Quantization of algebraic cones and Vogan's conjecture

Michèle Vergne

Introduction

Let C be a complex algebraic cone, provided with an action of a compact Lie group K. The symplectic form of the ambient complex Hermitian space induces on the regular part of C a symplectic form. Let $\mathfrak{k}$ be the Lie algebra of K. Let $f : C \to \mathfrak{k}^*$ be the Mumford moment map, that is $f(v)(X) = i(v, Xv)$, for $X \in \mathfrak{k}$ and $v \in C$. The space $R(C)$ of regular functions on C is a semi-simple representation of K. In this article, with the help of the moment map, we give some quantitative informations on the decomposition of $R(C)$ in irreducible representations of K. For λ a dominant weight, let $m(\lambda)$ be the multiplicity of the representation of highest weight λ in $R(C)$. Then, if the moment map $f : C \to \mathfrak{k}^*$ is proper, multiplicities $m(\lambda)$ are finite and with polynomial growth in λ. Furthermore, the study of the pushforward by f of the Liouville measure gives us an asymptotic information on the function $m(\lambda)$. For example, in the case of a faithful torus action, the pushforward of the Liouville measure by the moment map is a locally polynomial homogeneous function $\ell(\lambda)$ on the polyhedral cone $f(C) \subset \mathfrak{t}^*$, while the multiplicity function $m(\lambda)$ for large values of λ is given by the restriction to the lattice of weights of a quasipolynomial function, with highest degree term equal to $\ell(\lambda)$. If O is a nilpotent orbit of the coadjoint representation of a complex Lie group G, we show that the pushforward on $\mathfrak{k}^*$ of the G-invariant measure on O is the same that the pushforward of the Liouville measure on O associated to the symplectic form of the ambient complex vector space. Thus, this establishes for the case of complex reductive groups the relation, conjectured by D. Vogan, between the Fourier transform of the orbit O with multiplicities of the ring of regular functions on O.

Consider a complex vector space V of complex dimension ℓ with an action of a compact Lie group K. Let $\mathfrak{k}$ be the Lie algebra of K. Let C be a closed irreducible complex algebraic cone contained in V of complex dimension d and invariant under the action of K. The ring $R(C)$ of regular functions on C is a semi-simple representation of K.

Consider on the complex vector space V a K-invariant Hermitian form h. This determines a symplectic 2-form $\Omega(dv, dv) = -2\Im h(dv, dv)$ on V and a moment map $f : V \to \mathfrak{k}^*$ given by $f(v)(X) = ih(v, Xv)$ for $v \in V$ and $X \in \mathfrak{k}$. By restriction, the symplectic 2-form Ω determines a symplectic form on the regular part of C. If the restriction to C of the moment map $f : C \to \mathfrak{k}^*$ is proper, then the representation of K in $R(C)$ is trace class. According to the general philosophy of quantization, the quantized space $\mathcal{Q}(C)$ associated to the symplectic space C "is" the space of holomorphic functions on C. We should then obtain a character formula for the representation of K in $R(C)$ in terms of the symplectic data. As C is not smooth, such a formula is not easy to state and we will only obtain an asymptotic estimate. (In case $C - \{0\}$ is smooth, this estimate would indeed follow from the equivariant Riemann-Roch formula as stated in [2]). Let $d\beta_C$ be the Liouville measure on C and let $f_*(d\beta_C)$ be the Radon measure on $\mathfrak{k}^*$ which is the pushforward of $d\beta_C$ by the moment map. We prove the following theorem.

Theorem 1 *Assume that the restriction to C of the moment map $f : C \to \mathfrak{k}^*$ is proper. Then $R(C)$ is a trace class representation of K. The limit when t tend to 0 of the generalized functions $t^d \operatorname{Tr}(R(C))(\exp tX)$ of $X \in \mathfrak{k}$ exists. Furthermore, we have the equality of generalized functions on $\mathfrak{k}$:*

$$\lim_{t\to 0} t^d \operatorname{Tr}(R(C))(\exp tX) = \int_C e^{-i(f(v),X)} d\beta_C(v) = \int_{\mathfrak{k}^*} e^{-i(\xi,X)} f_*(d\beta_C)(\xi).$$

Let us indicate the proof. We denote by $\Omega(X)$ the equivariant symplectic form of V. The above integral can be written $(-2i\pi)^{-d} \int_C e^{-i\Omega(X)}$. It is not difficult to prove that $\det_V(X) \operatorname{Tr} R(C)(\exp X)$ is an analytic function on $\mathfrak{k}$. Furthermore, if the action of K in V is such that $\det_V(X) \neq 0$, then $\lim_{t\to 0} t^d \det_V(X) \operatorname{Tr} R(C)(\exp tX)$ is (up to multiplication by a constant) equal to the Joseph polynomial $J(C)(X)$ defined in [4]. We then use Rossmann's integral formula ([8], see also [11]) for $J(C)(X)$ in function of the equivariant Thom class of V. Thus the main technical tool is Lemma 14 which compares on a Hermitian space V the equivariant closed form $e^{-i\Omega(X)}$ and the equivariant Thom form.

