ARRANGEMENTS OF DOUBLE PSEUDOLINES

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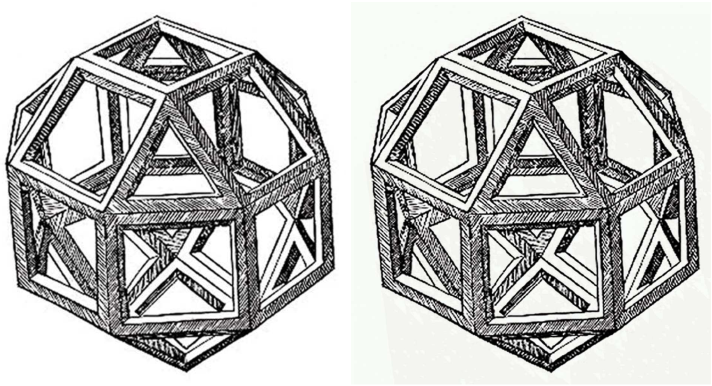


Figure 1: Première illustration d'un rhombicuboctaèdre en creux par Leonardo da Vinci pour la Divina proportione

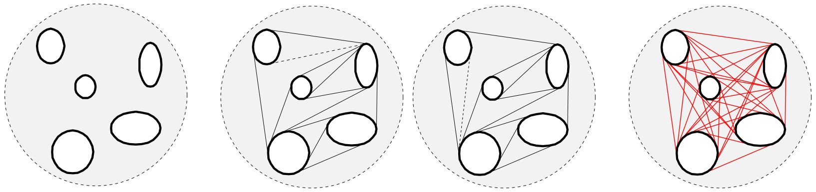
mainly joint work with Luc Habert, PhD student.

SUMMARY

We introduce a new class of arrangements of simple closed curves in the real projective plane defined by a certain condition on their subarrangements of size two. This new class, called the class of arrangements of double pseudolines, contains the well-studied class of arrangements of pseudolines and we extend several aspects of the classical theory of arrangements of pseudolines to the new class. Initially we were motivated in this work by the question of computing efficiently two-dimensional visibility graphs.

1. few words on the initial motivation
2. background material on arrangements of pseudolines and on geometric projective planes
3. definition of the class of arrangements of double pseudolines—examples et motivation
4. the representation theorem
5. the axiomatization theorem

INITIAL MOTIVATION



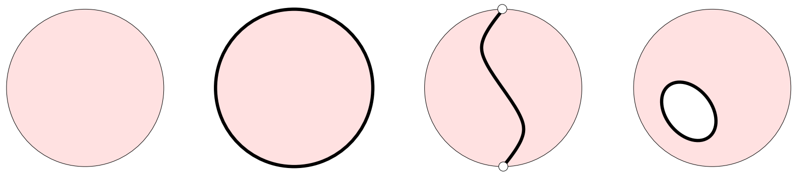
TH 1 (Habert & P. 04/06, Angelier & P. 03, P. & Vegter 96). *The convex hull, a (greedy) pseudotriangulation, the visibility graph of a finite planar family of pairwise disjoint convex bodies presented by its chirotope are each computable in time $O($size of the output $+$ size of the input $\times \log$ size of the input$)$ and linear space.* □

[1] L. Habert and M. Pocchiola. Computing pseudotriangulations and constrained pseudotriangulations via branched coverings. Submitted to Discrete Comput. Geom. (57 pages), January 2007.

[2] P. Angelier and M. Pocchiola. A sum of squares theorem for visibility complexes and applications. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, Discrete and Computational Geometry – The Goodman-Pollack Festschrift, Algorithms and Combinatorics, volume 25, pages 79–139. Springer-Verlag, June 2003.

[3] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. Discrete Comput. Geom., 16(4):419–453, December 1996.

