1. Cohomology of smooth projective (Shimura) varieties and the ℓ-adic comparison theorem.

For reasons having to do with the origin of our work, ℓ plays the role of p. Crystalline ℓ-adic representations, or more generally de Rham ℓ-adic representations, are the kinds of representations that arise in the ℓ-adic etale cohomology of algebraic varieties over number fields or ℓ-adic fields. This is a theorem, however: the condition of being crystalline or de Rham has an abstract definition, due to Fontaine, whose most important properties have been established in just the last few years. Our Galois representations will be realized in the cohomology of Shimura varieties, denoted $Sh(G)$, that are smooth and projective over $Spec(\mathcal{O})$ for some ℓ-adic integer ring $\mathcal{O}$ unramified over $\mathbb{Z}_\ell$. Thus our Galois representations will in fact by crystalline. In the proof of the Sato-Tate conjecture we have a great deal of freedom to choose ℓ arbitrarily large relative to other data, which means we can apply the easiest version of integral ℓ-adic Hodge theory due to Fontaine and Laffaille [FL].

We work over the base field $\mathbb{Q}_\ell$ for simplicity, although everything stated here is true over general unramified extensions of $\mathbb{Q}_\ell$ (and probably ramified extensions as well). From our point of view the main object is either an irreducible n-dimensional ℓ-adic representation ρ of $\Gamma_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ or a cuspidal automorphic representation π of $GL(n, \mathbb{Q})$, which is assumed unramified at ℓ. The Shimura variety $Sh(G)$ is needed for the construction of the ℓ-adic representation ρ_π associated to π but is not otherwise visible in the papers [CHT], [HST], [T]. In this section I refer to it explicitly because its properties are used to establish the properties of the restriction of ρ_π to a decomposition group at ℓ. The Shimura variety is attached to a certain unitary group G, split at ℓ. It is of dimension $n-1$ and its middle-dimensional cohomology is decomposed into n-dimensional pieces by the action of Hecke operators, and ρ_π is one of these pieces when the cohomology in question is ℓ-adic. Indeed, the Hecke algebra is the same as the one acting on modular forms on the definite unitary group discussed elsewhere in these notes, but this fact is never used explicitly.

Assume $Sh(G)$ has good reduction mod ℓ, i.e. extends to a proper smooth scheme $\mathbb{S}(G)$ over $Spec(\mathbb{Z}_\ell)$. (We are considering Shimura varieties of fixed level and this hypothesis is true if the level is prime to ℓ in a standard sense.) Then the Shimura variety $Sh(G)$ gives rise to two spaces of cohomology over $\mathbb{Q}_\ell$. The first is the ℓ-adic cohomology $H_\ell = H^{n-1}(Sh(G), \mathbb{Q}_\ell) = \varprojlim_N H^{n-1}_{et}(Sh(G), \mathbb{Z}/\ell^N/\mathbb{Z}) \otimes \mathbb{Q}_\ell$. The second is de Rham cohomology:

$$H_{dR} = H^{n-1}_{dR}(\mathbb{S}/Spec(\mathbb{Z}_\ell)) = \mathbb{H}^{n-1}(\mathbb{S}, \Omega^\bullet_{\mathbb{S}/\mathbb{Z}_\ell}).$$

The former is topological, the latter is computed by differential forms with coefficients in $\mathcal{O}_v$. Both have the same finite dimension and the eigenspaces for Hecke operators on the two spaces have the same dimension as well. However, they have different structures. The ℓ-adic cohomology carries an action of $\Gamma_{\mathbb{Q}}$, and in particular of the decomposition group Γ_ℓ. The second has two structures: a crystalline Frobenius:

$$\phi : H_{dR} \to H_{dR}$$

which is a bijective map that is $Frob$-linear

$$\phi(av) = Frob_\ell(a)\phi(v).$$

This doesn't look like anything more than a linear map but in fact it has the same property after base change to the completion of the maximal unramified extension of $\mathbb{Q}_\ell$. And a Hodge filtration: there is a filtration $\dots F^p H_{dR} \subset F^{p-1}H_{dR}\dots$ with

$$F^p/F^{p+1} \xrightarrow{\sim} H^q(\mathbb{S}_K, \Omega^{p-1}).$$

These two structures interact ("the Newton polygon lies above the Hodge polygon") but we don't need to know that.

What we do need to know, at least for a few seconds, is that there is a way to obtain H_{dR} from H_ℓ, with all the structure, and vice versa. The following theorem contains a part of ℓ-adic Hodge theory, and is due to many people.