Let G be a real semi-simple connected Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G. Let K be a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We consider a nilpotent orbit O of G in $\mathfrak{g}^*$ of dimension $2d$. Consider the Kirillov symplectic form σ_O on O. The corresponding Liouville measure $\beta_O = (2\pi)^{-d}\frac{\sigma_O^d}{d!}$ is a *tempered* measure on O and we can then define the G-invariant generalized function F_O of $X \in \mathfrak{g}$ by

$$F_O(X) = \int_O e^{-i(f,X)} d\beta_O(f).$$

This generalized function F_O has a restriction to $\mathfrak{k}$.

On the other hand, consider the nilpotent orbit $c(O)$ of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$ associated to O by Kostant-Sekiguchi correspondence. Then the closure $C(O)$ of $c(O)$ is a closed K-invariant complex cone in $\mathfrak{p}_{\mathbb{C}}$ of dimension d. The representation of K in the ring $R(C(O))$ of regular functions on $C(O)$ is a trace class representation of K.

Vogan conjectured the following equality of generalized functions on $\mathfrak{k}$. For $X \in \mathfrak{k}$:

$$F_O(X) = \lim_{t \mapsto 0} t^d \operatorname{Tr} R(C(O))(\exp tX).$$

When G is complex, the closure $\overline{O}$ of the orbit O is isomorphic to the complex cone $C(O)$. Thus, using a simple deformation argument between σ_O and the symplectic form on O induced from a K-invariant Hermitian form on $\mathfrak{g}$, we show that Vogan's conjecture follows from Theorem 1.

Theorem 2 *Vogan's conjecture holds if G is a complex semi-simple Lie group.*

Recall ([12]) that, when G is any real reductive group, there is a K-invariant diffeomorphism from O to $c(O)$. Thus Vogan's conjecture would follow immediately from Theorem 1 if a K-invariant *symplectic* diffeomorphism between (O, σ_O) and $c(O)$ equipped with the symplectic structure induced from the Hermitian structure of $\mathfrak{p}_{\mathbb{C}}$ would exist. Such a symplectic diffeomorphism is easily seen to exist in the case of minimal orbits. Thus Vogan's conjecture holds also for minimal orbits. This was already obtained by D. King [6] by direct calculations of both terms of Vogan's conjectural equality. We do not know if such a symplectic diffeomorphism exists in general. In fact we would only need a reasonable homotopy between these two symplectic structures to prove Vogan's conjecture. Several cases of Vogan's conjecture have been proved by D. King in [5], [6]. We hope our presentation shows that Vogan's conjecture is very natural.

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1 The moment map for Hermitian vector spaces

Let (V, B) be a finite dimensional symplectic vector space of dimension 2ℓ. Consider the 1-form $\omega = \frac{1}{2}B(v, dv)$ on V and the 2-form $\Omega = d\omega = \frac{1}{2}B(dv, dv)$. If p_j, q_j are symplectic coordinates on V, then $\omega = \frac{1}{2}\sum_{j=1}^{\ell}(p_j dq_j - q_j dp_j)$ and $\Omega = \sum_{j=1}^{\ell} dp_j \wedge dq_j$.

Let $\mathrm{Sp}(V)$ be the group of symplectic transformations of V. We denote by $\mathfrak{s}$ the Lie algebra of $\mathrm{Sp}(V)$. The action of $\mathrm{Sp}(V)$ on V is Hamiltonian. The moment map $f : V \to \mathfrak{s}^*$ is given , for $v \in V$, by $f(v)(X) = \frac{1}{2}B(v, Xv)$ for $X \in \mathfrak{s}$. Remark that $f : V \to \mathfrak{s}^*$ is homogeneous of degree 2. If X_V is the vector field on V tangent at v to the curve $(\exp -\epsilon X)v$, the moment map f satisfies the equation $(d_\xi f, X) = \Omega(X_V, \xi)$ for every tangent vector ξ.

