rho \notin \mathbb{Q}$,
- at most two bi-infinite minimizing orbit of rotation number ρ intersecting V_θ if $\rho \in \mathbb{Q}$,

in the latter case if there are two, one is α-asymptotic to $\big(y^{-i}(\theta)\big)_{i\in\mathbb{Z}}$ and ω-asymptotic to $\big(y^{+i}(\theta)\big)_{i\in\mathbb{Z}}$ and the other is ω-asymptotic to $\big(y^{-i}(\theta)\big)_{i\in\mathbb{Z}}$ and α-asymptotic to $\big(y^{+i}(\theta)\big)_{i\in\mathbb{Z}}$.

In our study of one-sided infinite minimizing sequence we obtain as a corollary a similar statement. The only difference is that instead of taking as reference the vertical foliation, we take its image by f.

As an application of the previous Theorem we obtain:

Theorem 4.5. *Let $\theta \in \mathbb{T}$ and $\rho_0 \in \mathbb{R}$, then*

- *if $\rho_0 \notin \mathbb{Q}$, there exists at most one $(x,p) \in f(V_\theta)$ such that $\big(\pi_1\circ f^i(x,p)\big)_{i\in\mathbb{Z}_-}$ is minimizing with rotation number ρ_0;*
- *if $\rho_0 \in \mathbb{Q}$, there exists at most two $(x,p) \in f(V_\theta)$ such that $\big(\pi_1\circ f^i(x,p)\big)_{i\in\mathbb{Z}_-}$ is minimizing with rotation number ρ_0.*

In the latter case, if there are two such points (x_1,p_1) and (x_2,p_2) with $x_1 < x_2$ then $\big(\pi_1\circ f^i(x_1,p_1)\big)_{i\in\mathbb{Z}_-}$ is α-asymptotic to $\big(y^{+(i-1)}(\theta)\big)_{i\in\mathbb{Z}}$ and $\big(\pi_1\circ f^i(x_2,p_2)\big)_{i\in\mathbb{Z}_-}$ is α-asymptotic to $\big(y^{-(i-1)}(\theta)\big)_{i\in\mathbb{Z}}$.

Proof. If $\rho_0 \notin \mathbb{Q}$ the only possible such point is $f\big(V_\theta \cap f^{-1}\big(\mathcal{PG}(c+u'_c)\big)\big)$ where $\{c\} = \rho^{-1}(\{\rho_0\})$.

If $\rho \in \mathbb{Q}$ the only possible such points are $f\big(V_\theta \cap f^{-1}\big(\mathcal{PG}(a+u'_a)\big)\big)$ and $f\big(V_\theta \cap f^{-1}\big(\mathcal{PG}(b+u'_b)\big)\big)$ where $[a,b] = \rho^{-1}(\{\rho_0\})$. $\square$

This proves the end of Theorem 1.1.

Appendix A. Examples

A.1. **An example a semi-concave function that is not a weak K.A.M. solution for $\widehat{T}^c$ and that satisfies $f^{-1}\big(\overline{\mathcal{G}(c+u')}\big) \subset \mathcal{G}(c+u')$.** Let us begin by introducing $g_t : \mathbb{A} \to \mathbb{A}$ as being the time t map of the Hamiltonian flow of the double pendulum Hamiltonian

$$H(\theta,r) = \frac{1}{2}r^2 + \cos(4\pi\theta).$$

If $t > 0$ is small enough, g_t is an ESTwD.

Observe that H is a so-called Tonelli Hamiltonian (see [27] for the definition) with associated Lagrangian $L(\theta,v) = \frac{1}{2}v^2 - \cos(4\pi\theta)$. The global minimum -1 of L is attained in $(0,0)$ and $(\frac{1}{2},0)$.

If G_t is the time t map of the lift of H to $\mathbb{R}^2$, then G_t is a lift of g_t and if $G_s(\theta,r) = (\theta_s,r_s)$, a generating function of G_t is

$$S_t(\theta,\theta_t) = \int_0^t L(\theta_s,\dot\theta_s)ds.$$

By using this formula, observe that the only ergodic minimizing measures for the cohomology class 0 are the Dirac measure at 0 and $\frac{1}{2}$.

Then we denote by $h : \mathbb{A} \to \mathbb{A}$ the map that is defined by $h(\theta, r) = (\theta + \frac{1}{2}, r)$. Then $f = h \circ g_t = g_t \circ h$ is again an ESTwD and H is an integral for f, which means that $H \circ f = H$.
It is easy to check that a generating function of a lift F of f is given by

$$S(\theta, \Theta) = S_t\big(\theta, \Theta - \frac{1}{2}\big).$$

From this, we deduce that the Mather set corresponding to the cohomology class zero (and the rotation number $\frac{1}{2}$) is the support of a unique ergodic measure, that is the mean of two Dirac measure $\frac{1}{2}(\delta_{(0,0)} + \delta_{(\frac{1}{2},0)})$.
As there is only one such minimizing measure, we know that there is a unique, up to constants, weak K.A.M. solution u with cohomology class 0. But there are a lot of graphs of v' with $v : \mathbb{T} \to \mathbb{R}$ semi–concave that are invariant by f. The first one we draw corresponds to the weak K.A.M. solution whose graph is strictly mapped into itself by f^{-1}. Perturbing slightly the pseudograph in the level $\{H = 1\}$, we obtain another backward invariant pseudograph that doesn't correspond to a weak K.A.M. solution.

In the right drawing 5, the perturbation of the pseudograph must be small enough so that, in the right eye on the upper manifold, the piece of pseudograph that goes beyond the vertical dotted line is mapped by f^{-1} in the upper piece of pseudograph of the left eye. With the notations of the figure, $f^{-1}\big(d, s^+(d)\big) = \big(e, s^+(e)\big)$.

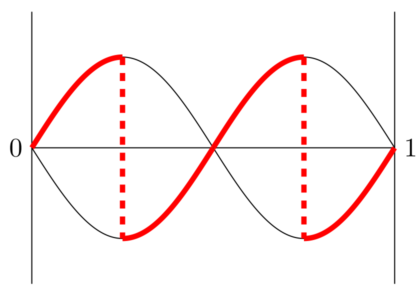

FIGURE 4. The pseudograph of the weak K.A.M. solution at cohomology 0

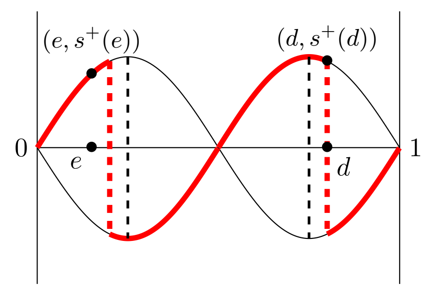

FIGURE 5. A backward invariant pseudograph that is not a weak K.A.M. solution

REMARK. The previous example also shows that Corollary 4.4 is not an equivalence in the sense that if the pseudograph of a semi–concave function $c + u_c'$ satisfies the inclusions of Corollary 4.4, then u_c is not necessarily a weak K.A.M. solution at cohomology c.

