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PÓLYA'S THEOREM BY SCHNEIDER'S METHOD

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Dedicated to Professor Th. Schneider on his 65th birthday

A well known theorem of G. Pólya states that 2^z is the smallest transcendental entire function with integral values at all positive integral points z; more precisely, if f is an entire function satisfying $f(n)\in\mathbf{Z}$ for all $n\in\mathbf{N}$, and

$$\limsup_{R\to\infty}\frac{1}{R}\operatorname{Log}|f|_R < \operatorname{Log} 2, \tag{1}$$

(where $|f|_R=\sup_{|z|=R}|f(z)|$), then f is a polynomial.

We give here a new proof of this theorem, with a somewhat worse constant in place of Log 2, but which allows some further generalisations.

Notations. We denote by $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{C}$ the non-negative rational integers, the rational integers, the rational numbers and the complex numbers, respectively. When α is an algebraic number, we denote by $s(\alpha)=\max\{\operatorname{Log}|\overline{\alpha}|, \operatorname{Log} d(\alpha)\}$ the size of α (see for instance [3], § 1.2). For $R>0$, B_R is the set $\{z\in\mathbf{C};\ |z|\leqq R\}$. Finally, when $h\in\mathbf{N}$ and $z\in\mathbf{C}$, we define $\binom{z}{h}$ by

$$\binom{z}{h}=\frac{z(z-1)\dots(z-h+1)}{h!}.$$

We shall use only the trivial bounds

$$\left|\binom{z}{h}\right|\leqq 2^{H+R}\quad\text{and}\quad\left|\binom{z}{h}\right|\leqq e^H\left(\frac{R}{H}+1\right)^H$$

for $|z|\leqq R$ and $1\leqq h\leqq H$.

The main result of this paper is the following.

THEOREM 1. *Let K be a number field, and γ_0, γ_1 two positive real numbers. Then there exists an effectively computable number C, depending only on γ_0, γ_1 and $[K:\mathbf{Q}]$, with the following property:*

Let S be a subset of $\mathbf{Z}$, with $\operatorname{Card} S\cap B_R\geqq\gamma_0 R$ *for all sufficiently large R; let f, g be two entire functions satisfying*

$$g(n)\neq 0\quad\textit{and}\quad\frac{f(n)}{g(n)}\in K\quad\textit{for all}\quad n\in S,$$

such that for all sufficiently large R,

$$\max_{n\in S\cap B_R} \operatorname{Log}\left\{\frac{1}{|g(n)|};\ s\left(\frac{f(n)}{g(n)}\right)\right\} \leqq \gamma_1 R,$$

and

$$\max\{\operatorname{Log}|g|_R;\ \operatorname{Log}|f|_R\} \leqq R/C.$$

Then f/g is a rational function.

We obtain Pólya's theorem (with the constant Log 2 in (1) replaced by $1/C$) by setting

$$S = \mathbf{N};\ \gamma_0 = \gamma_1 = 1;\ g = 1;\ K = \mathbf{Q}.$$

(When $m\in\mathbf{Z}$, then $s(m)=\operatorname{Log}|m|$). A computation[1] of C by the present method leads to $C=283$, and it is an interesting problem to obtain by this way the best possible constant $\dfrac{1}{\operatorname{Log} 2}=1.44\ldots$.

Proof of Theorem 1. Let k_0 be an integer with $k_0>2\delta/\gamma_0$, where $\delta=[K:\mathbf{Q}]$, and let h_0 be a real number with $2\delta/k_0<h_0<\gamma_0$ (for instance $k_0=[2\delta/\gamma_0]+1$, $h_0=(\gamma_0/2)+(\delta/k_0)$). Let N be a sufficiently large integer; c_1, c_2, c_3 will denote positive constants which are effectively (and easily) computable in terms of γ_0, γ_1, δ (and h_0, k_0).

First step. We construct rational integers

$$a_{h,k}\quad (0\leqq h<h_0N;\ 0\leqq k\leqq k_0-1),$$

of absolute value less than $\exp(c_1N)$, *not all zero, such that the meromorphic function*

$$F(z)=\sum_{0\leqq h<h_0N}\ \sum_{0\leqq k<k_0} a_{h,k}\binom{z}{h}\left(\frac{f(z)}{g(z)}\right)^k$$

satisfies

$$F(n)=0 \quad \textit{for all} \quad n\in S\cap B_N.$$

We have to solve a system of at most $2N+1$ linear equations, with at least h_0k_0N unknowns, and with coefficients in K; for $n\in S\cap B_N$, the numbers

$$\binom{n}{h}\left(\frac{f(n)}{g(n)}\right)^k \quad (0\leqq h<h_0N;\ 0\leqq k<k_0)$$

have a common denominator bounded by γ_1k_0N, and a size bounded by $(h_0+1+k_0\gamma_1)N$. Hence Lemma 1.3.1 of [3] gives a non trivial solution $a_{h,k}$ with $\operatorname{Log}\max_{h,k}|a_{h,k}|<c_1N$.