Theorem. *There are fields $B_{crys} \subset B_{dR}$ containing $\mathbb{Q}_\ell^{unr}$, the maximal unramified extension of $\mathbb{Q}_\ell$, with compatible actions of Γ_ℓ, and with the following additional structures:*

(1) *B_{dR} is a complete discrete valuation field, containing $\overline{\mathbb{Q}}_\ell$ and with residue field $\mathbb{C}_\ell$, the completion of $\overline{\mathbb{Q}}_\ell$ (via the residue map), and*
(a) *The valuation defines a Γ_ℓ-stable (decreasing) filtration $Fil^i B_{dR}$;*
(b) *There is a map $\mathbb{Q}_\ell(1) \to Fil^1 B_{dR}$ of G_v-modules whose image contains a uniformizer;*
(c) $B_{dR}^{G_v} = F_v$
(2) *B_{crys} is a subring containing $\mathbb{Q}_\ell(1)$ and endowed with a $Frob$-linear injective automorphism ϕ satisfying*
(a) *ϕ commutes with G_v;*
(b) *$\phi(t) = \ell \cdot t$ for $t \in \mathbb{Q}_\ell(1) \subset B_{crys} \cap Fil^1 B_{dR}$*
(c) $Fil^0 B_{dR} \cap B_{crys}^{\phi=1} = \mathbb{Q}_\ell$
(d) $B_{crys}^{G_v} = \mathbb{Q}_\ell$

These fields, constructed by Fontaine according to an explicit and very complicated recipe, turn out to be of the highest importance for a huge variety of applications. They are called the ℓ-adic period rings (usually called p-adic, but not in our papers) because their main application is comparison between ℓ-adic topological cohomology and ℓ-adic de Rham (analytic) cohomology. We state the main theorem only for cohomology with trivial coefficients:

Theorem (Fontaine-Messing, Faltings, Tsuji). *(a) There is a natural isomorphism*

$$[H_\ell(Sh(G)) \otimes B_{dR}]^{\Gamma_\ell} \xrightarrow{\sim} H_{dR}(Sh(G)/\mathbb{Q}_\ell)$$

where Γ_ℓ acts diagonally and the filtration on the right-hand side is inherited from B_{dR};

(b) Suppose $Sh(G)$ is smooth. Assume π contains a fixed vector under $GL(n, \mathbb{Z}_\ell)$. (This implies the existence of a smooth model $\mathbb{S}$ over $\mathbb{Z}_\ell$, as indicated above.) Then

there is a natural isomorphism

$$[H_\ell(Sh(G)) \otimes B_{crys}]^{\Gamma_\ell} \xrightarrow{\sim} H_{dR}(Sh(G)/\mathbb{Z}_\ell)$$

(actually with crystalline cohomology) and the action of ϕ on the right-hand side is inherited from B_{crys}.

(c) Assume $\ell > n$ and ℓ is unramified in F. Then (b) is even true integrally: there is a $\mathbb{Z}_\ell^{unr}$-subalgebra $A_{crys} \subset B_{crys}$ and the isomorphism in (b) is valid over $\mathbb{Z}_\ell$ (in a modified sense, see below).

2. Fontaine-Laffaille modules.

In (c) above, it is not true that

$$[H^{n-1}(Sh(G), \mathbb{Z}_\ell) \otimes A_{crys}]^{\Gamma_\ell} \xrightarrow{\sim} H_{dR}(Sh(G)/\mathbb{Z}_\ell).$$

because $H^{n-1}(Sh(G), \mathbb{Z}_\ell)$ is not the right lattice. What we need to know is that one can construct a lattice $M_{crys}(Sh(G)) \subset H_{dR}(Sh(G)/\mathbb{Z}_\ell)$ as a union of the Γ_ℓ-invariants in certain lattices in $H^{n-1}(Sh(G), \mathbb{Q}_\ell) \otimes A_{crys}$, and that the reduction modulo ℓ^m of M_{crys} for all m is a *Fontaine-Laffaille module*, to be discussed in this section. We will only use the Fontaine-Laffaille property to define the deformation ring and to determine its numerical invariants.

