Framed deformations, following Kisin and Taylor

I repeat some of the definitions from the lecture on the local level-raising problem:

Let $\mathcal{O}$ be an ℓ-adic integer ring with maximal ideal $\mathfrak{m}$ and residue field k. We consider the category $\mathcal{C}_{\mathcal{O}}$, also called $\widehat{\mathcal{AR}}_{\mathcal{O}}$, of complete local $\mathcal{O}$-algebras A with residue field k (such that the structure map $\mathcal{O} \mapsto A$ induces the identity map on residue fields), and define the functor F^{loc} on $\mathcal{C}_{\mathcal{O}}$ defined by

$$F^{loc}(A) = \{r : \Gamma \to GL(n, A) \mid \Gamma = 1 \pmod{\mathfrak{m}_A}\}$$

where $\mathfrak{m}_A$ is the maximal ideal of A. Such an r is obviously trivial on the wild inertia group, since $q \neq \ell$, and factors through the quotient $\Gamma_{(\ell)}$ of Γ which fits into a two-step exact sequence:

$$1 \to I_\ell \xrightarrow{\sim} \mathbb{Z}_\ell(1) \to \Gamma_{(\ell)} \to Gal(\bar{\mathbb{F}}/\mathbb{F}) \xrightarrow{\sim} \hat{\mathbb{Z}} \to 1, \tag{0.1}$$

where I_ℓ is the ℓ-adic part of tame inertia and $\hat{\mathbb{Z}}$ is topologically generated by geometric Frobenius $Frob_{\mathbb{F}}$. In other words, if we choose a generator $T \in I_\ell$ then $F^{loc}(A)$ is parametrized by the pairs of matrices

$$\Sigma = r(T), \Phi = r(Frob_{\mathbb{F}})$$

satisfying the relation

$$\Phi \Sigma \Phi^{-1} = \Sigma^q. \tag{0.2}$$

and the relation

$$\Sigma \equiv \Phi \equiv 1 \pmod{\mathfrak{m}}.$$

The functor F^{loc} is represented on $\mathcal{C}_{\mathcal{O}}$ by a ring R^{loc} in $\mathcal{C}_{\mathcal{O}}$.

We are interested in certain quotients of R^{loc}. Let $\chi = (\chi_1, \dots, \chi_n)$ be an n-tuple of characters of Γ with values in $1 + \mathfrak{m} \subset \mathcal{O}^\times$. Let R_χ be the maximal quotient of R^{loc} over which, for all $\sigma \in I_\ell$, the homomorphism r evaluated at σ has characteristic polynomial

$$P_{\chi,\sigma}(X) = \prod_{j=1}^{n} (X - \chi_i(\sigma)). \tag{0.3}$$

It is equivalent to impose the characteristic polynomial condition on the generator T.

1. Galois-cohomological identification of cotangent space.

We now give ourselves our usual absolutely irreducible representation $\bar{\rho} : \Gamma_F \to GL(n, k)$ and the corresponding homomorphism

$$\bar{r} : \Gamma_{F^+} \to \mathcal{G}_n(k).$$

We let R be a finite set of primes of F^+ that split in F, such that for all $v \in R$, $q_v = |k(v)| \equiv 1 \pmod{\ell}$. Let T_v be a chosen generator of $I_{\ell,v}$ for $v \in R$. We assume that, for all $v \in R$, the characteristic polynomial $P_{\chi,T}(X)$ satisfies (0.3) for some n-tuple $\chi_v = (\chi_{1,v}, \dots, \chi_{n,v})$. These are all primes of the sort considered in the lecture on level-raising. A *framed deformation* (of type $\mathcal{S}$) of $\bar{r}$ to a ring A in $\mathcal{C}$, of type χ, is a lifting

$$\tilde{r} : \Gamma_{F+} \to \mathcal{G}(A)$$

with the usual hypotheses $\mathcal{S}$ at S_ℓ (and S_{min} and Q, if relevant), together with a local representation r_v of Γ_v of type χ_v, in the sense described above, and equalities (local framings)

$$(\Box) \qquad \tilde{r} \ |_{\Gamma_v} = \alpha_v r_v \alpha_v^{-1},$$

for all $v \in R$, all taken up to $1 + \mathfrak{m}_A M(n, A)$-conjugacy. To be more precise, $\beta \in 1 + \mathfrak{m}_A M(n, A)$ acts on a tuple $(\tilde{r}, \{r_v\}, \{\alpha_v\})$ by

$$\beta(\tilde{r}, \{r_v\}, \{\alpha_v\}) = (\beta\tilde{r}\beta^{-1}, \{r_v\}, \beta\alpha_v).$$

