This is a continuation of the lecture on the Euler characteristic and Riemann-Roch formulas. As always, ℓ is an *odd* prime. In this section, $\bar{\rho}$ is an absolutely irreducible representation of Γ_F that extends to a homomorphism $\Gamma_{F+} \to \mathcal{G}_n(\bar{\mathbb{F}}_\ell)$. We set $M = \mathbf{A}(\bar{\rho})$, $\Gamma = \Gamma_{F+}$, and write $H^1(\Gamma, M)$ instead of $H^1(K, M)$, etc. We also write S as the disjoint union $S_\ell \cup Q \cup R \cup S_1 \cup S_\infty$, where S_∞ and S_ℓ are self-explanatory and the remaining sets will be described below. I have not yet explained the subspaces $L_v \subset H^1(\Gamma_v, M)$ introduced in the previous lecture. We will only consider v split in F/F^+, so that Γ_v can be considered a decomposition group for F or F^+. Recall that $H^1(\Gamma, M)$ is identified with the tangent space to the ring parametrizing deformations of $\bar{r}$ ramified at primes in S, with no restrictions on the type of ramification. The Selmer group $H^1_{\mathcal{S}}(\Gamma, M)$ is defined to be the subspace of the group $H^(\Gamma, M)$ whose deformation classes restrict to classes in L_v at $v \in S$. The L_v are chosen to correspond to certain well-defined types of deformations: a deformation ρ to $k[\varepsilon]$ is of type $\mathcal{S}$ (in the literature this is usually written $\mathcal{D}$ and we will gradually switch to this notation) if and only if the class of its restriction to Γ_v is in L_v for all $v \in S$. (The condition at S_∞ is empty because ℓ is odd.)

In subsequent lectures certain specific kinds of deformation types will be described for S_ℓ, Q, and R. The article [CHT] considers many more kinds of deformation types, but after writing [T], Taylor realized that most of these are unnecessary for applications in which there is a great deal of freedom in choosing the prime ℓ, and in particular for the Sato-Tate conjecture, and I will omit the (often extremely intricate) details. In the present lecture I will concentrate on the representability of the functor classifying deformations of type $\mathcal{S}$. I will describe a criterion due to Ramakrishna, based on Schlessinger's criterion for representability. There are alternative approaches based on the constructions of the deformation ring by Faltings and de Smit-Lenstra.

Ramakrishna's theorem.

Let $Rep_{\mathcal{O}}(\Gamma)$ be the category of $\mathcal{O}$-modules of finite length endowed with a continuous Γ-action. A *Ramakrishna subcategory* of $Rep_{\mathcal{O}}(\Gamma)$ is a full subcategory of $\mathcal{S}$ closed under passage to subobjects, quotients, and direct sums.

Let $\mathcal{S}$ be a Ramakrishna subcategory of $Rep_{\mathcal{O}}(\Gamma)$, and suppose $(\bar{\rho}, M)$ is in $\mathcal{S}$. Consider the subfunctor $Def_{\bar{\rho},\mathcal{S}}$ of $Def_{\bar{\rho}}$ such that, for A in $\mathcal{C}_{\mathcal{O}}$, a deformation ξ of $\bar{\rho}$ to A is in $Def_{\bar{\rho},\mathcal{S}}$ if, for some (any) lifting ρ of $\bar{\rho}$ to A representing the equivalence class ξ, ρ belongs to $\mathcal{S}$.

Theorem 1.1 [R]. *Under the above hypotheses, the functor $Def_{\bar{\rho},\mathcal{S}}$ is prorepresentable by a quotient $R_{\bar{\rho},\mathcal{S}}$ in $\hat{\mathcal{C}}$.*

The proof, following Mazur's survey article, is given in several steps.

Proposition 1.2. *Hypotheses as in Theorem 1.1. Then*

(1) *Let $A \to A_1$ be a morphism in $\mathcal{C}$, let (ρ, M) be a lifting of $\bar{\rho}$ to A, (ρ_1, M_1)*

the induced lifting to A_1 with $M_1 = M \otimes_A A_1$. If (ρ, M) is in $\mathcal{S}$, then so is (ρ_1, M_1).