Let F be an unramified extension of $\mathbb{Q}_\ell$, with residue field $k = \mathbb{F}_q$ with $q = \ell^f$ for some integer f. Let $\mathcal{O}_F$ its integer ring, which we also denote $W(k)$ (the Witt ring of k), σ the *absolute* Frobenius map on $\overline{\mathbb{Q}}_\ell$ or on any integer ring or quotient thereof, $\tau = \sigma^f$. Let A be a local $\mathcal{O}_F$ algebra with residue field k, M an A-module. A map $\phi : M \to M$ is τ-linear if it is additive and if

$$\phi(am) = \tau(a)\phi(m) \; \forall a \in A, m \in M.$$

Definition 2.1. *A* F*ontaine-Laffaille module over $W(k)$ ("module filtré", in the terminology of [FL]) is a torsion $W(k)$-module of finite type with a decreasing filtration $Fil^i M$, with $Fil^0 M = M$, $Fil^\ell M = 0$, and a family of τ-linear maps*

$$\phi^i : Fil^i(M) \to M$$

such that for all i

(F) $$\phi^i \mid_{Fil^{i+1}} = \ell \phi^{i+1}$$

and

(L) $$M = \sum_i Im(\phi^i).$$

For example, if $\ell > n$, the ℓ-torsion quotients of $H_{dR}(Sh(G)/\mathbb{Z}_\ell)$ have structures of this type. The ϕ^i are derived from $\ell^{-i} \cdot \phi$ in crystalline cohomology and the lattice in the remark in the first paragraph of this section is the smallest one for which this makes sense. Examples will be discussed in the next section.

The Fontaine-Laffaille modules over $W(k)$ form an abelian $W(k)$-linear category denoted MF^f_{tor} (or using more complicated notation). So is the category Rep_{Γ_F} of representations of $\Gamma_F = Gal(\overline{\mathbb{Q}}_\ell/F)$ on torsion $W(k)$-modules. One of the main points of [FL] is to construct an exact faithful functor from MF^f_{tor} to Rep_{Γ_F}. The representations belonging to the (essential) image of this functor are said to be of Fontaine-Laffaille type. For $\ell > n$, any subquotient of the representation of $\Gamma_{\mathbb{Q}_\ell}$ on $H^{n-1}(Sh(G), \mathbb{Z}/\ell^m\mathbb{Z})$ (any m) is of Fontaine-Laffaille type. For general ℓ (or for more general automorphic Galois representations, e.g. the representations attached to modular forms of weight $k > \ell$, the condition is still quite restrictive, and other approaches to ℓ-adic Hodge theory seem to be necessary (for example, Fontaine's theory of (ϕ, Γ)-modules).

To state this equivalence more precisely, we introduce the following notation. We let $\mathcal{O}_{\mathbb{C}_\ell}$ denote the ring of integers in the completion $\mathbb{C}_\ell$ of the algebraic closure of the fraction field of $W(k)$. Let $A_{cris} = A_{cris}(\mathcal{O}_{\mathbb{C}_\ell})$ be the ring defined in [Fo, 2.3], an ℓ-integral form of B_{cris}. The $W(k)$-algebra A_{cris} is endowed with a decreasing filtration $Fil^i A_{cris}$, $i \geq 0$ and with a σ-linear operator ϕ such that

$$\phi(Fil^i A_{cris}) \subset \ell^i A_{cris}, 0 \leq i \leq \ell - 1. \tag{2.2}$$

Let $\mathcal{O}$ be any ℓ-adic integer ring with residue field $k_{\mathcal{O}}$. To avoid complications we assume $\mathcal{O}$ is unramified over $\mathbb{Z}_\ell$, although this is certainly unnecessary. Let $\mathbf{Rep}_{W(k)\otimes\mathcal{O}}$ denote the category of $\mathcal{O}[\Gamma_F]$-modules of finite type. We define $\mathbf{Rep}_{W(k)\otimes\mathcal{O},cris,[0,\ell-1[}$ to be the full subcategory of $\mathbf{Rep}_{W(k)\otimes\mathcal{O}}$ whose objects are isomorphic to subquotients of crystalline $\mathbb{Q}_\ell\otimes_{\mathbb{Z}_\ell}\mathcal{O}[\Gamma_F]$-modules. For any object Λ of $\mathbf{Rep}_{W(k)\otimes\mathcal{O},cris,[0,\ell-1[}$ we can explicitly define the Fontaine-Laffaille module $M_{cris}(\Lambda)$. It is an object of the category $MF_{W(k)\otimes\mathcal{O},[0,\ell-1[}$ consisting of

(i) a $W(k) \otimes \mathcal{O}$-module M of finite type, a decreasing filtration $Fil^i(M)$ by $W(k) \otimes \mathcal{O}$-submodules which are direct factors, with $Fil^0 M = M$, $Fil^\ell M = 0$; and

(ii) a family $\phi^i : Fil^i(M) \to M$ of $\tau \otimes 1_{\mathcal{O}}$-linear maps such that

$$\phi^i(x) = \ell\phi^{i+1}(x) \forall x \in Fil^{i+1} M, \tag{F}$$

and such that

$$M = \sum_{i\in\mathbb{Z}} im(\phi^i). \tag{L}$$

It is further assumed that M contains no non-trivial subobject M' with $Fil^{\ell-1} M' = M'$.

