THE LOCAL LEVEL-RAISING PROBLEM

1. Moduli of certain pairs of matrices

When Mumford wrote an introduction to his approach to moduli via geometric invariant theory in 1970 [MS], his first example to show the importance of his stability criterion was the classification of endomorphisms of vector spaces; the presence of unipotents implies that no coarse moduli space exists. For the same reason, the functor of deformations of ℓ-adic representations of $\Gamma_K := Gal(\bar{K}/K)$, where K is a q-adic field, $q \neq \ell$, is in general not representable. The worst case is the one that arises in the problem of level raising. We consider an upper-triangular representation

$$\beta : \Gamma_K \to GL(n, \mathcal{O}); g \mapsto \begin{pmatrix} \chi_1(g) & * & \dots\dots & * \\ 0 & \chi_2(g) & \dots & * \\ 0 & 0 & \dots\dots & * \\ 0 & 0 & 0\dots & \chi_n(g) \end{pmatrix}$$

for some ℓ-adic integer ring $\mathcal{O}$. The diagonal entries are $\mathcal{O}^\times$-valued characters of Γ_K, whose reductions modulo $\mathfrak{m}$ are denoted $\bar{\chi}_i$, $i = 1, \dots, n$. N.B.: The deformation problem we consider imposes a restriction only on the inertial representation $\beta\mid_{I_K}$.

We will be assuming $K = F_v^+$, for some $v \in R$, so in particular (3.4.3) implies

$$q = q_v := \mathbf{N}_v \equiv 1 \pmod{\ell}. \tag{1.1}$$

We call (1.1) plus the running assumption $\ell > n$, *the classical limit mod* ℓ. After a finite cyclic extension – which makes no difference to the modularity problem, by Proposition 1. 9 – we can assume

$$\bar{\chi}_i = 1, 1, \dots, n; \tag{1.2}$$

this is the *degenerate case.* By hypothesis β is tamely ramified, and hence is determined up to isomorphism by an upper-triangular representation β_I of the tame inertia group I_K^{tame} and an upper-triangular invertible Frobenius element $\Phi = \beta(Frob_K)$, satisfying

$$\Phi\beta_I(x)\Phi^{-1} = \beta_I(x^q), \forall x \in I_K^{tame}$$

Again, one can assume (after a finite solvable extension) that tame inertia is purely ℓ-adic, and letting $x_0 \in I_K^{tame}$ denote a generator of ℓ-adic tame inertia, $\Sigma = \beta_I(x_0)$, the above equality becomes

$$\Phi\Sigma\Phi^{-1} = \Sigma^q. \tag{1.3}$$

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We are thus led to consider the moduli space of pairs of matrices (Φ, Σ) satisfying (1.3). More precisely, for any monic polynomial $P \in \mathcal{O}[X]$ of degree n, we let $\mathcal{M}(P,q)$ be the affine scheme over $\mathcal{O}$ representing pairs (Φ, Σ) as above, with Φ invertible, such that Σ has characteristic polynomial P. Note that (1.3) implies that, if $\mathcal{M}(P,q)$ is non-empty, P is invariant under the $q-th$ power operation applied to its roots. The following lemma is clear:

Lemma 1.4. *Suppose* $q \equiv 1 \pmod{\ell}$ *and* $P = \prod_{i=1}^{n}(X-\zeta_i)$, *where the* ζ_i *are distinct* ℓ*th roots of unity in* $\mathcal{O}$. *Then*

- $\mathcal{M}(P,q) \simeq \mathcal{M}(P,1)$
- $\mathcal{M}(P,q) \times Spec(k) \xrightarrow{\sim} \mathcal{M}((X-1)^n, q) \times Spec(k)$.

Proof. Note that P divides $X^\ell - 1$, which in turn divides $X^q - X$. Thus $\Sigma^q = \Sigma$, hence $\Phi\Sigma\Phi^{-1} = \Sigma^q$ if and only if $\Phi\Sigma = \Sigma\Phi$. The second part is obvious.

