AUTOMORPHIC REPRESENTATIONS OF $GL(n)$

1. Local theory of representations of GL(n)

$GL(n,F)$, F nonarchimedean.

In contrast to the case $n = 2$, there is no point in isolating three classes of representations at this stage. I will have more to say in later lectures, but for now I will be satisfied with a coarse classification. Let $\mathcal{O} \subset F$ be the ring of integers.

1.1.1 Principal series. Let $B \subset GL(n)$ be the upper-triangular Borel subgroup, $B(F) = AN$, with $A = F^{\times,n}$ the diagonal subgroup, with maximal compact subgroup $A^o = \mathcal{O}^{\times,n}$ The n-tuple $\{\alpha_1, \dots, \alpha_n\}$ defines a character $\chi : A = F^{\times,n} \to \mathbb{C}^\times$. Then χ lifts to a homomorphism $B \to \mathbb{C}^\times$ and we let $\pi(\chi) = Ind_{B(F)}^{GL(n,F)}\chi$, where the induction is normalized (as for $n = 2$ to make this generically independent of the order of the α_i. In most cases $\pi(\chi)$ is an irreducible representation, the exceptions generalizing the construction of the Steinberg (special) representation for $n = 2$.

Suppose each α_i is unramified. We identify $A/A^o = (F^\times/\mathcal{O}^\times)^n \xrightarrow{\sim} \mathbb{Z}^n$; then χ sends the i-th generator e_i of $\mathbb{Z}^n$ (an orientation is provided by the absolute value) to α_i. In this case

$$\dim \pi(\chi)^{GL(n,\mathcal{O})} = 1$$

as one checks directly; a basis is given by the function $f_0(k) = 1$ for all $k \in GL(n,\mathcal{O})$. When $\pi(\chi)$ is irreducible, it is then called a *spherical principal series*; in general, a spherical representation is the unique irreducible subquotient of $\pi(\chi)$ with non-trivial $GL(n,\mathcal{O})$-fixed subspace. More generally, representations of the form $\pi(\chi)$ for unramified χ are called *unramified principal series*; they have finite composition series.

Proposition 1.1.1.1. *(The character χ is not assumed unramified.) (a) $I(\chi)$ is irreducible unless $\alpha_i/\alpha_j = |\bullet|^{\pm 1}$ for some $i \neq j$. (b) If $\alpha_i/\alpha_j \neq |\bullet|^{\pm 1}$ for all i,j then $I(\chi) \xrightarrow{\sim} I(w(\chi))$ as irreducible admissible representations of $GL(n,F)$ for any permutation w of the α_i.*

One can define a spherical Hecke algebra for $GL(n)$ with coefficients in a ring A: it is the convolution algebra of compactly supported A-valued functions on $GL(n,F)$ bi-invariant under $GL(n,\mathcal{O})$, and is isomorphic to the polynomial

$$A[T_1, \dots, T_n, T_n^{-1}]$$

if A is a $\mathbb{Q}$-algebra. For any A, we will define the operators T_i as above and let $H_A(G,K)$ be the polynomial algebra on the T_i and T_n^{-1}. Explicit normalizations will be introduced later. Let ϖ be a uniformizing element in $\mathcal{O}$.

Proposition 1.1.1.2. *The set of spherical representations is in bijection (Satake isomorphism) with unordered n-tuples of non-zero complex numbers*

$$\pi \leftrightarrow \{\alpha_1(\varpi), \dots, \alpha_n(\varpi)\}$$

and we can define a local Euler factor

$$L(s,\pi) = \prod_{i=1}^{n}(1-\alpha_i(\varpi)Nv^{-s})^{-1}$$

where Nv is the order of $\mathcal{O}/(\varpi)$.

1.1.2 Induced representations.

M ore generally, if $n = \sum_{i=1}^{r} n_i$ is a partition, with $r > 1$, let $P \subset GL(n)$ be the parabolic subgroup with Levi factor $\prod_i GL(n_i)$. To any r-tuple of irreducible admissible representations σ_i of $GL(n_i,F)$ one can define $Ind_{P(F)}^{GL(n,F)}\sigma_1 \otimes \cdots \otimes \sigma_r$ (normalized induction). This induced representation has finite length as a representation of $GL(n,F)$. The irreducible representations of $GL(n,E_v)$ that do not occur in any such composition series are called supercuspidal. They are again in the discrete series, as for $n = 2$, and the remaining discrete series can be classified but the classification is more complicated.. Supercuspidal representations do not exist for archimedean v but they do exist for all finite v.

