1. Review of notation.

We refer to the notes "frameddeformations," "localclassicallimit", "patching," and "Hecke+TW" (reproduced in part below) for the notation. We assume $\bar{r}$ is the reduction of the homomorphism $r : \Gamma_{F+} \to \mathcal{G}_n(\mathcal{O})$ whose modularity we wish to prove. We assume $\bar{r}$, or rather its restriction $\bar{\rho}$ to Γ_F, is of the form $\bar{\rho}_{\Pi,\ell}$ for some cuspidal automorphic representation Π of $GL(n, F)$ satisfying the usual hypotheses, and so we need to show that all reasonable liftings of $\bar{r}$ are automorphic. (The problem of showing that the representation of interest is residually modular is independent and depends on the potential modularity techniques discussed separately.) We have chosen a set S which in the applications is just $S_\infty \cup S_\ell$ (possibly including the extra prime S_1 and an indeterminate set S_{min} that have no effect on the dimension count) including all primes at which our original automorphic representation Π has minimal-type ramification, as well as primes dividing ℓ. To S we add the Taylor-Wiles primes Q_N whose contribution to the dimension count was determined in previous lectures. Finally, we add the set R of primes where either the lifting r is ramified or where the ramification of $\bar{r}$ is not of minimal type. All finite primes in S are assumed split in F/F^+. At a prime $v \in R$ we require the ramification to be of type $\chi_v = \{\chi_{1,v}, \ldots, \chi_{n,v}\}$, where each $\chi_{i,v}$ is a tame character of $k(v)^\times$ (more precisely $k(\tilde{v})^\times$ for some lift $\tilde{v}$ of v to F) with values in μ_ℓ. The two extreme situations considered in "localclassicallimit" are where the χ_i are all distinct and non-trivial and where all the $\chi_i = 1$. In the former case the local lifting ring R^{loc}_χ is smooth and irreducible in characteristic zero; in the latter case, the irreducible components in characteristic zero and in characteristic ℓ are in one-to-one correspondence. The quotient $R^{loc}_\chi/m_{\mathcal{O}}$ is independent of χ. The deformation problem of interest is denoted $\mathcal{S}(Q_N)$ and includes all the conditions discussed previously.

As in "frameddeformations" we introduce a power series ring $\mathcal{T}_R$ over $\mathcal{O}$ in $|R| \cdot n^2$-variables, representing the matrix entries of local liftings at $v \in R$. We have the following functorialities:

$$(1.1) \qquad R^{loc}_\chi \to R^\square_{\chi,\mathcal{S}(Q_N)};\ ,\mathcal{T}_R \to R^\square_{\chi,\mathcal{S}(Q_N)};\ R^{univ}_{\chi,\mathcal{S}(Q_N)} \hat{\otimes} \mathcal{T}_R \xrightarrow{\sim} R^\square_{\chi,\mathcal{S}(Q_N)}$$

where the first map and the first coordinate of the third map are canonical and the remaining maps depend on a choice of matrix lifting representing the universal deformation r^{univ}.

Note that we have dropped the index $\bar{r}_\Pi$ from the universal deformation ring, for lack of space. As in "Hecke+TW" we still have a surjective map

$$(1.2) \qquad R^{univ}_{\chi,\mathcal{S},Q_N} \to \mathbb{T}_{\chi,\mathcal{S},Q_N}$$

A word about the Hecke algebra here. We need to specify level subgroups U_v for $v \in R$. Let I_v (denoted Iw_v in [CHT,T] be the Iwahori subgroup of integral matrices whose reduction modulo $\tilde{v}$ is upper-triangular. Let q_v be the order of the residue field $k(v)$, a power of the prime p_v. We let $U_v = I(1)_v \subset I_v$ be the p_v-Sylow subgroup, the matrices whose reduction modulo $\tilde{v}$ is upper-triangular unipotent; mapping to the diagonal entries thus identifies

$$I_v/I(1)_v \xrightarrow{\sim} (k(v)^{\times})^n \tag{1.3}$$

Our hypothesis is that $q_v = |k(v)| \equiv 1 \pmod{\ell}$ for any $v \in R$. The Hecke algebra $\mathbb{T}_{\chi,\mathcal{S},Q_N}$ is an appropriate localization (for $\bar{r}_{\Pi}$) of a module of of automorphic forms on the totally definite unitary group G_0 of level $U_{1,v}$ for $v \in Q_N$; for $v \in R$ we assume these forms are of level $I(1)_v$ and of character χ_v for $I_v/I(1)_v$. The localized module of automorphic forms is denoted $H_{\chi,N}$.

