1. The Dwork family.

Consider the equation

$$(f_\lambda) \qquad f_\lambda(X_0, X_1, \ldots, X_n) = \lambda(X_0^{n+1} + \cdots + X_n^{n+1}) - (n+1)X_0 \ldots X_n = 0,$$

where λ is a free parameter. This equation defines an $n-1$-dimensional hypersurface $Y_\lambda \in \mathbb{P}^n$ and, as λ varies, a family:

$$\begin{array}{ccc} Y & \subset & \mathbb{P}^n \times \mathbb{P}^1 \\ & \searrow & \downarrow \\ & & \mathbb{P}^1_\lambda \end{array}$$

Let

$$H = \mu_{n+1}^{n+1}/\Delta(\mu_{n+1}),$$

where Δ is the diagonal map, and let

$$H_0 = \{(\zeta_0, \ldots, \zeta_n) \mid \prod_i \zeta_i = 1\}/\Delta(\mu_{n+1}) \subset H.$$

The group H_0 acts on each Y_λ and defines an action on the fibration $Y/\mathbb{P}^1$. We examine the H_0-invariant part of the primitive cohomology $PH^{n-1}(Y_\lambda)$ in the middle dimension. The family Y was studied extensively by Dwork, who published articles about the p-adic variation of its cohomology when $n = 2$ (a family of elliptic curves) and $n = 3$ (a family of $K3$ surfaces).

Because f_λ is of degree $n + 1$, Y_λ, provided it is non-singular, is a Calabi-Yau hypersurface, which means that its canonical bundle is trivial (Y_λ has a nowhere vanishing $(n-1)$-form, unique up to scalar multiples). This follows from standard calculations of cohomology of hypersurfaces. When $n = 4$, Y is a family of quintic threefolds in $\mathbb{P}^4$. The virtual number n_d of rational curves (Gromov-Witten invariants) on Y_λ is determined by certain solutions of Picard-Fuchs equations describing monodromy on $H^3(Y_\lambda)^{H_0}$. This is the phenomenon of mirror symmetry, predicted by the physicists Candelas, de la Ossa, Green, and Parkes, relating the Gromov-Witten invariants of Y_λ with the Picard-Fuchs equation on $H^3((Y_\lambda/H_0)^\sim)$, where $(Y_\lambda/H_0)^\sim$ is a desingularization of (Y_λ/H_0), and proved mathematically in a number of situations, including this one.

When $\lambda = 0$ Y_λ is the union of coordinate hyperplanes; this is the totally degenerate case. In the arithmetic applications I will take $t = \lambda^{-1}$, so that this degeneration corresponds to the point $t = \infty$, which is the interesting singularity from the point of view of monodromy. When $t = 0$, f_λ is the Fermat hypersurface

$$X_0^{n+1} + \cdots + X_n^{n+1} = 0. \tag{1.1}$$

This point is of great importance in the applications.

For the purposes of this course, we are interested in the fact, highlighted by the mirror symmetry conjectures, that $PH^{n-1}(Y_\lambda)^{H_0}$ has Hodge numbers $H^{p,n-1-p}$

all equal to one, $p = 0, 1, \ldots, n-1$, provided Y_λ is nonsingular. This is calculated analytically, over $\mathbb{C}$.

The singular Y_λ are determined in the obvious way. The calculation is valid in any characteristic prime to $n+1$:

$$\frac{1}{n+1}\frac{\partial f_\lambda}{\partial X_i} = \lambda X_i^n - \prod_{j \neq i} X_j.$$

Thus

$$\frac{\partial f_\lambda}{\partial X_i} = 0 \Leftrightarrow \lambda X_i^{n+1} = \prod_j X_j. \tag{1.2}$$

If Y_λ is singular then there is a point $(x_0, \ldots, x_n) \in Y_\lambda$ satisfying the right-hand side of (1.2) for each i. In particular, if any $x_i = 0$ then all $x_j = 0$, which is impossible. Hence $\prod_j x_j \neq 0$. We multiply the equations in (1.2) over i and find

$$\lambda^{n+1} \prod_i X_i^{n+1} = (\prod_j X_j)^{n+1}$$

which is true if and only if $\lambda^{n+1} = 1$. Thus the map f_λ is smooth over $\mathbb{P}^* = \mathbb{P}^1 \setminus \{0, \mu_{\mathbf{n+1}}{}^{n+1}\}$.