Note that $\mathcal{M}(P,1)$ just parametrizes pairs of commuting matrices, one of which has fixed characteristic polynomial. The moduli problem makes sense over $(\mu_\ell^n)_{\mathbb{Z}}$ but only becomes interesting over $Spec(\mathbb{Z}_\ell)$, where μ_ℓ becomes connected over the closed point. The observation behind [T] is that the most degenerate case $P = (X-1)^n$ deforms to the least degenerate case $P = \prod_{i=1}^{n}(X-\zeta_i)$ with all ζ_i distinct. The affine algebra of $\mathcal{M}(P,q)$ is the ring of local liftings at $v \in R$ used in Kisin's version of the Taylor-Wiles method. To describe its geometric properties, we relate it to a Lie algebra variant. Let $\mathcal{N}(q)$ denote the moduli space of pairs of matrices (Φ, N), with Φ invertible, N nilpotent (characteristic polynomial X^n) and

$$\Phi N \Phi^{-1} = qN \tag{1.5.}$$

Lemma 1.6. *Assume* $\ell > n$. *Then*

(i) $\mathcal{N}(q)^{red}$ *is a union of reduced irreducible components parametrized by nilpotent conjugacy classes in* $Lie(GL(n))$; *i.e. by partitions of* n *(Jordan block decomposition).*

(ii) Each reduced irreducible component Z *of* $\mathcal{N}(q)$ *is equidimensional of dimension* n^2+1, Z_k *is irreducible of dimension* n^2 *and generically reduced, and each irreducible component of* $\mathcal{N}(q) \times Spec(k)$ *is contained in a unique irreducible component of* $\mathcal{N}(q)$ *which is not purely of characteristic* ℓ.

(iii) The logarithm and exponential (applied to Σ*) identify*

$$\mathcal{M}((X-1)^n, q)^{red} \xrightarrow{\sim} \mathcal{N}(q)^{red}.$$

In particular, the reduced irreducible components of $\mathcal{M}((X-1)^n, q)^{red}$ *have the properties (ii).*

At the other extreme:

Lemma 1.7. *Let* $P = \prod_{i=1}^{n}(X-\zeta_i)$ *with all* ζ_i *distinct. Then* $\mathcal{M}(P,1) \times K$ *is smooth and irreducible of dimension* n^2, *whereas* $\mathcal{M}(P,1) \times k \xrightarrow{\sim} \mathcal{N}(1) \times k$, *and hence has components indexed by partitions of* n *as in the previous lemma.*

Moreover, the completion of the affine ring of $\mathcal{M}(P,1)$ *at the closed point of the special fiber corresponding to* $\Sigma = 1$ *and* $\Phi = 1$ *has a unique minimal prime.*

In the second statement we just send Σ to $\Sigma - 1$, which is why there is no need to consider reduced components.

The ζ_i are the eigenvalues of $\beta(x_0)$. We can identify $Syl_\ell(k_v^\times)$ (ℓ-Sylow subgroup) with the subgroup of $Gal(K^{ab}/K)$ generated by x_0, and so we define χ_i to be the character of $k_v^\times$ of ℓ-power order whose image on x_0 is ζ_i. We let

$$R^{loc}_{v,\chi}$$

be the affine $\mathcal{O}$-algebra of $\mathcal{M}(P_\chi, 1)$, where $P_\chi = \prod_{i=1}^{n}(X - \zeta_i)$ as above. Thus χ and ζ are alternative notation for the same thing; we have already seen χ in the discussion of R in §3 in connection with Hecke algebras. The notation R^{loc} will be explained in the global discussion.

Complete proofs of these lemmas are somewhat delicate, because they concern functors on general $\mathcal{O}$-algebras, but the proofs on closed points over fields are quite clear.