Remark 1.4. It is simplest in practice to incorporate all primes ramified for Π in R (the primes dividing ℓ are not ramified for Π, by hypothesis). We need the primes to satisfy two hypotheses: $\Pi_v^{I(1)_v} \neq (0)$ and $|k(v)| \equiv 1 \pmod{\ell}$. The latter hypothesis can be guaranteed by replacing F^+ by an appoporiate cyclic extension unramified at v; the former hypothesis can be guaranteed by passage to an appropriate solvable extension of F^+. Our applications are indifferent to this sort of base change.

We have a good formula for the dimension of the cotangent space of $R^{\square}_{\chi,\mathcal{S}(Q_N)}$ relative to R^{loc}_{χ}:

Lemma 1.5. *$R^{\square}_{\chi,\mathcal{S}(Q_N)}$ can be topologically generated over R^{loc}_{χ} by*

$$|Q_N| - \varepsilon_\infty = r - \varepsilon_\infty$$

elements.

This is just Lemma 1.4 of "frameddeformations", where we have chosen Q_N as in "Hecke+TW" to eliminate the group $H^1_{\mathcal{S}(Q_N)^*,\chi}(\Gamma_{F^+}, ad\bar{r}(1))$ and where the remaining global term $H^0(\Gamma_{F^+}, ad\bar{r}(1))$ vanishes by hypothesis. However, this in itself does not help us compare R with $\mathbb{T}$, because $\mathbb{T}$ is not an algebra over $R^{\square}_{\chi,\mathcal{S}(Q_N)}$. For this we make another ad hoc construction, following Taylor (and Kisin). Let

$$\begin{aligned} H^{\square}_{\chi,N} &= R^{\square}_{\chi,\mathcal{S}(Q_N)} \hat{\otimes}_{R^{univ}_{\chi,\mathcal{S}(Q_N)}} H_{\chi,N} \\ &\xrightarrow{\sim} \mathcal{T}_R \hat{\otimes}_{\mathcal{O}} H_{\chi,N} \end{aligned} \tag{1.6}$$

Here $H_{\chi,N}$ is the localized module of automorphic forms on G_0 defined above, and the second isomorphism follows from (1.1).

Now we are ready to apply the Taylor-Wiles patching argument, in the version of Diamond-Fujiwara, but with a new ingredient. Instead of the algebra $A = \mathcal{O}[[S_1, \ldots, S_r]]$ which maps on $R^{univ}_{\chi,\mathcal{S}(Q_N)}$ by means of the diamond operators, we add new variables and construct an algebra

$$S_\infty = \mathcal{T}_R[[S_1, \ldots, S_r]] = \mathcal{T}_R \hat{\otimes}_{\mathcal{O}} A. \tag{1.7}$$

This maps to $R^{\square}_{\chi,\mathcal{S}(Q_N)}$ via (1.1) and the map $A \to R^{univ}_{\chi,\mathcal{S}(Q_N)}$ just recalled.

Recall the open ideal $J_N \subset A$, the kernel of the map $A \to \mathcal{O}[\Delta_{Q_N}]$. Let $\mathfrak{m}_{\mathcal{T}_R}$ be the kernel of the augmentation map $\mathcal{T}_R \to \mathcal{O}$ sending the variables $X_{v,i,j}$ to 0. As in the minimal case, $H_{\chi,N}$ is finite free over A/J_N, which implies the first part of

Lemma 1.8.

(a) $H^{\square}_{\chi,N}$ *is finite free over* $S_\infty/J_N S_\infty$.