Ignoring the framing, there is a universal deformation of type χ (and $\mathcal{S}$) over a ring $R^{univ}_{\chi,\mathcal{S}}$. That the condition (0.3) defines a functor that satisfies the Schlessinger criteria comes from the fact that the characteristic polynomial is assumed to have roots not merely in A but in the image of $\mathcal{O}^\times$ in $A^\times$. Thus the functor is relatively representable over the functor of deformations that are unrestricted at primes in v. In particular, if $A_3 = A_2 \times_{A_0} A_1$ is the typical fiber product, and if we have a deformation over A_3, then it is of type χ over A_3 if and only if its projections to A_2 and A_1 are of type χ, because the characteristic polynomials of T_v over A_2 and A_1 determine the characteristic polynomial over A_3.

There is also a universal framed deformation of $\bar{r}$ of type χ over an object $R^{\Box}_{\chi,\mathcal{S}}$ of $\mathcal{C}$. Once we have $R^{univ}_{\chi,\mathcal{S}}$, the framings at $v \in R$ are determined by additional data. Forgetting the framings defines a map $R^{univ}_{\chi,\mathcal{S}} \to R^{\Box}_{\chi,\mathcal{S}}$.

We drop the subscript $\mathcal{S}$ for the remainder of this section.

On the other hand, let

$$R^{loc}_\chi = \widehat{\otimes}_{v \in R} R^{loc}_{v,\chi}$$

(denoted R_R in [T]):

$$\mathcal{T}_R = \mathcal{O}[[X_{v,i,j} \mid v \in R\ , i, j = 1, \ldots, n]]$$

(matrix coefficients of liftings for each v to parametrize r_v). If r^{univ} is the universal deformation of $\overline{\rho}$, choose a lifting that represents it (i.e. in its equivalence class). Such a choice of matrix coordinates gives rise to maps

$$R^{loc}_\chi \to R^{\Box}_{\chi,\mathcal{S}}; \ \mathcal{T}_R \to R^{\Box}_{\chi,\mathcal{S}}.$$

The former is defined by forgetting everything except the framing, the latter by setting $\alpha_v = I_n + (X_{v,i,j})$ in the given matrix coordinates.

Lemma 1.1.

$$R^{univ}_{\chi,\mathcal{S}} \hat{\otimes} \mathcal{T}_R \xrightarrow{\sim} R^{\Box}_{\chi,\mathcal{S}}.$$

Proof. If we choose a matrix representation r^{univ}, then given any collection $\alpha_v \in 1 + \mathfrak{m}_A M(n, A)$ we can define $r_v = \alpha_v^{-1} r^{univ} \alpha_v$. The important point is that α_v and r_v determine each other uniquely, given r^{univ}, by the lemma below.

Lemma 1.2. *The centralizer of r^{univ} in $1+\mathfrak{m}_A M(n,A)$ is trivial.*

Proof. The centralizer of $\rho^{univ} = r^{univ} \mid_{\Gamma_F}$ is the group of scalar matrices, as we have already seen by successive approximation. and so the centralizer in $1+\mathfrak{m}_A M(n,A)$ is the group of scalars $a \in 1+\mathfrak{m}_A$. But such a scalar a commutes with $c \in \Gamma_{F+} \setminus \Gamma_F$ if and only if $a = a^{-1}$, hence $a = 1$, since $\ell > 2$.

Now we are in a position to calculate the embedding dimension of $R^{\square}_{\chi}$ as an R^{loc}_{χ}-algebra. We define $Z^1_{\mathcal{S},\chi}$ and $Z^1_{\mathcal{S},\chi}$ by the usual conditions for $v \in S, v \notin R$. For $v \in R$ we take $L_v = (0)$, meaning that deformations are assumed trivial at $v \in R$.

Lemma 1.3. *$R^{\square}_{\chi}$ can be topologically generated over R^{loc}_{χ} by*

$$\dim_k H^1_{\mathcal{S},\chi}(\Gamma_{F+}, ad\ \bar{r}) + \sum_{v\in R} \dim_k H^0(\Gamma_v, ad\ \bar{r})$$

elements.