Nilpotent conjugacy classes. Let $\mathcal{P}_n$ be the set of unordered partitions of n. It is standard (Jordan normal form) that the conjugacy classes of nilpotent matrices in $Lie(GL(n))$ are in bijection with $\mathcal{P}_n$. The set $\mathcal{P}_n$ is partially ordered by refinement: a partition $(\nu_1, \ldots, \nu_r)$ refines $(n_1, \ldots, n_s)$ if each n_i is a sum of some ν_j. Let Nil_n be the variety of nilpotent matrices over an algebraically closed field, say L, and for any $\sigma \in \mathcal{P}_n$ let $Nil_n(\sigma)$ be the reduced closed subscheme of nilpotent matrices whose blocks are a partition refining σ; let

$$Nil_n(\sigma)^0 = Nil_n(\sigma) \setminus \cup_{\sigma' > \sigma} Nil_n(\sigma').$$

For each σ let $N(\sigma) \in Nil_n(\sigma)$ be the standard upper triangular matrix in Jordan normal form. Let $Flag(\sigma)$ be the moduli space of flags in the free rank n module M_n of type σ, meaning the dimensions $d_i(\sigma)$ are the same as those of the *kernels* of successive powers of $N(\sigma)$. For example, if $n = 4$, $\sigma = (2, 1, 1)$, then the corresponding flag is of type $(0, 3, 4)$. Let $F(\sigma)$ be the standard flag of type σ. There is a universal flag over $Flag(\sigma)^0$ that (locally in the Zariski topology) is conjugate by a section of $GL(n)$ to $F(\sigma)$. Now let $Nil_n^F(\sigma)$ be the moduli space of pairs (F, N) where $F = (F_0 \subset F_1 \subset \cdots \subset M_n)$ is a flag of type σ and N is a nilpotent endomorphism such that $N(F_i) \subset F_{i-1}$. Let $Nil_n^F(\sigma)^0 \subset Nil_n^F(\sigma)$ be the open subset of maximal rank, i.e. where $N(F_{j+1}/F_j)$ is a direct summand of F_j/F_{j-1} for all j. There are natural maps

$$Nil_n^F(\sigma)^0 \hookrightarrow Nil_n^F(\sigma) \to Flag(\sigma)$$

where the second map takes (F, N) to F. Locally on the Zariski topology on $Flag(\sigma)$ this second map has a section (take a local section conjugating a given flag to the standard flag). Hence locally on $Flag(\sigma)$ there are isomorphisms

$$Nil_n^F(\sigma) \simeq Flag(\sigma) \times Q(\sigma); \;\; Nil_n^F(\sigma)^0 \simeq Flag(\sigma) \times Q^0(\sigma),$$

where $Q(\sigma) \subset M(n)$ is the (standard) set of $n \times n$ matrices taking $F(\sigma)_i$ to $F(\sigma)_{i-1}$ for all i and $Q^0(\sigma)$ is the open subset of $Q(\sigma)$ of maximal rank.

In this way we can calculate $\dim Nil_n^F(\sigma) = \dim Flag(\sigma) + \dim Q^0(\sigma)$. Now $\dim Flag(\sigma) = \dim GL(n) - \dim P(\sigma) = \frac{\dim(GL(n)) - \dim L(\sigma)}{2}$ where $P(\sigma)$ is the stabilizer of $F(\sigma)$ and $L(\sigma)$ is its Levi subgroup.

Exercise in notation.

$$\dim L(\sigma) = \sum_i [d_i(\sigma) - d_{i-1}(\sigma)]^2;$$

$$\dim Q^0(\sigma) = \dim Q(\sigma) = \dim P(\sigma) - \dim L(\sigma).$$

Hence $Nil_n^F(\sigma)^0$ *(resp.*$Nil_n^F(\sigma)$*) is smooth and connected (resp. integral and connected) of relative dimension*

$$\dim GL(n) - \dim L(\sigma) = n^2 - \sum_i [d_i(\sigma) - d_{i-1}(\sigma)]^2.$$

