In what follows, $\mathcal{O}$ will be an ℓ-adic integer ring with finite residue field k. We let $\mathcal{C} = \mathcal{O}$ be the category of artinian local $\mathcal{O}$-algebras with residue field k (such that the structure map $\mathcal{O} \mapsto A$ induces the identity map on residue fields), and $\hat{\mathcal{C}}$ the category of complete noetherian local $\mathcal{O}$-algebras with residue field k as above. Schlessinger has given a criterion that guarantees that a functor from $\mathcal{C}$ to sets is (pro)representable by an object in $\hat{\mathcal{C}}$ that is similar to but more flexible than the general criterion due to Grothendieck. Schlessinger's criterion is the basis of Mazur's approach to deformation theory of Galois representations which is the framework used by Wiles and all those who have followed him. The original paper [S] of Schlessinger is easy to read and of course much more general than the case needed to study ℓ-adic Galois representations, but I will nevertheless present the proof in some detail.

1. Basic notions. As usual, $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. If A is in $\hat{\mathcal{C}}$ then $\mathfrak{m}_A$ is its maximal ideal and the Zariski cotangent space of A over $\mathcal{O}$ is

$$t_A^* = \mathfrak{m}_A/(\mathfrak{m}_A)^2 + \mathfrak{m}.$$

For $p : A \to B$, $q : C \to B$ in $\mathcal{C}$, let

$$A \times_B C = \{(a, c) \in A \times C \mid p(a) = q(c)\}.$$

Lemma 1.1. *A morphism $p : B \to A$ in $\hat{\mathcal{C}}$ is surjective if and only if the induced map $dp : t_B^* \to t_A^*$ is surjective.*

Proof. Exercise (one proves that the induced map on graded rings is surjective and then applies a lemma from Bourbaki's *Algèbre Commutative*.)

Definition 1.2. *Let $p : B \to A$ be a surjection in $\mathcal{C}$.*

(a) The map p is called a small extension *if* $\ker p$ *is a nonzero principal ideal (t) such that $\mathfrak{m}_B \cdot t = (0)$.*

(b) The map p is called essential *if for any $q : C \to B$ in $\mathcal{C}$, surjectivity of pq implies surjectivity of q.*

Note that if p is a small extension then $length(B) = length(A) + 1$.

Corollary 1.3. *Let $p : B \to A$ be a surjection in $\mathcal{C}$. Then*

(i) p is essential if and only if $dp : t_B^* \to t_A^*$ is an isomorphism;

(ii) If p is a small extension, then p is not essential if and only of p has a section $s : A \to B$ with $p \circ s = Id_A$.

Proof. (i) If dp is an isomorphism then by Lemma 1.1 p is essential. Conversely, let t_i, $i = 1, \ldots, r$ be a basis of t_A^*, lift them to $x_i \in B$, and set $C = \mathcal{O}[x_1, \ldots, x_r] \subset B$, with $i : C \to B$ the inclusion The natural map $C \to A$ is an isomorphism on cotangent spaces, hence surjective. Since $p \circ i$ is surjective if p is essential then i is also surjective, hence $B = C$, and it follows that dp is surjective.

(ii) If p has a section s, then since $\ker p \neq (0)$ s is not surjective, but $p \circ s$ is surjective, so p is not essential. Conversely, if p is not essential then the subring C of B constructed in (i) is not equal to B. Hence $length(C) \leq length(B) - 1 = length(A)$. But since $C \to A$ is surjective, $length(C) = length(A)$, hence the map $C \to A$ is an isomorphism and defines a section.

2. The representability criterion.

In what follows, F will be a covariant functor $\mathcal{C} \to Sets$ such that $F(k)$ contains just one element. Our primary example will be the deformations of a fixed $\bar{\rho} : \Gamma \to G(k)$, where Γ is some Galois group and $G = GL(n)$ or a related group. The deformations are assumed to satisfy a variety of additional properties, but $F(k)$ is just the original $\bar{\rho}$, which is a genuine homomorphism (not up to equivalence).

