Introduction.

The Taylor-Wiles method to prove that the surjective map $R \to T$ is an isomorphism is based on the analysis of a system of diagrams

$$\begin{array}{ccc} R_Q & \longrightarrow & T_Q \\ \downarrow & & \downarrow \\ R & \longrightarrow & T \end{array}$$

where Q varies over sets of auxiliary primes, all of the same cardinality u, and all arrows are surjective. One lets Q "tend to infinity" in an ad hoc way, and in the limit R_∞ turns out to be an algebra over a power series ring A in u variables. Because R_∞ is known to be of dimension u, and because each T_Q is known to be free over an appropriate quotient of A, this leads to the conclusion that $R \xrightarrow{\sim} T$.

This was the original theorem, as proved by Taylor and Wiles (and improved by Faltings). I will state this theorem below. A more versatile theorem of the same sort was found a few years later independently by F. Diamond and K. Fujiwara. I will give a complete proof of the latter theorem, in Diamond's version. More recently still, Kisin discovered a relative version of this theorem over deformation rings at primes of bad reduction. This was strengthened by Taylor and is at the heart of the last paper on the Sato-Tate conjecture.

0. Complete intersections and commutative algebra.

We fix ℓ and the ring $\mathcal{O}$ that at first approximation one may think of as the ring of integers in a finite extension of $\mathbb{Q}_\ell$. However, for the time being it suffices that $\mathcal{O}$ be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic ℓ. In particular, $\mathcal{O}$ need not be a discrete valuation ring. This is the generality in which we ultimately work, following Kisin, and everything is true in this generality. All the rings in this section will be local $\mathcal{O}$ algebras with residue field k.

Another warning: it is standard in the literature to let $\mathcal{O}$ be a fixed ℓ-adic integer ring, and so the ring labelled $\mathcal{O}$ here will be called $\mathcal{O}_\bullet$, where the subscript varies depending on the situation. We introduce this convention in §2.2.

Definition 0.1. *Let R be a local $\mathcal{O}$-algebra that is of finite type as $\mathcal{O}$-module. Let $B = \mathcal{O}[[X_1, \ldots, X_n]]$. We say that R is a* **complete intersection** *over $\mathcal{O}$ if it is of the form*

$$R \xrightarrow{\sim} B/(f_1, \ldots, f_n)$$

where f_i belongs to the maximal ideal of B.

More generally, a complete intersection R in B is a quotient of B by $\dim B - \dim R$ elements. We only need R of dimension $= \dim \mathcal{O}$.

Definition 0.2. *Let R be a ring and M an R-module. Let $(x_1, \ldots, x_r)$ be a sequence of elements of R, J_i the ideal $(x_1, \ldots, x_i)$ and $M_i = M/J_iM$. Then $(x_1, \ldots, x_r)$ is said to be an* M**-regular sequence** *if the map*

$$x_{i+1} : M_i \to M_i$$

(multiplication) is injective for $i = 0, \ldots, r-1$.

Definition 0.3. *Let R be a noetherian ring, I an ideal of R, and M a finitely generated R-module. The* **depth** *of M with respect to I, denoted $depth_I(M)$, is the supremum of integers r such that there exists an M-regular sequence of length r of elements in I.*

If R is a local ring with maximal ideal $\mathfrak{m}$, we write $depth_R(M) = depth_{\mathfrak{m}}(M)$.

Definition 0.4. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then R is* **regular** *if $\mathfrak{m}$ is generated by r elements, where $r = \dim R$.*

In particular, $k[[x_1, \ldots, x_r]]$ is a regular local ring if k is a field.

Theorem 0.5 (Auslander-Buchsbaum). *Let R be a regular local ring and M a non-zero finite R-module. Then*

$$depth_R(M) + proj.dim_R(M) = dim(R).$$

Regarding projective dimension, we need only

Proposition 0.6. *Let R be a regular local ring and M a non-zero finite R-module. Then $proj.dim_R(M) = 0$ if and only if M is free.*

Here we note that any projective module over a noetherian local ring is free (by Nakayama's Lemma).