If $\lambda = \zeta \in \mu_{n+1}$, then we have seen that, setting $p = \prod_i X_i$,

$$X_i^{n+1} = \zeta^{-1} p$$

for all i, hence $x_i/x_j \in \mu_{n+1}$ if $(x_0, \ldots, x_n) \in Y_\lambda$. We have also seen that $x_i \neq 0$ for all i. Scaling, we may thus assume $x_0 = 1$, and then each $x_i \in \mu_{n+1}$ and satisfy

$$\prod_i x_i = \zeta.$$

Conversely, any such point is a singular point. In particular, the singular points in Y_λ are isolated if $\lambda \in \mu_{n+1}$ and form a single orbit under H_0. Moreover, as λ varies in μ_{n+1}, the set of all singularities of all the singular fibers form a single orbit under H. In particular, all the singularities are isomorphic to the one for $\lambda = 1$ at the point $X_0 = X_1 = X_2 = \cdots = X_n = 1$. Writing $x_i = 1 + t_i$ for $i > 0$ we obtain the local equation

$$1 + \sum_i^n (1+t_i)^{n+1} = (n+1)\prod (1+t_i)$$

and one checks that the constant and linear terms vanish but the term of degree two is a non-degenerate quadratic form. Thus the singularities are ordinary quadratic singularities and can be analyzed by Picard-Lefschetz theory. We return to this point below.

2. Variation of Hodge structure.

Suppose $p : Y \to X$ is a smooth projective morphism of complex algebraic varieties, $\tilde{X}$ the universal cover of X, which we view as a complex analytic space and therefore a C^∞-manifold, and $\tilde{p} : \tilde{Y} = Y \times_X \tilde{X} \to \tilde{X}$ the pullback map. Since p is smooth, the implicit function theorem shows it is locally constant as a C^∞-map. In particular, as x varies in X, the cohomology spaces $H^i(Y_x, \mathbb{Z})$ form a locally constant sheaf. This means that if X is replaced by $\tilde{X}$, its universal cover, the sheaf $R^i\tilde{p}_*(\mathbb{Z})$ on $\tilde{X}$ is constant, for any i. This is a purely topological argument. The sheaf $R^ip_*(\mathbb{Z})$ on X is determined up to isomorphism by the representation of the fundamental group $\pi_1(X, x_0)$ on the fiber $H^i(Y_{x_0}, \mathbb{Z})$. This means in particular that one can differentiate sections with respect to parameters on the base X. If the base is one-dimensional and if one chooses a local coordinate λ on the base, one obtains an explicit first-order matrix differential equation

$$\frac{dF}{d\lambda} = A(\lambda)F$$

a basis of whose local solutions is just the (constant) cohomology with coefficients in $\mathbb{Z}$. This is the *Picard-Fuchs equation* which we will calculate for the Dwork family in §5.

Now since each Y_x is a smooth projective variety, its cohomology is endowed with a Hodge structure, which we describe as follows:

$$H^i(Y_x, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{p+q=i} H^q(Y_x, \Omega^p). \tag{2.1}$$

Here Ω^p is the sheaf of (algebraic) p-forms, and the spaces $H^{p,q} = H^q(Y_x, \Omega^p)$ are calculated as sheaf cohomology in the Zariski topology. Complex conjugation acts on the coefficients on the left-hand side of (2.1) and we have $\bar{H}^{p,q} = H^{q,p}$.

The Hodge decomposition (2.1) is valid for any Kähler manifold and is proved analytically, but it also has an algebraic version. Namely, we consider the de Rham complex

$$0 \to \mathbb{C} \to \Omega^0 = \mathcal{O}_{Y_x} \to \Omega^1 \to \ldots \to \ldots \Omega^d \to 0 \tag{2.2}$$

where $d = \dim Y - \dim X$. Since the cohomology of coherent sheaves on a projective variety is the same in the complex topology as in the Zariski topology (Serre's GAGA), one can compute the cohomology of $\mathbb{C}$ in terms of the cohomology of the Ω^j. The precise statement is that there is a spectral sequence

$$E_1^{p,q} = H^q(Y_x, \Omega^p) \Rightarrow H^{p+q}(Y_x, \mathbb{C}). \tag{2.3}$$

The Hodge decomposition is then the fact that this spectral sequence degenerates at E_1.