Definition 2.1. *A morphism $F \to G$ of functors is formally smooth if for any surjection $B \to A$ in $\mathcal{C}$, the morphism*

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective.

Since every surjection can be written as a composition of small extensions, it suffices to check the criterion in (2.1) for small extensions.

The property of formal smoothness is preserved under composition and base change. If R is in $\hat{\mathcal{C}}$, let h_R be its functor of points

$$h_R(A) = Hom_{\mathcal{O}}(R, A).$$

The functor $R \to h_R$ is a (contravariant) equivalence of categories (Yoneda's Lemma, and an excellent exercise, including the exact definition of the category to which h_R belongs).

Proposition 2.2. *Let $R \to S$ be a morphism in $\hat{\mathcal{C}}$. Then $h_S \to h_R$ is formally smooth if and only if R is a power series ring over S.*

Proof. This is proved by successive approximation, starting from the obvious choice of power series ring (the number of variables being $\dim t_{R/S}^*$, and is left as an exercise.

Definition 2.3. *The ring of dual numbers is $k[\varepsilon]/(\varepsilon^2)$, written just $k[\varepsilon]$. The tangent space t_F to F is $F(k[\varepsilon])$.*

An easy and important exercise is to show that $t_{h_R} \equiv t_R$ (the dual to t_R^*).

Definition 2.4. *A pair $(R, \xi \in F(R))$ is called a* hull *of F (resp.* prorepresents *F) if (a) the map $h_R \to F$ induced by*

$$f \in Hom(R, A) \mapsto f_*(\xi)$$

is formally smooth and (b) the induced map $t_R \to t_F$ is a bijection (resp. if $h_R \to F$ is an isomorphism).

Proposition 2.5. *Two hulls of F are (noncanonically) isomorphic. If (R, ξ) and (R', ξ') prorepresent F then they are canonically isomorphic*

Proof. Another good exercise.

Lemma 2.6. *Suppose F is a functor satisfying*

$$F(k[\varepsilon] \times_k k[\varepsilon]) \xrightarrow{\sim} F(k[\varepsilon]) \times F(k[\varepsilon]).$$

Then t_F has a canonical vector space structure.

Proof. Exercise.

Theorem 2.7. *Let F be a functor from $\mathcal{C}$ to Sets such that $F(k)$ is a single point. For morphisms $A' \to A$, $A" \to A$ in $\mathcal{C}$, consider the map*

$$(*) \qquad F(A' \times_A A") \to F(A') \times_{F(A)} F(A").$$

Then F has a hull (resp. is prorepresentable) if and only if F has properties (H_1), (H_2), (H_3) (resp. and also (H_4)) below:

- (H_1) *(*) is a surjection whenever $A" \to A$ is a small extension;*
- (H_2) *(*) is a bijection when $A = k$ and $A" = k[\varepsilon]$;*
- (H_3) $\dim_k(t_F) < \infty$*;*
- (H_4) *(*) is a bijection whenever $A' = A"$ is a small extension.*

Note that (H_2) implies in particular that F satisfies the hypothesis of Lemma 2.6, so t_F is a vector space. Also (H_1) implies by induction that (*) is always a surjection whenever $A" \to A$ is a surjection.

Proof. That any representable functor $F = h_R$ satisfies $(H_1) - (H_4)$ is a useful exercise and we assume it below.