1. Isomorphism theorems.

In this section $\mathcal{O}$ is an ℓ-adic integer ring. We introduce another copy of B called $A = \mathcal{O}[[S_1, \ldots, S_n]]$. In A we define a sequence of ideals J_m, $m = 0, 1, \ldots$,

$$J_m = (((1+S_1)^{\ell^m} - 1), ((1+S_2)^{\ell^m} - 1) \ldots, ((1+S_n)^{\ell^m} - 1)).$$

We have

$$J_0 = (S_1, \ldots, S_n),\ A/J_0 = \mathcal{O}; \tag{1.1}$$

$$\cdots \subset J_m \subset J_{m+1} \subset \ldots, \tag{1.2}$$

because $((1+S_i)^{\ell^m} - 1)$ divides $((1+S_i)^{\ell^{m+1}} - 1)$ for each i;

$$\bigcap_{m \geq 0} J_m = (0). \tag{1.3}$$

Here is the Taylor-Wiles isomorphism criterion, used to prove $R = T$ theorems.

Theorem 1.4 (Taylor-Wiles). *Suppose that for every $m > 0$ there is a commutative diagram of complete $\mathcal{O}$-algebras*

$$\begin{tikzcd} A \arrow[r] & R_m \arrow[r, "\phi_m"] \arrow[d] & T_m \arrow[d] \\ & R \arrow[r, "\phi"] & T \end{tikzcd}$$

with the following properties:

(i) *There is a surjection of $\mathcal{O}$-algebras $B \to R_m$ for all m.*
(ii) *The map $\phi_m : R_m \to T_m$ is surjective for all m.*
(iii) *The vertical arrows are surjective and induce isomorphisms*

$$R_m/J_0R_m \xrightarrow{\sim} R;\ T_m/J_0T_m \xrightarrow{\sim} T.$$

(iv) *The quotient ring T_m/J_mT_m is finite flat over A/J_m for each m.*

Then $\phi : R \to T$ is an isomorphism of complete intersections over $\mathcal{O}$.

Note that (ii) and (iii) imply already that ϕ is surjective; the theorem asserts that ϕ is injective and that both R and T are complete intersections.

Let $\bar{B} = B/\mathfrak{m} = k[[X_1, \ldots, X_n]]$, $\bar{A} = k[[S_1, \ldots, S_n]]$, $\mathfrak{n}$ the maximal ideal of $\bar{A}$. Here is Diamond's version of the isomorphism theorem.

Theorem 1.5 (Diamond). *Suppose R is a k-algebra and H is a non-zero R-module, finite-dimensional over k, T the image of R in $End(H)$. Suppose that for each $m > 0$, there is a quotient $\bar{B}_m$ of $\bar{B}$, a $\bar{B}_m$-module H_m and a commutative diagram*

$$\begin{tikzcd} \bar{A} \arrow[r] & \bar{B}_m \arrow[r, "\phi_m"] \arrow[d, "\psi_m"'] & T_m \subset End(H_m) \arrow[d, "\pi_m"] \\ & R \arrow[r, "\phi"] & T \subset End(H) \end{tikzcd}$$

with the following properties:

(a) *The horizontal maps ϕ_m and ϕ are surjections;*
(b) *There is a homomorphism $c_m : \bar{A} \to \bar{B}$ so that $\bar{B}_m = \bar{B}/c_m(\mathfrak{n}^m)$, and surjection $\psi_m : \bar{B}_m \to R$ such that $\psi_m \circ c_m((n)) = 0$;*
(c) *π_m is a $\bar{B}_m$-homomorphism that induces an isomorphism $H_m/\mathfrak{n}H_m \xrightarrow{\sim} H$;*
(d) *$Ann_{\bar{A}}H_m = \mathfrak{n}^m$ and H_m is free over $\bar{A}_m = \bar{A}/\mathfrak{n}^m$.*

Here H_m is an A-module via c_m.

Then R is a complete intersection of dimension zero, and H is free over R.