This version makes sense in families. Let

$$F^qH^i(Y_x, \mathbb{C}) = \oplus_{q' \geq q} H^{i-q',q'}.$$

For each i these subspaces define a decreasing filtration of the sheaf on X

$$R^i p_* \mathbb{C} = F^0 \supset F^1 \supset \cdots \supset F^q \supset \cdots \supset F^i \supset 0 \ldots \tag{2.4}$$

where the fiber $F^q_x = F^q H^i(Y_x, \mathbb{C})$ defined as above. Since the de Rham complex is a resolution of $\mathbb{C}$ in the complex topology (by the holomorphic Poincaré lemma), we can write $R^i p_* \mathbb{C} \simeq R^i p_*(\Omega^\bullet_{Y/X})$ as the cohomology of a complex of coherent sheaves (algebraic vector bundles), and (2.4) is a filtration by algebraic vector bundles. This is the variation of Hodge structure studied by Griffiths and others. We will not develop its general properties, notably the Griffiths transversality property that is a condition on the action of differentiation with respect to parameters on the base.

3. Griffiths' theory of cohomology of hypersurfaces.

Suppose $Y^{n-1} \hookrightarrow V^n$ is an embedding of smooth projective varieties of the indicated dimensions. For any integer p, we have the following commutative diagram

$$\begin{tikzcd}
H_p(Y) \arrow[r, "\tau"] \arrow[d, "\simeq", "\text{Poincaré}"'] & H_{p+1}(V - Y) \arrow[d, "\text{Alexander}", "\simeq"'] \\
H^{2n-2-p}(Y) \arrow[r, "\delta"] & H^{2n-1-p}(V, Y) = H_c^{2n-1-p}(V - Y)
\end{tikzcd}$$

Here τ is the tube map and δ is the connecting homomorphism. We are interested in $p = n - 1$. Suppose $V = \mathbb{P}^n$, so the bottom line continues.

$$\ldots H^{n-1}(\mathbb{P}^n) \to H^{n-1}(Y) \to H^n_c(\mathbb{P}^n - Y) \to H^n(\mathbb{P}^n) \ldots$$

If $n - 1$ is odd, δ is injective because the cohomology of $\mathbb{P}^n$ is concentrated in even degrees, and surjective because the map $H^n(\mathbb{P}^n) \to H^n(Y)$ is non-zero on the appropriate power of the Kähler class. If $n - 1$ is even, δ is surjective with one-dimensional kernel, for the same reasons. In any case, δ is an isomorphism on primitive cohomology, by definition.

Here is Griffiths' calculation of the Hodge filtration on a hypersurface. Suppose Y is given by the equation $f = 0$, with f a homogeneous polynomial of degree d. Let

$$S = \mathbb{C}[X_0, \ldots, X_n], \; J = (\frac{\partial f}{\partial X_i}, i = 0, \ldots n) \subset S.$$

Let R be the quotient ring S/J. Consider the Hodge filtration on primitive cohomology

$$PH^{n-1}(Y) = F^0 \supset F^1 \supset \cdots \supset F^n = 0.$$

Let $t_a = (n-1)d - (n+1)$.

Theorem 3.1. *Let $R^j \subset R$ be the homogeneous piece of degree j. Then for any a, there is an isomorphism*

$$R^{t_a} \overset{r_a}{\simeq} F^a / F^{a+1}$$

which is equivariant under $Aut(Y)$.

The isomorphism is given as follows. The *residue map*

$$Res : H^n(\mathbb{P}^n - Y) \to H^{n-1}(Y)$$

is adjoint to the tube map

$$\tau : H_{n-1}(Y) \to H_n(\mathbb{P}^n - Y).$$

Any $[\alpha] \in H^n(\mathbb{P}^n - Y)$ is represented by the differential form $\alpha = A\Omega/f^n$ where f is the defining equation,

$$\Omega = \sum_i (-1)^i X_i dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n,$$

and $A \in S$ is a homogeneous polynomial of the appropriate degree so that $\deg(\alpha) = 0$. Then

$$Res[\alpha] \in F^a \Leftrightarrow f^a \mid A.$$

Now suppose $Y = Y_\lambda$. By our earlier calculation (1.2), the ring R has a basis of monomials

$$X^{(j)} := X_0^{j_0} \dots X_n^{j_n} \qquad j_i \leq n+1,$$

where $(j) = (j_0, \dots, j_n)$ can be regarded as an element of $(\mathbb{Z}/(n+1)\mathbb{Z})^{n+1}$, the Pontryagin dual of μ_{n+1}^{n+1}. The annihilator of H_0 in $(\mathbb{Z}/(n+1)\mathbb{Z})^{n+1}$ is $\mathbb{Z}/(n+1)\mathbb{Z}$ diagonally embedded in $(\mathbb{Z}/(n+1)\mathbb{Z})^{n+1}$. In the above calculation, $d = n+1$, so $t_a = (n-a-1)(n+1)$. We are interested in the H_0-invariants of $R^{t_a} = R^r(n+1)$ if $r = n-a-1$.