The definition is

$$M_{cris}(\Lambda) = \bigcup M \mid M \subset A_{cris} \otimes_{W(k)} \Lambda)^{\Gamma_F};\ M \in MF_{W(k),[0,\ell-1[} \tag{2.3}$$

(cf. [Niz, p. 750]; [Wa,Remarque 2.4.4]). The filtration is inherited from the filtration on A_{cris}, the $\mathcal{O}$-action on Λ is left undisturbed, and ϕ^i is inherited from $\ell^{-i}\phi$ on $Fil^i A_{cris}$, which makes sense by condition (F). Then $M_{cris} : \mathbf{Rep}_{W(k),cris,[0,\ell-1[} \to MF_{W(k),[0,\ell-1[}$ is an equivalence of categories. An inverse equivalence [FL] is given by the formula

$$\Lambda(M) = Fil^0(A_{cris} \otimes_{W(k)} M)^{\phi=1}. \tag{2.4}$$

For our purposes, crystalline Galois representations are those of the form $\Lambda(M)$, for some $M \in MF_{W(k),[0,\ell-1[}$. More generally, let A be a $W(k)$ algebra of finite type with a σ-linear automorphism σ_A. Then we can define the categories $MF_{A,[0,\ell-1[}$ and $\mathbf{Rep}_{A,cris,[0,\ell-1[}$ by analogy with (i), (ii) above, taking A as coefficient ring. The functors M_{cris} and Λ can be defined as inverse equivalences between these two categories. In the applications, A will be a $W(k) \otimes \mathcal{O}$-algebra, where $\tau_{W(k)\otimes\mathcal{O}}$ is defined to be $\tau \otimes 1 : W(k) \otimes \mathcal{O} \to W(k) \otimes \mathcal{O}$.

Let $MF_{tor,[0,\ell-1[}$ denote the subcategory of $MF_{W(k),[0,\ell-1[}$ of objects of finite length (as $W(k)$-modules). By [FL,Prop. 1.8], $MF_{tor,[0,\ell-1[}$ is an abelian category. Let

$$\mathbf{Rep}_{tor,cris,[0,\ell-1[} \subset \mathbf{Rep}_{W(k),cris,[0,\ell-1[}$$

denote the essential image of the functor Λ, restricted to $MF_{tor,[0,\ell-1[}$. Then $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$ is a full subcategory of $\mathbf{Rep}_{W(k),cris,[0,\ell-1[}$, itself a full subcategory of $\mathbf{Rep}_{W(k)}$. The functors M_{cris} and Λ define inverse equivalences between the abelian categories $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$ and $MF_{tor,[0,\ell-1[}$. The important point is the following Lemma, an immediate consequence of the equivalence of categories:

Lemma 2.5. *Given two objects $M, N \in MF_{tor,[0,\ell-1[}$, there is a natural isomorphism*

$$Ext^1_{MF}(M,N) \xrightarrow{\sim} Ext^1_{cris}(\Lambda(M),\Lambda(N)), \tag{2.6}$$

where Ext^1_{MF} (resp. Ext^1_{cris}) is shorthand for extensions in $MF_{tor,[0,\ell-1[}$ (resp. $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$).

3. Examples in dimensions 1 and 2.

Let $f \geq 1$ be an integer and let F_f be the unramified extension of $\mathbb{Q}_\ell$ of degree f. Let M be a one-dimensional vector space over $k = k_f = \mathbb{F}_{\ell^f}$. In order to endow it with the structure of Fontaine-Laffaille module, the only variable is the filtration: there is a unique i such that $Fil^i M = M$, $Fil^{i+1} M = (0)$, the map ϕ^i being uniquely determined up to isomorphism by τ-linearity. We call the corresponding module $M_{k,i}$.