For $X \in \mathfrak{s}$, we denote by $\mu(X)$ the function on V given by $\mu(X)(v) = (f(v), X) = \frac{1}{2}B(v, Xv)$. Let

$$\Omega(X) = \mu(X) + \Omega \tag{1}$$

be the equivariant symplectic form of V. It is an exact equivariant form on V. We have

$$\Omega(X) = d_X\omega. \tag{2}$$

Consider a complex structure J on V compatible with B, that is such that $B(Jv, Jw) = B(v, w)$ for all $v, w \in V$ and $B(v, Jv) > 0$ for $v \neq 0$. Let $Q(v, w) = B(v, Jw)$. This is a Euclidean scalar product on V. The form $h = \frac{1}{2}(Q - iB)$ is a Hermitian form on the complex vector space (V, J). The unitary group $U(V)$ is a maximal compact subgroup of $\mathrm{Sp}(V)$. We denote by $\mathfrak{u}$ the Lie algebra of $U(V)$. Reciprocally, if (V, h) is a Hermitian space of complex dimension ℓ, we define on V the symplectic form

$$B(v, w) = -2\mathrm{Im}\, h(v, w).$$

The moment map $f : V \to \mathfrak{u}^*$ is given by

$$f(v)(X) = ih(v, Xv)$$

for any skew hermitian transformation X of V (as X is antihermitian, the number $h(v, Xv)$ is purely imaginary).

Let G be a complex connected reductive Lie group acting on a complex vector space V. We choose a maximal compact subgroup K of G and a Hermitian K-invariant inner product $\|v\|^2$ on V. Thus K is a subgroup of $U(V)$. Let $\mathfrak{k} \subset \mathfrak{u}$ be the Lie algebra of K. The group K is a real form of G and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. The moment map $f_K : V \to \mathfrak{k}^*$ for the action of K on V is simply the composition of the map $f : V \to \mathfrak{u}^*$ with the natural restriction $\mathfrak{u}^* \to \mathfrak{k}^*$. If K is understood, we again denote this map by f.

We denote by V^* the dual vector space of V and by $S(V^*)$ the symmetric algebra of V^*. We identify $S(V^*)$ with the ring of polynomial functions on V. We write

$$S(V^*) = \oplus_{n=0}^{\infty} S_n(V^*)$$

where $S_n(V^*)$ is the space of homogeneous polynomials of degree n.

Let C be a complex algebraic irreducible closed cone in V stable by G. We denote by $R(C)$ the ring of regular functions on C. An element of $R(C)$ is the restriction to C of a polynomial function on V, thus we will also call an element of $R(C)$ a polynomial function on C. We write

$$R(C) = \oplus_{n=0}^{\infty} R_n(C).$$

We denote by $R(C)^G$ the ring of G-invariant polynomial functions on C.

The following lemma due to Mumford is a straightforward consequence from the fact that the geometric quotient $C//G$ constructed from the graded algebra $R(C)^G$ can be realized as the reduced symplectic manifold $f^{-1}(0)/K$. For completeness, we give a proof of this corollary.

Lemma 3 *(Mumford) The two following conditions are equivalent:*
1) We have $f^{-1}(0) \cap C = \{0\}$.
2) We have $R(C)^G = \mathbb{C}$.

Proof. We show first that, when $f^{-1}(0) \cap C = \{0\}$, invariant polynomial functions on V take constant values on C. Let P be a G-invariant polynomial and let v be a point of C. Consider the level set L of P passing through v, that is

$$L = \{m \in C; P(m) = P(v)\}.$$

Then L is a closed subset of V and is G-invariant. Take a point $v_0 \in L$ at minimum distance from the origin. Thus the restriction of the function $\|v\|^2$

to the G-orbit of v_0 has a minimum at v_0. Let $Y \in \mathfrak{k}$. We have necessarily

$$\frac{d}{d\epsilon}\| \exp i\epsilon Y \cdot v_0\|^2|_{\epsilon=0} = 0$$

for all $Y \in \mathfrak{k}$. This gives us the relation $2i < Y \cdot v_0, v_0 >= 0$ for all $Y \in \mathfrak{k}$, that is $f(v_0) = 0$. Our condition $f^{-1}(0) \cap C = \{0\}$ implies that $v_0 = 0$. As v_0 is in the same level set than v, we obtain $P(v) = P(v_0) = P(0)$. Thus P is constant on C.