Proposition 3.2. *Assume $\lambda \neq 0$, $\lambda \notin \mu_{n+1}$. Then*

$$(R^{r(n+1)})^{H_0} = \mathbb{C}(X_0 \dots X_n)^r, r = 0, 1, \dots, n-1.$$

Proof. The right hand side is obviously contained in the left-hand side. I will show that it defines a non-zero element of R. In the next section I will show that

(3.3) $$\dim F^a/F^{a+1} = 1 \forall 0 \leq a \leq n-1,$$

which will complete the proof.

First, setting $Z = \prod_i X_i$, $W_j = X_j^{n+1}$, one verifies that

(3.4) $$\mathbb{C}[X_0, \dots, X_n]^{H_0} = \mathbb{C}[Z, W_0, \dots, W_n]/(Z^{n+1} - \prod_j W_j).$$

Indeed, if $X^{(j)}$ is an H_0-invariant monomial, dividing by powers of Z we may assume $j_0 = 0$. Then as polynomial in $X_1, \dots, X_n$, the result is invariant under μ_{n+1}^n, hence every j_i must be divisible by $n+1$. Now if $Z^r = 0$ as an element of R, it must be in the ideal generated by the partial derivatives of f_λ. Thus Z^{r+1} is in the ideal of R generated by the $W_j - tZ$, where we have set $t = \lambda^{-1}$. Using (3.4), we find that Z^{r+1} is in the ideal of $\mathbb{C}[Z, W_0, \ldots, W_n]$ generated by the $W_j - tZ$ and by $Z^{n+1} - \prod_j W_j$. But since $r+1 \leq n$, the homogeneous polynomial Z^{r+1}, a priori a sum

$$Z^{r+1} = \sum \phi_j \cdot (W_j - tZ) + g(Z^{n+1} - \prod_j W_j)$$

must in fact lie in the ideal of $\mathbb{C}[Z, W_0, \ldots, W_n]$ generated by the $W_j - tZ$ (look at the homogeneous part of degree $r+1$). Setting $Z = 1$ and $W_0 = W_1 = \ldots W_n = t$, we find that $1 = 0$, a contradiction.

Proposition 3.5. *The cohomology $H^{n-1}(Y_\lambda, \mathbb{Z}[\frac{1}{n+1}])^{H_0}$ is torsion free.*

This is proved in the following section.

We can also work with the case where $n-1$ is even, but it suffices for the applications to consider $n-1$ odd. Then Poincaré duality defines a perfect symplectic pairing on $H^{n-1}(Y_\lambda, \mathbb{Z})$, restricting to a perfect pairing on the n-dimensional space $PH^{n-1}(Y_\lambda, \mathbb{Q})^{H_0}$ (or even with coefficients in $\mathbb{Z}[\frac{1}{n+1}]$).

4. Cohomology of the Fermat hypersurface.

We consider the point $\lambda^{-1} = t = 0$, so Y_∞ is defined by the Fermat equation

$$\sum X_i^{n+1} = 0.$$

The calculation that follows works for more general Fermat hypersurfaces, but we simplify the notation by considering only this one, and moreover taking $n-1$ odd, so that the cohomology in the middle dimension is all primitive. The action of H_0 on Y_∞ extends to a natural action of H. The cohomology $H^i(Y_\infty, \mathbb{Z})$ is calculated by Deligne (Milne's notes) by an elementary method. Let $P^{n-1} \subset \mathbb{P}^n$ be the hyperplane defined by the equation $\sum X_i = 0$. There is a finite surjective H-equivariant map

$$\pi : Y_\infty \to P^{n-1};\ (x_0, x_1, \ldots, x_n) \mapsto (x_0^{n+1}, x_1^{n+1}, \ldots, x_n^{n+1}).$$

The cohomology of Y_∞ is calculated by the Leray spectral sequence, and since π is finite, this is just an isomorphism

$$H^i(Y_\infty, \mathbb{Z}[\frac{1}{n+1}]) \xrightarrow{\sim} H^i(P^{n-1}, \pi_*\mathbb{Z}[\frac{1}{n+1}]). \tag{4.1}$$