If $F = \mathbb{Q}_\ell$, the functor Λ takes $M_{\mathbb{F}_\ell,i}$ to the representation $\mathbb{F}_\ell(i) = \mathbb{F}_\ell(1)^{\otimes i}$, the ith power of the cyclotomic character. In general, the field $F(p^{\frac{1}{q-1}})$ is a cyclic abelian tamely ramified extension of F of degree $q-1$, and if ϖ is a chosen $q-1$st root of p, then there is an isomorphism

$$\chi_h : Gal(F(p^{\frac{1}{q-1}})/F) \xrightarrow{\sim} k^\times;\ g \mapsto g(\varpi)/\varpi.$$

The character χ_h is called the hth fundamental character, and $\Lambda(M_{k,i}) = \chi_h^{\otimes i}$.

Let M be a rank 2 Fontaine-Laffaille module over $\mathbb{F}_\ell$. First, suppose M is reducible, hence contains a rank 1 submodule N. Thus there is a short exact sequence

$$0 \to N = M_{\mathbb{F}_\ell,i} \to M \xrightarrow{h} N' = M_{\mathbb{F}_\ell,i'} \to 0.$$

Shifting all filtration indices by a fixed constant j is equivalent to tensoring by $M_{\mathbb{F}_\ell,j}$, which on the Galois side is equivalent to twisting by $\mathbb{F}_\ell(j)$. We thus lose no generality by assuming $M = Fil^0 M$, $M \neq Fil^1 M$. Let e be a basis of N and $e' \in M$ an element not in N, hence mapping to a basis of N'. Without loss of generality (up to extending coefficients to $\overline{\mathbb{F}}_\ell$) we may assume

$$\phi^i(e) = e, \phi^{i'}_{N'}(h(e')) = h(e'), \tag{3.1}$$

where $\phi^*_{N'}$ is the map ϕ^* on N'. We leave the case $M = gr^0 M$ as an exercise; thus there is a maximum index $j > 0$ such that $0 \neq Fil^j M \neq M$; in particular $Fil^j M$ is of dimension 1 . Then $\phi : gr^\bullet M = gr^0 M \oplus gr^j M \to M$ is an isomorphism. If $Fil^j M$ is in the kernel of the map $h : M \to N'$, then $Fil^j M = N$, $i = j$, $i' = 0$, and the extension is determined up to isomorphism by the scalar a such that $\phi^0(e') = e' + ae$.

If $N \neq Fil^j M$, then we may take $e' \in Fil^j M$, and then $\phi^j(e')$ is not equal to zero, say $\phi^j(e') = ae + be'$. Moreover, $\phi^0(e') = \ell^j \phi^j(e') = 0$, which means $\phi^0(e) \neq 0$ by condition (L). On the other hand, $\phi^i(e) = e$, so $i = 0$. Then

$$bh(e') = h \circ \phi^j(e') = \phi^j_{N'} \circ h(e').$$

If $j \neq i'$ then $b = 0$; but then (3.1) implies $Im(\phi^j + \phi^i) \subset N$ which contradicts condition (L). Thus $j = i'$. Then we may assume $b = 1$, and the extension is determined up to isomorphism by a.

In either case, the functor Λ takes reducible 2-dimensional M to reducible 2-dimensional Galois representations; the coefficient a equals zero if and only if $\Lambda(M)$ is split.

Now suppose M is irreducible, with $j > 0$ maximal such that $\dim Fil^j M = \dim Fil^1 M = 1$, $M = Fil^0 M$. Take F a basis of $Fil^j M$, and let $e = \phi^j(f)$. Since M is irreducible, $e \notin Fil^j M$, so $\phi^0(e) = ae + bf$. Suppose $a \neq 0$; then over the algebraic closure we may arrange that $\phi^0(e) = e + bf = e^*$. But $\phi^0(f) = \ell^j \phi^j(f) = 0$, hence $\phi^0(e^*) = e^*$, and then e^* generates a non-trivial subobject. Thus $a = 0$, and we may assume $b = 1$. Thus the only invariant of M is the index j, and we denote the corresponding M by $M(2, j)$. Then for $0 < j \leq p - 1$ $\Lambda(M(2, j)) = Ind_{F/\mathbb{Q}_\ell} \chi_2^j$ where F is the unramified quadratic extension of $\mathbb{Q}_\ell$ and χ_2 is the fundamental character defined above.