Second step. For $m\in S$, either $F(m)=0$, or $\operatorname{Log}|F(m)|\geqq -c_2|m|$.

The denominator of $F(m)$ is bounded by $\gamma_1k_0|m|$, and the size of $F(m)$ is bounded by

$$\operatorname{Log}[k_0(h_0N+1)]+c_1N+(|m|+h_0N)\operatorname{Log}2+\gamma_1k_0|m|.$$

[1] Made by A. Escassut and M. Mignotte.

The basic inequality

$$-2\delta s(\alpha) \leqq \operatorname{Log}|\alpha| \quad \text{for all} \quad \alpha \in K,\ \alpha \neq 0$$

(see [3], (1.2.3)) for $|m| \geqq N$, and the first step for $|m| \leqq N$, give the result.

Third step: induction. Define $G(z) = (g(z))^{k_0} \cdot F(z)$. *Then, for all integers* $M \geqq N$,

$(\mathrm{I})_M$: $$F(m) = 0 \quad \text{for all} \quad m \in S \cap B_M,$$

and

$(\mathrm{II})_M$: $$\operatorname{Log}|G|_M < -c_3 M, \quad \text{with} \quad c_3 = k_0 \gamma_1 + c_2.$$

The first step proves $(\mathrm{I})_N$, and $(\mathrm{II})_M \Rightarrow (\mathrm{I})_M$ is a consequence of the second step and of the hypothesis on the lower bound for $|g(m)|$. The property "$(\mathrm{II})_M$ for all M" implies $F=0$, which means that f/g is an algebraic function (and consequently a rational function, because f/g is meromorphic in $\mathbf{C}$). Now, to conclude the proof of Theorem 1, it is sufficient to prove $(\mathrm{I})_M \Rightarrow (\mathrm{II})_{M+1}$.

Assume $(\mathrm{I})_M$ is true. Then, for $R > M$, we get from Schwarz lemma

$$\operatorname{Log}|G|_{M+1} \leqq \operatorname{Log}|G|_R - \gamma_0 M \operatorname{Log} \frac{R^2 + (M+1)^2}{2R(M+1)}.$$

(Cf. Lemma 6.2.1 of [3], where the inequality

$$\left| \frac{R_2^2 - z\bar{z}_j}{R_2(z - z_j)} \right| \geqq \frac{R_2^2 - R_1 \varrho}{R_2(R_1 + \varrho)}, \quad |z| = R_1, \quad |z_j| \leqq \varrho,$$

can be sharpened to[2]

$$\left| \frac{R_2^2 - z\bar{z}_j}{R_2(z - z_j)} \right| \geqq \frac{R_2^2 + R_1 \varrho}{R_2(R_1 + \varrho)}.\Big)$$

We bound $|G|_R$ for $R > M$:

$$\operatorname{Log}|G|_R \leqq \operatorname{Log}[(h_0 N + 1) k_0] + c_1 N + h_0 N \left[1 + \operatorname{Log}\left(\frac{R}{h_0 N} + 1 \right) \right] + 2k_0 \frac{R}{C}.$$

Choose $R = l_0(M+1)$, with l_0 sufficiently large, say

$$\frac{\gamma_0 - h_0}{3} \operatorname{Log} l_0 \geqq \max\{c_3;\ 2h_0 + c_1 + \gamma_0 - h_0 \operatorname{Log} h_0\}.$$

Then

$$\gamma_0 \operatorname{Log} \frac{R^2 + (M+1)^2}{2R(M+1)} \geqq \gamma_0 \operatorname{Log} \frac{l_0}{2},$$

and we obtain

$$\operatorname{Log}|G|_{M+1} < -\left\{ \frac{2}{3} (\gamma_0 - h_0) \operatorname{Log} l_0 - \frac{2k_0 l_0}{C} \right\} M,$$

which is $< -c_3 M$ when C is sufficiently large. This proves Theorem 1.

[2] This was pointed out to me by J. Dufresnoy and H. L. Montgomery.