Since π is H-equivariant, $\pi_*\mathbb{Z}[\frac{1}{n+1}]$ breaks up according to $Gal(\mathbb{Q}(\mu_{n+1})/\mathbb{Q})$-orbits of characters of H. We extend scalars temporarily and let $A = \mathbb{Z}[\mu_{n+1}, \frac{1}{n+1}]$. The character group $X(H)$ of H can be identified explicitly:

$$X(H) = \{\underline{a} = (a_0, \ldots a_n) \in (\mathbb{Z}/(n+1)\mathbb{Z})^{n+1} \ \mid\ \sum a_i = 0\}.$$

The $\underline{a}$-isotypic subspace of cohomology is denoted by $[\underline{a}]$. Then we have

$$H^i(Y_\infty, A)[\underline{a}] \xrightarrow{\sim} H^i(P^{n-1}, \pi_* A[\underline{a}]). \tag{4.2}$$

Now the map π is étale outside the union of the hyperplanes L_i defined by $X_i = 0$, hence $\pi_* A[\underline{a}]$ is locally constant and of dimension one away from the union of these hyperplanes. (It is of dimension one because H acts transitively on the fiber over any point. The calculation is local in the complex topology.) group But the group H is a product of $H_{(i)} \xrightarrow{\sim} \mu_{n+1}$, corresponding to the different coordinates. The $H_{(i)}$-invariants in $\pi_* A$ are unramified over the hyperplane L_i. Thus $\pi_* A[\underline{a}]$ only ramifies over the L_i such that $a_i \neq 0$, and at such hyperplanes the stalk is 0. In other words, if $j_{\underline{a}}$ is the inclusion of $P^{n-1} - \cup_{a_i \neq 0} L_i \hookrightarrow P^{n-1}$, then

$$\pi_* A[\underline{a}] = j_{\underline{a},!} j_{\underline{a}}^* \pi_* A[\underline{a}] \tag{4.1}$$

We first consider $\underline{a} = 0$. Obviously $\pi_* A[0] = A$, so

$$H^i(P^{n-1}, \pi_* A[0]) = H^i(P^{n-1}, A) = A \text{ if } i \text{ is even }, = 0 \text{ otherwise.} \tag{4.2}$$

This is the same as $H^i(Y_\infty, A) \simeq \oplus_{\underline{a}} H^i(P^{n-1}, \pi_* A[\underline{a}])$ if $i \neq n-1$, hence

$$H^i(P^{n-1}, \pi_* A[\underline{a}]) = 0 \text{ if } [\underline{a}] \neq 0 \text{ unless } i = n-1.$$

It follows that

Lemma 4.3. *Suppose $\underline{a} \neq 0$. Then $(-1)^{n-1} \dim H^{n-1}(P^{n-1}, \pi_* A[\underline{a}])$ equals the Euler-Poincaré characteristic of $\pi_* A[\underline{a}]$, which is the same as the Euler-Poincaré characteristic of the constant sheaf A on $P^{n-1} - \cup_{a_i \neq 0} L_i$.*

Proof. We have already seen the first point. The second point is a general fact, and is asserted in [DMOS] (first article, §7) without explanation, so there may be an elementary proof. I can only come up with the following argument: the fundamental group of the hyperplane complement $P^{n-1} - \cup_{a_i \neq 0} L_i$ acts on $\pi_* A[\underline{a}]$ through a character whose values on generating loops around the hyperplanes are $(n+1)$st roots of unity. One can define a continuous family of local systems by letting the eigenvalues of these generating loops vary (in $\mathbb{C}/\mathbb{Z}$). The Euler characteristic is constant on a continuous family, thus it can be calculated on the trivial local system.

Now we identify $\pi_* A[\underline{a}]$ more generally. First suppose $a_i \neq 0$ for all i. Then $\pi_* A[\underline{a}]$ is ramified over all coordinate hyperplanes, We identify P^{n-1} with $\mathbb{P}^{n-1}$ by taking the first n coordinates $(x_0, \ldots, x_{n-1})$ and setting $x_n = -\sum_{i=0}^{n-1} x_i$ For example, if $n = 2$, $P^{n-1} - \cup_{i=0}^{2} L_i$ is then identified with $\mathbb{P}^1 - \{0, \infty, -1\} = \mathbb{P}^1 - [\cup_{i=0}^{1} L_i \cup P^0]$. In general,

$$P^{n-1} - \cup_{i=0}^{n} L_i \xrightarrow{\sim} \mathbb{P}^{n-1} - [\cup_{i=0}^{n-1} L_i \cup P^{n-2}]$$

and inversely

$$[P^{n-1} - \cup_{i=0}^{n} L_i] \coprod [P^{n-2} - \cup_{i=0}^{n-1} L_i] = \mathbb{P}^{n-1} - [\cup_{i=0}^{n-1} L_i \simeq (\mathbb{C}^*)^{n-1}.$$

