MONOMIAL REPRESENTATIONS

M. Harris, Eilenberg lecture 3

In the previous lecture I described Hecke's construction of automorphic representations of $GL(2)$ starting with a Hecke character χ of an imaginary quadratic field, say $\mathcal{K}$. The Hecke character χ is a continuous homomorphism

$$\chi : \mathbf{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times} \to \mathbb{C}^{\times}$$

to which one can associate an L-function

$$L(s,\chi) = \prod_{v} L(s,\chi_v)$$

the product taken over prime ideals of $\mathcal{K}$. If χ is unramified at v, its restriction χ_v to $\mathcal{K}_v^{\times}$ factors through $\mathcal{K}_v^{\times}/\mathcal{O}_v^{\times} \xrightarrow{\sim} \varpi_v^{\mathbb{Z}}$, where ϖ_v is a generator of the maximal ideal, and then $L(s,\chi_v) = (1-\chi_v(\varpi_v)Nv^{-s})^{-1}$. Otherwise $L(s,\chi_v) = 1$. The simplest way to describe the automorphic induction $AI(\chi)$ as an automorphic representation of $GL(2)$ is by the identity

$$L(s, AI(\chi)) = L(s,\chi).$$

Here

$$L(s, AI(\chi)_p) = \prod_{v|p} L(s,\chi_v).$$

This construction generalizes to quadratic extensions of any field, as shown in Jacquet-Langlands, but is not limited to quadratic extensions.

Theorem. *Let M/L be a cyclic extension of number fields of degree n, and let χ be a Hecke character of $\mathbf{A}_M^{\times}/M^{\times}$. Then there is an automorphic representation $AI(\chi) = AI_{M/L}(\chi)$ of $GL(n,\mathbf{A}_L)$, with the following properties*

1. *$L(s,AI(\chi)) = L(s,\chi)$ as Euler products (over places of L)*
2. *Let $\eta_{M/L}$ be the character of the idèles of L corresponding to the cyclic extension M/L. Then*

$$AI(\chi)\otimes \eta_{M/L}\circ \det \xrightarrow{\sim} AI(\chi).$$

3. *Conversely, if Π is an automorphic representation of $GL(n,\mathbf{A}_L)$ satisfying the isomorphism in (2), then Π is of the form $AI(\chi)$ for some Hecke character χ of M.*
4. *$AI(\chi)$ is cuspidal if and only if $\chi \neq \chi\circ\sigma$ for any $\sigma \in Gal(M/L)$.*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(5) *Suppose* χ *is associated to a compatible family*

$$\chi_\ell \to \Gamma_M \to \bar{\mathbb{Q}}_\ell^\times$$

of one-dimensional ℓ*-adic representations (say* χ *is of motivic type). Then* $AI(\chi)$ *is associated to a compatible family* $\rho_{AI(\chi),\ell}$ *of* n*-dimensional* ℓ*-adic representations, and*

$$\rho_{AI(\chi),\ell} \xrightarrow{\sim} Ind_{M/L}\chi_\ell.$$

In particular, $\rho_{AI(\chi),\ell}$ *is irreducible (for any* ℓ*) if and only if* $AI(\chi)$ *is cuspidal.*

The representations $Ind_{M/L}\chi_\ell$ are induced from characters and are therefore called *monomial*. Correspondingly, the representations $AI(\chi)$ are called *monomial automorphic representations*. The first four assertions of the theorem above are due to Kazhdan and were generalized by Arthur-Clozel, who replaced χ by an automorphic representation of $GL(m)$ for any m, which we recall below. Part (5) was proved by me in some cases and in general follows from the results of my book with Taylor. This book also provides the definition of "associated" used in the statement: the relation $L(s, AI(\chi)) = L^*(s, \rho_{AI(\chi),\ell})$ implies that the local factor at v of $AI(\chi)$ at almost all primes v is associated by the local Langlands correspondence to the restriction of $\rho_{AI(\chi),\ell}$ at Γ_v, and my book with Taylor, completed by his article with T. Yoshida, shows that this remains true for all v prime to ℓ.

Monomial automorphic representations have the following fundamental property:

Theorem. *Let* L'/L *be any extension of number fields and let* $\Pi = AI(\chi)$ *be a monomial automorphic representation of* $GL(n, \mathbf{A}_L)$*. Then* $\Pi' = AI(\chi \circ N_{L'/L})$ *is isomorphic to the formal base change of* Π.

In particular, suppose χ *is of motivic type, and let* $\rho_{AI(\chi),\ell}$ *be as in the previous theorem. Then for any extension* L'/L, $\rho_{AI(\chi),\ell}\mid_{\Gamma_{L'}}$ *is associated to an automorphic representation of* $GL(n, \mathbf{A}_{L'})$*. In other words,* $\rho_{AI(\chi),\ell}$ *is universally automorphic.*

I remind you that base change of automorphic representations is defined for cyclic extensions, and by induction for arbitrary solvable extensions, but certainly not for non-solvable extensions (see below) However, one can define formal base change $BC_{L'/L}(\Pi)$ of Π as an irreducible admissible representation of $GL(n, \mathbf{A}_{L'})$ for any extension L'/L by using the local Langlands parametrization to define local factors. The previous theorem asserts that $BC_{L'/L}(\Pi)$ is an automorphic representation for any L' if Π is monomial. This property is not known for any other class of automorphic representation. Moreover, if Π is cuspidal and automorphically induced from M as above, then it is easy to see that $BC_{L'/L}(\Pi)$ is again cuspidal if L' is linearly disjoint from M.

Base change and automorphic induction in general.

Let now π be a cuspidal automorphic representation of $GL(m, M)$, with M/L as above cyclic of degree n. Then it is shown by Arthur-Clozel that there exists an automorphic representation $AI(\pi) = AI_{M/L}(\pi)$ of $GL(nm, L)$. Moreover,

(1) $L(s, \pi) = L(s, AI(\pi)$ as Euler products
(2) We have the relation

$$AI(\pi) \otimes \eta_{M/L} \circ \det \xrightarrow{\sim} AI(\pi); \qquad (\eta)$$

(3) Conversely, if Π is an automorphic representation of $GL(nm, L)$ that satisfies (η), then it is of the form $AI_{M/L}(\pi)$ for some π.
(4) $AI(\pi)$ is cuspidal if and only if $\pi \neq \pi \circ \sigma$ for any $\sigma \in Gal(M/L)$.
(5) If π is associated to $\rho_{\pi,\ell}$ as above then $AI(\pi)$ is associated to $Ind_{M/L}(\rho_{\pi,\ell})$.

Again, if π is a cuspidal automorphic representation of $GL(m, L)$, then there exists an automorphic representation $BC(\pi) = BC_{M/L}(\pi)$ of $GL(m, M)$. This is determined by the local relation

$$BC_{M/L}(\pi)_w = BC_{M_w/L_v}(\pi_v), \text{ if } w \mid v. \tag{loc}$$

In other words, the abstract representation $BC(\pi)$ of $GL(m, \mathbf{A}_M)$ defined by the relation (loc) is automorphic. It is cuspidal if and only if $\pi \neq \pi \otimes \eta^a_{M/L}$ for any power a.