1.1.3 Steinberg representations. We also need to consider the Steinberg representations $St(n,\alpha)$, where α is a character of $F^\times$ (not necessarily unramified). These are discrete series and can be defined in various ways. The simplest is to consider that the one-dimensional representation $\alpha \circ \det$ of $GL(n,F)$ can be written uniquely as a quotient of a certain principal series. The Steinberg representation $St(n,\alpha)$ is then the unique irreducible subrepresentation (not subquotient!) of this induced representation. The Steinberg representations are the only constituents of principal series that belong to the discrete series.

1.1.4 Distribution characters. The theory is valid as for $GL(2)$.

1.2. Langlands parametrization.

The most conceptually useful parametrization of $\mathcal{A}$. Let $\mathcal{G} = \mathcal{G}_{n,F}$ denote the set of equivalence classes of (Frobenius-semisimple) two-dimensional representations of the Weil-Deligne group WD_F. Let

$$r_F : (WD_F)^{ab} \xrightarrow{\sim} F^\times$$

be the (normalized) reciprocity isomorphism.

1.2.1. Local Langlands correspondence. *There is a bijection*

$$\mathcal{L} : \mathcal{A}_{n,F} \to \mathcal{G}_{n,F}$$

with the following natural properties, among others;

(i) *If $I(\chi)$ is irreducible, then*

$$\mathcal{L}(I(\chi)) = \oplus_{i=1}^{n} \alpha_i \circ r_F$$

(ii) *π is supercuspidal if and only if $\mathcal{L}(\pi)$ is irreducible.*

(iii) *$\mathcal{L}$ preserves local L and ε factors of pairs of representations, where the local factors for $\mathcal{A}$ are those defined by Jacquet-Piatetski-Shapiro-Shalika or Shahidi.*

(iv) $\det \mathcal{L}(\pi)) = r_F \circ \xi_\pi$*, where ξ_π is the central character of π.*

1.2.2. Local base change and automorphic induction.

Let F'/F be a finite extension. The inclusion $WD_{F'} \subset WD_F$ defines a natural restriction map

$$\mathcal{G}_{n,F} \to \mathcal{G}_{n,F'},$$

hence by the local Langlands parametrization a natural base change map

$$BC = BC_{F'/F} : \mathcal{A}_{n,F} \to \mathcal{A}_{n,F'}.$$

When F'/F is a cyclic Galois extension of prime degree, the map $BC_{F'/F}$ coincides with the Arthur-Clozel construction and is determined by an explicit relation between χ_π and $\chi_{BC\pi}$.

Let F'/F be of degree d. Induction of finite-dimensional representations defines a natural map

$$\mathcal{G}_{m,F'} \to \mathcal{G}_{md,F},$$

hence by the local Langlands parametrization a natural map called (local) **automorphic induction**:

$$AI_{F'/F} : \mathcal{A}_{m,F'} \to \mathcal{A}_{md,F}.$$

This can also determined by an explicit relation of distribution characters (Henniart-Herb).

Let α be an unramified character of $F^\times$ The Steinberg representation $St(n, \alpha)$ corresponds under $\mathcal{L}$ to a Weil-Deligne representation $Sp(n, \alpha)$ in $\mathcal{G}_{n,F}$. A representation π of $GL(n, F)$ is called *monodromic* if $\mathcal{L}(\pi) = \oplus_{i=1}^{r} Sp(n_i, \alpha_i)$ for some unramified characters α_i and some partition $n = n_1 + \cdots + n_r$. Since the Galois group of any finite extension of F is solvable, it follows that

Proposition 1.2.3. *Any $\pi \in \mathcal{A}_{n,F}$ becomes monodromic after a finite series of cyclic base changes.*

This is very important in applications of the Taylor-Wiles technique.

1.2.4. Other realizations.

The Bushnell-Kutzko theory of types is complete for $GL(n)$. Supercuspidal representations of $GL(n, F)$ can be constructed by rigid geometry in two ways, one contained in my 1997 article in Inventiones, the other in my book with Richard Taylor. Faltings (and Fargues) have proved that the two constructions are equivalent.

1.3. Classification of irreducible representations of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$..

For the purposes of this lecture, we will fix a single Harish-Chandra module π for $GL(n, F)$ when $F = \mathbb{R}$ or $\mathbb{C}$. It is a *cohomological* representation, which means it has an explicit relation to cohomology of locally symmetric spaces, and indeed to the cohomology with constant coefficients. It is the generalization of the representation $\pi_{2,0}$ of $GL(2)$, corresponding to holomorphic modular forms of weight 2 on the upper half-plane.