It would be interesting to generalize Theorem 1 to more general sets S, for example to $S\subset\mathbf{Z}[i]$; the corresponding generalisation of Pólya's theorem is due to FUKASAWA and GEL'FOND [2]: if f is an entire function satisfying $f(a+ib)\in\mathbf{Z}[i]$, when $a+ib\in\mathbf{Z}[i]$, and

$$\limsup_{R\to\infty}\frac{1}{R^2}\operatorname{Log}|f|_R<\frac{\pi}{2(1+e^{164/\pi})^2},\tag{2}$$

then f is a polynomial.

With the present method, we can deal only with a stronger hypothesis (where $1/R^2$ is replaced by $(\operatorname{Log}R)/R^2$ in (2)), because we do not know interpolation polynomials in $\mathbf{Z}[i]$ generalizing the polynomials $\binom{z}{n}$ in $\mathbf{Z}$; this problem[3] is connected with those of the measure of irrationality (or transcendence) of e^π, and of the algebraic independence of π and e^π.

On the other hand, we can consider more general sets of algebraic numbers. Using the method of proof of Theorem 1, we get:

THEOREM 2. *Let K be a number field, γ_1,γ_2 positive real numbers, and $\Psi:[0,+\infty)\to[0,+\infty)$ a positive real valued function satisfying* $\limsup_{R\to\infty}\frac{\Psi(\lambda R)}{\Psi(R)}<\infty$ *for all $\lambda\geqq 1$. Then there exists a constant $C>0$ with the following property: Let S be a subset of K with*

$$\operatorname{Card}S\cap B_R\geqq\Psi(R)$$

and

$$\max_{\alpha\in S\cap B_R}s(\alpha)\leqq\gamma_2\operatorname{Log}R$$

for all sufficiently large R. Let f,g be two entire functions, satisfying

$$g(\alpha)\neq 0\quad and\quad \frac{f(\alpha)}{g(\alpha)}\in K\quad for\quad \alpha\in S,$$

such that for all sufficiently large R,

$$\max_{\alpha\in S\cap B_R}\left\{\operatorname{Log}\frac{1}{|g(\alpha)|};\ s\left(\frac{f(\alpha)}{g(\alpha)}\right)\right\}\leqq\gamma_1\cdot\frac{\Psi(R)}{\operatorname{Log}R},$$

and

$$\max\{\operatorname{Log}|f|_R;\ \operatorname{Log}|g|_R\}\leqq\frac{\Psi(R)}{C\operatorname{Log}R}.$$

Then f/g is a rational function.

We obtain as a corollary Gel'fond Schneider's theorem on the transcendence of a^b (choose: $f(z)=a^z$; $g(z)=1$; $\Psi(R)=R^2$; $S\subset\{h+kb,\ h,k\in\mathbf{Z}\}$.)

The proof of Theorem 2 is essentially the same as that of Theorem 1; first we assume that the function $R\mapsto\Psi(R)/\operatorname{Log}R$ is non decreasing (otherwise we replace $\Psi(R)$ by $(\operatorname{Log}R)\cdot\inf_{R'\geqq R}\Psi(R')/\operatorname{Log}R'$); then we replace the polynomials

[3] Concerning this problem, see a forthcoming paper by Douglas Hensley: "Polynomials with Gaussian integer values at Gaussian integers."

$\binom{z}{h}$ by z^h in the preceeding proof, and the parameters h_0N, k_0 by $[h_0\Psi(N)/\mathrm{Log}\,N]$, $[k_0\,\mathrm{Log}\,N]$, respectively.

Finally, we mention two possible generalisations of Theorems 1 and 2. Firstly Pólya's theorem has been generalized to functions of several variables by A. BAKER [1]; using the interpolation formulas in [1], it is easy to derive the corresponding generalisation of the present paper. Secondly, it is possible to replace the number field K by the field of algebraic numbers, provided that we assume a growth condition on the function

$$R \mapsto \max_{\alpha \in S \cap B_R} [\mathbf{Q}(\alpha, f(\alpha)\colon \mathbf{Q}],$$

(see [3], Exercise 2.2.f).

References

[1] A. BAKER, A note on integral integer-valued functions of several variables, *Proc. Camb. Phil. Soc.*, **63** (1967), 715—720.
[2]A. O. GEL'FOND, Sur les propriétés arithmétiques des fonctions entières, *Tohoku Math. J.*, **30** (1929), 280—285.
[3] M. WALDSCHMIDT, *Nombres transcendants;* Lecture Notes in Math., Springer Verlag. 402 (1974).

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