PSEUDOLINES VERSUS DOUBLE PSEUDOLINES



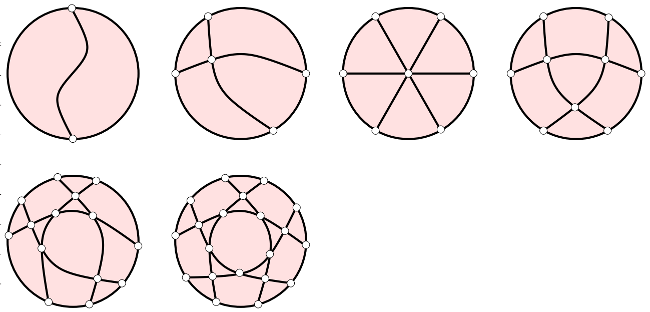
$$\mathbb{P}^2 = \{\mathbb{R}u \mid u \in \mathbb{R}^3, u \neq 0\} \approx \mathbb{S}^2/\{x \sim -x\} \approx \mathbb{D}^2/\{x \sim -x \mid x \in \partial\mathbb{D}^2\}$$

The real projective plane $\mathbb{P}^2$ - its representation by circular diagrams - classification up to homeomorphism of the simple closed curves embedded in the projective plane - (simple or ordinary) pseudolines - double pseudolines

ARRANGEMENTS OF PSEUDOLINES

Levi 1926, Ringel 1956, Grünbaum 1972, etc.

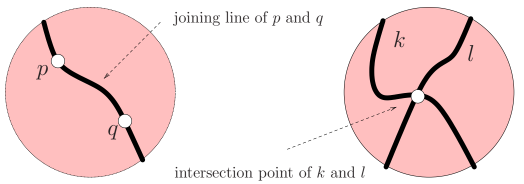
n	a_n
2	1
3	1
4	1
5	1
6	4
7	11
8	135
9	4 382
10	312 356
11	41 848 591



Definition - simple and non-simple arrangements - the number a_n of (isomorphism classes of) simple arrangements of n pseudolines is known up to $n = 11$ - according to the On-Line Encyclopedia of Integer Sequences (id: A006248) the value of a_{11} is due to F. Aurenhammer, 2002 - `http://www.research.att.com/~njas/sequences/A006248` -

GEOMETRIC PROJECTIVE PLANES

Hilbert 1899, Kolmogoroff 1932, Köthe 1939, Skornjakov 1954, Salzmann 1955, Freudenthal 1957

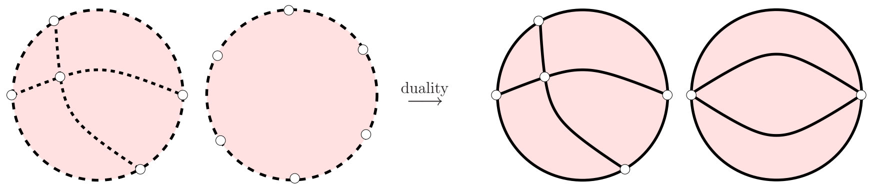


DF 1. *A <u>geometric projective plane</u> is a topological point-line incidence geometry $(\mathcal{P}, \mathcal{L})$ whose point space $\mathcal{P}$ is a projective plane and whose line space $\mathcal{L}$ is a subspace of the space of pseudolines of $\mathcal{P}$ such that*
(1) Any two distinct points are contained in exactly one line—called their joining line—which depends continuously on the two points;
(2) Any two distinct lines intersect in exactly one point—called their intersection point—which depends continuously on the two lines. □

Finite families of lines of geometric projective planes are examples of arrangements of pseudolines.

TH 2 (J-E. Goodman, R. Pollack, R. Wenger and T. Zamfirescu, 1994). *Any arrangement of pseudolines is isomorphic to a finite family of lines of a geometric projective plane.* □

DUALITY IN GEOMETRIC PROJECTIVE PLANES



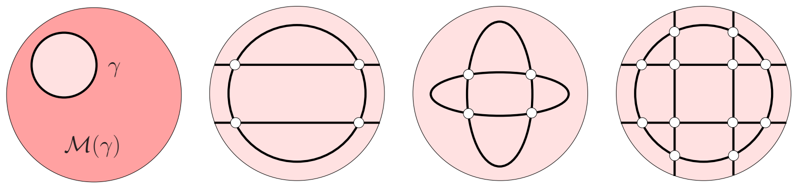
TH 3. *The line space of a geometric projective plane is a projective plane and the pencil of lines through a point of a geometric projective plane is a pseudoline of its line space.* □

The <u>dual</u> of a point of a geometric projective plane is the pencil of lines through that point.
<u>Dual</u> families of finite families of points of geometric projective planes are examples of arrangements of pseudolines.