1.4 Automorphic representations.

I reproduce the paragraph from the notes on $GL(2)$, in modified form:

We let $G = GL(n)_F$. The theory of representations of $G(F_v)$ is as above. For v non-archimedean it is roughly analogous to the case already considered of $GL(2)$.

On the other hand, for v archimedean there is no natural collection of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules comparable to the holomorphic representations. Here $K = \prod_{v|\infty} K_v$ is a maximal compact subgroup of $G_\infty = \prod_{v|\infty} G(F_v)$, so K_v is either isomorphic to $\mathbb{R}^\times \cdot O(n)$ (v real) or $\mathbb{C}^\times \cdot U(n)$, ($v$ complex). If π is an (irreducible cuspidal) automorphic representation of G, then it factors

$$\pi = \otimes'_v \pi_v$$

(restricted tensor product over places v of F), where for each v, π_v is an admissible irreducible representation of $G(F_v)$ in the sense described in §1. For almost all v π_v is necessarily spherical, which allows us to define the restricted tensor product. The function $L(s,\pi) = \prod_{v\not|\infty} L(s,\pi_v)$ converges in a right half plane and extends to an entire function with functional equation (Godement-Jacquet).

2. Associated global Galois representations

For the remainder of these notes I insert the notes of a talk I gave in Montreal two years ago. For the case $n = 2$ this overlaps with the other notes for this week, but gives more details. The book project has advanced considerably over the last two years, as I will explain during the course.

Review of the case $n = 2$

I begin with the situation with which I assume you are all familiar. Let f be an elliptic modular normalized newform of weight $k \geq 2$. Equivalently, let π be the corresponding cuspidal automorphic representation of $GL(2,\mathbf{A})$; the weight k condition corresponds by the Shimura isomorphism to the condition I will label

(i) (Regularity) The archimedean component π_∞ of π is the discrete series representation π_k of $GL(2,\mathbb{R})$ which has cohomology with coefficients in the irreducible representation $Sym^{k-2}\mathbb{R}^2$.

The L-function $L(s,f)$ in the usual normalization is absolutely convergent for $Re(s) > \frac{k+1}{2}$ (Ramanujan-Petersson; otherwise $\mathfrak{k} + \mathfrak{z}2$) and satisfies a functional equation relating $L(s,f)$ to $L(k-s,f)$. We adopt the unitary normalization identifying $L(s,f) = L(s,\pi)$, where the abcissa of convergence is at $Re(s) = 1$ and the functional equation relates $L(s,\pi)$ to $L(1-s,\pi)$, and we do so without comment in the rest of the talk. The L-functions of compatible systems of λ-adic Galois representations are also given the unitary normalization.

The theorem is the following:

Theorem I. (Eichler, Shimura, Deligne, Langlands, Carayol,T. Saito). *Let $E(\pi) = E(f)$ be the field generated by the Fourier coefficients of f. There is a compatible system of λ-adic representations (of geometric type), as λ runs through non-archimedean completions of $E(\pi)$: $\rho_{\lambda,\pi} = \rho_{\lambda,f} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(2, E(\pi)_\lambda)$ that is associated to π in the sense that, for every ℓ and every place v of $\mathbb{Q}$ prime to the residue characteristic ℓ of λ,*

$$\rho_{\lambda,f} \mid_{G_v} = \mathcal{L}(\pi_v) \tag{1}$$

where $\mathcal{L}$ is the (motivically normalized) local Langlands correspondence.

Moreover, the representation $\rho_{\lambda,f} \mid_{G_\ell}$ is potentially semistable, the Hodge-Tate numbers of $\rho_{\lambda,f}$ at ℓ are 0 *and* $k-1$, *with multiplicity one, and*

$$D(\rho_{\lambda,f} \mid_{G_\ell}) = \mathcal{L}(\pi_\ell) \tag{2}$$

where D is the Fontaine functor associating to any potentially semistable representation of G_ℓ a representation of the Weil-Deligne group of $\mathbb{Q}_\ell$.