A.2. **Cases where the discounted solution doesn't depend continuously on c.** Let us start this appendix of counterexamples with a positive result. We will show that even if discounted solutions may depend in a discontinuous way on c, the same is not true for their derivative. In what follows we use the notion of Clarke sub-derivative introduced earlier in Definition 2.3.

Let us recall that by Proposition 2.2, if $g_n : \mathbb{T} \to \mathbb{R}$ are equi-semi-concave functions converging to $g : \mathbb{T} \to \mathbb{R}$, then $\mathcal{PG}(g_n')$ converges to $\mathcal{PG}(g')$ for the Hausdorff distance.

Let us now state our result:

Proposition A.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be an ESTwD. For $c \in \mathbb{R}$, we denote by $\mathcal{U}_c$ the weak K.A.M. discounted solution. Then the map $c \mapsto \mathcal{PG}(\mathcal{U}_c')$ is continuous.*

As a straightforward corollary, we deduce for instance that if $c_n \to c$ and $x_n \to x$ and if the $\mathcal{U}_{c_n}'(x_n)$ exist, as well as $\mathcal{U}'(c)(x)$, then $\mathcal{U}_{c_n}'(x_n) \to \mathcal{U}'(c)(x)$.

Proof of Proposition A.1. If $\rho(c_0) \in \mathbb{R} \setminus \mathbb{Q}$, there is a unique weak K.A.M. solution up to constants, hence continuity of $\mathcal{PG}(\mathcal{U}_c')$ at c_0 follows from Proposition 2.2.

If $\rho(c) = r \in \mathbb{Q}$, let us denote $\rho^{-1}(r) = [c_1, c_2]$. Again, continuity at c_1 and c_2 is obvious as there is a unique weak K.A.M solution at these cohomology classes (see Proposition 2.4).

It remains to study what happens inside (c_1, c_2) and we will prove that in this interval, the map $c \mapsto \mathcal{U}_c$ is concave. Let us set $\mathcal{M}_r$ the set of Mather measures corresponding to any cohomology class $c \in (c_1, c_2)$. Recall that as seen in (1) page 15, this set does not depend on c. Moreover, the function α is affine on (c_1, c_2).

From [22], we know that $\mathcal{U}_c(x) = \sup_u u(x)$, where the supremum is taken amongst (continuous) c-dominated functions $u : \mathbb{T} \to \mathbb{R}$ such that $\int u(x) d\mu(x,y) \leqslant 0$ for all $\mu \in \mathcal{M}_r$. Moreover, it is proven that $\int \mathcal{U}_c(x) d\mu(x,y) \leqslant 0$ for all $\mu \in \mathcal{M}_r$. Let now $c, c' \in (c_1, c_2)$ and $\lambda \in [0,1]$. Let us set $v = \lambda \mathcal{U}_c + (1-\lambda)\mathcal{U}_{c'}$.

As $\int \mathcal{U}_c(x) d\mu(x,y) \leqslant 0$ and $\int \mathcal{U}_c'(x) d\mu(x,y) \leqslant 0$ for all $\mu \in \mathcal{M}_r$ we deduce that $\int v(x) d\mu(x,y) \leqslant 0$ for all $\mu \in \mathcal{M}_r$.

Moreover, passing to lifts (with the same $\sim$ notation as previously), from

$$\forall \theta, \theta' \in \mathbb{R}, \quad \widetilde{\mathcal{U}}_c(\theta) - \widetilde{\mathcal{U}}_c(\theta') \leqslant S(\theta', \theta) + c(\theta' - \theta) + \alpha(c);$$
$$\forall \theta, \theta' \in \mathbb{R}, \quad \widetilde{\mathcal{U}}_{c'}(\theta) - \widetilde{\mathcal{U}}_{c'}(\theta') \leqslant S(\theta', \theta) + c'(\theta' - \theta) + \alpha(c');$$

and recalling that $\alpha\big(\lambda c + (1-\lambda)c'\big) = \lambda\alpha(c) + (1-\lambda)\alpha(c')$, we get

$$\forall \theta, \theta' \in \mathbb{R}, \quad \tilde{v}(\theta) - \tilde{v}(\theta') \leqslant S(\theta', \theta) + \big(\lambda c + (1-\lambda)c'\big)(\theta' - \theta) + \alpha\big(\lambda c + (1-\lambda)c'\big).$$

Hence v is $\big(\lambda c + (1-\lambda)c'\big)$-dominated. We conclude that $v \leqslant \mathcal{U}_{\lambda c + (1-\lambda)c'}$, proving the claim, and the Proposition.

$\square$

Remark. The previous proof is intimately linked to the 1-dimensional setting we work with. Indeed, it was communicated to us by Patrick Bernard that as soon as we move up to dimension 2, there are examples on $\mathbb{T}^2$ for which it is not possible to construct a function $c \mapsto u_c$ that maps to each cohomology class a weak K.A.M. solution and such that $c \mapsto u_c'$ is continuous (in any possible way).

We obtain as a corollary:

Corollary A.1. *The function $\mathcal{U}(x,c) = \mathcal{U}_c(x) - \mathcal{U}_c(0)$ also satisfies the conclusions of Theorem 1.2.*

We now give a C^∞ integrable example for which the discounted method doesn't select a transversely continuous weak K.A.M. solution.

Example. We use the notation of Theorem 1.6. We define $F_0, H : \mathbb{A} \to \mathbb{A}$ by $F_0(\theta, r) = (\theta + r, r)$ and $H(\theta, r) = (h(\theta), \frac{r}{h'(\theta)})$ where $h : \mathbb{T} \to \mathbb{T}$ is a smooth orientation preserving diffeomorphism of $\mathbb{T}$ such that $h(t) = t + d(t)$ and $d : \mathbb{T} \to \mathbb{R}$ satisfies $d(0) = 0$ and