Let us prove the converse. We assume that $R(C)^G = \mathbb{C}$ and we will deduce that $f^{-1}(0) \cap C = \{0\}$. Let $x \in C$ be such that $f(x) = 0$. Let us show that the distance of 0 to the G-orbit of x is $\|x\|$. Let $G = KP$ with $P = \exp i\mathfrak{k}$. By K-invariance of the Hermitian norm, it is sufficient to prove that for all $Y \in \mathfrak{k}$, then $\|(\exp iY) \cdot x\|^2 \geq \|x\|^2$. Consider the function on the real line $\mathbb{R}$ given by $g(t) = \|(\exp itY) \cdot x\|^2$. As $f(x)(Y) = 0$, the function $g'(t)$ vanishes at 0. We have $g''(0) = 2\|Yx\|^2 \geq 0$. Thus $g(0) = \|x\|^2$ is a local minimum for the function $g(t)$. The same calculation shows that if $g'(a) = 0$ at another value $a \in \mathbb{R}$, then $g''(a) \geq 0$. It follows that any critical point of g is a local minima. This obviously implies that 0 is the unique critical point and $\|x\|^2$ is the absolute minimum of the function $g(t)$. The orbit $G \cdot x$ is thus at distance $\|x\|$ of 0. If x was not equal to zero, this would imply that 0 does not belong to the closure $\overline{G \cdot x}$ of $G \cdot x$. The Zariski closure of $G \cdot x$ coincides with its topological closure. Thus we could find a polynomial $P \in R(C)$ such that $P = 1$ on $G \cdot x$ and equal to 0 on 0. Averaging P under K, we could find a G-invariant polynomial equal to 1 on $G \cdot x$ and 0 at 0. Thus $R(C)^G$ would not be reduced to the constant functions. ∎

2 Joseph polynomials

2.1 Definition of Joseph polynomial

Let T be a torus and let $\mathfrak{t}$ be the Lie algebra of T. We denote by $P \subset i\mathfrak{t}^*$ the set of weights of T. If $\mu \in P$, we denote by $e_\mu \in \hat{T}$ the character of T such that $e_\mu(\exp X) = e^{(\mu,X)}$ for $X \in \mathfrak{t}$. We denote by $R(T)$ the set of virtual characters on T. We have

$$R(T) = \{\sum_{\mu \in I} a_\mu e_\mu\}$$

where I is a finite subset of P and $a_\mu \in \mathbb{Z}$. We consider the space $R(T)^{-\infty}$ of trace class virtual representations of T. We have

$$R(T)^{-\infty} = \{\sum_{\mu\in P} a_\mu e_\mu\}$$

where a_μ is of at most polynomial growth. Then $R(T)^{-\infty}$ is a module over $R(T)$.

For $\xi \in P$, we denote by Θ_ξ the element of $R(T)^{-\infty}$ given by

$$\Theta_\xi = \sum_{n=0}^{\infty} e_{n\xi}.$$

Remark that

$$(1 - e_\xi)\Theta_\xi = 1. \tag{3}$$

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$ be a set of weights. We say that S is contained in a half space if there exists $X_0 \in \mathfrak{t}$ such that $(\alpha_j, iX_0) > 0$ for all $\alpha_j \in S$. In this case we can multiply the series Θ_{α_k} and still obtain an element of $R(T)^{-\infty}$.

Definition 4 *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$ be a set of weights contained in a half space. Define*

$$\Theta(S) = \Theta_{\alpha_1}\Theta_{\alpha_2}\cdots\Theta_{\alpha_\ell}.$$

Let $\mathbb{Z}_+ = \{0, 1, 2, 3....\}$. We thus have

$$\Theta(S)(\exp X) = \sum_{(n_j)\in\mathbb{Z}_+^\ell} e^{(\sum n_j\alpha_j, X)}.$$

Let M be a semi-simple T-module. We write

$$M = \oplus_{\mu\in P} M_\mu$$

where $M_\mu = \{m \in M, t\cdot m = e_\mu(t)m\}$. If $\dim M_\mu$ is finite for all μ, we can associate to M its formal character

$$\mathrm{ch}(M) = \sum_\mu (\dim M_\mu) e_\mu.$$

If $\dim M_\mu$ is of at most polynomial growth, then we associate to M its character $\mathrm{Tr}(M)$ which is a generalized function on T.

