In case I've forgotten to mention this, ℓ is henceforth an odd prime. We begin with the n-dimensional ℓ-adic representation ρ. For the remainder of these lectures I will usually assume it has coefficients in $\mathbb{Q}_\ell$, to simplify the exposition; this is by no means a necessary hypothesis. Since the Galois group is compact, ρ stabilizes a lattice, say $\Lambda \subset \mathbb{Q}_\ell^n$. Let $\bar{\rho}$ denote the representation on $\Lambda/\ell\Lambda$. This is an n-dimensional representation of G_F with coefficients in $\mathbb{F}_\ell$. A priori it depends on the choice of Λ, but we will always assume

Hypothesis. *$\bar{\rho}$ is absolutely irreducible.*

Then the Brauer-Nesbitt theorem implies $\bar{\rho}$ is independent of the choice of lattice, up to equivalence. The residual representation $\bar{\rho}$ is the basic object that allows us to define the universal deformation ring $R_{\bar{r}}$. One could work with the n-dimensional representation $\bar{\rho}$ itself, but the additional structure coming from the polarization is essential. We let $\mathcal{G}_n$ denote the algebraic group (group scheme over $\mathbb{Z}$) whose identity component $\mathcal{G}_n^o$ is $GL(n) \times GL(1)$, and which is a semi-direct product of $GL(n) \times GL(1)$ by the group $\{1, j\}$ acting by

$$j(g,\mu)j^{-1} = (\mu\, {}^t g^{-1}, \mu), \;\; g \in GL(n), \mu \in GL(1).$$

There is a homomorphism $\nu : \mathcal{G}_n \to GL(1)$ sending (g, μ) to μ and j to -1. We let $\mathfrak{g}_n = Lie(GL(n)) \subset Lie(\mathcal{G}_n)$, $\mathfrak{g}_n^0$ the trace zero subspace.

We consider a topological group Γ with a closed subgroup Δ of index 2 and an element $c \in \Gamma - \Delta$ with $c^2 = 1$.

Lemma. *Let R be any commutative ring. There is a natural bijection between the following two sets:*

1. *Homomorphisms $r : \Gamma \to \mathcal{G}_n(R)$ such that $\Delta = r^{-1}\mathcal{G}_n^o(R)$,*
2. *Pairs $(\rho, <,>)$, where $\rho : \Delta \to GL(n, R)$ and*

$$<,>: R^n \otimes R^n \to R$$

is a perfect bilinear pairing such that

- *$< x, y >= -\mu(c) < y, x >$ for some $\mu(c) \in R$, for all $x, y \in R^n$, and*
- *$\mu(\delta) < \delta^{-1}x, y >=< x, c\delta c y >$ for any $\delta \in \Delta$, some $\mu(\delta) \in R^\times$. Under this correspondence $\mu(\gamma) = \nu \circ r(\gamma)$ for all $\gamma \in \Gamma$.*

Proof. Given r, let $\rho = r \mid_\Delta$. Write

$$r(c) = (A, -(\nu \circ r)(c)) \times j.$$

Then we can define

$$< x, y >= {}^t x A^{-1} y.$$

The correspondence is now a simple calculation.

Since $c^2 = 1$, we have

$$1 = r(c)^2 = (A, -(\nu \circ r)(c)) \cdot j(A, -(\nu \circ r)(c))j^{-1} = (A \cdot (-(\nu \circ r)(c))^t A^{-1}, 1),$$

i.e.

$$^tA = -(\nu \circ r)(c) \cdot A.$$

Thus A is either symmetric or alternating, as $\nu \circ r(c) = -1$ or $+1$.

Corollary. *Under the above hypotheses,*

$$\dim Lie(GL(n))^{c=1} = \frac{n(n + \nu \circ r(c))}{2}.$$

In other words, $\dim Lie(GL(n))^{c=1} = \frac{n(n-1)}{2}$ *(resp* $= \frac{n(n+1)}{2}$ *if* A *is symmetric (resp. if* A *is alternating).*

In the end, we will always find that A is symmetric.

We let ω denote the cyclotomic character acting on $\mathbb{Q}_\ell(1)$ or $\mathbb{F}_\ell(1)$.

Corollary. *Let* $k = \mathbb{Q}_\ell$ *(resp.* $\mathbb{F}_\ell$*). There is a homomorphism*

$$r : G_{F^+} \to \mathcal{G}_n(\mathbb{Q}_\ell)$$

(resp. $\bar{r} :: G_{F^+} \to \mathcal{G}_n(\mathbb{F}_\ell)$ *such that* $r\mid_{G_F} = \rho$ *(resp.* $\bar{\rho}$*),* $\nu \circ r\mid_{G_F} = \omega^{1-n}$*,* $r(c) \in \mathcal{G}_n(k) - GL(n,k)$*.*

The possible extensions r of ρ are classified up to isomorphism by elements of $k^\times/(k^\times)^2$. We will ignore this issue.

Now $k = \mathbb{F}_\ell$ (though this may not always be legitimate), $\mathcal{O} = \mathbb{Z}_\ell$. Let $S \supset S_\ell \cup S_\infty$ be as before. We write $\Gamma = G_{F^+,S}$, $\Delta = G_{F,S}$, where the subscript S means "the Galois group of the maximal extension unramified outside S. Let c denote complex conjugation, and assume $\bar{r}$ is absolutely irreducible. For $v \in S$ we let Δ_v be the decomposition group G_v. All places in S are assumed split in F/F^+. We write $S_f = S_\ell \cup S_{min} \cup R \cup Q \cup S_1$, where at places in S_{min} Π_v is ramified, at Q Π_v is unramified, and at R the ramification, if any, is of principal series type. The set S_1 is present for technical reasons. For $v \in S_1$, $\rho_v = \rho\mid_{\Delta_v}$ is then absolutely irreducible. We always assume that the residual representation $\bar{\rho}_v$ is also absolutely irreducible.