TH 4 (Representation Theorem for PA). *Any arrangement of pseudolines is isomorphic to the dual family of a finite family of points of a geometric projective plane.* □

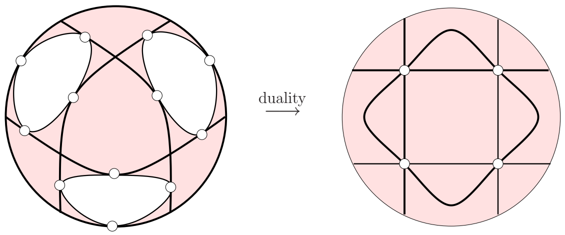
Proof. Rephrasing of the Goodman et al. enlargement theorem. □

DEFINITION OF ARRANGEMENTS OF DOUBLE PSEUDOLINES



DF 2. *Let $\mathcal{P}$ be a projective plane. An <u>arrangement of double pseudolines</u> in $\mathcal{P}$ is a finite family of at least two double pseudolines in $\mathcal{P}$, with the property that any two intersect in exactly four points—where they cross—and induce a cell structure on $\mathcal{P}$, that is, the connected components of the complement of the union of the double pseudolines are two-cells.* □

EXAMPLES

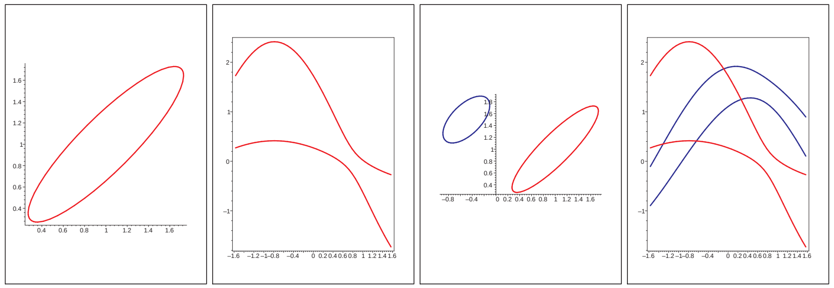


Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a geometric projective plane.

Define a convex body in $\mathcal{G}$ as a compact subset of $\mathcal{P}$ with nonempty interior whose intersection with any line is an interval of that line, and define its dual as its set of tangent lines. (A line is a said to be tangent to a convex body if the line intersects the body but avoids its interior.)

TH 5. *The dual family of a finite family of at least two pairwise disjoint convex bodies of a geometric projective plane is an arrangement of double pseudolines.* □

DUAL OF AN ELLIPSIS OF THE EUCLIDEAN PLANE

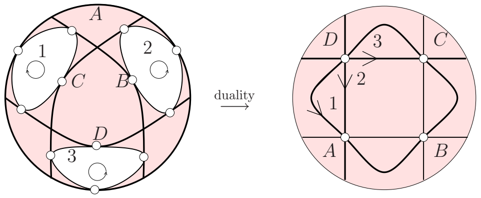


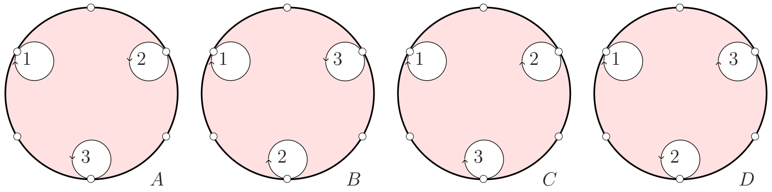
The space of lines of the Euclidean plane $\mathbb{R}^2$ is homeomorphic to the Möbius strip $\mathbb{M} = \mathbb{R}^2/\{(x,y) \sim (x+\pi, -y)\}$ via the map that assigns to the line with equation $-x\sin\theta + y\cos\theta - u = 0$ the pair (θ, u)

The dual of the point (x,y) of the Euclidean plane $\mathbb{R}^2$ is the range of the curve that assigns to $t \in [0,1]$ the point $(\pi t, -x\sin\pi t + y\cos\pi t) \in \mathbb{M}$.

The dual of the ellipse with center (x,y), half-axis a, b and angle φ of the Euclidean plane is the range of the curve that assigns to $t \in [0,1]$ the point $(2\pi t, -x\sin 2\pi t + y\cos 2\pi t - R(2\pi t - \varphi)) \in \mathbb{M}$ where $R(\theta) = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}$.