Since the (topological) Euler-Poincaré characteristic of the constant sheaf is additive in disjoint unions, we find

$$EP(P^{n-1} - \cup_{i=0}^{n} L_i) + EP(P^{n-2} - \cup_{i=0}^{n-1} L_i) = EP((\mathbb{C}^*)^{n-1}).$$

But $EP(\mathbb{C}^*) = 0$ and the same is true for any power of $\mathbb{C}^*$, hence

$$(-1)^{n-1} EP(P^{n-1} - \cup_{i=0}^{n} L_i) = (-1)^{n-2} EP(P^{n-2} - \cup_{i=0}^{n-1} L_i),$$

which by induction is $EP(P^0) = 1$. It follows from Lemma 4.3 that

$$\dim H^{n-1}(P^{n-1}, \pi_* A[\underline{a}]) = 1, \tag{4.4}$$

provided all $a_i \neq 0$.

On the other hand, if $a_i = 0$ for some, but not all i, then $P^{n-1} - \cup_{a_i \neq 0} L_i$ is topologically simpler. Note that if all but one $a_i = 0$ then all $a_i = 0$, since they sum to 0. For example, if $n = 2$, we have $P^1 - \{0, \infty\} = \mathbb{C}^*$, whose Euler-Poincaré characteristic is zero. More generally, if $r < n + 1$ is the number of i such that $a_i \neq 0$, then

$$P^{n-1} - \cup_{a_i \neq 0} L_i \simeq (\mathbb{C}^*)^{r-1} \times \mathbb{C}^{n-r}$$

which has Euler-Poincaré characteristic zero. Thus (4.4) implies

Theorem 4.5. *If all $a_i \neq 0$, or if all $a_i = 0$, then* $\dim H^{n-1}(P^{n-1}, \pi_* A[\underline{a}]) = 1$. *Otherwise,* $\dim H^{n-1}(P^{n-1}, \pi_* A[\underline{a}]) = 0$.

Now the set of characters $\underline{a}$ of H trivial on H_0 is of cardinality $[H : H_0] = n + 1$. This includes the character $\underline{a} = 0$, but $H^{n-1}(P^{n-1}, A) = 0$ because $n - 1$ is odd (cf. (4.2)). It follows that

Corollary 4.6. *$H^{n-1}(Y_\infty, A)^{H_0}$ is a free A-module of rank n.*

Since $H^{n-1}(Y_\lambda, A)^{H_0}$ forms a local system over $\mathbb{P}^1 \setminus \{0, \mu_{n+1}\}$, it follows that

Corollary 4.7. *For all $\lambda \notin \{0, \mu_{n+1}\}$, $H^{n-1}(Y_\lambda, A)^{H_0}$ is a free A-module of rank n. In particular,* $\dim H^{n-1}(Y_\lambda, \mathbb{C})^{H_0} = n$.

Now we can complete the discussion in §3. Proposition 3.5 is an immediate consequence of Corollary 4.7. As for (3.3), we know that

$$n = \dim H^{n-1}(Y_\lambda, \mathbb{C})^{H_0} = \sum_{a \geq 0} \dim F^a / F^{a+1} \geq \sum_{a=0}^{n-1} \dim F^a / F^{a+1}.$$

In the proof of Proposition 3.2, we have shown that $\dim F^a / F^{a+1} \geq 1$ for $a = 0, \ldots, n - 1$. Thus we have equality, which completes the proof of Proposition 3.2.

5. Calculation of the Picard-Fuchs equation. The main theorem of this lecture is the following

Theorem 5.1. *The monodromy representation of $\pi_1(\mathbb{P}^1 - \{0, \mu_{n+1}\})$ on $PH^{n-1}(Y_\lambda, \mathbb{Q})^{H_0}$ has Zariski dense image.*

This is proved by calculating the Picard-Fuchs equation. We begin by finding a solution. This is done in a neighborhood of the singular point $\lambda = 0$. Let D be a small disk around $\lambda = 0$, $D^* = D \setminus 0$, so that $T := \pi_1(D^*, \lambda_0) \xrightarrow{\sim} \mathbb{Z}$ for any base point λ_0. This fundamental group acts on $H^{n-1}(Y_\lambda, \mathbb{Z})$ as well as $H_{n-1}(Y_\lambda, \mathbb{Z})$.