Suppose F satisfies conditions (H_1), (H_2), (H_3). We construct a hull R by successive approximation. Let $t_1, \ldots, t_r$ be a basis of t_F^* with dual basis $\{t_1^*, \ldots, t_r^*\}$, and set $S = \mathcal{O}[T_1, \ldots, T_r]$. Let

$$R_2 = S/(\mathfrak{m}_S)^2 + \mathfrak{m} \cdot S \simeq k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$$

(with r factors on the right). By (H_2)

$$F(R_2) \xrightarrow{\sim} F(k[\varepsilon]) \times_{F(k)} \cdots \times_{F(k)} F(k[\varepsilon]) \xrightarrow{\sim} t_F \times \cdots \times t_F,$$

because $F(k)$ is a point; but the right hand side is canonically $t_F \otimes t_F^* = Hom(t_F, t_F)$ by construction. The trace (identity element) then defines an element $\xi_2 \in F(R_2)$ such that, if $x_i \in t_{R_2} = Hom(R_2, k[\varepsilon])$ is dual to T_i, then $(x_i)_*(\xi_2) = t_i^*$ in t_F; hence ξ_2 defines an isomorphism $t_{R_2} \xrightarrow{\sim} t_F$.

Now suppose we have found (R_q, ξ_q) with $R_q = S/J_q$ for some ideal. We seek J_{q+1} minimal among ideals $J \subset S$ satisfying $\mathfrak{m}_S J_q \subset J \subset J_q$ and ξ_q lifts to S/J. Each such J corresponds to a vector subspace of $J_q/\mathfrak{m}_S J_q$ hence it suffices to show that the set $\mathcal{S}$ of such J is stable under pairwise intersection.

So let J, K be two such ideals, and enlarge J so that $J + K = J_q$ without changing $J \cap K$. Then

$$S/J \times_{S/J_q} S/K \simeq S/(J \cap K)$$

(check this) and then by (H_1)

$$F(S/J \cap K) = F(S/J \times_{S/J_q} S/K) \to F(S/J) \times_{F(S/J_q)} F(S/K)$$

is surjective. Thus $J \cap K$ is also in $\mathcal{S}$. Now let J_{q+1} be the intersection of all ideals in $\mathcal{S}$ and let $R_{q+1} = S/J_{q+1}$ and ξ_{q+1} any lifting of ξ_q.

Let $J = \cap_q J_q$, $R = S/J$, a complete noetherian quotient of S, and set $\xi = \operatorname{proj\,lim} \xi_q$. (In the case of interest, it's just a group representation, hence exists as a genuine element of $F(R)$.) We know that $t_R \xrightarrow{\sim} t_F$ because this is already true for R_2, hence to show that R is a hull of F it suffices to show that the map $h_R \to F$ is formally smooth. It is enough to check that $h_R(A') \to F(A') \times_{F(A)} h_R(A)$ is surjective for any small extension $p : A' \to A$, say $A = A'/I$. In other words, let $\eta \in F(A)$ be the image of $\eta' \in F(A')$, $f : R \to A$ taking ξ to η; we have to show f lifts to $f' : R \to A'$ taking ξ to η'.

It is time to interpret condition (H_1). We have

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]; (x, y) \mapsto (x, x_0 + y - x); \tag{2.8}$$

here $k[I]$ is the algebra $k \oplus I$ with $I^2 = 0$ and $x_0 = x \pmod{\mathfrak{m}}_{A'} \in k$. (This is obviously a bijection; one has to check it is an isomorphism of algebras. The trick is to show that if (a, b) and (c, d) are in $A' \times_A A'$ then

$$bd - ac = a_0(d - c) + c_0(b - a) = a(d - c) + c(b - a)$$

which is true because $(b - a)(d - c)$, $(a - a_0)(d - c)$, and $(c - c_0)(b - a)$ are all in $I^2 = 0$. This is just the proof of Leibniz' rule, which we see again below in another form.

Now if $p : A' \to A$ is a small extension then $k[I] \xrightarrow{\sim} k[\varepsilon]$; It then follows from (H_2) that

(2.9)
$$F(A') \times t_F \otimes I = F(A') \times_{F(k)} F(k[\varepsilon]) \otimes I = F(A' \times_k k[I]) = F(A' \times'_A A') \to F(A') \times_{F(A)} F(A')$$

where the third equality is (2.8) and the last arrow is surjective by (H_1) (resp. bijective if we assume (H_4)). By the formal properties of this map one sees that, for any $\eta \in F(A)$, (2.9) defines a group action of $t_F \otimes I$ on the fiber $F(p)^{-1}(\eta)$ of the right-hand map $F(A') \to F(A)$, which is transitive because the composition of the maps in (2.9) is surjective. Moreover, if we assume (H_4), then this makes $F(p)^{-1}(\eta)$ a principal homogeneous space under $t_F \otimes I$.