This is about as complete a result as one could wish. I have only omitted Scholl's theorem that the compatible system $\rho_{\lambda,\pi}$ is actually associated to a Grothendieck motive. This guarantees, for example, the validity of the Katz-Messing theorem, that the characteristic polynomial of crystalline Frobenius at p coincides with the characteristic polynomial of Frobenius at p acting on ℓ-adic cohomology. The analogue of Scholl's theorem cannot be proved in full generality in other situations, This is already the case for Hilbert modular forms. Let now F be a totally real field, and π a cuspidal automorphic representation of $GL(2, \mathbf{A}_F)$. We consider the analogue of condition (i) above:

(i) (Regularity) The local component π_{sigma} of π, as σ runs over archimedean places of F is the discrete series representation $\pi_{k(\sigma)}$ of $GL(2,\mathbb{R})$; moreover, all the $k(\sigma)$ are congruent modulo 2.

We also add an optional condition:

(iii) **local monodromy** For some finite prime v of F, π_v is square integrable.

The reason for this numbering will be apparent when I treat $n > 2$. The square-integrable representations of $GL(2, F_v)$ are the Steinberg representations, parametrized by characters of $F_v^\times$, and the supercuspidal representations. The analogue of the Eichler-Shimura et al. theorem is divided into several steps. We can define $E(\pi)$ to be the field generated by the eigenvalues of Hecke operators, acting in the classical space of Hilbert modular forms, or to be the field of definition of the non-archimedean representation π_f of $GL(2, \mathbf{A}^f{}_F)$. It is not difficult to see that $E(\pi)$ is always a number field, in fact a CM field or a totally real field, though it is less trivial than for elliptic modular forms. Let $d = [F : \mathbb{Q}]$, $G_F = Gal(\overline{\mathbb{Q}}/F)$.

Theorem II (Rogawski-Tunnell, Ohta, Carayol, T. Saito). *Suppose either d is odd or π satisfies condition (iii). Then there is a compatible system of λ-adic representations (of geometric type), as λ runs through non-archimedean completions of $E(\pi)$: $\rho_\lambda = \rho_{\lambda,\pi} : G_F \to GL(2, E(\pi)_\lambda)$ satisfying the analogue of (1) above at primes v of F not dividing ℓ. Moreover, the Hodge-Tate numbers of ρ_λ at primes dividing ℓ are explicitly determined by the weights k_σ (by a formula that depends on comparing real and ℓ-adic embeddings of F), and ρ_λ satisfies the analogue of (2) above.*

Note that I have chosen to ignore Hilbert modular forms and to express the result directly in terms of the representation π. The most important remark concerning the proof is that one cannot work directly with the automorphic representation π of $GL(2)$. The classical Hilbert modular form is a holomorphic function on a product of d copies of the upper half plane, and contributes to cohomology of the Hilbert modular variety, a Shimura variety of dimension d. More precisely, it contributes to intersection cohomology of the Baily-Borel compactification with twisted coefficients, provided the parity property is satisfied. Brylinski and Labesse decomposed this cohomology into Hecke eigencomponents. The result associates to

π a representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of dimension 2^d which arises a posteriori as the tensor product of the 2-dimensional representations corresponding to different real embeddings $\tau : F \to \mathbb{R}$.

We assume $d > 1$, since the case $d = 1$ is the subject of the earlier theorem. If $d > 1$ is odd, then for each fixed choice of real embedding τ, there is a quaternion algebra B_τ unramified at τ and at all finite primes, and ramified at the remaining real places. This is an additional parity condition imposed by the theory of algebraic groups over number fields: a quaternion algebra over a number field has to be ramified at exactly an even number of places. The multiplicative group J_τ of B_τ is an algebraic group over $\mathbb{Q}$ which defines an (adelic) Shimura curve S_τ each connected component of which is a compact quotient of a single copy of the upper half plane. The set of weights k_σ defines ℓ-adic local systems $L_{(k_\sigma),\ell}$ on S_τ and the ρ_λ are defined as

$$Hom_{J_\tau(\mathbf{A}^f)}(\pi_f, H^1(S_\tau, L_{(k_\sigma),\ell})$$

These are already the ℓ-adic realizations of a Grothendieck motive (with coefficients in $E(\pi)$. Note that $J_\tau(\mathbf{A}^f) = GL(2, \mathbf{A}^f)$, so the Hom makes sense. The fact that this space is 2-dimensional (rather than 0-dimensional) expresses a relation between automorphic representations of $GL(2)$ and of J_τ, which is the global Jacquet-Langlands correspondence. It is proved by means of the Selberg trace formula, which is the main technique for relating automorphic representations of different groups.