(b) *We have an isomorphism*

$$R^{\square}_{\chi,\mathcal{S}(Q_N)}/(\mathfrak{m}_{\mathcal{T}_R}+J_0)R^{\square}_{\chi,\mathcal{S}(Q_N)} \xrightarrow{\sim} R^{univ}_{\chi,\emptyset}$$

(c) $H^{\square}_{\chi,N}/(\mathfrak{m}_{\mathcal{T}_R}+J_0)H^{\square}_{\chi,N}$ *is isomorphic as* $R^{univ}_{\chi,\emptyset}$*-modules to* $H_{\chi,\emptyset}$.

Proof. For (b), we use the isomorphism (1.1), so that

$$R^{\square}_{\chi,\mathcal{S}(Q_N)}/(\mathfrak{m}_{\mathcal{T}_R}+J_0)R^{\square}_{\chi,\mathcal{S}(Q_N)} \xrightarrow{\sim} R^{univ}_{\chi,\mathcal{S}(Q_N)}/J_0 R^{univ}_{\chi,\mathcal{S}(Q_N)}\hat{\otimes}\mathcal{T}_R/\mathfrak{m}_{\mathcal{T}_R} \xrightarrow{\sim} R^{univ}_{\chi,\emptyset}$$

as in the minimal case. The assertion (c) is a combination of two points: On the one hand,

$$H^{\square}_{\chi,N}/(\mathfrak{m}_{\mathcal{T}_R})H^{\square}_{\chi,N} \xrightarrow{\sim} H_{\chi,N}$$

by the second line of (1.6). On the other hand, $H_{\chi,N}/J_0 H_{\chi,N}$ is just $H_{\chi,\emptyset}$ as in the minimal Taylor-Wiles argument (using the subtleties of the Taylor-Wiles primes to establish Diamond's point (c).

Now consider first the case of $\{\chi_v\}$ *generic.* Diamond's patching argument does not go over to this situation directly, because it is based on patching modulo ℓ, and even in the generic case R^{loc} is singular and highly reducible modulo ℓ. Taylor introduces a new patching argument purely in characteristic zero. To avoid obscuring the main point we assume the archimedean error term $\varepsilon_\infty = 0$. Let $B = \mathcal{O}[[X_1,\ldots,X_r]]$ as in "patching." This is the algebra that maps onto $R^{univ}_{\chi,\mathcal{S}_{Q_N}}$ by the Riemann-Roch calculation in the minimal case. Correspondingly, if we let $B^{\square}_{\chi} = R^{loc}_{\chi}\hat{\otimes}_{\mathcal{O}} B$, then Lemma 1.5 implies

Lemma 1.9. *For all* N*, there are surjective homomorphisms*

$$\phi^{\square}_N : B^{\square}_{\chi} \to R^{\square}_{\chi,\mathcal{S}(Q_N)}$$

of R^{loc}_{χ}*-algebras.*

I remind you that these maps are in no way canonical and the patching is based on a finiteness argument. In any case, these maps, together with the S_∞-structure on $H^{\square}_{\chi,N}$, yields:

Lemma 1.10. *For all* N*,* $H^{\square}_{\chi,N}$ *is a module over* $B^{\square}_{\chi}\hat{\otimes}_{\mathcal{O}} S_\infty$ *which is finite free over* $S_\infty/J_N S_\infty$*. Moreover, the image of* S_∞ *in* $End(H^{\square}_{\chi,N})$ *is contained in the image of* $B^{\square}_{\chi}$*.*

The second point follows from Lemma 1.9, the inclusion (1.1) $\mathcal{T}_R \to R^{\square}_{\chi,\mathcal{S}(Q_N)}$, and the inclusion of the diamond operators in $R^{univ}_{\chi,\mathcal{S}_{Q_N}}$.