$$\int_{\mathbb{T}} d(t)dt > \frac{d(\frac{1}{2})}{2}. \tag{10}$$

Observe that $h^{-1}(t) = t - d \circ h^{-1}(t)$. As the symplectic diffeomorphism H maps a vertical $\{\theta\} \times \mathbb{R}$ onto a vertical $\{h(\theta)\} \times \mathbb{R}$ and preserves the transversal orientation, the smooth diffeomorphism[16] $F = H \circ F_0 \circ H^{-1}$ is also a symplectic C^∞ integrable ESTwD. The new invariant foliation is the set of the graphs of $\eta_c(\theta) = \frac{c}{h'(h^{-1}(\theta))} = c(h^{-1})'(\theta)$. Hence we have $u_c(\theta) = -cd \circ h^{-1}(\theta)$. Observe that the function u is smooth.
Then $H_c(\theta) = \theta + \frac{\partial u_c}{\partial c}(\theta) = \theta - d \circ h^{-1}(\theta) = h^{-1}(\theta)$. Hence the measure defined on $\mathbb{T}$ by $\mu([0, \theta]) = h^{-1}(\theta)$, i.e. the measure with density $\frac{1}{h' \circ h^{-1}}$, is invariant by the restricted-projected Dynamics g_c. When the rotation number $\rho(c)$ of g_c is irrational, this is the only measure invariant by g_c.
Let us recall that the discounted solution $\mathcal{U}_c$ that is selected in [46] and [22] is the weak K.A.M. solution that is the supremum of the subsolutions that satisfy for every minimizing g_c-invariant measure μ: $\int u_c d\mu \leqslant 0$. When c is irrational, we deduce that

$$\mathcal{U}_c(\theta) = u_c(\theta) - \int u_c(t) d\mu(t) = c\left(\int_{\mathbb{T}} d \circ h^{-1}(t)(h^{-1})'(t)dt - d \circ h^{-1}(\theta)\right);$$

i.e.

$$\mathcal{U}_c(\theta) = c\left(\int_{\mathbb{T}} d(t)dt - d \circ h^{-1}(\theta)\right) = u_c(\theta) + c\int_{\mathbb{T}} d(t)dt. \tag{11}$$

Assume now that $c = \frac{1}{2}$. Then

$$g_{\frac{1}{2}}(0) = h \circ R_{\frac{1}{2}} \circ h^{-1}(0) = h\Big(\frac{1}{2}\Big) = \frac{1}{2} + d\Big(\frac{1}{2}\Big) \quad \text{and} \quad g_{\frac{1}{2}}\left(\frac{1}{2} + d\Big(\frac{1}{2}\Big)\right) = 0.$$

The mean of the two Dirac measures

$$\nu = \frac{1}{2}\left(\delta_0 + \delta_{\frac{1}{2} + d(\frac{1}{2})}\right)$$

is a measure that is invariant by $g_{\frac{1}{2}}$. Hence $\mathcal{U}_{\frac{1}{2}}(\theta) = u_{\frac{1}{2}}(\theta) - K$ with $K \geqslant \int_{\mathbb{T}} u_{\frac{1}{2}} d\nu$. We deduce that

$$K \geqslant \frac{1}{2}\left(u_{\frac{1}{2}}(0) + u_{\frac{1}{2}}\left(\frac{1}{2} + d\Big(\frac{1}{2}\Big)\right)\right) = -\frac{1}{4}\left(d \circ h^{-1}(0) + d \circ h^{-1}\left(\frac{1}{2} + d\Big(\frac{1}{2}\Big)\right)\right);$$

i.e.

$$K \geqslant -\frac{1}{4} d\Big(\frac{1}{2}\Big).$$

By Inequality (10), we know that $\varepsilon = \int_{\mathbb{T}} d(t)dt - \frac{d(\frac{1}{2})}{2} > 0$. We have then

$$\mathcal{U}_{\frac{1}{2}}(\theta) \leqslant u_{\frac{1}{2}}(\theta) + \frac{1}{4} d\Big(\frac{1}{2}\Big) = u_{\frac{1}{2}}(\theta) + \frac{1}{2}\int_{\mathbb{T}} d(t)dt - \frac{\varepsilon}{2}$$

---

[16]Note that F_0 is the time-1 map of the Hamiltonian function $f_0(\theta, r) = \frac{1}{2}r^2$. It follows that F, being conjugated to F_0 by a symplectic map, is itself the time-1 map of the Tonelli Hamiltonian $f_0 \circ H^{-1}$.

Using Equation (11), we deduce that

$$\limsup_{c\to\frac12}\mathcal{U}_c(\theta)\geqslant \mathcal{U}_{\frac12}(\theta)+\frac{\varepsilon}{2}.$$

Hence $(\theta,c)\mapsto\mathcal{U}_c(\theta)$ is not continuous.
Observe that in the integrable case, there exists a unique weak K.A.M. solution in each cohomology class up to the addition of a constant. Hence selecting a weak K.A.M. solution in every cohomology class is reduced in this case to choosing a constant. Using this remark, it can be proved that for the integrable case, the discounted choice is lower semi-continuous.

A.3. **An example of weak K.A.M. solution with a calibrating orbit starting from the interior of a vertical bar.** We have seen that for an ESTwD f, if u_c is a weak K.A.M. solution for $\widehat{T}^c$ and $(\tilde\theta_k)_{k\in\mathbb{Z}_-}$ calibrates its lift $\tilde u_c$, then setting for $k\in\mathbb{Z}_-$, $r_k=\frac{\partial S}{\partial\Theta}(\tilde\theta_{k-1},\tilde\theta_k)$, the sequence $(\tilde\theta_k,r_k)_{k\in\mathbb{Z}_-}$ is a piece of orbit of F such that $(\tilde\theta_0,r_0)\in\mathcal{PG}(c+\tilde u_c')$ and for all $k<0$, $(\tilde\theta_k,r_k)\in\mathcal{G}(c+\tilde u_c')$. We now construct an example of such a situation where $(\tilde\theta_0,r_0)\notin\overline{\mathcal{G}(c+\tilde u_c')}$. It can be proven that such a situation cannot happen if f is the time–t map of an autonomous Tonelli Hamiltonian flow, for any $t>0$.

Let us start from the classical pendulum Hamiltonian $H:T^*\mathbb{T}\to\mathbb{R}$ defined by

$$\forall(\theta,p)\in T^*\mathbb{T},\quad H(\theta,p)=\frac12|p|^2+\cos(2\pi\theta).$$

Let $s^+:\theta\mapsto\sqrt{2-2\cos(s\pi\theta)}$ be the function whose graph, $\mathcal{S}^+$, is the upper part of the level set $H^{-1}(\{1\})$ and $c_0=\int_0^1 s^+(\theta)d\theta$. Finally, let $t_0>0$ be a small enough real number. It is then known that if Φ_H denotes the Hamiltonian flow of H, for t_0 small enough, $\Phi_H^{t_0}$ is an ESTwD that we denote by f_0. We also denote by $S_0:\mathbb{R}^2\to\mathbb{R}$ a generating function associated to the lift $F_0:\mathbb{R}^2\to\mathbb{R}^2$ of f_0 that fixes $(0,0)$. It can be proven that at cohomology c_0, there is a unique weak K.A.M. solution u_0 for $\widehat{T}^{c_0}$ such that $u_0(0)=0$ and it is given by

$$\forall\theta\in\mathbb{R},\quad u_0(\theta)=\int_0^\theta s^+(t)dt-c_0\theta.$$

This function is C^1 and $\mathcal{G}(c_0+u_0')=\mathcal{PG}(c_0+u_0')=\mathcal{S}^+$. Moreover, $\mathcal{S}^+$ is invariant by f_0.