On the other hand, there is a forgetful map $Nil_n^F(\sigma) \to Nil_n$ (forget F). One can show (using the valuative criterion) that this map is proper, so its image $Z(\sigma)$ is integral and connected. Let $Z^0(\sigma) \subset Z(\sigma)$ be the open dense subset where $\ker N^j$ is locally free of rank $d_j(\sigma)$ for all j. Over $Z^0(\sigma)$ the filtration F is thus unique, hence the forgetful map

$$Nil_n^F(\sigma)^0 \to Z^0(\sigma)$$

is an isomorphism, i.e. $\dim Z^0(\sigma) = n^2 - \sum_i [d_i(\sigma) - d_{i-1}(\sigma)]^2$. But over L, the map

$$GL(n)/Z_{GL(n)}(N(\sigma)) \to Z^0(\sigma)$$

is a bijection on matrices. This shows that

$$\dim Z_{GL(n)}(N(\sigma)) = \sum_i [d_i(\sigma) - d_{i-1}(\sigma)]^2 = \dim L(\sigma).$$

With these preliminaries out of the way, we can sketch the proofs of the lemmas.

Sketch of proof of 1.6, following Taylor. Let Pol_n be the affine space of monic polynomials of degree n. For each $\sigma = (n_1, \ldots, n_r)$, let $Pol_n(q, \sigma) \subset Pol_n$ be the reduced closed subscheme corresponding to the set of polynomials whose roots can be partitioned into r sub multisets of the form $\{\alpha, q\alpha, \ldots, q^{n_j - 1}\alpha\}$. There are maps

$$n : \mathcal{N}(q) \to Nil_n$$

(forget Φ) and

$$\pi : \mathcal{N}(q) \to Pol_n$$

((Φ, N) goes to the characteristic polynomial of Φ). For each σ, let

$$\mathcal{N}(q, \sigma)^0 = n^{-1}(Z^0(\sigma)),$$

and let $\mathcal{N}(q, \sigma)$ be the reduced subscheme of the closure of $\mathcal{N}(q, \sigma)^0$ in $\mathcal{N}(q)$. Let $\mathcal{N}(q, \sigma)'$ be the reduced subscheme of $n^{-1}(Z(\sigma)) \cap \pi^{-1}(Pol_n(q, \sigma))$; thus

$$\mathcal{N}(q, \sigma)' \supset \mathcal{N}(q, \sigma) \supset \mathcal{N}(q, \sigma)^{0,red}.$$

For any σ as above and any r-tuple $a = (a_1, \ldots, a_r)$, we construct an element $\Phi(\sigma, a, q)$ such that

$$(\Phi(\sigma, a, q), N(\sigma)) \in \mathcal{N}(q, \sigma) \tag{*.}$$

as the explicit diagonal matrix

$$diag(a_1q^{n_1-1}, a_1q^{n_1-2}, \dots, a_1; a_2q^{n_2-1}, \dots, a_2; \dots; a_rq^{n_r-1}, \dots, a_r).$$

One verifies (*) by explicit calculation. Moreover, if $(\Phi, N(\sigma)) \in \mathcal{N}(q,\sigma)$ is any element then it is clear that

$$(**) \qquad \Phi = \Phi(\sigma,q) \cdot z \text{ for some } z \in Z_{GL(n)}(N(\sigma)$$

where $\Phi(\sigma,q) = \Phi(\sigma,(1,\dots,1),q)$.

Now locally in the Zariski topology, the map

$$\mathcal{N}(q,\sigma)^0 \to Z^0(\sigma)$$

splits as a map

$$Z^0(\sigma) \times Z_{GL(n)}(N(\sigma) \to Z^0(\sigma).$$

Indeed, on an open subset U the universal N over $Z^0(\sigma)$ is of the form $gN(\sigma)g^{-1}$; then by (**) the preimage of U in $\mathcal{N}(q,\sigma)^0$ is just

$$U \times g\Phi(\sigma,q)Z_{GL(n)}(N(\sigma))g^{-1}.$$

Thus $\mathcal{N}(q,\sigma)$ is a smooth variety over L of dimension

$$\dim Z^0(\sigma) + \dim Z_{GL(n)}(N(\sigma)) = [n^2 - \dim L(\sigma)] + \dim L(\sigma)] = n^2.$$

Over $\mathcal{O}$ it is an integral scheme of dimension n^2+1.