Take an affine piece $X_0 \neq 0$ and set $x_i = X_i/X_0$,

$$S = \{|x_1| = \cdots = |x_n| = 1\}$$

(a real torus). For $0 < |\lambda| << 1$ we have $S \cap Y_\lambda = \emptyset$. Indeed, for $(x_i) \in S \cap Y_\lambda$

$$|\lambda| = \frac{n+1}{|1 + \sum x_i^{n+1}|} \geq \frac{n+1}{1 + \sum |x_i^{n+1}|} = 1.$$

Thus S defines a constant family of cycles in $H_n(\mathbb{P}^n - Y_\lambda, \mathbb{Z})^{H_0}$ as λ varies in a small circle around 0. In other words,

$$[S] \in H_n(\mathbb{P}^n - Y_\lambda)^T.$$

Now the tube map $\tau : H_{n-1}(Y_\lambda) \to H_n(\mathbb{P}^n - Y_\lambda)$ is equivariant under T and H_0, and since we have seen it is an isomorphism $\tau^{-1}[S] = \gamma_\lambda$ is a T-invariant cycle in $H_{n-1}(Y_\lambda, \mathbb{Z})^{H_0}$. We have seen that the tube map is adjoint to the residue map. Interpreting this in terms of periods of integrals, we have

(5.1)
$$F(\lambda) = -\frac{1}{(2\pi i)^n} \int_S \frac{dx_1 \dots dx_n}{\lambda(1 + \sum_{i=1}^n x_i^{n+1}) - (n+1)(x_1 \dots x_n)} = -\frac{1}{(2\pi i)^n} \int_{\gamma_\lambda} \omega_\lambda,$$

where ω_λ is the Poincaré residue of the integrand.

However, we can also calculate the integral explicitly, since it is an integral over a torus, hence an iterated series of residues in $\mathbb{C}^n$. Letting c_m denote the coefficient of $\prod x_i^m$ in the expression $(1 + \sum x_i^{n+1})^m$, and expanding the integrand in a geometric series, we find the integral equals

$$F(\lambda) = \sum_{m \geq 0} \frac{\lambda^m}{(n+1)^{m+1}} c_m = \sum_p \frac{\lambda^{(n+1)p}}{(n+1)^{(n+1)p+1}} \frac{[(n+1)p]!}{(p!)^{n+1}}. \tag{5.2}$$

It is known a priori that the monodromy operator T, acting on $H^{n-1}(Y_\lambda, \mathbb{C})^{H_0}$, is quasi-unipotent, which means that some power of T can be realized as a unipotent $n \times n$-matrix. More generally, ω_λ can be integrated over any cycle in $H_{n-1}(Y_\lambda)^{H_0}$. Choosing a basis (in which an appropriate power of T is unipotent) we obtain an n-vector of functions $F_1(\lambda), F_2(\lambda), \dots F_n(\lambda)$, in which $N = log(T)$ is upper-triangular nilpotent.

Now there is a classical dictionary identifying the local system on D^* given by an n-dimensional unipotent representation of $\pi_1(D^*)$, in other words a unipotent matrix (the action of T) and a differential equation of order n with regular singular points. (A matrix differential equation of order 1 corresponds to a linear ordinary differential operator of order n in the usual way.) The Picard-Fuchs equation is the corresponding equation for action on cohomology. Under this dictionary, the integral obtained by integrating a cohomology class against an invariant cycle is a solution of the Picard-Fuchs equation, and determines the monodromy matrix. We have found a solution $F(\lambda)$. Note that $F(\lambda)$ can be written $\phi_0(z)$, where $z = \lambda^{n+1}$:

$$\phi_0(z) = \sum_p \frac{z^p}{(n+1)^{(n+1)p+1}} \frac{[(n+1)p]!}{(p!)^{n+1}}. \tag{5.3}$$

Proposition 5.4. *Let $\theta = z\frac{d}{dz}$, and let D be the differential operator*

$$D = \theta^n - z(\theta + \frac{1}{n+1}) \ldots (\theta + \frac{n}{n+1}).$$

Then $D\phi_0 = 0$.

Proof. Explicit calculation.

Verify (Katz: *Exponential sums and differential equations*, p. 94) that this D is irreducible. Since it is of degree $n = \dim H^{n-1}(Y_\lambda, \mathbb{C})^{H_0}$, it must be the Picard-Fuchs equation.

Note that D is a polynomial in θ, hence has regular singular points at 0. Now D is of the form $\theta^n - zQ(\theta)$ for some polynomial Q: this means that D is *hypergeometric.* Such equations have been studied classically. It is known (cf. Whittaker-Watson) that D has singularities only at $0, 1, \infty$, and ϕ_0 is the only solution not involving $log(z)$. It follows by the classification of ordinary differential equations that T acting on $H^{n-1}(Y_\lambda, \mathbb{C})^{H_0}$ has a single unipotent Jordan block.