4. Galois cohomology of Fontaine-Laffaille representations.

The advantage of the Fontaine-Laffaille category is that it can be used to simplify calculations of Galois cohomology. More precisely, in the study of deformations one needs to work with $H^i(\Gamma_F, \rho_\pi \otimes \mathbb{F}_\ell) = Ext^i_{\Gamma_F}(\mathbb{F}_\ell, \rho_\pi \otimes \mathbb{F}_\ell)$ where Γ_F acts trivially on $\mathbb{F}_\ell$. But these Ext groups, or at least their Euler characteristics, which are all we really need, can be calculated explicitly in terms of the Fontaine-Laffaille data, using Lemma 2.5. The result is quite simple, using the following observation:

Lemma 4.1. *Let k be a finite field of characteristic ℓ. Then the category of Fontaine-Laffaille modules over k is equivalent to the category of k-vector spaces M with decreasing filtrations Fil^iM and σ-linear isomorphisms*

$$\phi : gr^\bullet M = \oplus_j Fil^jM/Fil^{j+1}M \xrightarrow{\sim} M.$$

Proof. The condition (F) that $\phi^j \mid_{Fil^{j+1}} = \ell\phi^{j+1}$ just means that $\phi^j \mid_{Fil^{j+1}} = 0$, so the collection of ϕ^j is equivalent to a single ϕ as indicated. But condition (L) implies that ϕ is surjective, and since the two sides have the same dimension the lemma is clear.

Write $k \otimes k_{\mathcal{O}} = k \otimes_{\mathbb{Z}_\ell} k_{\mathcal{O}}$, and suppose now that M and N are free $k \otimes k_{\mathcal{O}}$-modules. In particular, $\ell \cdot M = 0$, $\ell \cdot N = 0$. Let $Ext^1_{MF,k\otimes k_{\mathcal{O}}}(M,N) \subset Ext^1_{MF,k\otimes k_{\mathcal{O}}}(M,N)$ denote the subgroup of extensions in the category of Fontaine-Laffaille modules which are free $k \otimes k_{\mathcal{O}}$-modules. Likewise, let $\mathbf{Rep}_{k\otimes k_{\mathcal{O}}}$ denote the category of $k \otimes k_{\mathcal{O}}[\Gamma_F]$-modules free over $k \otimes k_{\mathcal{O}}$, and define

$$\mathbf{Rep}_{k\otimes k_{\mathcal{O}},cris,[0,\ell-1[} \subset \mathbf{Rep}_{k\otimes k_{\mathcal{O}}}$$

to be the intersection of $\mathbf{Rep}_{k\otimes k_{\mathcal{O}}}$ with $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$.

By the usual arguments, we can calculate Ext^1 of group representations in terms of group cohomology:
(4.2)

$$\begin{aligned} Ext^1_{\mathbf{Rep}_{k\otimes k_{\mathcal{O}}}}(\Lambda(M),\Lambda(N)) &\xrightarrow{\sim} Ext^1_{\mathbf{Rep}_{k\otimes k_{\mathcal{O}}}}(k\otimes k_{\mathcal{O}}, Hom_{k\otimes k_{\mathcal{O}}}(\Lambda(M),\Lambda(N))) \\ &\xrightarrow{\sim} H^1(\Gamma_F, Hom_{k\otimes k_{\mathcal{O}}}(\Lambda(M),\Lambda(N))). \end{aligned}$$

On the other hand, composing the isomorphism (2.6) with the forgetful functor, we obtain a homomorphism for any Fontaine-Laffaille modules M, N:

$$Ext^1_{MF}(M,N) \to Ext^1_{\mathbf{Rep}_{W(k)}}(\Lambda(M),\Lambda(N)). \tag{4.3}$$

Now let $V, V' \in \mathbf{Rep}_{k\otimes k_{\mathcal{O}},cris,[0,\ell-1]}$, so that $V = \Lambda(M)$, $V' = \Lambda(M')$ for two $k \otimes k_{\mathcal{O}}$-Fontaine-Laffaille modules in the range $[0, \ell-1]$. With respect to (4.3) and (4.2), we let

$$H^1_f(\Gamma_F, Hom_{k\otimes k_{\mathcal{O}}}(V,V')) \subset H^1(\Gamma_F, Hom_{k\otimes k_{\mathcal{O}}}(V,V'))$$

denote the image of $Ext^1_{MF,k\otimes k_{\mathcal{O}}}(M,M')$, which is also the image of $Ext^1_{\mathbf{Rep}_{W(k),cris,[0,\ell-1]}}(V,V')$ as in Lemma 2.5. The notation H^1_f is taken from Bloch-Kato. ‘

Say $M \in MF_{W(k)\otimes\mathcal{O},[0,\ell-1[}$ is *regular* if $gr^i(M) = Fil^i(M)/Fil^{i+1}(M)$ is a free $k \otimes k_{\mathcal{O}}$-module of rank ≤ 1 for all i.