It remains to show that there are no other irreducible components. For this we choose sufficiently general a; it suffices to assume $a_iq^j \neq a_{i'}q^{j'}$ for $i \neq i'$ and $0 \leq j \leq n_i$, $0 \leq j' \leq n_{i'}$. Then calculating the characteristic polynomial of $\Phi(\sigma,a,q)$, we find that $(\Phi(\sigma,a,q),N(\sigma)) \notin \mathcal{N}(q,\sigma')'$ if $\sigma' \neq \sigma$. In particular,

$$(\Phi(\sigma,a,q),N(\sigma)) \in \mathcal{N}(q,\sigma) \setminus \bigcup_{\sigma'\neq\sigma} \mathcal{N}(q,\sigma).$$

It follows that the $\mathcal{N}(q,\sigma)^{red}$ exhaust the reduced irreducible components of $\mathcal{N}(q)$. Moreover, the final assertion of 1.6 (ii) follows because the construction is independent of characteristic.

Finally, 1.6 (iii) is obviously true over a field of characteristic 0 or $\ell > n$, where the logarithm and exponential maps are well defined, and that will be enough for us.

Proof of 1.7 (sketch).

I will be more brief. Recall that $\mathcal{M}(P,q) = \mathcal{M}(P,1)$ when the roots are distinct and $q \equiv 1 \pmod{\ell}$, so we are only concerned with the case $q=1$. Let λ be the maximal ideal of $\mathcal{O}$ and let $\alpha \in (1+\lambda)^n$ be any n-tuple of distinct elements of $\mathcal{O}^\times$; let $d(\alpha)$ be the corresponding diagonal matrix and P_α its characteristic polynomial. Let T_n be the group of invertible diagonal matrices and $Flag(n)$ the total flag variety (non-canonically isomorphic to $GL(n)/B$ where B is a Borel subgroup). Let $0 = F_0 \subset F_1 \subset \dots F_n = L^n$ be the standard flag. Let $\mathcal{M}^{Flag}(\alpha)$ be the incidence space of triples $(\{F_i\}, \Phi, \Sigma)$ where $\{F_i\} \in Flag(n)$, $(\Phi,\Sigma) \in \mathcal{M}(P_\alpha, 1)$, both Φ and Σ preserve each F_i, and Σ acts by α_j on F_j/F_{j-1}.

Lemma. *The maps*

$$GL(n)/T_n \times T_n \to \mathcal{M}^{Flag}(\alpha) \to \mathcal{M}(P_\alpha, 1)$$

where the first map takes (gT_n, t) *to* $\{gF_j\}, gtg^{-1}, gd(\alpha)g^{-1})$ *and the second map forgets the filtration, are isomorphisms over the field* L *of characteristic zero.*

In particular, the generic fiber of $\mathcal{M}(P_\alpha, 1)$ *is smooth and connected of dimension* n^2*.*

This is an easy matrix calculation – in particular, note that the flag is uniquely determined by the distinct eigenvalues in characteristic zero – and completes the first part of the proof of Lemma 1.7. The second part is more technical and I refer you to [T] for the proof (dispersed among Lemmas 1.4 (7), 1.5, and 1.6).

2. Local lifting rings in the degenerate classical limit

The previous section concerned moduli spaces of matrices of certain forms. These are not the same as local deformation rings; in particular, they are not ℓ-adically complete. The process of completion can potentially disturb the properties established in the previous section; for example, the completion of the localization at ℓ of the ring of integers of a number field in general is semilocal rather than local. The present subsection defines the local deformation rings and states the analogues of the theorems of the previous section without proof.