Remark. The T_m are completely superfluous in the statement and indeed are absent from Diamond's formulation. The present version has been adapted to emphasize the parallel with Taylor-Wiles; the commutative diagram is an icon of the theory and is recycled in each of its variants. Moreover, Diamond's theorem is stated entirely in characteristic ℓ, without reference to $\mathcal{O}$.

Proof. Let $d = \dim_k H$ and choose a k-basis $x_1, \ldots, x_d$ of H. For each $m \geq 1$ and $i = 1, \ldots, d$, choose an element $x_{i,m} \in \pi_m^{-1}(x_i) \subset H_m$. By Nakayama's lemma, $\{x_{i,m}, i = 1, \ldots, d\}$ is an $\bar{A}_m$-basis of H_m. The algebra $T_m \subset End(H_m)$ is in fact a subalgebra of the matrix algebra $M(d, \bar{A}_m)$ and thus ϕ_m can be viewed as a homomorphism

$$\mu_m : \bar{B} \to M(d, \bar{A}_m)$$

of $\bar{A}$-algebras. Now for each $m \geq 1$ and $j = 1, \ldots, n$, choose a lift $\nu_m(X_j)$ of $\mu_m(X_j)$ in $M(d, \bar{A})$. Now $\bar{B}^n \times R^n \times M(d, \bar{A})^n$ is compact, hence the sequence

$$s_m = (c_m(S_1), \ldots, c_m(S_n), \psi_m(X_1), \ldots, \psi_m(X_n), \nu_m(X_1), \ldots, \nu_m(X_n))$$

has a convergent subsequence s_{m_α}. Denote the limit of this convergent subsequence

$$(c_\infty(S_1), \ldots, c_\infty(S_n), \psi_\infty(X_1), \ldots, \psi_\infty(X_n), \nu_\infty(X_1), \ldots, \nu_\infty(X_n)).$$

Obviously c_∞ and ψ_∞ define k-algebra homomorphisms $c_\infty : \bar{A} \to \bar{B}$ and $\psi_\infty : \bar{B} \to R$. I claim that ν_∞ also extends, necessarily uniquely, to a k-algebra homomorphism $\nu_\infty : \bar{B} \to M(d, \bar{A})$. It suffices to show that the $\nu_\infty(X_j)$ commute as operators on H, and this is clear, because modulo any m_α power of the maximal ideal of $\bar{A}$, $\nu_\infty(X_j)$ is congruent to $\mu_{m_\alpha}(X_j)$ for all j. We thus have a $\bar{A}$-algebra action of $\bar{B}$ on $H_\infty = \bar{A}^d$; this just says that

$$\nu_\infty \circ \phi_\infty : \bar{A} \to M(d, \bar{A})$$

is the natural inclusion, which is obvious by construction. Moreover ψ_∞ is surjective, $\psi_\infty \circ c_\infty(\mathfrak{n}) = 0$, and $H_\infty/\mathfrak{n}H_\infty$ is isomorphic to H as $\bar{B}$-module.

Now H_∞ is a free $\bar{A}$-module, hence $c_\infty(S_1), \ldots, c_\infty(S_n)$ is an H_∞-regular sequence. Thus the $\bar{B}$ depth of H_∞ is at least n; but since $\bar{B}$ is itself regular of dimension n, $depth_{\bar{B}} H_\infty = n$. The Auslander-Buchsbaum-Serre theorem implies H_∞ has finite projective dimension over $\bar{B}$. Now we apply the Auslander-Buchsbaum formula:

$$depth_{\bar{B}} H_\infty + proj. \dim_{\bar{B}} H_\infty = depth\ \bar{B}$$

which implies $proj. \dim_{\bar{B}} H_\infty = 0$, hence that H_∞ is free over $\bar{B}$. But this in turn implies that H is free over $\bar{B}/\phi_\infty(\mathfrak{n})\bar{B}$, thus that the latter ring is zero-dimensional, since H is finite-dimensional over k. In particular, $Ann_{\bar{B}} H = \phi_\infty(\mathfrak{n})\bar{B}$, hence the map $\psi_\infty : \bar{B} \to R$ factors through an isomorphism

$$\bar{B}/\phi_\infty(\mathfrak{n})\bar{B} \xrightarrow{\sim} R$$

which means both that H is free over R and that R is the zero-dimensional quotient of the n-dimensional local ring $\bar{B}$ by an ideal generated by n-elements, hence that R is a complete intersection.