2. Completion of the proof. As in the proof of Diamond's Theorem 1.5 in "patching," the patching in the limit yields the following

(2.1) A module $H^{\square}_{\chi,\infty}$ over $B^{\square}_{\chi}\hat{\otimes}_{\mathcal{O}}S_{\infty}$

(2.2) The action of S_{∞} factors through $B^{\square}_{\chi}$ (since S_{∞} is a power series ring over $\mathcal{O}$, any action can be made to lift) and makes $H^{\square}_{\chi,\infty}$ a free S_{∞}-module

(2.3) An isomorphism $H^{\square}_{\chi,\infty}/J_0H^{\square}_{\chi,\infty} \xrightarrow{\sim} H_{\chi,\emptyset}$.

(2.4) The isomorphism (2.2) is compatible with surjective maps

$$B^{\square}_{\infty} \to R^{univ}_{\chi,\mathcal{S}_{\emptyset}} \to \mathbb{T}_{\chi,\emptyset}.$$

The first map in (2.4) is defined by a simultaneous patching over the map $S_{\infty} \to R^{univ}_{\chi,\mathcal{S}_{\emptyset}}$ which factors through $\mathcal{O}$, so is patched together from finite quotients of $R^{univ}_{\chi,\mathcal{S}_{\emptyset}}$. The novelty of this point, compared to Taylor-Wiles, Diamond, and Fujiwara, is somewhat concealed in the middle of the proof of the main theorem of [T].

By (2.2) the $B^{\square}_{\chi}$-depth of $H^{\square}_{\chi,\infty}$ is at least $\dim S_{\infty} = 1+r+\dim \mathcal{T}_R = 1+r+n^2|R|$. Thus

$$\dim B^{\square}_{\chi}/Ann(H^{\square}_{\chi,\infty}) \geq 1+r+n^2|R| \tag{2.5}$$

However, Taylor's determination of R^{loc}_{χ} in the generic case implies that $\dim B^{\square}_{\chi} = \dim R^{loc}_{\chi} + r = 1+r+n^2|R|$, and that $B^{\square}_{\chi}$ has a *unique minimal ideal.* It follows that

Proposition 2.6. *$Ann(H^{\square}_{\chi,\infty})$ is contained in the unique minimal ideal of $B^{\square}_{\chi}$; in other words, $H^{\square}_{\chi,\infty}$ is a* **nearly faithful** *module over $B^{\square}_{\chi}$, in the terminology of [T]. In particular, $Ann(H^{\square}_{\chi,\infty})$ is* **nilpotent**.

Now J_0 defines an ideal in $B^{\square}$ by (2.2), contained in the kernel of the surjection (2.4). Now (2.3) and (2.4), together with Proposition 2.6, imply that

Proposition 2.7. *$H_{\chi,\emptyset}$ is a nearly faithful $R^{univ}_{\chi,\mathcal{S}_{\emptyset}}$-module* **in the generic case**.

Recalling that our Hecke algebras are reduced, Proposition 2.7 implies that

$$R^{univ,red}_{\chi,\mathcal{S}_{\emptyset}} \to \mathbb{T}_{\chi,\mathcal{S}_{\emptyset}}$$

is an isomorphism in the generic case, which is strong enough to imply the modularity lifting theorem in the generic case. But this is not what we need to prove! We need to work with the stable situation, the worst possible case, namely when $\chi_{i,v} = 1$ for all i and all v. This is stable because one can always reduce to this case by passing to an appropriate abelian extension of F^+. However, Proposition 2.6 does have this additional consequence:

Corollary 2.8. *$H_{\chi,\emptyset}/\mathfrak{m}_{\mathcal{O}}H_{\chi,\emptyset}$ is a nearly faithful $B^{\square}_{\chi}/\mathfrak{m}_{\mathcal{O}}B^{\square}_{\chi}$-module. In particular, $H_{1,\emptyset}/\mathfrak{m}_{\mathcal{O}}H_{1,\emptyset}$ is a nearly faithful $B^{\square}_{1}/\mathfrak{m}_{\mathcal{O}}B^{\square}_{\chi}$-module.*

The first part is an argument in commutative algebra, using Nakayama's lemma. The second part follows from the first because the reductions mod $\mathfrak{m}_{\mathcal{O}}$ of either H_χ or B_χ^{square} are isomorphic for all χ. But now recall Taylor's analysis of R_1^{loc}, namely the following (reprinted from "localclassicallimit"):