Let $\mathcal{C}^f_{\mathcal{O}}$ be the category of Artinian local $\mathcal{O}$-algebras A for which the map $\mathcal{O} \to A$ induces an isomorphism on residue fields, $\mathcal{C}_{\mathcal{O}}$ the full subcategory of topological $\mathcal{O}$-algebras whose objects are inverse limits in $\mathcal{C}^f_{\mathcal{O}}$. For A an object of $\mathcal{C}^f_{\mathcal{O}}$ or $\mathcal{C}_{\mathcal{O}}$ we want to classify liftings of $\bar{\rho}$ to homomorphisms $\rho' : \Delta \to GL(n, A)$ satisfying the properties of 2 of the Lemma, or more properly homomorphisms $r' : \Gamma \to \mathcal{G}_n(A)$

lifting $\bar{r}$. Moreover, we only consider liftings up to equivalence: two liftings are *equivalent* if they are conjugate by an element of $GL(n, A)$ that reduces to 1 in $GL(n, A/m_A) = GL(n, k)$, where m_A is the maximal ideal of A.

Suppose A is an object of $\mathcal{C}_\mathcal{O}$ with closed ideal I, and suppose r_1 and r_2 are two liftings of $\bar{r}$ to A that are equivalent mod I. By induction on the length of A/I we can reduce to the case where $m_A \cdot I = (0)$. Thus there is a short exact sequence

$$1 \to M(n,k) \to \mathcal{G}_n(A) \to \mathcal{G}_n(A/I) \to 1$$

where $M(n,k) = 1 + M(n,I) \subset \mathcal{G}_n(A)$. Then

$$\gamma \mapsto r_2(\gamma) r_1(\gamma)^{-1} - 1 \in M(n,k)$$

defines a cocycle $[r_2 - r_1] \in Z^1(\Gamma, M(n,k))$ where the action of Γ on $M(n,k)$ is given by conjugation in $\mathcal{G}_n(A)$, i.e. by $ad\ \bar{r}$. We have a cocycle because the liftings are group homomorphisms; and two cocycles give rise to equivalent liftings if and only if they define the same class in $H^1(\Gamma, ad\ \bar{r})$.

Without much difficulty we can prove that the functor classifying liftings of $\bar{r}$ to $\mathcal{C}_\mathcal{O}$ is representable by a ring R^{univ}, in the sense that homomorphisms $R^{univ} \to A$ are canonically in bijection with liftings of $\bar{r}$ to A. Since $\Gamma = G_{F^+,S}$, the resulting liftings are automatically unramified outside $S \coprod S_\ell$. However, we need additional conditions, for example to guarantee that the liftings are geometric in the sense of Fontaine-Mazur. The only liftings of interest are thus those that satisfy certain conditions upon restriction to Δ_v, $v \in S \coprod S_\ell$. This makes representability more delicate. We begin with the minimal conditions. We always assume ρ comes from cohomology with trivial coefficients:

Hypotheses (minimal case). *We only consider liftings r' of $\bar{r}$ with the following properties:*

1. *For $v \in S_{min}$, the natural map $r'(I_v) \to \bar{r}(I_v)$ is an isomorphism.*
2. *For $v \in R$, there is no restriction on $r'(I_v)$, but since this is the minimal case, R is now assumed empty.*
3. *For $v \in S_\ell$, $r' \mid_{\Delta_v}$ is crystalline (Fontaine-Laffaille) with Hodge-Tate weights $0, 1, \ldots, n-1$, each with multiplicity one.*
4. *For the moment, Q is empty.*

Let $\rho' = r' \mid_\Delta$. We always assume

Polarization hypothesis. *We assume $\nu \circ \rho' = \omega^{1-n}$.*

Condition (3) means that that the Fontaine-Laffaille functor $M_{crys}(\rho')$ attached to $\rho' \mid_{\Delta_v}$ is a free A-module of rank n with Fil^i/Fil^{i+1} free of rank 1 for $i = 0, 1, \ldots, n-1$. One of the main open questions in the theory is what condition to use when $\ell < n$. If we restrict attention to ordinary representations, or even "nearly ordinary" in Hida's sense, there is a practical substitute. Otherwise, it's quite mysterious except for $n = 2$, where Kisin has defined a workable theory.

Theorem. *The functor classifying minimal liftings is representable in $\mathcal{C}_{\mathcal{O}}$ by a noetherian $\mathbb{Z}_\ell$-algebra $R_{\bar{r}}^{min}$ with residue field k.*

The proof, which follows the arguments of Mazur and Ramakrishna, is based on Schlessinger's criterion for pro-representability of functors on categories like $\mathcal{C}_{\mathcal{O}}^f$. In the next lecture I will say more about $R_{\bar{r}}^{min}$ and the non-minimal variants, and the relations with Galois cohomology and Selmer groups. The goal of the theory is to prove that $R_{\bar{r}}^{min}$ and its non-minimal variants are isomorphic to certain Hecke algebras, acting on automorphic forms on the *definite* unitary group G'. This is sufficient to prove that every lifting of $\bar{r}$ of geometric type, in the sense of Fontaine-Mazur, comes from automorphic forms on $GL(n, F)$.