There is a minor point: π_f is a complex representation, so one needs to replace it by an ℓ-adic version in order to define the Hom space. This becomes more problematic when working with p-adic Banach spaces!

If d is even, but π satisfies (iii), we consider B_τ^v which is ramified at real primes other than τ, and at the finite prime v, but unramified elsewhere. One constructs J_τ^v and S_τ^v as before, and then ρ_λ is then

$$Hom_{J_\tau(\mathbf{A}^f)}(\pi_f^*, H^1(S_\tau, L_{(k_\sigma),\ell})$$

Since $J_\tau(\mathbf{A}^f)$ and $GL(2, \mathbf{A}^f)$ differ at the prime v, we need to define a π_f^*. The definition is provided by the local Jacquet-Langlands correspondence, which requires hypothesis (iii).

I ignore the construction of ℓ-adic representations in weight 1. These are no longer associated to discrete series representations at archimedean places, and other techniques (Deligne-Serre, etc.) are needed.

Hypothesis (iii) can be eliminated in two ways. The first, due to Taylor, is to construct the individual ρ_λ independently by constructing increasingly close (mod ℓ^i) approximations of π by π_i, with the same weights as π, which do satisfy (iii), then pasting together the resulting residual representations by the method of pseudorepresentations. A related earlier method of Wiles used ordinary Hida families and applied only at primes λ where the Hilbert modular form associated to π is ordinary; here the weight varies. The starting point for Taylor's construction is the Jacquet-Langlands correspondence from $GL(2)$ to forms on the multiplicative group $J_\emptyset$ of the quaternion algebra unramified at all finite places but definite at all archimedean places – this is possible because d is even. The group $J_\emptyset$ defines a zero-dimensional Shimura variety which has no interesting cohomology, but which can be used to define interesting modules over the Hecke algebra, whose congruence properties can be studied. We return to the analogues of $J_\emptyset$ when considering $n > 2$.

The second method, which provides more information when it works, is due to Blasius and Rogawski, and relies on the results in Rogawski's book and the Montreal conference. I will concentrate on this method here and return to the method of congruences after considering the general case. In the Blasius-Rogawski method, the missing two-dimensional representations are a byproduct of the construction of compatible families of three-dimensional representations, using the cohomology of Picard modular surfaces, which are Shimura varieties of dimension 2. Generally speaking, let F be a totally real field of degree d, E/F a totally imaginary quadratic extension $c \in Gal(E/F)$ complex conjugation. Let V be an n-dimensional vector space over E with a non-degenerate hermitian form $(*,*)$, and define the unitary similitude group

$$GU(V) = \{g \in GL(V) \mid (gv, gv') = \nu(g)(v,v'), \forall v, v' \in V\}$$

where $\nu(g)$ is a scalar. The map $g \mapsto \nu(g)$ is a homomorphism $GU(V) \to \mathbb{G}_m$ with kernel $U(V)$, the usual unitary group. These are naturally reductive groups over F but they can also be considered reductive algebraic groups over $\mathbb{Q}$.

The same definition is valid for hermitian spaces over local fields. When v is a place of F that splits in E, say $v = w \cdot w^c$, then $U(V)(F_v) \xrightarrow{\sim} GL(V \otimes_E E_w)$. If σ is a real place of F, $U(V)(F_\sigma)$ is determined by the signature of the hermitian form on $V \otimes E_\sigma$ (as groups, $U(a,b) \xrightarrow{\sim} U(b,a)$, so the choice of prime of E above σ isn't important. Suppose for every place v of F we have a unitary group (or similitude group) G_v of dimension n. If n is odd, there is exactly one G_v, up to isomorphism, for finite places v, and there is a unitary group G over F which localizes to G_v at each v. If n is even, one assigns to each G_v a Hasse invariant $\varepsilon_v \in \{\pm 1\}$, which is automatically 1 if v splits. If almost all $\varepsilon_v = 1$ and if $\prod_v \varepsilon_v = 1$ then there is a global G localizing to the given G_v everywhere; if not, there is no such global G. When $n = 2$ this just comes down to the condition that a quaternion algebra has to be ramified at an even set of places.