Lemma 4.4. *Let Λ be a crystalline $k \otimes k_{\mathcal{O}}[\Gamma_F]$-module, and let $Ad(\Lambda)$ denote the $k \otimes k_{\mathcal{O}}[\Gamma_F]$-module $Hom_{k\otimes k_{\mathcal{O}}}(\Lambda, \Lambda)$. Suppose $\Lambda = \Lambda(M)$, with M a regular Fontaine-Laffaille module of rank n over $k \otimes k_{\mathcal{O}}$. Then*

$$rank_{k\otimes k_{\mathcal{O}}} H^1_f(\Gamma_F, Ad(\Lambda)) - \dim H^0(\Gamma_F, Ad(\Lambda)) = \frac{1}{2}n(n-1).$$

Proof. By Lemma 2.5 and (4.2), the left-hand side equals

$$rank_{k\otimes k_{\mathcal{O}}} Ext^1_{MF,rank_{k\otimes k_{\mathcal{O}}}}(M,M) - rank_{k\otimes k_{\mathcal{O}}} Hom_{MF,rank_{k\otimes k_{\mathcal{O}}}}(M,M).$$

This is unchanged when we extend scalars from $k_{\mathcal{O}}$ to a finite extension k' and replace M by $M_{k'}$. We thus may assume $k \subset k'$, and then by projecting on irreducible components we may replace $k \otimes k_{\mathcal{O}}$ by k'.

Since M is a k'-module, its structure as Fontaine-Laffaille module reduces to a triple consisting of a k'-vector space V, a decreasing filtration $Fil^{\bullet}V$, and an isomorphism $\phi_V : gr^{\bullet}V \xrightarrow{\sim} V$. Let $gr^{\bullet}V = \oplus_{i=1}^n gr^{a_i}V$, where $0 \leq a_1 < a_2 \cdots < a_n \leq \ell - 1$ is an n-tuple of positive integers. Fix a basis $e_1, \ldots, e_n$ of V such that $Fil^i V$ is the span of $e_{i+1}, \ldots, e_n$. Suppose $\mathcal{E}$ is an extension of M by itself in $Ext^1_{MF,k'}(M,M)$ and let E be the underlying k'-module. There is a short exact sequence

$$0 \to V \xrightarrow{i} E \xrightarrow{\pi} V \to 0$$

compatible with filtrations and the morphisms ϕ_E and ϕ_V. Let $s : V \to E$ denote any splitting of π as filtered module; s is determined uniquely up to an element $\alpha \in F^0End(V)$, where $F^0End(V) \subset End_k(V)$ denotes the subspace of filtration-preserving endomorphisms. Then $gr^{\bullet}E = gr(i)(gr^{\bullet}V) \oplus gr(\pi)(gr^{\bullet}V)$. In terms of this basis, ϕ_E can be written $\begin{pmatrix} \phi_V & \mu \\ 0 & \phi_V \end{pmatrix}$, for some $\mu \in Hom(gr^{\bullet}V, V)$. We have

$$\mu = \phi_E \circ gr(s) - s \circ \phi_V.$$

Moreover, replacing s by $s + \alpha$ changes μ to $\mu + \phi_V \circ gr(\alpha) - \alpha \circ \phi_V$. Thus the map $M \mapsto \mu$ (mod equivalence) defines an isomorphism

$$Ext^1_{MF,k'}(M,M) \to \{\phi \in Hom(V,V)\}/\{\phi \circ gr(\alpha) - \alpha \circ \phi | \alpha \in F^0End(V))\}.$$

Moreover,

$$Hom_{MF,k'}(M,M) \to \{\alpha \in F^0End(V) | \phi \circ gr(\alpha) = \alpha \circ \phi\}.$$

This yields an exact sequence

$$0 \to Hom_{MF,k'}(M,M) \to F^0End(V) \xrightarrow{j} Hom_k(gr^{\bullet}V, V) \to Ext^1_{MF,k'}(M,M) \to 0,$$

where the map j takes α to $\phi \circ gr(\alpha) - \alpha \circ \phi$.