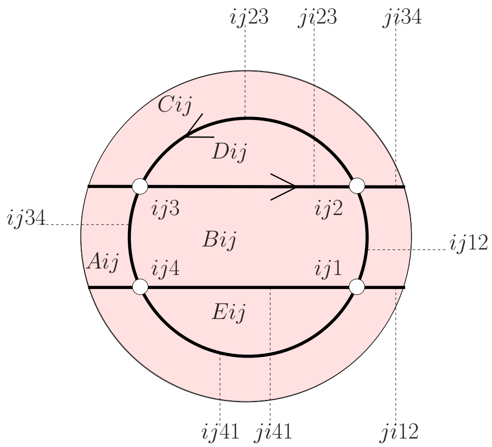
CONVEXITY IN GEOM. PROJ. PLANES - COCYCLES

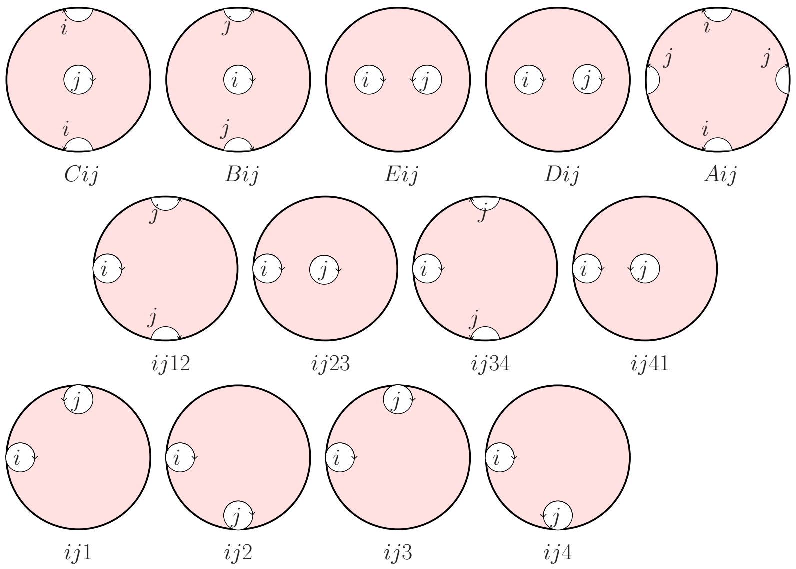


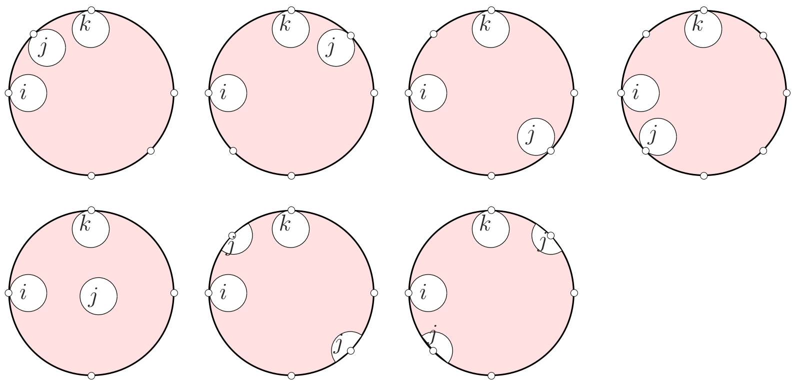


TH 6. *Two indexed families of disjoint oriented convex bodies of geometric projective planes have isomorphic dual arrangements if and only if they have the same family of cocycles.* □

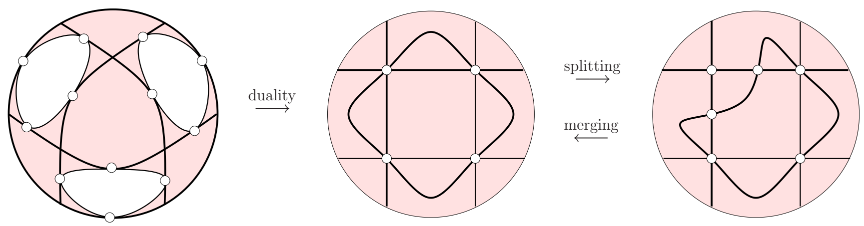
CO-CYCLE MAP







REPRESENTATION THEOREM



TH 7 (Representation Theorem). *Any arrangement of double pseudolines is isomorphic to the dual family of a finite family of disjoint convex bodies of a geometric projective plane.* □

Two proofs.