For $n = 3$ we have seen that G_v is unique for finite v; for $v = \tau$ we take $G_v = U(2,1)$, and $G_\sigma = U(3)$ (compact unitary group) for real places $\sigma \neq \tau$. The global G is associated to a Shimura variety $Sh(G)$, a Picard modular surface, whose cohomology (with twisted coefficients L) is computed in terms of automorphic forms on G. Assume $d > 1$. Then $Sh(G)$ is projective, and the cohomology $H^*(Sh(G), L)$ is a sum of three kinds of contributions:

(a) Non-tempered cohomology.. I will ignore these, though these can be used to construct elements in Selmer groups;

(b) Stable (tempered) cohomology. Consider cuspidal automorphic representations Π of $GL(3)_E$ which satisfy the following two hypotheses:

(i) (Regularity) The local component Π_{sigma} of Π, as σ runs over archimedean places of F (or of E) is (tempered and) of cohomological type; i.e., there is a finite dimensional representation W_σ of $GL(3, E_\sigma)$ such that W_σ and Π_{sigma} have the same infinitesimal character.

(ii) **Polarization** $\Pi^\vee = \Pi \circ c$, where ^{vee} is contragredient

One way to obtain (i) and (ii) is to start with a cuspidal cohomological automorphic representation Π_1 of $GL(3)_F$ and apply Arthur-Clozel base change to define $\Pi = \Pi_{1,E}$. If Π_1 is self-dual, then Π satisfies (ii).

Since Π is cohomological, it turns out we can define a number field $E(\Pi)$ as before. One of the main theorems of the Montreal volume is

Theorem III (collective). *Let Π satisfy (i) and (ii). There is a compatible system of λ-adic representations (of geometric type), as λ runs through non-archimedean completions of $E(\Pi)$:*

$$\rho_\lambda = \rho_{\lambda,\pi} : G_E \to GL(3, E(\Pi)_\lambda)$$

satisfying the analogue of (1) above at primes v of F not dividing ℓ. Moreover

$$\rho_\lambda^\vee \xrightarrow{\sim} \rho_\lambda \circ c \otimes \xi$$

where ξ is an explicit Hecke character depending on choices I have not mentioned. (Usually, ξ is the -2 power of the cyclotomic character.) Finally, the Hodge-Tate numbers of ρ_λ at primes dividing ℓ are explicitly determined by the finite-dimensional representations W_σ.

Blasius and Rogawski actually proved the ρ_λ are irreducible.

(c) Endoscopic (tempered) cohomology.

The representations in (b) occur in $H^2(Sh(G), L)$, where the coefficients L are defined in terms of the W_σ (basically, they are dual to W_σ). But they do not exhaust $H^2(Sh(G), L)$. One can summarize the information provided by the stable trace formula as a decomposition of the set of cohomological automorphic representations of G:

$$(\mathcal{E}) \qquad Coh(G) = Coh(G)_{nt} \oplus Coh(G)_{GL(3)} \oplus Coh(G)_{GL(2)\times GL(1)}$$

corresponding to the three cases (a), (b), (c). The missing pieces come from pairs of representations (Π_2, η), where Π_2 is a representation of $GL(2)_E$ satisying an analogue of the polarization condition (ii), and η is a Hecke character of $GL(1)_E$ that factors through the antinorm $x \mapsto x/x^c$. To the pair (Π_2, η), Rogawski associates a collection (L-packet) of automorphic representations $(\pi_i, i \in I)$ of the unitary (similitude) group G; this packet is called the *endoscopic transfer* of (Π_2, η). We make a simplifying assumption

Hypothesis A. *(a) E/F is unramified at all finite places. (b) Π_v is spherical (unramified) at all non-split non-archimedean places v of E.*

Condition (a) looks restrictive – it's never satisfied with $F = \mathbb{Q}$ – but in fact one can always use quadratic base change to reduce to Hypothesis A.

Under Hypothesis A, there is a unique representation π_f of $G(\mathbf{A}^f)$ such that $\pi_i = \pi_\infty \otimes \pi_f$ for all $i \in I$. Moreover, I is a finite set and one can define a system of ℓ-adic coefficients L_ℓ, for all ℓ, such that

$$\dim_{\overline{\mathbb{Q}}_\ell} Hom_{G(\mathbf{A}^f)}(\pi_f, H^2(Sh(G), L) \otimes \overline{\mathbb{Q}}_\ell) = |I|.$$

and is an $|I|$-dimensional representation of $Gal(\overline{\mathbb{Q}}/E)$. In fact, if Π_2 is fixed then, depending on the choice of η, the Galois representation is of dimension either 1 or 2. Manipulating this construction, Blasius and Rogawski proved the following theorem.