Now we return to our lifting problem. We need to lift the map $f : (R, \xi) \to (A, \eta)$ to a map $f' : (R, \xi) \to (A', \eta')$. Claim it suffices to lift f to a map f', i.e. to find f' such that $p \circ f' = f$. Indeed, given such an f',

$$F(p) \circ F(f')(\xi) = F(f)(\xi) = \eta$$

thus $F(f')(\xi)$ is in the fiber $F(p)^{-1}(\eta)$, which by the previous paragraph, equals the orbit of η' under $t_F \otimes I$. Meanwhile, $t_F \otimes I$ also acts transitively on $h_R(p)^{-1}(\eta)$, which is thus the orbit of f'. In other words, there exists $\sigma \in t_F \otimes I$ such that $\sigma[F(f')(\xi)] = \eta'$, and we can replace f' by $g' = \sigma \circ f'$ to obtain

$$F(g')(\xi) = \eta', p \circ g' = f.$$

To lift f to f', note that f factors modulo R_q for some q. Thus it suffices to complete the diagram

$$\begin{array}{ccccc} \mathcal{O}[T_1, \ldots, T_r] & \xrightarrow{w} & R_q \times_A A' & \longrightarrow & A' \\ \downarrow & & {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ R_{q+1} & \longrightarrow & R_q & \xrightarrow{f} & A \end{array}$$

by lifting to a map $R_{q+1} \to R_q \times_A A'$. Now either pr_1 has a section, in which case the lift is obvious, or we have seen pr_1 is essential, in which case w is surjective by definition. Now (H_1) implies the map

$$F(R_q \times_A A') \to F(R_q) \times_{F(A)} F(A')$$

is surjective, hence $\xi_q \in F(R_q)$ lifts back to $F(R_q \times_A A')$. This implies $J_{q+1} \subset ker w$ by minimality of J_{q+1}, hence w factors through R_{q+1}, which completes the proof of formal smoothness, hence that (R, ξ) is a hull.

Finally, under (H_4), we prove $h_R(A) \xrightarrow{\sim} F(A)$ by induction on the length of A. Suppose it's true for A and let $p : A' \to A = A'/I$ be a small extension. Let $\eta \in F(A)$. Now the action of $t_F \otimes I$ is simply transitive on $h_R(p)^{-1}(\eta)$ and under (H_4) it is also simply transitive on $F(p)^{-1}(\eta)$. Finally, $h_R(A') \to F(A')$ is surjective since $h_R \to F$ is formally smooth, and it follows that $h_R(A') \to F(A')$ is bijective, which completes the induction.

3. Moduli of Galois representations.

In what follows, G is either (i) $Gal(K_S/K)$, where K is a number field, S is a finite set of places of K, and K_S is the maximal extension of K unramified outside S, or (ii) $Gal(\bar{K}/K)$, where K is a p-adic field. In case (i), if L is a finite extension of K, let L_S be the maximal extension of L unramified outside the primes of L above S.

Lemma 3.1. *For any finite extension L/K, let $G_L = Gal(L_S/L)$ in case (i), resp. $G_L = Gal(\bar{K}/L)$ in case (ii). Then $Hom(G_L, k)$ is a finite set.*

Proof. This is a consequence of class field theory. If L is p-adic then

$$Hom(G_L, k) = Hom(L^\times, k) = Hom(L^\times/(L^\times)^\ell, k)$$

which is finite by the structure theory of p-adic fields (and of dimension no bigger than 2 if $\ell \neq p$). If L is a number field then

$$G_L^{ab} \xrightarrow{\sim} L_{\mathbf{A}}^\times/L^\times \cdot \prod_{v \notin S} U_v \cdot L_\infty^{+,\times}.$$