The dynamics of f_0 restricted to $\mathcal{S}^+\backslash\{(0,0)\}$ is going from the left to the right with a fixed point $(0,0)=(1,0)$. Let $[a_{-1},a_0)\subset(0,1)$ a fundamental domain of the projected dynamics restricted to $\mathcal{S}^+\backslash\{(0,0)\}$. This means that if $\mathcal{S}^+_{|[a_{-1},a_0)}=\big\{\big(\theta,s^+(\theta)\big),\ \theta\in[a_{-1},a_0)\big\}$, then $\mathcal{S}^+$ is the disjoint union of $\{(0,0)\}$ and of the $f_0^n(\mathcal{S}^+_{|[a_{-1},a_0)})$ when $n\in\mathbb{Z}$. In particular, $f_0\big(a_{-1},s^+(a_{-1})\big)=\big(a_0,s^+(a_0)\big)$.

Let $\varphi:\mathbb{T}\to[0,+\infty)$ be a C^2 function supported in $[a_{-1},a_0]$, we define the diffeomorphism $v_\varphi:\mathbb{A}\to\mathbb{A}$ by $(\theta,r)\mapsto\big(\theta,r+\varphi'(\theta)\big)$ and then $f_\varphi=v_\varphi\circ f_0$ that is also an EStwD. A direct computation shows that $S_\varphi:\mathbb{R}^2\to\mathbb{R}$, defined by $(\tilde\theta,\widetilde\Theta)\mapsto S_0(\tilde\theta,\widetilde\Theta)+\varphi(\widetilde\Theta)$ is the generating function of F_φ, the lift of f_φ that fixes $(0,0)$ (we still denote by $\varphi:\mathbb{R}\to\mathbb{R}$ the lift of $\varphi:\mathbb{T}\to\mathbb{R}$). As $\varphi\geqslant0$, it follows that $S_\varphi\geqslant S_0$.

For F_0, the projected Mather set at cohomology c_0 is $\{k,\ \ k\in\mathbb{Z}\}$, as $(k,0)$, $k\in\mathbb{Z}$ are the only fixed points of F_0 in $\mathcal{S}^+$. Also, the rotation number at cohomology c_0 is

0. We deduce that if $k \in \mathbb{Z}$, then $S_0(k,k) = \min_{y\in\mathbb{R}} S_0(y,y)$ by [9]. It follows that for F_φ, the projected Mather set for the 0 rotation number is also $\{k, \ \ k \in \mathbb{Z}\}$, by [9], as if $x \in \mathbb{R}$ is in this projected Mather set, then $S_\varphi(x,x) = \min_{y\in\mathbb{R}} S_\varphi(y,y) = \min_{y\in\mathbb{R}} S_0(y,y)$.

Let now $c_\varphi \in \mathbb{R}$ be the biggest cohomology class such that $\rho(c_\varphi) = 0$ for F_φ. Let $\tilde u_\varphi : \mathbb{R} \to \mathbb{R}$ be the corresponding weak K.A.M. solution at cohomology c_φ such that $\tilde u_\varphi(0) = 0$. In the following lemmas, we study properties of $\tilde u_\varphi$. We introduce $a_{-2} \in [0, a_{-1})$ such that $F_0\big(a_{-2}, c_0 + \tilde u_0'(a_{-2})\big) = \big(a_{-1}, c_0 + \tilde u_0'(a_{-1})\big)$.

Lemma A.1. *The function $\tilde u_\varphi$ is C^1 on $[0, a_{-1}]$ and equality $(c_0 + \tilde u_0')_{|[0,a_{-1}]} = (c_\varphi + \tilde u_\varphi')_{|[0,a_{-1}]}$ holds.*

Proof. Let $\tilde\theta_0 \in [0, a_{-1}]$ and let $(\tilde\theta_k)_{k\leqslant 0}$ be the unique minimizing chain starting at $\tilde\theta_0$ that calibrates $\tilde u_0$ (for S_0). From what was recalled above, $\tilde\theta_k \to 0$ and is non–decreasing with k. Moreover, as the θ_k's are not in the support of φ, the sequence $(\tilde\theta_k)_{k\leqslant 0}$ is also minimizing for S_φ (because $S_\varphi \geqslant S_0$). Hence, by Proposition 4.5, $(\tilde\theta_k)_{k\leqslant 0}$ calibrates $\tilde u_\varphi$ (for S_φ). It follows that for $k \leqslant -1$, $\tilde u_\varphi$ is derivable at $\tilde\theta_k$ and

$$c_\varphi + \tilde u_\varphi'(\tilde\theta_k) = \frac{\partial S_\varphi}{\partial \widetilde\Theta}(\tilde\theta_{k-1}, \tilde\theta_k) = \frac{\partial S_0}{\partial \widetilde\Theta}(\tilde\theta_{k-1}, \tilde\theta_k) = c_0 + \tilde u_0'(\tilde\theta_k).$$

When $\tilde\theta_0$ sweeps $[0, a_{-1}]$, $\tilde\theta_{-1}$ takes all values in $[0, a_{-2}]$.

We then extend what was just established to $(a_{-2}, a_{-1}]$. Let $\tilde\theta_0 \in (a_{-2}, a_{-1}]$ a point where $\tilde u_\varphi$ is derivable. The previous argument shows that if $(\tilde\theta_k)_{k\leqslant 0}$ is the unique calibrating chain for u_φ, then is is also the unique calibrating chain for $\tilde u_0$. Hence using the previous result,

$$\big(\tilde\theta_0, c_\varphi + \tilde u_\varphi'(\tilde\theta_0)\big) = F_\varphi\big(\tilde\theta_{-1}, c_\varphi + \tilde u_\varphi'(\tilde\theta_{-1})\big) = F_0\big(\tilde\theta_{-1}, c_0 + \tilde u_0'(\tilde\theta_{-1})\big) = \big(\tilde\theta_0, c_0 + \tilde u_0'(\tilde\theta_0)\big).$$

In the above, we used the fact that $\tilde\theta_{-1} \in [a_{-2}, a_{-1}]$ lies away from the support of φ and then F_0 and F_φ coincide on the fiber above $\tilde\theta_{-1}$. As a conclusion, restricted to $[a_{-2}, a_{-1}]$, the Lipschitz functions $\tilde u_\varphi$ and $t \mapsto \tilde u_0 + (c_0 - c_\varphi)t$ have the same derivative almost everywhere and same value at a_{-2}, then they are equal. □

The next lemma provides the values of $\tilde u_\varphi$ on $(a_{-1}, a_0]$.