Theorem IV (Blasius-Rogawski). *Let π be a cuspidal automorphic representation of $GL(2, \mathbf{A}_F)$, F totally real. Suppose π satisfies (i) and $k_\sigma > 2$ for at least one σ. Then there exists a compatible system $\rho_{\lambda,\pi}$ of two-dimensional representations as in the earlier theorem. Moreover, up to twisting by an appropriate Hecke character, $\rho_{\lambda,\pi}$ are the λ-adic realizations of a Grothendieck motive (with coefficients in $E(\pi)$).*

The complete local-global compatibility for these representations was only recently proved by Blasius. The condition $k_\sigma > 2$ for some σ is needed in order to guarantee the possibility of choosing an η for which $|I| = 2$ and not 1; it was missing in their first paper.

What remains is the case $k_\sigma = 2$ for all σ, d even, when π fails to satisfy (iii). This case, which conjecturally includes the Hilbert modular forms associated to elliptic curves over F with good reduction everywhere, was treated by Taylor. One of the biggest open questions in number theory is to construct the ellliptic curves over F associated to such π.

Modularity.

The methods inaugurated by Wiles are well adapted to the representations of Theorems I and II and prove under various sets of hypotheses that if a mod ℓ Galois representation admits one modular lifting, then all its liftings are modular. As far as I know, no one has tried to apply these techniques to the three-dimensional representations of Theorem III nor to the 2-dimensional representations of Theorem IV. For the latter it is surely unnecessary, since one can use Taylor's construction. I mention this here because one of the goals of the book project is to proving whatever is necessary to apply the Wiles techniques in higher dimensions.

$$n > 2$$

For n odd, the model is Theorem III; for n even the model is Theorem IV. As matters stand, we have the analogue of Theorem II. Let E/F be as before, and for simplicity we always assume Hypothesis A; in particular, $d > 1$. Consider a cuspidal automorphic representation Π of $GL(n)_E$. First assume Π satisfies Hypotheses (i), (ii), and (iii), which can be stated just as before; Regularity means having cohomology with respect to a specific finite-dimensional representation $W(\Pi_\infty)$.

Theorem V. *Assume Π satisfies (i)-(iii)*

(a) (Clozel, Kottwitz, H-Taylor, Taylor-Yoshida) There is a compatible system of λ-adic representations (of geometric type):

$$\rho_\lambda = \rho_{\lambda,\Pi} : G_E \to GL(n, E(\Pi)_\lambda)$$

satisfying the analogue of (1) above at primes v of F not dividing ℓ. Moreover

$$\rho_\lambda^\vee \xrightarrow{\sim} \rho_\lambda \circ c \otimes \chi^{1-n}$$

where χ is the cyclotomic character. The Hodge-Tate numbers of ρ_λ at primes dividing ℓ are explicitly determined by the highest weights of the finite-dimensional representations $W(\Pi_\infty)$, and occur with multiplicity ≤ 1.

(b) (H-Labesse) The ρ_λ can be realized in $H^{n-1}(Sh(GU(V), L(\Pi_\infty)$ (cohomology with twisted coefficients), where V is a hermitian space of signature $(n-1, 1)$ at

one real place τ of F, definite at all other real places, and quasi-split at all finite places other than possibly the place v where Π_v is square integrable.

In (a), there was a Shimura variety attached to a twisted unitary group. The proof of (b) is a reduction by generalized Jacquet-Langlands transfer to the twisted case, and to the analysis of bad reduction in [HT].

The goal of the book project is to dispense with hypothesis (iii). Henceforward $n = 2m + 1$ is odd, and V is a hermitian space of signature $(n - 1, 1)$ at τ and definite at other real places. Our starting point is the following recent theorem of Arthur.

Theorem (Arthur, plus Labesse base change). *Assume two fundamental lemmas for the group $G = GU(V)$: the standard fundamental lemma for endoscopy, and the fundamental lemma for weighted orbital integrals. Then there is an endoscopic decomposition of the space of cohomological cuspidal automorphic forms on G, analogous to the case $n = 3$:*

$$(\mathcal{E}) \qquad Coh(G) = Coh(G)_{nt} \oplus \oplus_{i=0} m Coh(G)_{GL(n-i)\times GL(i)}$$

Restricting to representations satisfying (iii), the theorem is known unconditionally (H-Labesse, using the fundamental work of Kottwitz) and the only non-trivial factor corresponds to $i = 0$.