(2) *Let A_1, A_2, A_0 be in $\mathcal{C}$, $\alpha : A_1 \to A_0$, $\beta : A_2 \to A_0$ morphisms in $\mathcal{C}$, with $A_3 = A_1 \times_{A_0} A_2$. Let (ρ_3, M_3) be a lifting of $\bar{\rho}$ to A_3, and let (ρ_i, M_i) denote the induced liftings over A_i, $i = 1, 2$, with respect to the projections of A_3 on A_i. Then (ρ_3, M_3) is in $\mathcal{S}$ if and only if (ρ_i, M_i) is for $i = 1, 2$.*

(3) *Let $A \to A_1$ be an injection in $\mathcal{C}$, and suppose (ρ_1, M_1) is a lifting of $\bar{\rho}$ to A_1 that is in $\mathcal{S}$. Then (ρ_1, M_1), viewed as an $A[\Gamma]$-module, is in $\mathcal{S}$.*

Proof. Note that in (3), the $A[\Gamma]$-module (ρ_1, M_1) is not a lifting of $\bar{\rho}$. Of course (3) is obvious because the property of being in $\mathcal{S}$ does not depend on the coefficient ring. As for (1): we may assume A_1 is of the form $A[X_1, \ldots, X_m]/J$ for some integer m and some ideal J. Passage from A to $A[X_1, \ldots, X_m]$ replaces M by a direct sum of copies of M, which is still in $\mathcal{S}$; passage from $A[X_1, \ldots, X_m]$ to $A[X_1, \ldots, X_m]/J$ replaces an object in $\mathcal{S}$ by a quotient, which is therefore also in $\mathcal{S}$.

Finally, in the situation of (2), suppose (ρ_3, M_3) is in $\mathcal{S}$. Since A_i is a quotient ring of A_3 for $i = 1, 2$, the argument used for (1) implies that each (ρ_i, M_i) is in $\mathcal{S}$. Conversely, as $\mathcal{O}[\Gamma]$-module, (ρ_3, M_3) is a submodule of $(\rho_1 \times \rho_2, M_1 \times M_2)$. Thus the implication in the reverse direction follows because $\mathcal{S}$ is closed under passage to subobjects.

Proposition 1.3. *Let $\mathcal{D}$ be a subfunctor of the covariant functor D from $\mathcal{C}$ to the category of sets, with $\mathcal{D}(k) = D(k)$ a singleton. Suppose for all triples A_1, A_2, A_0 in $\mathcal{C}$, with $\alpha : A_1 \to A_0$, $\beta : A_2 \to A_0$ morphisms in $\mathcal{C}$, and $A_3 = A_1 \times_{A_0} A_2$, the square*

$$
\begin{array}{ccc}
\mathcal{D}(A_3) & \longrightarrow & \mathcal{D}(A_1) \times_{\mathcal{D}(A_0)} \mathcal{D}(A_2) \\
\downarrow & & \downarrow \\
D(A_3) & \longrightarrow & D(A_1) \times_{D(A_0)} D(A_2)
\end{array}
\tag{1.4}
$$

is Cartesian, where the vertical maps are the natural inclusions. Suppose D is prorepresentable by an algebra R_D in $\hat{\mathcal{C}}$. Then $\mathcal{D}$ is prorepresentable by a quotient $R_{\mathcal{D}}$ of R_D.

If $\mathcal{D}$ satisfies the hypothesis of the proposition, it is called *relatively representable.*

Proof. I sketch a proof. If D is prorepresentable by R, then D satisfies the Schlessinger conditions (H_i), $i = 1, 2, 3, 4$. In particular, the bottom row of the diagrom (1.4) is surjective or bijective under certain hypotheses on A_1 and A_2. Now if $\mathcal{D}$ is a relatively representable subfunctor, then the surjectivity (resp. bijectivity) of the top row of (1.4) is an immediate consequence of surjectivity of the bottom row. In particular, $\mathcal{D}$ is prorepresentable by some $R_{\mathcal{D}}$. It remains to show that $R_{\mathcal{D}}$ is a quotient of R_D, but this is a formal consequence of the fact that $\mathcal{D}$ is a subfunctor of D. Indeed, there is a natural morphism $R_D \to R_{\mathcal{D}}$ corresponding to the inclusion $\mathcal{D} \to D$ by Yoneda's lemma. Since both rings are complete, noetherian, and local, to show surjectivity it suffices to show that the map on Zariski tangent spaces is surjective. Equivalently, it suffices to show that $\mathcal{D}(k[\varepsilon]) \to D(k[\varepsilon])$ is injective, but this is true by definition.

It remains to show that

Lemma 1.5. *The morphism of functors $Def_{\bar{\rho},\mathcal{S}} \to Def_{\bar{\rho}}$ is relatively representable.*

Proof. First of all, we need to know that the inclusion $Def_{\bar{\rho},\mathcal{S}} \to Def_{\bar{\rho}}$ really is a morphism of functors, and this follows from Proposition 1.2, (1). Next, we need to show that the diagram (1.4) is Cartesian for the pair $D = Def_{\bar{\rho}}$, $\mathcal{D} = Def_{\bar{\rho},\mathcal{S}}$, and this is an immediate consequence of Proposition 1.2 (2).

References.

[R] R. Ramakrishna, On a variation of Mazur's deformation functor, *Compositio Math.*, **87** 269-286 (1993).