Here U_v is the group of units in the v-adic completion of L.

$$[L_{\mathbf{A}}^\times : L^\times \cdot \prod_v U_v \cdot L_\infty^\times]$$

is the order of the ideal class group of L which is finite. It thus suffices to show that

$$\prod_{v \in S} U_v/U_v^\ell \times L_\infty^\times/L_\infty^{+,\times}$$

is a finite group, but the archimedean part is a finite two group and the non-archmedean part was already used for case (i).

Now let $\bar{r} : G \to GL(n, k)$ be a finite-dimensional representation. For A in $\mathcal{C}$, a *lifting* of $\bar{r}$ to A is a homomorphism

$$\rho : G \to GL(n, A); \rho = \bar{r} \pmod{\mathfrak{m}_A}.$$

For all N, let $\Gamma(\mathfrak{m}_A^N)$ be the principal congruence subgroup of $GL(n, A)$:

$$\Gamma(\mathfrak{m}_A^N) = \{\gamma \in GL(n, A) \mid \gamma \equiv 1 \pmod{\mathfrak{m}_A^N}\}.$$

A *deformation* of $\bar{r}$ to A is an equivalence class of liftings ρ, where ρ_1 and ρ_2 are equivalent if there exists a matrix $\gamma \in GL(n, A)$, with $\gamma \in \Gamma(\mathfrak{m}_A)$, such that $\rho_2 = \gamma \circ \rho_1 \circ \gamma^{-1}$. Define the functor $Def(\bar{r})$ on $\hat{\mathcal{C}}$ for which $Def(\bar{r})(A)$ is the set of deformations of $\bar{r}$ to A.

The functor of liftings is obviously prorepresentable by some sort of ring (take generators and relations). Now Lemma 3.1 implies G is (in a certain weak sense) topologically finitely generated, so the functor of liftings is prorepresentable by a noetherian local ring. (A more precise formulation of this point is given at the end of the lecture on Carayol's theorem.) More difficult is

Theorem 3.2 (Mazur). *Suppose $\bar{r}$ is absolutely irreducible. Then $Def(\bar{r})$ is prorepresentable by a ring $R_{\bar{r}}$ in $\hat{\mathcal{C}}$.*

This is the first and simplest of the theorems of this kind. It suffices to show that the functor $Def(\bar{r})$ satisfies hypotheses $(H_1) - (H_4)$ of Schlessinger.

Lemma 3.3. *Let A be in $\mathcal{C}$ and ρ a lifting of $\bar{r}$ to A. Then $Aut(\rho) = \mathbb{G}_m(A)$ is the group of scalar matrices in $GL(n, A)$, and this isomorphism is functorial if one is given compatible liftings.*

Proof. Let $\alpha \in Aut(\rho)$. If $A = k$, then because $\bar{r}$ is absolutely irreducible we know that $\alpha \in \mathbb{G}_m(k)$ by Schur's Lemma. Now we may lift any element in $\mathbb{G}_m(k)$ to $\mathbb{G}_m(A)$ for any fixed A (i.e., $\mathbb{G}_m$ is formally smooth), so we may assume $\alpha = 1 \pmod{()\mathfrak{m}_A}$. The matrix of $\alpha - 1$ belongs to $End_G(\bar{r} \otimes \mathfrak{m}_A/(\mathfrak{m}_A)^2) \pmod{\mathfrak{m}_A^2}$. Inducting on the dimension of t_A^* we find by Schur's lemma again that $\alpha - 1$ is a scalar (in $\mathfrak{m}_A/(\mathfrak{m}_A)^2$) up to $\mathfrak{m}_A^2$. By induction we find that α is scalar.