The first fundamental lemma in Arthur's theorem has been proved by Laumon and Ngô. Arthur's proof involves a complicated double induction on the two sides of the trace formula that requires the second fundamental lemma, still unproved. The expectation of the book project is that the Laumon-Ngô fundamental lemma suffices to obtain a version of $(\mathcal{E})$ adequate for deduction of the analogues of Theorems III and IV. The odd- (resp. even-) dimensional Galois representations will be in the part corresponding to $i = 0$ (resp. $i = 1$), as for $n = 3$. The remaining (tempered) components will involve no new Galois representations but presumably have applications no one has yet considered. Arthur's conjectures on the structure of non-tempered contribution have been analyzed by Bellaiche and Chenevier.

Book I is devoted to the stable trace formula. We depart from Arthur and from the mainstream of work on the stable trace formula by writing our endoscopic decomposition in terms of anisotropic (partially definite) unitary groups rather than quasi-split endoscopic groups, hoping that this can replace the fundamental lemma for weighted orbital integrals. A year of work on Book I has reduced the goal to five explicit questions primarily about the archimedean local terms in the geometric and spectral sides of the trace formula, and I hope these will be resolved by the spring.

Book II will be devoted to the ℓ-adic cohomology of the Shimura varieties $Sh(GU(V))$. The new element, which we hope to address this year, is the determination of the local Galois representations at all finite places. With our choice of signatures the analysis of bad reduction should be identical to that in my book with Taylor. However, we were able to avoid problems of endoscopy, whereas in the present situation endoscopy is central. Kottwitz determined the local representation at (most) unramified places, assuming the fundamental lemma which has now been proved, but we didn't see how to adapt his approach to bad reduction, and the main new contribution of Book II will be to reconcile the two approaches.

At the end of Book II, we then expect to have the complete analogues of Theorems III and IV for all n. Corresponding to the condition $k(\sigma) > 2$ will be a condition like

Hypothesis B. *Let $W(\Pi_\infty) = \otimes_\sigma W(\Pi_\sigma)$ where $W(\Pi_\sigma)$ is an irreducible finite-dimensional representation of $GL(n+1,\mathbb{C})$. The highest weight of $W(\Pi_\tau)$ is* **regular**.

This condition may be too strong, but it should suffice.

Up to now, I expect the $\rho_{\lambda.\Pi}$ to have all the properties necessary for application of the methods of my paper with Clozel and Taylor, in particular residual modularity of a minimal representation (together with technical hypotheses about the size of the image and the residue characteristic) should imply modularity. The goal of Book III, of which Eric Urban has agreed be the main editor, is to develop the theory of p-adic families of automorphic representations of $GU(V)$ and the associated Galois representations. I hope this will suffice to allow construction of Galois representations even in the absence of Hypothesis B. Chenevier's thesis constructed eigenvarieties under the local monodromy condition (iii), but I don't see why it shouldn't work in the more general setting, and specialization at points where Hypothesis B fails should provide p-adic representations that can be shown to be associated to the cohomological Π on $GL(n)$, in the sense that they are compatible with the local Langlands correspondence at primes of residue characteristic $\neq p$. It is known that in general specializations of families of p-adic representations in low weight can fail to be Hodge-Tate. However, specializations corresponding to cohomological cuspidal automorphic representations of $GL(n)_E$ that just happen not to transfer to Shimura varieties should nevertheless be potentially semistable at primes dividing p. This is known (in most cases?) for $n = 2$, thanks to work of Taylor, Breuil, and Kisin. In general only even n pose a problem, for the simple reason that when d and n are even there is no unitary group G that is quasi-split at all finite primes and of signature $(n-1,1)$ at τ and definite at remaining archimedean primes. However, one can find a unitary group G' that is quasi-split at all finite primes and of signature $(n-2,2)$ at τ and definite at remaining archimedean primes. After stabilization of the trace formula, we should find that, up to an abelian twist, the exterior square of the Galois representation associated to any cohomological Π on $GL(n)$ can be realized in the cohomology of the Shimura variety associated to G'. In particular, the exterior square of the p-adic representation $\rho_{\lambda,\Pi}$ is potentially semistable at all primes dividing p. The original representation $\rho_{\lambda,\Pi}$ will correspond to specialization of an analytic family at a point where the Sen weights are integral It then follows from work of Bellaiche-Chenevier and from a general theorem of Wintenberger that $\rho_{\lambda,\Pi}$ is itself potentially semistable. See http://www.institut.math.jussieu.fr/projets/fa/bp0.html (the second article by M. Harris).