ALGÈBRES DE LIE

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1. Day 2

- (a) Homomorphisms of Lie algebras
- (b) Representations
- (c) Reducible representations
- (d) Examples (adjoint representation)

1.1. **Homomorphisms of Lie algebras.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras over K. A *homomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ is a K-linear map such that

$$[\phi(X), \phi(Y)] = \phi([X, Y]).$$

One has the expected isomorphism theorems.

Theorem 1.1. *(a) Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then* $\ker(\phi)$ *is a Lie ideal in $\mathfrak{g}$.*

(b) Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie ideal in $\mathfrak{g}$. Then the bracket on $\mathfrak{g}$ defines a Lie algebra structure on the quotient space $\mathfrak{g}/\mathfrak{k}$, and $\mathfrak{g}/\mathfrak{k}$ is the universal object for Lie algebra homomorphisms from $\mathfrak{g}$ that are trivial on $\mathfrak{k}$.

(c) Suppose $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$ is a sequence of ideals of $\mathfrak{g}$. Then $\mathfrak{h}/\mathfrak{k}$ is an ideal of $\mathfrak{g}/\mathfrak{k}$ and there is an isomorphism

$$(\mathfrak{g}/\mathfrak{k})/(\mathfrak{h}/\mathfrak{k}) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}.$$

(d) If $\mathfrak{k}$ and $\mathfrak{h}$ are ideals of $\mathfrak{g}$ then so are $\mathfrak{k} + \mathfrak{h}$ and $\mathfrak{k} \cap \mathfrak{h}$, and there is an isomorphism of Lie algebras

$$\mathfrak{k}/\mathfrak{k} \cap \mathfrak{h} \xrightarrow{\sim} (\mathfrak{k} + \mathfrak{h})/\mathfrak{h}.$$

One already knows the isomorphisms for vector spaces, and one has to show that the maps respect the Lie algebra structure. But this is easy.

If $\phi : G \to H$ is a homomorphism of Lie groups, then $d\phi : T_{G,e} \to T_{H,e}$ is a homomorphism of Lie algebras (and we have seen that the converse holds in the semisimple case.

1.2. **Representations.**

Definition 1.2. Let V be a vector space over K, and endow $End(V)$ with the structure of Lie algebra defined by the bracket. Let $\mathfrak{g}$ be a Lie algebra over K. A representation of $\mathfrak{g}$ on V is a homomorphism

$$\phi : \mathfrak{g} \to End(V)$$

of Lie algebras.

The definition immediately gives rise to the usual categorical constructions: subrepresentation, quotient representation, etc. For example

Definition 1.3. Let $\phi : \mathfrak{g} \to End(V)$, $\phi' : \mathfrak{g} \to End(V')$ be two representations of V. A homomorphism $f : (\phi, V) \to (\phi', V')$ is a linear map $f : V \to V'$ that commutes with the action of $\mathfrak{g}$ (it makes the obvious triangle commute).

In the above definition, we can define $\phi \oplus \phi' : \mathfrak{g} \to End(V \oplus V')$ by setting

$$(\phi \oplus \phi')(X)(v, v') = (\phi(X)(v), \phi'(X)v'),\ X \in \mathfrak{g}, v \in V, v' \in V'.$$

If $f : (\phi, V) \to (\phi', V')$ is a homomorphism then so is λf for any $\lambda \in K$. Thus $Hom(V, V')$ (we drop the ϕ's) is a K-vector space. Representations of $\mathfrak{g}$ thus form a K-linear additive category; and it is even an abelian category.

Moreover, there is an action on $V \otimes V'$:

$$(\phi \otimes \phi')(X)(v \otimes v') = \phi(X)(v) \otimes v' + v \otimes \phi'(X)v',\ X \in \mathfrak{g}, v \in V, v' \in V'.$$

Proof. We have to show that

$$(\phi \otimes \phi')([X, Y]) = [(\phi \otimes \phi')(X), (\phi \otimes \phi')(Y)].$$

In other words

(1.4)

$$\phi([X, Y])(v) \otimes v' + v \otimes \phi'([X, Y])v'$$
$$= (\phi \otimes \phi')(X)(\phi(Y)v \otimes v' + v \otimes \phi(Y)v') - (\phi \otimes \phi')(Y)(\phi(X)(v) \otimes v' + v \otimes \phi'(X)v').$$

This is a simple computation; however, the left-hand side has four terms upon expansion, whereas the right-hand side has eight terms, so some terms must cancel in order to obtain an equality. □

So $Rep(\mathfrak{g})$ is a monoidal category.