The hypotheses can be relaxed in various ways. For example, it suffices to have the modules H_m and the maps π_m and so forth as m runs over powers of p, the missing H_m being obtained by reducing an appropriate one modulo $\mathfrak{n}^m$, so that conditions (c) and (d) remain valid. In particular, one can restore the ℓ-adic integer ring $\mathcal{O}$ to the construction to obtain the following variant:

Corollary 1.6 (Diamond). *Suppose R is a complete $\mathcal{O}$-algebra and H is a non-zero R-module, finite and flat over $\mathcal{O}$, T the image of R in $End(H)$. Suppose that for each $m > 0$, there is a quotient R_m of B, a B_m-module H_m and a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & R_m & \xrightarrow{\phi_m} & T_m \subset End(H_m) \\ & & \psi_m \big\downarrow & & \big\downarrow \pi_m \\ & & R & \xrightarrow{\phi} & T \subset End(H) \end{array}$$

with the following properties:

(a) *The horizontal maps ϕ_m and ϕ are surjections;*
(b) *There is a homomorphism $c_m : A \to B$ so that $R_m = B/c_m(J_m)$, and surjection $\psi_m : R_m \to R$ such that $\psi_m \circ c_m(J_0) = 0$;*
(c) *π_m is a B_m-homomorphism that induces an isomorphism $H_m/J_0H_m \xrightarrow{\sim} H$;*
(d) *$Ann_{\bar{A}} H_m = J_m$ and H_m is free over $\bar{A}_m = \bar{A}/J_m$.*

Here H_m is an A-module via c_m.

Then R is a complete intersection **over** *$\mathcal{O}$ in B of dimension $= \dim \mathcal{O}$, and H is free over R.*

I have changed B_m to R_m to conform with Taylor-Wiles. Note that the image of J_m in $\bar{A} = A/\ell A$ is $\mathfrak{n}^{\ell^m}$, so that reduction modulo ℓ recovers the hypotheses of Theorem 1.5, with H_m replaced by $H_m/\ell H_m$. The conclusion then follows from the conclusion of Theorem 1.5 by Nakayama's Lemma.

2. Variants.

2.1. Patching. It should be clear from the statement that A arises in the applications naturally, in terms of group theory, whereas B is *ad hoc.* In practice, the local $\mathcal{O}$-algebras R_m are given naturally, together with maps $c_m^A : A \to R_m$ for all m factoring through A/J_m. We assume all have the numerical property that

$$\dim_k \mathfrak{m}_m/(\mathfrak{m} + (\mathfrak{m}_m)^2) = n, \tag{2.2}$$

where $\mathfrak{m}_m$ is the maximal ideal of R_m. The set of isomorphism classes of Artinian $\mathcal{O}$-algebras of fixed dimension is finite, because k is finite. By the pigeonhole principle it follows that one can extract a subsequence R_{m_α} of R_m such that each $R_{m_\alpha}/\mathfrak{m}^{m_{\alpha-1}}$ maps surjectively onto $R_{m_{\alpha-1}}/\mathfrak{m}^{m_{\alpha-1}}$. Here one can replace $m_{\alpha-1}$ by any smaller integer u_α; the point is just that we are working with objects of finite cardinality. Since each R_m has embedding dimension n, as in (2.2), one can choose a compatible system of generators $X_{m_\alpha,j}$ of the $\mathfrak{m}_{m_\alpha}$ (modulo the maximal ideal of $\mathcal{O}$) and define maps $B \to R_{m_\alpha}$ by sending each X_j to $X_{m_\alpha,j}$. In this way we construct B and the maps ψ_m; the maps c_m are any lifts to B of c_m^A.