The proof in [T] is analogous to the usual calculations of

$$\dim \mathfrak{m}_{R^{\square}_{\chi,\mathcal{S}}}/(\mathfrak{m}^2_{R^{\square}_{\chi,\mathcal{S}}} + \mathfrak{m}_{R^{loc}_{\chi}} R^{\square}_{\chi,\mathcal{S}}),$$

which are deformations over $k[\epsilon]/(\epsilon)^2$ with the R-data fixed (equal to $\bar{r} \mid_{\Gamma_v}$ at v). I copy the argument from [T]. The space $\mathfrak{m}_{R^{\square}_{\chi,\mathcal{S}}}/(\mathfrak{m}^2_{R^{\square}_{\chi,\mathcal{S}}} + \mathfrak{m}_{R^{loc}_{\chi}} R^{\square}_{\chi,\mathcal{S}})$ is dual to the set of equivalence classes of framed deformations

$$((I_n + \varepsilon\phi)\bar{r}, \{\bar{r} \mid_{\Gamma_v}\}, \{1+\varepsilon a_v)\})$$

of $\bar{r}$ to $k[\varepsilon]/(\varepsilon)^2$ of type $\mathcal{S},\chi$, satisfying the compatibility ($\square$) of §1. As in the unframed calculation, we have $\phi \in Z^1_{\mathcal{S},\chi}(\Gamma_{F+}, \mathrm{ad}\ \bar{r})$, where the condition at $v \in R$ is guaranteed by the condition $r_v = \bar{r} \mid_{\Gamma_v}$ is fixed. The number of generators is equal to the dimension over k of

$$[\ker(Z^1_{\mathcal{S},\chi}(\Gamma_{F+}, \mathrm{ad}\ \bar{r}) \ \oplus \bigoplus_{v\in R} \mathrm{ad}\ \bar{r} \rightarrow \bigoplus_{v\in R} Z^1(\Gamma_v, \mathrm{ad}\ \bar{r}))]/\mathrm{ad}\ \bar{r}.$$

Here the map sends

$$(\phi, (a_v)) \mapsto (\phi \mid_{\Gamma_v} - \partial a_v),$$

and the fact that our class lies in the kernel is an expression of ($\square$). Note that for fixed ϕ, the dimension of the space of solutions to $\phi \mid_{\Gamma_v} - \partial a_v = 0$ is the same as for $\phi = 0$, i.e. it equals $h^0(\Gamma_v, \mathrm{ad}\ \bar{r})$. Because the conjugation action is faithful (by lemma 1.2), the action of ad $\bar{r}$) is also faithful, hence the number of generators is just

$$\dim Z^1_{\mathcal{S},\chi}(\Gamma_{F+}, \mathrm{ad}\ \bar{r}) + \sum_v h^0(\Gamma_v, \mathrm{ad}\ \bar{r}) - n^2$$

which can be rewritten

$$[\dim Z^1_{\mathcal{S},\chi}(\Gamma_{F+}, \mathrm{ad}\ \bar{r}) - n^2] + \sum_v h^0(\Gamma_v, \mathrm{ad}\ \bar{r})$$

which (again because the action is faithful) equals

$$\dim_k H^1_{\mathcal{S},\chi}(\Gamma_{F+}, \mathrm{ad}\ \bar{r}) + \sum_{v\in R} \dim_k H^0(\Gamma_v, \mathrm{ad}\ \bar{r}).$$

Using the Riemann-Roch formula as in the unframed calculation (which may not yet have been presented), we find that

Lemma 1.4. $R_\chi^\square$ *can be topologically generated over* R_χ^{loc} *by*

$$|Q| + \dim_k H^1_{\mathcal{S}^*,\chi}(\Gamma_{F^+}, ad\ \bar{r}^*) - \dim_k H^0(\Gamma_{F^+}, ad\ \bar{r}^*) - n \sum_{v \in S_\infty} \frac{c_v + 1}{2}$$

elements.

Indeed, the term $\dim_k H^0(\Gamma_{F^+}, \mathrm{ad}\ \bar{r})$ vanishes by Lemma 1.2. On the other hand, for $v \in R$ $\dim L_v = 0$ and so $\chi_{\mathcal{S},v} = -\dim_k H^0(\Gamma_v, \mathbf{A}\bar{r})$, which exactly cancels the local terms in Lemma 1.3. Finally, the term $|Q|$ is a consequence of the calculation, yet to be presented, that $\chi_{\mathcal{S},v} = 1$ for $v \in Q$.