There is also the contragredient: if $V^\vee = Hom(V, K)$ then $\phi^\vee(X)(f)(v) = -f(\phi(X)(v))$. More generally, on $Hom(V, V')$ we define

$$s(X)(f)(v) = \phi'(X)(f(v)) - f(\phi(X)v), f \in Hom(V, V'), v \in V$$

The letter s means nothing and is only introduced for convenience in the next few lines. The proof that this is a representation deserves writing out, because there is a delicate point in the middle of the proof. Here is what has to be checked: for all $X, Y \in \mathfrak{g}$, all $f \in Hom(V, V')$, and all $v \in V$, we have the relation

$$s([X,Y])(f)(v) = s(X)(s(Y)(f))(v) - s(Y)(s(X)(f))(v).$$

The left hand side is given by $\phi'([X,Y])(f(v)) - f(\phi([X,Y])(v))$. Since ϕ and ϕ' are both representations, each of these terms can be expanded. We have the unproblematic term

$$\phi'([X,Y])(f(v)) = \phi'(X) \circ \phi'(Y)(f(v)) - \phi'(Y) \circ \phi'(X)(f(v))$$

and then

$$-f(\phi(X) \circ \phi(Y)v) + f(\phi(Y) \circ \phi(X)v)$$

(since f is a linear map).

The right hand side has two parts, of which the first is

(1.5)

$$\begin{aligned} & s(X)(\phi'(Y)(f(v)) - f(\phi(Y)v)) \\ =& \phi'(X) \circ \phi'(Y)(f(v)) - \phi'(Y)(f(\phi(X)v)) - \phi'(X)f(\phi(Y)v) + f(\phi(Y) \circ \phi(X)v). \end{aligned}$$

Note that the $\phi(X)$ has been inserted between the $\phi(Y)$ and the v, and not in front of $\phi(Y)$!!! This is because X is acting on the function $v \mapsto f(\phi(Y)v)$. One should take the time to understand why this is the correct formula.

The second part of the right hand side is obtained from the first by exchanging the role of X and Y, but appears with a $-$ sign:

(1.6)

$$\begin{aligned} & - s(Y)(s(X)(f))(v) \\ =& - \phi'(Y) \circ \phi'(X)(f(v)) + \phi'(X)(f(\phi(Y)v)) + \phi'(Y)f(\phi(X)v) - f(\phi(X) \circ \phi(Y)v). \end{aligned}$$

Adding the two parts of the right hand side, one sees that the mixed terms cancel, and what remains is

$$\phi'(X) \circ \phi'(Y)(f(v)) + f(\phi(Y) \circ \phi(X)v) - \phi'(Y) \circ \phi'(X)(f(v)) - f(\phi(X) \circ \phi(Y)v).$$

This coincides with the left-hand side and completes the proof that

$$s([X,Y]) = [s(X), s(Y)].$$

Proposition 1.7. *Let V, V' be two finite-dimensional representations of $\mathfrak{g}$. Then the natural isomorphism of vector spaces*

$$V^\vee \otimes V' \xrightarrow{\sim} Hom(V, V'); \lambda \otimes v' \mapsto [v \mapsto \lambda(v)v']$$

is an isomorphism of representations.

Proof. Exercise. □

In particular,

$$End_{\mathfrak{g}}(V) = Hom(V,V)^{\mathfrak{g}} = \{\in Hom_K(V,V) \mid \phi(X)f(v) = f((\phi(X)v)) \forall X \in \mathfrak{g}, v \in V\}.$$

1.3. **Examples.**

Example 1.8. *Let $V = K$, and $\phi : \mathfrak{g} \to K = End(K)$ be the zero map. This is the trivial representation.*

More generally, if (ϕ, V) is any representation, we set

$$V^{\mathfrak{g}} = \{v \in V \mid Xv = 0 \forall X \in \mathfrak{g}\}.$$

Example 1.9. *The map $ad : \mathfrak{g} \to End(\mathfrak{g})$ is a representation, called the adjoint representation.*