Lemma 2.9. *Suppose $\chi_i = 1$ for all i. Then $R_\chi = R_1$ is equidimensional of dimension $n^2 + 1$ and no minimal prime contains $\mathfrak{m}_{\mathcal{O}}$. Moreover, every minimal prime is contained in a prime which is minimal over $\mathfrak{m}_{\mathcal{O}} \cdot R_1$, and every prime which is minimal over $\mathfrak{m}_{\mathcal{O}} \cdot R_1$ contains a unique minimal prime.*

Under these conditions, Taylor proves (by another commutative algebra argument) that the second claim of Corollary 2.8 implies that

Corollary 2.10. *$H_{1,\emptyset}$ is a nearly faithful $B_1^{\square}$-module.*

Finally, just as in Proposition 2.7, we obtain the main theorem of [T]:

Theorem 2.10. *$H_{1,\emptyset}$ is a nearly faithful $R_{1,\mathcal{S}_\emptyset}^{univ}$-module* **(i.e., in the totally degenerate case)**. *In particular, the map*

$$R_{1,\mathcal{S}_\emptyset}^{univ,red} \to \mathbb{T}_{1,\mathcal{S}_\emptyset}$$

is an isomorphism.

This is the optimal modularity lifting theorem and suffices for all applications, provided of course the ramification at ℓ is controlled (of Fontaine-Laffaille type, with ℓ unramified in F).

Remarks 2.11.

(1) The proof of Theorem 2.10 actually requires simultaneous patching for the modules $H_{\chi,N}^{\square}$ with generic χ and for $H_{1,N}^{\square}$, as well as for the maps $S_\infty \to R_{\chi,\mathcal{S}_\emptyset}^{univ}$ used to construct (2.4). This does not pose any new difficulty but the notation is much more cumbersome.

(2) The hypothesis $\varepsilon_\infty = 0$ was made for simplicity. In [T] the algebra $B_\chi^{\square}$ is defined to have dimension $R_\chi^{loc} + r - \varepsilon_\infty$, which suffices to define surjective maps to the framed deformational algebras. In the end the depth calculation shows that $\varepsilon_\infty = 0$, just as in the minimal case.

Appendix: Taylor-Wiles patching in the minimal case (reprise).

To apply the above calculations to obtain the situation described in the notes on patching, we replace n by r and m by N. The modules H_m are what we have called $S_{\{\chi_v\}}(U(Q_N), \mathcal{O})_{\mathfrak{m}}$ above, where $U(Q_N)$ is the open compact subgroup satisfying condition (3.5.4) at primes in Q_N. We write H_{Q_N} for H_m Ignore the χ_v for the time being, since the set R of non-minimal primes of (3.5.2) is here assumed empty. The ring R_m of Diamond's Corollary 1.6 is our ring $R_{\bar{\rho},\mathcal{S}(Q_N)}$ which we write more simply $R_{\bar{r}_\Pi,Q_N}$. The ring T_m is just the image of $R_{\bar{r}_\Pi,Q_N}$ in $End(H_{Q_N})$, and this is just $\mathbb{T}^T_{\{\chi_v\}}(U(Q_N))_{\mathfrak{m}}$, which we denote $\mathbb{T}_{\bar{r}_\Pi,Q_N}$ for consistency. Indeed, the only way R^{univ} acts on modular forms is through its (surjective) homomorphism to the corresponding Hecke algebra.

We have not yet constructed the maps involving A and B. Recall that A and B are power series ring in r variables, denoted S_i and X_i, respectively. It follows from Proposition 3.14.5 that each $R_{\bar{r}_\Pi,Q_N}$ is generated over $\mathcal{O}$ by $r - n\sum_{v\in S_\infty}\frac{1+c_v}{2}$. elements, hence is a quotient of a power series ring in r variables. One can therefore construct the maps $B \to R_{\bar{r}_\Pi,Q_N}$ ad hoc.