Let $q \neq \ell$ be a prime and F a q-adic field with residue field $\mathbb{F}$, $\Gamma = Gal(\bar{F}/F)$. As in §1, we assume $q \equiv 1 \pmod{\ell}$ and $\ell > n$. Let $\mathcal{O}$ be an ℓ-adic integer ring with maximal ideal $\mathfrak{m}$ and residue field k. We consider the category $\mathcal{C}_\mathcal{O}$, also called $\widehat{\mathcal{AR}}_\mathcal{O}$, of complete local $\mathcal{O}$-algebras A with residue field k (such that the structure map $\mathcal{O} \mapsto A$ induces the identity map on residue fields), and define the functor F^{loc} on $\mathcal{C}_\mathcal{O}$ defined by

$$F^{loc}(A) = \{r : \Gamma \to GL(n, A) \mid \Gamma = 1 \pmod{\mathfrak{m}_A}\}$$

where $\mathfrak{m}_A$ is the maximal ideal of A. Such an r is obviously trivial on the wild inertia group, since $q \neq \ell$, and factors through the quotient $\Gamma_{(\ell)}$ of Γ which fits into a two-step exact sequence:

$$1 \to I_\ell \xrightarrow{\sim} \mathbb{Z}_\ell(1) \to \Gamma_{(\ell)} \to Gal(\bar{\mathbb{F}}/\mathbb{F}) \xrightarrow{\sim} \hat{\mathbb{Z}} \to 1, \tag{2.1}$$

where I_ℓ is the ℓ-adic part of tame inertia and $\hat{\mathbb{Z}}$ is topologically generated by geometric Frobenius $Frob_\mathbb{F}$. In other words, if we choose a generator $T \in I_\ell$ then $F^{loc}(A)$ is parametrized by the pairs of matrices

$$\Sigma = r(T), \Phi = r(Frob_\mathbb{F})$$

satisfying relation (1.3) and the relation

$$\Sigma \equiv \Phi \equiv 1 \pmod{\mathfrak{m}}.$$

However, F^{loc} is represented on $\mathcal{C}_\mathcal{O}$ by a ring R^{loc} in $\mathcal{C}_\mathcal{O}$ and the rings considered in §1 are not complete.

We are interested in certain quotients of R^{loc}. Let $\chi = (\chi_1, \ldots, \chi_n)$ be an n-tuple of characters of Γ with values in $1 + \mathfrak{m} \subset \mathcal{O}^\times$. Let R_χ be the maximal quotient of R^{loc} over which, for all $\sigma \in I_\ell$, the homomorphism r evaluated at σ has characteristic polynomial

$$P_{\chi,\sigma}(X) = \prod_{j=1}^{n}(X - \chi_i(\sigma)). \tag{2.2}$$

It is equivalent to impose the characteristic polynomial condition on the generator T.

Lemma 2.3. *Suppose the χ_i are distinct characters of order ℓ, in other words, that $\chi_i(T)$ are distinct ℓth roots of unity in $\mathcal{O}^\times$. Then R_χ has a unique minimal prime ideal and this prime does not contain $\mathfrak{m}$. Moreover, R_χ has dimension $n^2 + 1$ (the generic fiber has dimension n^2).*

Lemma 2.4. *Suppose $\chi_i = 1$ for all i. Then $R_\chi = R_1$ is equidimensional of dimension $n^2 + 1$ and no minimal prime contains $\mathfrak{m}$. Moreover, every minimal prime is contained in a prime which is minimal over $\mathfrak{m} \cdot R_1$, and every prime which is minimal over $\mathfrak{m} \cdot R_1$ contains a unique minimal prime.*

Here a prime of R_1 is "minimal over $\mathfrak{m} \cdot R_1$" if it is the inverse image of a minimal prime in the reduction modulo $\mathfrak{m} \cdot R_1$.

The ring R_1 is the formal completion at $\mathfrak{m} \cdot R_1$ of the ring denoted $\mathcal{M}((X - 1)^n, q)$ in §1; likewise, R_χ is the formal completion of $\mathcal{M}(P_{\chi,T}, q) \simeq \mathcal{M}(P_{\chi,T}, q)$ in the notation of where $P = P_{\chi,T}$ is the polynomial (2.2). The two lemmas are derived from the corresponding properties of moduli spaces (Lemma 1.7 and Lemma 1.6, respectively) by general arguments about completions in commutative algebra, namely Lemmas 1.6 and 1.7 of [T].