Lemma A.2. *The function $\tilde u_\varphi$ is C^1 on $(a_{-1}, a_0]$ and for all $t \in (a_{-1}, a_0]$, $c_\varphi + \tilde u_\varphi'(t) = c_0 + \tilde u_0'(t) + \varphi'(t)$.*

Proof. Let us now consider the chain (a_{-2}, a_{-1}, a_0), that calibrates $\tilde u_0$ and is minimizing for S_0. By the same arguments used in the previous lemmas, the same chain (a_{-2}, a_{-1}, a_0) is also minimizing for S_φ and calibrates $\tilde u_\varphi$. It follows from Lemma 2.1 that if $(\tilde\theta_k)_{k\leqslant 0}$ calibrates $\tilde u_\varphi$, and if $\tilde\theta_0 \in (a_{-1}, a_0)$, then $\tilde\theta_{-1} \in (a_{-2}, a_{-1})$. If now $\tilde\theta_0$ is a derivability point of $\tilde u_\varphi$, then $\big(\tilde\theta_0, c_\varphi + \tilde u_\varphi'(\theta_0)\big) = F_\varphi\big(\tilde\theta_{-1}, c_\varphi + \tilde u_\varphi'(\theta_{-1})\big) \in \mathcal{G}(c_0 + \tilde u_0' + \varphi')_{|(a_{-1}, a_0)}$. Indeed, $F_\varphi\big(\mathcal{G}(c_0 + \tilde u_0')_{|(a_{-2}, a_{-1})}\big) = v_\varphi\big(\mathcal{G}(c_0 + \tilde u_0')_{|(a_{-1}, a_0)}\big) = \mathcal{G}(c_0 + \tilde u_0' + \varphi')_{|(a_{-1}, a_0)}$.

We conclude that $\mathcal{G}(c_\varphi + \tilde u_\varphi')_{|(a_{-1}, a_0)} \subset \mathcal{G}(c_0 + \tilde u_0' + \varphi')_{|(a_{-1}, a_0)}$. As previously, this implies that the inclusion must be an equality and this proves the lemma. □

We now specify how to chose φ in order to obtain our example:

Hypothesis: Let $a_1 > a_0$ such that $F_0\big(a_0, c_0 + \tilde u_0'(a_0)\big) = \big(a_1, c_0 + \tilde u_0'(a_1)\big)$. We assume that φ is chosen as follows: there exists $d \in (a_0, a_1)$ such that $F_\varphi\big(\mathcal{G}(c_\varphi +$

$\tilde{u}'_\varphi)_{|(a_{-1},a_0)}\big) = F_0\big(\mathcal{G}(c_\varphi + \tilde{u}'_\varphi)_{|(a_{-1},a_0)}\big)$ is the union of a graph above (a_0, d), a graph above (d, a_1) and a non trivial vertical interval above $\{d\}$.

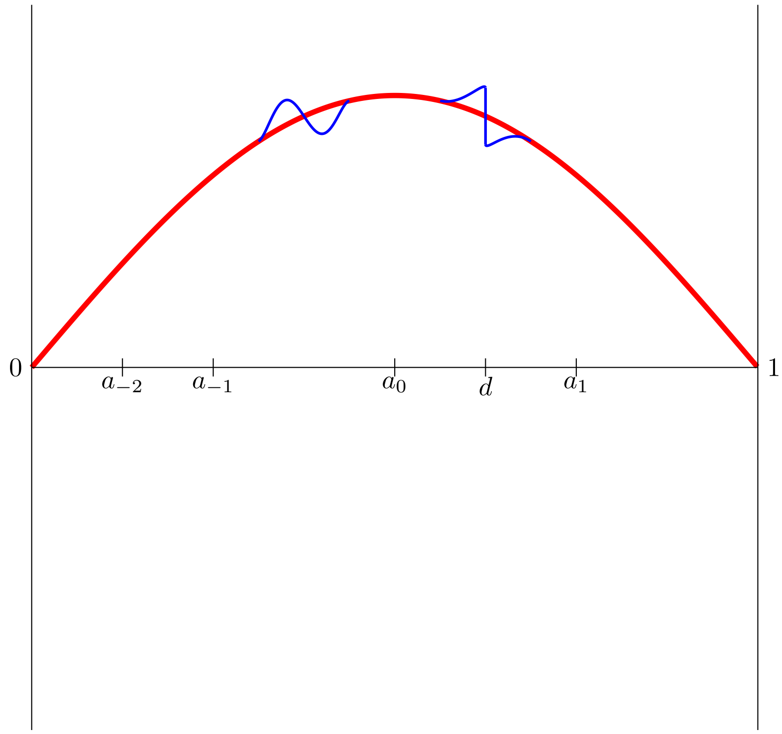

FIGURE 6. In red the graph of $c_0 + u'_0$. In blue, the perturbation giving $c_\varphi + u'_\varphi$ on $[0, a_1]$.

The next lemma provides a description of the weak K.A.M. solution for S_φ on (a_0, a_1).

Lemma A.3. *Under the previous hypothesis,*

$$\mathcal{PG}(c_\varphi + u'_\varphi)_{|(a_0,a_1)} = F_\varphi\big(\mathcal{G}(c_\varphi + \tilde{u}'_\varphi)_{|(a_{-1},a_0)}\big).$$

Proof. Arguing as in the two previous lemmas, we find that $\mathcal{G}(c_\varphi + u'_\varphi)_{|(a_0,a_1)} \subset F_\varphi\big(\mathcal{G}(c_\varphi + \tilde{u}'_\varphi)_{|(a_{-1},a_0)}\big)$. As the right hand side set is a graph above $(a_0, d) \cup (d, a_1)$, again arguing as previously we obtain that u_φ is C^1 on $(a_0, d) \cup (d, a_1)$ and that $\mathcal{G}(c_\varphi + u'_\varphi)_{|(a_0,d)\cup(d,a_1)} = F_\varphi\big(\mathcal{G}(c_\varphi + \tilde{u}'_\varphi)_{|(a_{-1},a_0)}\big) \setminus \{d\} \times \mathbb{R}$. The result follows. $\square$

Next we prove that this construction indeed yields the desired example. To that end, we prove that any negative orbit of F_φ starting from the vertical bar of $\mathcal{PG}(c_\varphi + \tilde{u}'_\varphi)$ above d calibrates $\tilde{u}_\varphi$.