It follows that

$$\begin{aligned} \dim\ Ext^1_{MF,k'}(M,M) - \dim\ Hom_{MF,k'}(M,M) &= \dim\ Hom_{k'}(gr^{\bullet}V, V) - \dim\ F^0End(V) \\ &= n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1). \end{aligned}$$

5. Crystalline deformations of Fontaine-Laffaille representations.

Let $A = k[\varepsilon]/(\varepsilon^2)$, let $\Lambda \in \mathbf{Rep}_{k,cris,[0,\ell-1[}$, and let $\tilde{\Lambda}$ be a deformation of Λ to A. Multiplication by ε defines an isomorphism

$$\Lambda \cong \tilde{\Lambda}/\varepsilon\tilde{\Lambda} \xrightarrow{\sim} \varepsilon\tilde{\Lambda}$$

of $k[\Gamma_F]$-modules. Thus $\tilde{\Lambda}$ defines an extension

$$0 \to \Lambda \to \tilde{\Lambda} \to \Lambda \to 0, \tag{5.1}$$

and this correspondence defines a bijection between the equivalence classes of deformations of Λ to A and $Ext^1_{\mathbf{Rep}_k}(\Lambda, \Lambda)$ (cf. [DDT], p. 67). By definition, the deformation $\tilde{\Lambda}$ is crystalline if and only if (5.1) is an extension in the category of crystalline representations; in other words, crystalline deformations are in bijection with $Ext^1_{cris}(\Lambda, \Lambda) \cong H^1_f(\Gamma_F, Ad(\Lambda)$. We have thus verified the equivalence of ??.2.3.1 and ??.2.1.1 in the definition of $H^1_{\mathcal{D}}(E, adr_{\bar{r}})$.

6. Action of tame inertia for regular Fontaine-Laffaille modules (bonus).

For $M \in MF_{W(\overline{\mathbb{F}}_\ell),[0,\ell-1[}$ the action of tame inertia on $\Lambda(M)$ is calculated explicitly in [FL]. As a consequence of that calculation, we can prove

Lemma 6.1. *Let $M \in MF_{W(k),[0,\ell-1[}$ and suppose M is regular. Then the eigenspaces of tame inertia on $\Lambda(M)$ have dimension* 1*. More precisely, let $\overline{\Lambda}(M)_{ss}$ denote the semi-simplification of the $k[\Gamma_F]$-module $\Lambda/\ell\Lambda(M)$. Then the action of the inertia subgroup on $\overline{\Lambda}(M)_{ss}$ factors through the tame quotient, and the latter has* $\dim M/\ell M$*-distinct eigenvalues.*

Proof. The proof is a simple combinatorial exercise, using the results of [FL,§§4-5]. Without loss of generality we may assume M to be a semi-simple ℓ-torsion module, so that $\Lambda(M) = \overline{\Lambda}(M)_{ss}$. Let $n = dim_{\overline{\mathbb{F}}_\ell} M$ and let b_i, $i = 1, \ldots, n$ be the integers such that $\dim gr^{b_i}(M) = 1$; let $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}$. Let $q = |k| = \ell^r$, for some r. Then M is determined, up to isomorphism, by the following set of data:

(a) A partition $n = h_1 + \cdots + h_s$, with $h_j \geq 1$ for all j;

(b) For each j a map $\iota_j : \mathbb{Z}/h_j\mathbb{Z} \to B$, with image B_j, so that $B = \cup_j B_j$;

(c) The period of the map ι_j is exactly h_j; i.e. for any $h < h_j$ the map $\iota_j(a) \mapsto \iota_j(a+h)$ is a non-trivial permutation of B_j.

Indeed, we can write $M = \oplus_j M(j)$ as a sum of simple objects, and the object $M(j)$ is determined up to isomorphism by a pair (h_j, ι_j) as above. Let $\Lambda(j) = \Lambda(M(j))$. For any positive integer h, let $\alpha \in \overline{\mathbb{Q}}_\ell$ satisfy

$$\alpha^{q^h-1} = \ell$$

and set $\chi_h(g) = g(\alpha)/\alpha \pmod{\ell} \in \overline{\mathbb{F}}_\ell^\times$, for g in the tame inertia group I^t. Let

$$C(j) = \{\sum_{a=0}^{h_j-1} \iota_j(a+t)q^a \mid t = 0, 1, \ldots, h_j - 1\}. \tag{5.2}$$

It follows from (c) above, and from the fact that $B \subset \{0, \ldots, p-1\}$, that $C(j)$ has h_j distinct elements. The calculation in [FL] shows that the action of I^t on $\Lambda(j)$ factors through the character χ_{h_j}, and $g \in I^t$ has eigenvalues

$$\chi_{h_j}(g)^c, \; c \in C(j).$$

The exponents in $C(j)$ being distinct, the action of I^t on $\Lambda(j)$ is multiplicity-free. On the other hand, it is easy to see, that I^t has no common eigenvalues on $\Lambda(j_1)$ and $\Lambda(j_2)$, first if $h_{j_1} \neq h_{j_2}$, then in the general case.

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