Proof. We have to show that $ad_{[X,Y]} = [ad_X, ad_Y]$, in other words for all $Z \in \mathfrak{g}$,

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

Equivalently,

$$0 = [[X, Y], Z] + [[Y, Z], X] + [Y, [X, Z]] = [[X, Y], Z] + [[Y, Z], X] - [[X, Z], Y]$$

$$= [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$$

and you recognize once again the Jacobi identity. □

1.4. **Reducible representations.** A representation (ϕ, V) is *irreducible* if there is no subrepresentation $W \subset V$ other than $W = 0$ or $W = V$. We say V is *reducible* if it is not irreducible, and V is *completely reducible* if V is isomorphic to $\oplus_{i \in I} V_i$ with each V_i an irreducible representation.

A representation (ϕ, V) is *indecomposable* if it is not isomorphic to the direct sum of two or more representations. A representation can be reducible but indecomposable. For example, if $\mathfrak{g} = K$, the one-dimensional abelian Lie algebra, let X be a generator, $V = K^2$, and $\phi(X) = e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, K)$. Then

$$\ker \phi(X) = K \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is the only non-trivial subrepresentation. This is the typical example.

In general, if K is algebraically closed, and $\mathfrak{g} = K$ as above, then any representation (ϕ, V) has a one-dimensional invariant subspace. Indeed, $\phi(X)$ has a non-zero eigenvector v such that $\phi(X)(v) = \lambda v$. Then (ϕ, V) is completely reducible if and only if $\phi(X)$ is a diagonalizable matrix. The classification of representations of $\mathfrak{g}$ is then given by the Jordan decomposition.

Definition 1.10. The Lie algebra $\mathfrak{g}$ is semisimple if every representation of $\mathfrak{g}$ is completely reducible.

Lemma 1.11 (Schur's Lemma)**.** *Let K be algebraically closed and let (ϕ, V) be a finite-dimensional irreducible representation of $\mathfrak{g}$. Then $End_{\mathfrak{g}}(V) = Hom(V, V)^{\mathfrak{g}} = K \cdot Id_V$.*

Proof. Let $f \in End_{\mathfrak{g}}(V)$. Since K is algebraically closed, f has a non-zero eigenvector v say $f(v) = \lambda v$. Let $V_\lambda \subset V$ be the λ-eigenspace of V. Since V is irreducible it suffices to show that V_λ is invariant under $\mathfrak{g}$. But since f commutes with $\mathfrak{g}$ this is clear. □

Lemma 1.12. *Let (ϕ, V) be a finite-dimensional representation of $\mathfrak{g}$. The following conditions are equivalent.*

(a) *If $W \subset V$ is a subrepresentation then there is a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*
(b) *V is completely reducible.*
(c) *V is the sum of its irreducible subrepresentations.*

Proof. This is an elementary induction on dimension but it is worth seeing it once.

(a) ⇒ (b): Induction on $\dim V$. If V is irreducible, in particular if $\dim V \leq 1$, there is nothing to show. Suppose it is known up to $\dim V - 1$. Let $0 \neq W \subset V$ be an irreducible representation, which we may assume to be of codimension > 0. Write $V = W \oplus V'$ with $V' \subset V$ of codimension $= \dim W > 0$. It suffices to show that V' satisfies condition (a). Let $V' \supset W'$ and let $V'' = W + W' = W \oplus W' \subset V$. Then $V = V'' \oplus W''$ for some $W'' \subset V$, with codimension $= \dim V'/W'$. Consider $pr : V \to V'$ with kernel W, and let $U = pr(W'')$. Since $W'' \cap W = 0$, $\dim U = \dim W''$. Since $W'' \cap W' = 0$, $U \cap W' = 0$, and therefore $V' = W' \oplus U$. Thus V' satisfies (a), and so is a sum of irreducibles by induction; thus so is V.

(b) ⇒ (c): Indeed, V is a direct sum of some of its irreducible subrepresentations, so this is a tautology.

(c) ⇒ (a): Let $W \subset V$ be a subrepresentation and find W'. We may assume W of positive codimension a. By (c) there is an irreducible subrepresentation $U \subset V$ that is not contained in W. Thus $U \cap W = 0$ and $W' = W \oplus U \subset V$ is of codimension $< a$. By induction it has a supplement: $V = W' \oplus U' = W \oplus (U \oplus U')$ and we are done. □