Proposition A.2. *Let* $(\tilde{\theta}_0, r_0) \in \mathcal{PG}(c_\varphi + \tilde{u}'_\varphi) \cap \{d\} \times \mathbb{R}$ *and* $(\tilde{\theta}_{-1}, r_{-1}) = F_\varphi^{-1}(\tilde{\theta}_0, r_0)$. *Then*

$$\tilde{u}_\varphi(\tilde{\theta}_0) - \tilde{u}_\varphi(\tilde{\theta}_{-1}) = S_\varphi(\tilde{\theta}_{-1}, \tilde{\theta}_0) + c_\varphi(\tilde{\theta}_{-1} - \tilde{\theta}_0) + \alpha_\varphi(c_\varphi),$$

where α_φ *is Mather's function associated to* F_φ.

Proof. If $\tilde\theta_0 \in (a_0, a_1)$ and $\tilde\theta_0 \neq d$, then $\tilde u_\varphi$ is derivable at $\tilde\theta_0$. Setting $(\tilde\theta_{-1}, r_{-1}) = F_\varphi^{-1}\big(\tilde\theta_0, c_\varphi + \tilde u'_\varphi(\tilde\theta_0)\big)$, then

$$\tilde u_\varphi(\tilde\theta_0) - \tilde u_\varphi(\tilde\theta_{-1}) = S_\varphi(\tilde\theta_{-1}, \tilde\theta_0) + c_\varphi(\tilde\theta_{-1} - \tilde\theta_0) + \alpha_\varphi(c_\varphi),$$

using classical results on weak K.A.M. solutions recalled page 12 $\big($see Equation (5)$\big)$.

Let now $[\tilde\theta(0), \tilde\theta(1)] = \pi_1\big((\pi_1 \circ F_{\varphi|\mathcal{G}(c_\varphi + \tilde u'_\varphi)})^{-1}(\{d\})\big) \subset (a_{-1}, a_0)$. For $t \in [0,1]$ we define $\tilde\theta(t) = (1-t)\tilde\theta(0) + t\tilde\theta(1)$ and $\big(d, R(t)\big) = F_\varphi\big(\tilde\theta(t), c_\varphi + \tilde u'_\varphi\big(\tilde\theta(t)\big)\big)$. As for $i \in \{0,1\}$, $F_\varphi\big(\tilde\theta(i), c_\varphi + \tilde u'_\varphi\big(\tilde\theta(i)\big)\big) \in \overline{\mathcal{G}(c_\varphi + \tilde u'_\varphi)}$, equality

$$\tilde u_\varphi(d) - \tilde u_\varphi\big(\tilde\theta(i)\big) = S_\varphi(\tilde\theta(i), d) + c_\varphi(\tilde\theta(i) - d) + \alpha_\varphi(c_\varphi),$$

still holds. Let now $t \in (0,1)$, one computes using the definition of the generating function S_φ (see Equations (1)) that

$$\begin{aligned}
&\tilde u_\varphi(d) - \tilde u_\varphi\big(\tilde\theta(t)\big) - c_\varphi(\tilde\theta(t) - d)\\
&\qquad = \tilde u_\varphi(d) - \tilde u_\varphi\big(\tilde\theta(0)\big) - c_\varphi(\tilde\theta(0) - d) - \int_0^t \big[\tilde u'_\varphi\big(\tilde\theta(s)\big) + c_\varphi\big]\theta'(s)ds\\
&\qquad = S_\varphi(\tilde\theta(0), d) + \alpha_\varphi(c_\varphi) + \int_0^t \frac{\partial S}{\partial\tilde\theta}(\tilde\theta(s), d)\tilde\theta'(s)ds\\
&\qquad = S_\varphi(\tilde\theta(t), d) + \alpha_\varphi(c_\varphi).
\end{aligned}$$

This finishes the proof. □

In order to conclude, we explain how to construct the function φ as desired. In fact, we rather construct $F_\varphi\big(\mathcal{G}(c_\varphi + \tilde u'_\varphi)_{|(a_{-1}, a_0)}\big)$. To that end, let $\varepsilon_0 > 0$ be a small real number to be specified through the construction. Let $d \in (a_0, a_1)$ and assume $\varepsilon_0 < \min(d - a_0, a_1 - d)$. We set $R = \{(\tilde\theta, s^+(\tilde\theta) + r),\ \ \theta \in [d - \varepsilon_0, d + \varepsilon_0], |r| \leqslant \varepsilon_0\}$. Let $\Psi_0 : R \mapsto [d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0]$ defined by $\Psi_0(\tilde\theta, r) = (\tilde\theta, r - s^+(\tilde\theta))$. Obviously, Ψ_0 preserves each vertical $V_{\tilde\theta}$.

Let $\tilde\theta \in (a_{-1}, a_0)$, and let $\tilde\theta_0 \in (a_0, a_1)$ such that $F_0(V_{\tilde\theta})$ and $\mathcal{S}^+$ intersect at $\big(\tilde\theta_0, s^+(\tilde\theta_0)\big)$. As $\mathcal{S}^+$ is F_0 invariant and F_0 is a twist map, by [3], at this intersection point, the slope of $F_0(V_{\tilde\theta})$ is greater than $(s^+)'(\tilde\theta_0)$. If $(\tilde\theta_0, r) \in [d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0]$, and $\tilde\theta$ is the unique real number such that $(\tilde\theta_0, r) \in \Psi_0 \circ F_0(V_{\tilde\theta})$ let $v_1(\tilde\theta_0, r)$ be the slope of $\Psi_0 \circ F_0(V_{\tilde\theta})$ at $(\tilde\theta_0, r)$. Then up to taking ε_0 smaller, by the previous fact and continuity, we may assume that $v_1(\tilde\theta_0, r) > 0$ for all $(\tilde\theta_0, r) \in [d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0]$. Let then $\varepsilon_1 > 0$ such that $v_1(\tilde\theta_0, r) > \varepsilon_1$ for all $(\tilde\theta_0, r) \in [d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0]$.

Finally, let $\varepsilon_2 > 0$ and $\Psi_1 : (\tilde\theta, r) \mapsto (\tilde\theta - \varepsilon_2 r, r)$. Each vertical is sent by Ψ_1 to a straight line of slope $-\varepsilon_2^{-1}$. We assume that ε_2 is chosen small enough so that in $\Psi_1([d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0])$ the curves $\Psi_1 \circ \Psi_0 \circ F_0(V_{\tilde\theta})$ still are graphs of increasing functions with derivative greater than ε_1.