The first proof proceeds by mutation : we show that the property "to be the dual family of a family of disjoint convex bodies" is stable by mutation and that the graph of mutations is connected.

The second proof proceeds by reduction to the Representation Theorem for arrangements of pseudolines.

Both proofs are based on the Pumping Lemma.

AXIOMATIZATION THEOREM

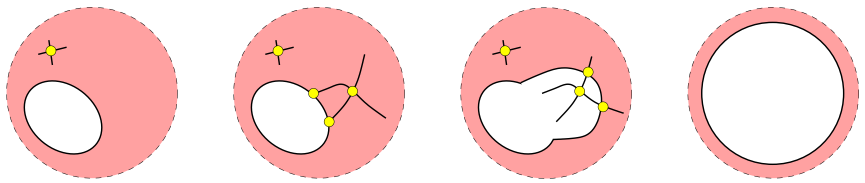
DF 3. *The chirotope of an indexed arrangement of oriented double pseudolines is the map that assigns to each subset of indices of size at most three the isomorphism class of the subarrangement indexed by this subset (note that two isomorphic arrangements have the same chirotope).* □

TH 8. *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one.* □

TH 9 (Axiomatization Theorem). *Let I be a finite set of indices, and let χ be a map that assigns to each triple of I the isomorphism class of an arrangement of three oriented double pseudolines indexed by this triple. Then the two following assertions are equivalent:*

1. *χ is the chirotope of an arrangement of oriented double pseudolines indexed by I;*
2. *for any subset J of I of size at most five the restriction of χ to the set of triples of J is the chirotope of an arrangement of oriented double pseudolines indexed by J.* □

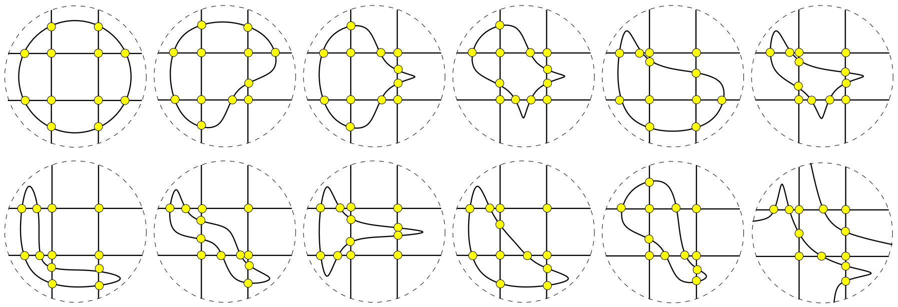
THE PUMPING LEMMA

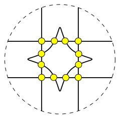


TH 10 (Pumping Lemma). *Let Γ be a simple arrang. of double pseudolines and let $\gamma \in \Gamma$. Assume that there is a vertex of Γ lying in the interior of the Möbius strip $\mathcal{M}(\gamma)$ bounded by γ. Then there is a triangular face of Γ included in $\mathcal{M}(\gamma)$ with a side supported by γ.* □

TH 11 (Homotopy Theorem). *Let $\mathcal{P}$ be a projective plane. Any two arrangements of double pseudolines in $\mathcal{P}$ of the same size are homotopic via a finite sequence of mutations followed by an isotopy.* □

SIMPLE ARRANG. THREE DOUBLE PSEUDOLINES





n	2	3	4	5
a_n	1	13	6570	181 403 533

[1] Julien Ferté, Vincent Pilaud, and Michel Pocchiola. On the number of simple arrangements of five double pseudolines. In Abstracts 18th fall Workshop Comput. Geom., 2008. `http://www.di.ens.fr/~pocchiol/pdffiles/fpp-nsafd-08.pdf`.

RESEARCH PROJECTS

1. Our axiomatic characterization theorem for the isomorphism classes of indexed arrangements of oriented double pseudolines provides a 'meaningless' system of a huge number of axioms—namely the full list of arrangements of five double pseudolines. Is it possible to reduce this system of axioms to a system of raisonnable size with a clear geometric or combinatorial interpretation of each axiom?
2. Design a quadratic time algorithm to compute the cell lattice of an arrangement of double pseudoline presented by its chirotope.
3. Arrangements of double pseudolines generalize in many aspects arrangements of pseudolines. From this perspective what are the similar generalizations for arrangements of pseudohyperplanes of dimensions 4, 5, etc.?