The maps $A \to R_m$ are more intrinsic. For each $v \in Q_N$ let Δ_v be the quotient of order ℓ^N of $k(v)^\times$. By (3.14.1) there is such a quotient. Let

$$U_{0,v} = U_{1,v} := \{g \in GL(n, \mathcal{O}_{F,\tilde{v}}) \mid g \equiv \begin{pmatrix} g_{n-1} & *_{n-1} \\ 0_{n-1} & * \end{pmatrix} \pmod{\mathfrak{m}_{\tilde{v}}}\}$$

by analogy to (3.5.4), so that $U_{0,v}/U_{1,v} \xrightarrow{\sim} k(v)^\times$. Let $U^+_{1,v} \subset U_{0,v}$ be the subgroup containing $U_{1,v}$ such that

$$U_{0,v}/U^+_{1,v} \xrightarrow{\sim} \Delta_v \xrightarrow{\sim} \mathbb{Z}/\ell^N\mathbb{Z}$$

We modify our modular forms H_{Q_N} and consider only the submodule of $\prod_{v\in Q_N} U^+_{1,v}$-fixed vectors, but we do not change notation. Let $\Delta_{Q_N} = \prod_{v\in Q_N}\Delta_v$. This group acts on H_{Q_N}, and we have the important

(not quite true) Principle 3.15.1. *For any N, the module H_{Q_N} is free over $\mathcal{O}[\Delta_{Q_N}]$.*

This principle is almost true because the $S_{\{\chi_v\}}(U(Q_N), \mathcal{O})$ are spaces of functions on finite sets on which the group Δ_{Q_N} acts almost freely, and the localization that produces H_{Q_N} does not affect the condition of being free over the group algebra. Since we don't know that Δ_{Q_N} acts freely, we follow Taylor and Wiles and introduce an additional prime of potential ramification, denoted S_1 above. Adding S_1 to the level makes the action of Δ_{Q_N} free, and S_1 is chosen so that no constituent of the localization at $\mathfrak{m}$ is actually ramified at S_1, so the Riemann-Roch calculation is unchanged. The existence of an appropriate S_1 is another condition guaranteed by the hypothesis that the image of $\bar{\rho}$ is "big." I will not dwell on this point.

On the other hand, for $v \in Q_N$, consider the action of inertia $I_{\tilde{v}}$ on the universal deformation r^{univ} of type $\mathcal{S}(Q_N)$ of $\bar{r}_\Pi$. We can restrict our attention to the homomorphism $\rho^{univ} : \Gamma_F \to GL(n, R_{\bar{r}_\Pi,Q_N}$. Then in an appropriate basis, $\rho^{univ}\ |_{I_{\tilde{v}}}$ can be written as the sum of a trivial $n-1$-dimensional representation (lifting $\bar{s}_v$) and a one-dimensional character $\xi_v : I_{\tilde{v}} \to R^\times_{\bar{r}_\Pi,Q_N}$ on the lifting of the α_v-eigenspace. The character ξ_v is well-defined and independent of the choice of basis, and is tame, hence factors through the tame inertia group $k(\tilde{v})^\times$. Moreover, we have

Principle 3.15.2. *The character ξ_v factors through the quotient Δ_v of $k(\tilde{v})^\times$, and the action of Δ_v on H_{Q_N} induced by the composition of ξ_v with the homomorphism $R_{\bar{r}_\Pi,Q_N} \to End(H_{Q_N})$ is the natural group-theoretic action described above.*

Both parts of this principle follow from the compatibility of the local and global Langlands correspondences for the representation ρ_Π, proved in my book with Taylor.

Let $A_N = A/J_N$. Choose a generator $\delta_v \in \Delta_v$ for each $v \in Q_N$. The variables S_i in $A = \mathcal{O}[S_1, \ldots S_r]$ are indexed by the elements $v \in Q_N$ for some ordering of the latter – say we write $i = i(v), i = 1, \ldots, r$ – and we identify $A_N = \mathcal{O}[\Delta_N]$ by identifying δ_v with the image of $1 + S_i(v)$ in A_N. In this way, there is a natural map

$$A \to A_N \to R^{\times}_{\bar{r}_\Pi, Q_N}$$

where the second arrow is the product of the ξ_v of 3.15.2. In this way H_N becomes an A-module for each N, and Diamond's condition (d) is satisfied:

3.15.3. *$Ann_A(H_N) = J_N$ and H_N is a free A_N-module for each N..*

To simplify the notation further, we write R_N and $\mathbb{T}_N$ instead of $R_{\bar{r}_\Pi, Q_N}$ and $\mathbb{T}_{\bar{r}_\Pi, Q_N}$ We have already seen Diamond's condition (a) (surjectivity of the maps $R_N \to \mathbb{T}_N$). Condition (b) is not quite true as stated. We have chosen ad hoc maps $B \to R_N$ and we can lift the maps $A \to R_N$ to maps $c_N : A \to B$ in such a way that the map $B \to R_N$ factors through $B_N = B/c_N(J_N)$. In (b) we can replace R_N by B_N, as Diamond did, and then (b) remains true.

Condition (c) is a subtle point. It is not hard to see that $H_N/J_0 H_N \xrightarrow{\sim} H_0$ which is the localization at $\mathfrak{m}$ of the automorphic forms invariant under the group $U_0(Q_N)$, which are fixed by $\prod_{v \in Q_N} U_{0,v}$, in the above notation. But condition (c) requires an identification of H_0 with $H_{\Pi,\emptyset}$. There are two independent points, one global, one local, discussed in the appendix.

The global point – see Lemma A.2 of the appendix – is that the condition at $\mathfrak{m}$ guarantees that any representation Π' of type $\mathcal{S}(Q_N)$, with $\bar{\rho}_{\Pi'} \xrightarrow{\sim} \bar{\rho}_\Pi$, and with $(\Pi')^{U_0(Q_N)} \neq 0$, is necessarily unramified at Q_N. This is the group-theoretic equivalent of the Galois-theoretic condition (3.13.6) that says that any deformation of $\bar{\rho}_\Pi$ of type $\mathcal{S}(Q_N)$ necessarily breaks up as a sum of the unramified $n-1$-dimensional piece and the potentially ramified one-dimensional piece. This heuristic argument can be made rigorous by considering the classification of admissible representations of $GL(n, F_{\tilde{v}})$ with $U_{0,v}$-fixed vectors.

The second point is that H_0 is naturally a space of U_{0,Q_N}-invariant automorphic forms in the space of automorphic forms unramified at Q_N. For each v, the space of $U_{0,v}$-invariant forms in $\Pi_{\tilde{v}}$ is of dimension n, and one needs to pick out a submodule of rank one over $R_\emptyset$ and construct an isomorphism with the module of $\prod_v GL(n, \mathcal{O}_{\tilde{v}})$-invariant forms. It is for this reason that we need the additional operators V_v for $v \in Q_N$ and to include $V_v - \alpha_v$ in the ideal $\mathfrak{m}$. This can be done by means of Hensel's lemma, but the construction depends on an analysis of the reduction modulo $\mathfrak{m}_\mathcal{O}$ of principal series representations of $GL(n, F_{\tilde{v}})$ when $Nv - 1$ is divisible by ℓ. This was considered by Vignéras and the results are described in part in the notes entitled modularprincipalseries.pdf.

Admitting this last step, we have completed the verification of Diamond's conditions (or Fujiwara's equivalent conditions). We may therefore conclude as in Taylor-Wiles:

Theorem 3.15.4. *The map*

$$\phi_{\emptyset} : R_{\bar{r}_{\Pi},\emptyset} \to \mathbb{T}_{\Pi}$$

is an isomorphism of complete intersections, and H_{Π} *is a free module over* $\mathbb{T}_{\Pi}$*. In particular, any deformation of* $\bar{r}_{\Pi}$ *of minimal type* $\mathcal{S}$ *is of the form* $r_{\Pi'}$ *for some automorphic representation* Π' *of* $GL(n)$ *of cohomological unitary type.*

Finally, the error term $\epsilon_{\infty} = n \sum_{v \in S_{\infty}} \frac{1+c_v}{2}$ *necessarily vanishes. In other words,* $c_v = -1$ *for all* $v \in S_{\infty}$*.*