Let now $\rho : \mathbb{R} \to \mathbb{R}$ be the C^∞ function supported in $[-1,1]$ defined by

$$\forall x \in [-1,1], \quad \rho(x) = \left(\int_{-1}^1 \exp[(s^2-1)^{-1}]ds\right)^{-1} \exp[(x^2-1)^{-1}],$$

and if $\varepsilon > 0$ we define $\rho_\varepsilon : x \mapsto \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. For $s > 0$ small enough, we define the function g_s as the continuous piecewise affine function that vanishes outside of $[d - \varepsilon_0 + s, d + \varepsilon_0 - s]$ and that is $x \mapsto \frac{d-x}{\varepsilon_2}$ for $x \in [d - s, d + s]$ and that is affine on each remaining connected component of $\mathbb{R}$. Finally, we set $h_s = \rho_{s/2} * g_s$ where $*$ stands for the regular convolution product.

There exists $\varepsilon_3 > 0$ such that for $s < \varepsilon_3$ the following hold:

(1) h_s is C^∞ and well defined and the non–vanishing part of its graph is included in $\Psi_1([d - \varepsilon_0, d + \varepsilon_0] \times [-\varepsilon_0, \varepsilon_0])$,
(2) h_s coincides with $x \mapsto \frac{d-x}{\varepsilon_2}$ on $[d - s/2, d + s/2]$ and has derivative greater than $-\varepsilon_2^{-1}$ elsewhere,
(3) $h_s' < \varepsilon_1$ on $\mathbb{R}$.

The second point implies that $(\Psi_1 \circ \Psi_0)^{-1}\big(\mathcal{G}(h_{s|[a_0,a_1]})\big)$ coincides with V_d on a non trivial segment, it coincides with $\mathcal{S}^+$ on neighborhoods of a_0 and a_1, and it is the graph of a smooth function apart for the vertical part on V_d.

The third point implies that $F_0^{-1}\Big((\Psi_1 \circ \Psi_0)^{-1}\big(\mathcal{G}(h_{s|[a_0,a_1]})\big)\Big)$ is transverse to the vertical foliation. Hence it is the graph of a smooth function $(s^+ + \varphi_s)$ where φ_s is supported on $[a_{-1}, a_0]$. We now wish to set $\varphi(\tilde\theta) = \int_{a_{-1}}^{\tilde\theta} \varphi_s(t)dt$. The problem is that there is a priori no reason that $\int_{a_{-1}}^{a_0} \varphi_s(t)dt = 0$ so that φ would be compactly supported.

To remedy this, we slightly modify our construction. If $s, s' < \varepsilon_3$ we set $h_{s,s'} = h_s$ on $[a_0, d]$ and $h_{s,s'} = h_{s'}$ on $[d, a_1]$. Again, it coincides with V_d on a non trivial segment, it coincides with $\mathcal{S}^+$ on neighborhoods of a_0 and a_1, and it is the graph of a smooth function apart for the vertical part on V_d. Then, $F_0^{-1}\Big((\Psi_1 \circ \Psi_0)^{-1}\big(\mathcal{G}(h_{s|[a_0,a_1]})\big)\Big)$ is the graph of a smooth function $(s^+ + \varphi_{s,s'})$ where $\varphi_{s,s'}$ is supported on $[a_{-1}, a_0]$. Now, by the intermediate value theorem, it is possible, given s small, to find s' such that $\int_{a_{-1}}^{a_0} \varphi_{s,s'}(t)dt = 0$ and defining $\varphi(\tilde\theta) = \int_{a_{-1}}^{\tilde\theta} \varphi_{s,s'}(t)dt$ on $[a_{-1}, a_0]$ we obtain the desired function.

Remark.

(1) In the constructed example, all backward orbits starting on the vertical bar of $\mathcal{PG}(c_\varphi + \tilde u_\varphi')$ above d calibrate the weak K.A.M. solution. Modifying slightly the example is is also possible to have a unique backward orbit starting on the vertical bar of $\mathcal{PG}(c_\varphi + \tilde u_\varphi')$ above d calibrate the weak K.A.M. solution.
(2) The same construction can be made, starting from an invariant circle of arbitrary rotation number.

Appendix B. Some results concerning the full pseudographs

Most of the results that follow are standard and even hold in all dimension. One can find them in similar of different formulations in [17]. However, we provide proofs for the reader's convenience.

B.1. An equivalent definition.

DEFINITION. Let $u : \mathbb{R} \to \mathbb{R}$ be a K semi-concave function. Then $p \in \mathbb{R}$ is a super-derivative of u at $x \in \mathbb{R}$ if

$$\forall y \in \mathbb{R}, \quad u(y) - u(x) - p(y-x) \leqslant \frac{K}{2}(y-x)^2.$$

We denote the set of super-derivatives of u at x by $\partial^+ u(x)$. It is a convex set.

Observe that a derivative is always a super-derivative. If $u : \mathbb{R} \to \mathbb{R}$ is K-semi-concave, then $x \mapsto u(x) - \frac{K}{2}x^2$ is concave and thus locally Lipschitz, and $x \mapsto u'(x) - Kx$ is non-increasing. Hence a 1-periodic K-semi-concave function is K-Lipschitz.

Observe also that $\bigcup\limits_{x\in\mathbb{T}} \{x\} \times \partial^+ u(x)$ is compact.

Proposition B.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a K-semi-concave function. Then, for every $x \in \mathbb{R}$, we have*

$$\partial u(x) = \{x\} \times \partial^+ u(x).$$

Hence the full pseudograph of u is also the subbundle of all the super-derivatives of u.

Proof. Let us prove the inclusion $\partial u(x) \subset \{x\} \times \partial^+ u(x)$. Let us consider $(x,p) \in \partial u(x)$. Then there exist $(x,p_-), (x,p_+) \in \overline{\mathcal{G}(u')}$ such that $p_- \leqslant p \leqslant p_+$ and there exist two sequences $(x_n,p_n), (y_n,q_n) \in \mathcal{G}(u')$ that respectively converge to (x,p_-), (x,p_+). Every derivative is a super-derivative and a limit of super-derivatives is a super-derivative. Hence, we have $p_-, p_+ \in \partial^+ u(x)$. By convexity of $\partial^+ u(x)$, we deduce that $p \in \partial^+ u(x)$.