2.3. Double patching.

An important variant was introduced by Taylor in his article [T]. In this situation it is convenient to fix an ℓ-adic integer ring $\mathcal{O}$ and two complete (commutative) $\mathcal{O}$-algebras $\mathcal{O}_\bullet$ and $\mathcal{O}'_\bullet$ of finite type such that $\mathcal{O}_\bullet/\mathfrak{m} \xrightarrow{\sim} \mathcal{O}'_\bullet/\mathfrak{m}$ as $k = \mathcal{O}/\mathfrak{m}$-algebras.

We define $A = \mathcal{O}_\bullet[[S_1, \dots, S_n]]$, $A' = \mathcal{O}'_\bullet[[S_1, \dots, S_n]]$, so that we have a fixed isomorphism

$$a : \bar{A} \xrightarrow{\sim} \bar{A}' \tag{2.3.1}$$

Finally, we assume we are given two families of diagrams:

$$A \xrightarrow{c_m^A} R_m \xrightarrow{\phi_m} End(H_m)$$

$$A' \xrightarrow{c_m^{A,\prime}} R'_m \xrightarrow{\phi'_m} End(H'_m)$$

where c_m^A (resp. $c_m^{A,\prime}$) factors through A/J_m (resp. A'/J'_m), together with natural isomorphisms

$$r_m : \bar{R}_m \xrightarrow{\sim} \bar{R}'_m$$

satisfying

$$\bar{c}_m^{A,\prime} \circ a = \bar{c}_m^A$$

for all m, with the obvious notation. We also assume we are given isomorphisms

$$h_m : \bar{H}_m \xrightarrow{\sim} \bar{H}'_m$$

that are compatible with the previous maps in the obvious sense that

$$eh_m \circ \bar{\phi}_m = \bar{\phi}'_m \circ r_m$$

where $eh_m : End(\bar{H}_m) \xrightarrow{\sim} End(\bar{H}'_m)$ is the map induced from h_m. Finally, we assume that for all m,

$$\dim_k \mathfrak{m}_m/(\mathfrak{m}_\bullet + (\mathfrak{m}_m)^2) = \dim_k \mathfrak{m}'_m/(\mathfrak{m}'_\bullet + (\mathfrak{m}'_m)^2) = n, \tag{2.3.2}$$

where here $\mathfrak{m}'_m$ is the maximal ideal of R'_m.

Proposition 2.3.3. *Under the above hypotheses, we can find an infinite sequence of integers m_α such that there are surjective maps*

$$R_{m_\alpha} \to R_{m_{\alpha-1}};\ R'_{m_\alpha} \to R'_{m_{\alpha-1}};$$

$$H_{m_\alpha} \to H_{m_{\alpha-1}};\ H'_{m_\alpha} \to H'_{m_{\alpha-1}};$$

compatible with each other and with the isomorphisms r_{m_α}, h_{m_α} in characteristic ℓ. In this way, as m runs through the m_α, we obtain compatible families of diagrams

$$\begin{tikzcd} A \arrow[r, "c_m^A"] & R_m \arrow[r, "\phi_m"] \arrow[d, "\psi_m"'] & End(H_m) \arrow[d, "\pi_m"] \\ & R \arrow[r, "\phi"] & End(H) \end{tikzcd}$$

$$\begin{array}{ccccc}
A' & \xrightarrow{c_m^{A,\prime}} & R'_m & \xrightarrow{\phi'_m} & End(H'_m) \\
 & & \psi'_m \downarrow & & \downarrow \pi_m \\
 & & R' & \xrightarrow{\phi} & End(H')
\end{array}$$

satisfying

$$\bar{\psi}'_m \circ r_m = \bar{\psi}_m.$$

Finally, R_m and R'_m can be replaced by quotients B_m and B'_m of B and B', respectively, with the obvious notation.

Since $\mathcal{O}_\bullet$ and $\mathcal{O}'_\bullet$ are not local rings, the depth arguments above cannot be applied as such. Taylor's patching construction is more complicated than the one indicated here, since additional structures need to be respected, but the main point is that he considers a situation in which the depth argument does not apply to the R_m but does apply to the R'_m. This will be explained later.