Lemma 3.4. *Let A be in $\hat{\mathcal{C}}$. Then*

$$Def(\bar{r})(A) = \varprojlim_n Def(\bar{r})(A/\mathfrak{m}_A^n).$$

In particular, the definition of $Def(\bar{r})$ on $\hat{\mathcal{C}}$ is the same as the tautological definition obtained from its restriction to $\mathcal{C}$.

Proof. Suppose ρ_1 and ρ_2 are two liftings of $\bar{r}$ to A, and suppose that, for all N there exists $\gamma_N \in \Gamma(\mathfrak{m}_A)$ such that Any two such γ_N differ by an element of $Aut(\rho_2) \pmod{\mathfrak{m}_A^N}$, and so in particular

$$\gamma_{N+1} \equiv \gamma_N \cdot \alpha_N \pmod{\mathfrak{m}_A^N}$$

for some $\alpha \in Aut(\rho_2 \pmod{\mathfrak{m}}_A^n) = \mathbb{G}_m(A/\mathfrak{m}_A^n)$ where the equality follows from Lemma 3.3. Thus we can modify γ_{N+1} by a scalar to obtain

$$\gamma_{N+1} \equiv \gamma_N \pmod{\mathfrak{m}_A^N}$$

for all N, hence that ρ_1 and ρ_2 are equivalent over A.

We now verify Schlessinger's conditions. For simplicity, write $F = Def(\bar{r})$.

(H_3). By definition

$$t_F = Hom_{\bar{r}}(G, GL(n, k[\varepsilon])/(1 + \varepsilon M(n, k))$$

where $Hom_{\bar{r}}$ denotes liftings and the quotient by $(1 + \varepsilon M(n, k))$ is for the conjugation action. I claim this is isomoprhic to $H^1(G, Ad(\bar{r}))$ where $Ad(\bar{r})$ is the representation of G on $End_k(\bar{r})$. Here

$$H^1(G, Ad(\bar{r})) = \{C : G \to M(n, k) : c(g_1 g_2) = c(g_1) + Ad(\bar{r})(g_1)c(g_2)\}.$$

Indeed, consider a lifting ρ of $\bar{r}$ to $GL(n, k[\varepsilon])$. Define

$$d\rho(g) = \frac{d}{d\varepsilon}\rho(g),\ c_\rho(g) = d\rho(g)\bar{r}g^{-1}.$$

Now by Leibniz' rule,

$$d\rho(g_1g_2) = d\rho(g_1)\bar{r}(g_2) + \rho(g_1)d\bar{r}(g_2),$$

hence

$$\begin{aligned} c_\rho(g_1g_2) &= [d\rho(g_1)\bar{r}(g_2) + \bar{r}(g_1)d\rho(g_2)](\bar{r}(g_2)^{-1}\bar{r}(g_1)^{-1}) \\ &= d\rho(g_1)\bar{r}(g_1)^{-1} + \bar{r}(g_1)[d\rho(g_2)(\bar{r}(g_2)^{-1}]\bar{r}(g_1)^{-1} \\ &= c_\rho(g_1) + Ad(\bar{r}(g_1))c_\rho(g_2) \end{aligned}$$

which is the cocycle relation. I leave as an exercise the verification that the quotient by $(1 + \varepsilon M(n,k))$ is exactly the quotient of cocycles by coboundaries.

Now let L be a finite Galois extension of K such that $\bar{r}$ is trivial on G_L. We have the exact inflation-restriction sequence

$$1 \to H^1(Gal(L/K), Ad(\bar{r})) \to H^1(G, Ad(\bar{r})) \to H^1(G_L, Ad(\bar{r})) = Hom(G_L, Ad(\bar{r}))$$

The first group is finite because $Gal(L/K)$ and $Ad(\bar{r})$ are finite groups, and the last group is finite by Lemma 3.1. This completes (H_3).

(H_1).