Let us now prove the reverse inclusion. Being K-semi-concave, u is K-Lipschitz, hence the set of all its super-derivatives is bounded (by K). If $x \in \mathbb{R}$, we have then $\partial^+ u(x) = [p_-, p_+]$ with $-K \leqslant p_- \leqslant p_+ \leqslant K$. We will prove that $(x,p_-), (x,p_+) \in \partial u(x)$. We have

$$\forall y \in \mathbb{R}, \quad u(y) - u(x) - p_-(y-x) \leqslant \frac{K}{2}(y-x)^2$$
$$\text{and} \quad u(y) - u(x) - p_+(y-x) \leqslant \frac{K}{2}(y-x)^2.$$

This implies that

- for $y > x$, we have
$$\frac{u(y)-u(x)}{y-x} \leqslant p_- + \frac{K}{2}(y-x);$$
- for $y < x$, we have
$$\frac{u(y)-u(x)}{y-x} \geqslant p_+ + \frac{K}{2}(y-x).$$

Recall that $\frac{u(y)-u(x)}{y-x} = \frac{1}{y-x}\int_x^y u'(t)dt$. This gives the existence of two sequences $(x_n) \in (-\infty, x)$ and $(y_n) \in (x, +\infty)$ that converge to x where u is differentiable and

$$\limsup u'(x_n) \geqslant p_+ \quad \text{and} \quad \liminf u'(y_n) \leqslant p_-.$$

As we know that a derivative is a super-derivative, that the set of super-derivatives is closed and that $\partial^+u(x) = [p_-, p_+]$, we deduce that

$$\big(x, \lim u'(x_n)\big) = (x, p_+) \in \partial u(x) \quad \text{and} \quad \big(x, \lim u'(y_n)\big) = (x, p_-) \in \partial u(x).$$

$\square$

B.2. **Proof of Lemma 2.5.** We just recall the argument of the proof of

Lemma B.1. *For all $c \in \mathbb{R}$, $\mathcal{PG}(c + u'_c)$ is a Lipschitz one dimensional compact manifold that is an essential circle.*

Proof. It is proved in [4], that for every $c \in \mathbb{R}$ and every K-semi-concave function $u : \mathbb{T} \to \mathbb{R}$, there exists $\tau > 0$ such that $\varphi_{-\tau}\big(\mathcal{PG}(c+u')\big)$ is the graph of a Lipschitz function, where (φ_t) is the flow of the pendulum. This gives the wanted result. $\square$

B.3. **Proof of Proposition 2.2.** Let us now prove the following proposition[17].

Proposition B.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of equi-semi-concave functions from $\mathbb{T}$ to $\mathbb{R}$ that converges (uniformly) to a function f (that is hence also semi-concave).*
Then $\big(\mathcal{PG}(f'_n)\big)$ converges to $\mathcal{PG}(f')$ for the Hausdorff distance.

Proof. Let us prove that the lim sup of the $\mathcal{PG}(f'_n)$ is in $\mathcal{PG}(f')$. Up to a subsequence, we consider $(x_n, p_n) \in \mathcal{PG}(f'_n)$ that converges to some (x, p), and we want to prove that $(x, p) \in \mathcal{PG}(f')$. We have

$$\forall n, \forall y \in \mathbb{R}, \quad f_n(y) - f_n(x_n) - p_n(y - x_n) \leqslant \frac{K}{2}(y - x_n)^2.$$

Taking the limit, we deduce that $(x, p) \in \mathcal{PG}(f')$.

Let us now assume that $\big(\mathcal{PG}(f'_n)\big)$ doesn't converge to $\mathcal{PG}(f')$. There exists a point $(x, p) \in \mathcal{PG}(f')$, $r > 0$ and $N \geqslant 1$ such that, up to a subsequence,

$$\forall n \geqslant N, \quad \mathcal{PG}(f'_n) \cap B\big((x, p), r\big) = \varnothing.$$

Hence, for n large enough, $\mathcal{PG}(f'_n)$ is contained in a small neighbourhood of a simple arc (and not loop). This implies that for n large enough, $\mathcal{PG}(f'_n)$ doesn't separate the annulus into two unbounded connected components, a contradiction.

$\square$

Appendix C. Sketch of the proof of point 3 page 16

We wish to explain why if $u : \mathcal{M}\big(\rho(c)\big) \to \mathbb{R}$ is dominated, then there exists only one extension U of u to $\mathbb{T}$ that is a weak K.A.M. solution for $\widehat{T}^c$ that is given by

$$\forall x \in \mathbb{T}, \quad U(x) = \inf_{\substack{\pi(\theta) \in \mathcal{M}(\rho(c)) \\ \pi(\theta') = x}} \tilde{u}(\theta) + \mathcal{S}^c(\theta, \theta')$$

where $\mathcal{S}^c(\theta, \Theta) = \inf\limits_{n\in\mathbb{N}} \big(\mathcal{S}^c_n(\theta, \Theta) + n\alpha(c)\big)$.

- It is a general fact that if $\pi(\theta) \in \mathcal{M}\big(\rho(c)\big)$ the function $\theta' \mapsto \mathcal{S}^c(\theta, \theta')$ is a weak K.A.M solution that vanishes at $\theta' = \theta$ (see [48, Definition 2.1 and Proposition 2.8] recalling that the function $\mathcal{S}^c$ corresponds to the lift of the Mañé potential φ in the reference and that our Mather set $\mathcal{M}\big(\rho(c)\big)$ is included in the Aubry set). As the set of weak K.A.M. is invariant by

[17] The statement holds in arbitrary dimension and follows from the same result for concave functions. We present here a simple proof relying on the 1-dimensional setting.

addition of constants and an infimum of weak K.A.M. solutions is a weak K.A.M. solution ([48, Lemma 2.33]) it follows that U is a weak K.A.M. solution.

- To prove that $U = u$ on $\mathcal{M}\big(\rho(c)\big)$ just notice that as u is dominated, if $x \in \mathcal{M}\big(\rho(c)\big)$ and $\pi(\theta) = x$

$$\forall \theta' \in \pi^{-1}\left(\mathcal{M}\big(\rho(c)\big)\right), \quad \tilde{u}(\theta') + \mathcal{S}^c(\theta', \theta) \geqslant \tilde{u}(\theta) = u(x) + \mathcal{S}^c(\theta, \theta).$$

- It remains to prove that U is unique. This follows from the fact that if two weak K.A.M. solutions U_1 and U_2 coincide on $\mathcal{M}\big(\rho(c)\big)$ they are equal.

Let $x_0 \in \mathbb{T}$. One constructs inductively a sequence $(x_n)_{n\leqslant 0}$ such that

$$\forall n < 0, \quad U_1(x_0) = U_1(x_n) + \sum_{k=n}^{-1} S^c(x_k, x_{k+1}).$$

As U_2 is a weak K.A.M. (hence dominated) one also has

$$\forall n < 0, \quad U_2(x_0) \leqslant U_2(x_n) + \sum_{k=n}^{-1} S^c(x_k, x_{k+1}).$$

Hence $U_2(x_0) - U_1(x_0) \leqslant U_2(x_n) - U_1(x_n)$. To conclude, one proves, using a Krylov-Bogoliubov type argument that there exists a subsequence $(x_{\varphi(n)})$ that converges to a point $x \in \mathcal{M}\big(\rho(c)\big)$, hence proving that $U_2(x_0) - U_1(x_0) \leqslant 0$. Then the result follows by a symmetrical argument.

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