Proposition 5.5. *The representation of $\pi_1(\mathbb{P}^1_z \setminus \{0, 1, \infty\}, z_0)$ coming from the local system of solutions to the Picard-Fuchs equation is infinite and primitive (cannot be broken up as a sum).*

This is already clear from the monodromy at $z = \lambda = 0$. Note that replacement of λ by z replaces the singular points $\lambda \in \mu_{n+1}$ by the single singular point $z = 1$, but introduces new ramification at $z = \lambda = \infty$, corresponding to the Fermat hypersurface of §4. This ramification is of *finite order* $n + 1$ and in particular becomes trivial over the cover $\mathbb{P}^1_\lambda \setminus \{0, \mu_{n+1}\}$, as we have already seen.

Theorem (Beukers-Heckman, 1995). *If a hypergeometric differential equation on $(\mathbb{P}^1_z \setminus \{0, 1, \infty\})$ has primitive monodromy, then (up to homotheties), the Zariski closure of the image is one of the following:*

1. *A finite group;*
2. *$SL(V)$;*
3. *$SO(V)$;*
4. *$Sp(V)$.*

The first option has been eliminated. We know our cohomology has a symplectic pairing (Poincaré duality), hence we are in case (4).

The proof of Beukers-Heckman is based on the following principles. We know that monodromy around $z = 0$ is of infinite order and indeed is given by a single unipotent Jordan block. On the other hand, it is a general consequence of Hodge theory that the Zariski closure of the monodromy of the Picard-Fuchs equation is *semisimple*. Since it is also irreducible and symplectic, this leaves very few options, by the general classification of irreducible representations of Lie algebras: indeed, it can only be either $Sp(V)$ or the $n-1$st symmetric power of the standard 2-dimensional representation of $SL(2)$. But now one can calculate monodromy around the point $z = 1$. Recall from §1 that the singularities are ordinary quadratic. Thus the monodromy around $z = 1$ can be calculated by the Picard-Lefschetz formula, which shows that it contains a unipotent element U such that $U - 1$ has rank 1. Such a U is called a *symplectic transvection*. Since it obviously does not belong to the image of $SL(2)$ under a symmetric power representation if $n > 2$, the only remaining option is $Sp(V)$.

6. Monodromy mod ℓ, for $\ell >> 0$.

I briefly describe an argument explained to me by Nick Katz. We still assume $n-1$ odd. Let z_0 be a point of $\mathbb{P}^1_z \setminus \{0,1,\infty\}$, and for any $\mathbb{Z}[\frac{1}{n+1}]$ algebra R, write $V_{z_0,R} = H^{n-1}(Y_{\lambda_0}, R)^{H_0}$ for some point λ_0 satisfying $\lambda_0^{n+1} = z_0$. We have seen that the image of $\pi_1(\mathbb{P}^1_z \setminus \{0,1,\infty\}, z_0)$ in $Aut(V_{z_0,\mathbb{Q}})$ is irreducible. Thus by Burnside's theorem, the linear span of the image is the full matrix algebra. In other words, each generator of the matrix algebra is a $\mathbb{Q}$-linear combination of elements of $\pi_1(\mathbb{P}^1_z \setminus \{0,1,\infty\}, z_0)$. It follows that there is an integer N_0 such that each generator of the matrix algebra is a $\mathbb{Z}[\frac{1}{N_0}]$-linear combination of elements of $\pi_1(\mathbb{P}^1_z \setminus \{0,1,\infty\}, z_0)$. Thus for all primes $\ell > N_0$, the image G_ℓ of $\pi_1(\mathbb{P}^1_z \setminus \{0,1,\infty\}, z_0)$ in $Sp(V_{z_0,\mathbb{F}_\ell})$ acts irreducibly.

On the other hand G_ℓ contains the reduction modulo ℓ of monodromy at 0 and at 1. For ℓ sufficiently large, the monodromy mod ℓ at 0 still is a single unipotent Jordan block, whereas the Picard-Lefschetz theorem shows again that monodromy mod ℓ at 1 is a symplectic transvection. The same argument as used by Beukers-Heckman shows that

Theorem. *There exists an integer N_0 such that, for $\ell > N_0$, $G_\ell = Sp(V_{z_0,\mathbb{F}_\ell})$.*

References

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