Let A_i, $i = 0, 1, 2, 3$ be four rings in $\mathcal{C}$, with $A_3 = A_2 \times_{A_0} A_1$, and let E_i be the set of liftings of $\bar{r}$ to A_i for each i, $F_i = F(A_i) = E_i/\Gamma(\mathfrak{m}_{A_i})$. We have clearly that

$$E_3 \to E_2 \times_{E_0} E_1$$

is bijective, because the lifting functor is representable . Assume $A_2 \to A_0$ is small. In particular it is surjective, and this implies that $\Gamma(\mathfrak{m}_{A_2}) \to \Gamma(\mathfrak{m}_{A_0})$ is surjective. Now suppose we have liftings ρ_i, $i = 0, 1, 2$, such that $\rho_1 = \gamma_0\rho_2\gamma_0^{-1}$ in $GL(n, A_0)$. We lift γ_0 to $\gamma_2 \in GL(n, A_2)$ and define

$$\rho_3 = (\rho_1, \gamma_2\rho_2\gamma_2^{-1}).$$

This gives surjectivity of

$$F_3 \to F_2 \times_{F_0} F_1.$$

To prove injectivity, we use the following lemma:

Lemma 3.5. *If for all $\phi_2 \in E_2$, $\phi_0 = Im(\phi_2) \in E_0$, the map*

$$Cent_{A_2}(\phi_2) \to Cent_{A_0}(\phi_0)$$

is surjective, then

$$F_3 \to F_2 \times_{F_0} F_1$$

is injective.

Since the hypothesis is always satisfied for small extensions, so is the injectivity. Thus this lemma implies (H_1).

Proof. Suppose ρ_3, ρ_3' are elements of E_3 such that there exist γ_1, γ_2 for which

$$\gamma_1 \rho_3 \gamma_1^{-1} = \rho_3' \text{ on } A_1;$$

$$\gamma_2 \rho_3 \gamma_2^{-1} = \rho_3' \text{ on } A_2.$$

Since $\Gamma(\mathfrak{m}_{A_3}) \to \Gamma(\mathfrak{m}_{A_1})$ is surjective, we may assume $\gamma_1 = 1$. Now recall $A_2 \to A_0$ is a small extension, i.e. $A_2/(t) = A_0, \mathfrak{m}_{A_2} \cdot (t) = (0)$. Let $p_{ij} : A_i \to A_j$ be the maps when they exist. We know $\rho_3 \in E_3$, which means

$$p_{10} \circ p_{31}(\rho_3) = p_{20} \circ p_{32}(\rho_3)$$

and likewise with ρ_3 replaced by ρ_3'. But we also have

$$p_{10} \circ p_{31}(\rho_3') = p_{10} \circ p_{31}(\rho_3) = p_{20} \circ p_{32}(\rho_3); \tag{3.5.1}$$

$$p_{20}(\gamma_2 p_{32}(\rho_3)\gamma_2^{-1} = p_{20} \circ p_{32}(\rho_3') = p_{10} \circ p_{31}(\rho_3') = p_{20} \circ p_{32}(\rho_3)$$

where the last equality is (3.5.1). It follows that $p_{20}(\gamma_2) \in Cent_{A_0}(\rho_0)$. By hypothesis, it lifts to an element $\delta \in Cent_{A_2}(\rho_2)$, and multiplying γ_2 by δ^{-1} we may assume $p_{20}(\gamma_2) = 1$. In other words, the element $(1, \gamma_2) \in GL(n, A_1 \times A_2)$ belongs to $GL(n, A_3)$ and can be used to conjugate γ_3 to γ_3'.

(H_2). If $A_0 = k$, $A_2 = k[\varepsilon]$, then $Cent_{A_2}(\phi_2) \to Cent_{A_0}(\phi_0)$ is obviously surjective (because the map $A_2 \to A_0$ is split.

(H_4).

Here $\bar{r}$ is assumed absolutely irreducible and $A_1 = A_2$. But Lemma 3.3 implies the surjectivity of $Cent_{A_2}(\phi_2) \to Cent_{A_0}(\phi_0)$, hence Lemma 3.5 implies that the map is bijective.

This completes the proof of Theorem 3.2