Let V be a complex vector space where T acts. We write ℓ for the complex dimension of V. We write Δ^+ for the set of weights of T in V. We count then with multiplicities: $|\Delta^+| = \ell$. We assume that the set Δ^+ of weights of the action of T on V is contained in some half-space. Consider the space $S(V^*)$ of polynomial functions on V. Then $S(V^*)$ is a trace class representation of T. We have

$$\mathrm{Tr}(S(V^*)) = \prod_{\alpha\in\Delta^+} \Theta_{-\alpha} = \Theta(-\Delta^+).$$

Let us consider more generally a $(T, S(V^*))$-module M. This means that the module M is a semi-simple T-module and that we have compatible actions of T and $S(V^*)$ on M:

$$t \cdot (Pm) = (t \cdot P)t \cdot m.$$

If M is finitely generated under $S(V^*)$, we say that M is an admissible $(T, S(V^*))$-module.

Lemma 5 *Let M be an admissible $(T, S(V^*))$-module. Then M is a trace class representation of T and there exists an element $U \in R(T)$ such that*

$$\mathrm{Tr}(M) = U \,\mathrm{Tr}\, S(V^*) = U\Theta(-\Delta^+)$$

Proof. Indeed we can choose a resolution of M by free $S(V^*)$-modules. ∎

Let M be a $(T, S(V^*))$-admissible module. Using Equation 3, we have

$$\prod_{\alpha\in\Delta^+} (1 - e_{-\alpha}) \,\mathrm{Tr}(M) = U = \sum_a c_a e_{\mu_a}$$

with $c_a \in \mathbb{Z}$ and μ_a a finite number of weights of T. Consider the analytic function

$$X \mapsto \prod_{\alpha\in\Delta^+} (1 - e^{-(\alpha,X)}) \,\mathrm{Tr}(M)(\exp X)$$

on $\mathfrak{t}$ and its Taylor series

$$\sum_{a,n} c_a \frac{\mu_a^n}{n!}$$

at the origin. Let r be the smallest integer n such that $\sum_a c_a \frac{\mu_a^n}{n!}$ is non zero. By definition Joseph's polynomial $J(M)$ is the homogeneous polynomial of degree r given by

$$J(M) = i^r \sum_a c_a \frac{\mu_a^r}{r!}.$$

Let C be a T-invariant complex closed cone in V of dimension d. Then $R(C)$ is a $(T, S(V^*))$- admissible module. We write $J(C)$ for $J(R(C))$. It is easy to prove that $J(C)$ is a polynomial of degree $\ell - d$. Let us recall Rossmann's integral formula ([8]) for $J(C)$. We choose on V a T-invariant Hermitian inner product. Then V is an oriented Euclidean space and we can construct an equivariant Thom class $[a_V]$ of V. By definition, a representative a_V of $[a_V]$ is a $SO(V)$-equivariant differential form on V compactly supported on V and such that $\int_V a_V(X) = 1$ for all $X \in \mathfrak{so}(V)$. The total equivariant degree of a_V is equal to 2ℓ. Thus $a_V(X) = \sum_a P_a(X)\omega_a$ where $P_a(X)$ are homogeneous polynomials of degree p_a and ω_a are differential forms with compact support on V of exterior degree $2\ell - 2p_a$. Let C' be the regular part of C. This is a smooth manifold and by Lelong integrability's theorem of algebraic cycles, $\int_{C'} \omega_a$ is convergent. We write this integral simply as $\int_C \omega_a$. We always choose as orientation on the regular part of C the orientation given by the complex structure. We can thus define $\int_C a_V(X) = \sum_a P_a(X) \int_C \omega_a$. If C is of pure complex dimension d, this is a homogeneous polynomial of degree $\ell - d$. Furthermore, this polynomial is independent of the choice of the representative a_V of the Thom class. The following proposition is proved in [11]. It follows easily from Rossmann's formula for $J(C)$.

Proposition 6 *Let C be a cone of pure dimension d. We have*

$$J(C)(X) = (-1)^d (2\pi)^{\ell - d} \int_C a_V(X).$$

2.2 Asymptotic formulas for multiplicities. The case of a Torus.

Let T be a torus. We introduce some generalized functions on T and $\mathfrak{t}$.

Let ξ be a non zero element of $\mathfrak{t}^*$ and let θ_ξ be the generalized function on $\mathfrak{t}$ given by

$$\theta_\xi(X) = \int_0^\infty e^{i(t\xi, X)} dt.$$

If $S = \{\xi_1, \xi_2, ..., \xi_\ell\}$ is a finite subset of elements of $\mathfrak{t}^*$ contained in a half-space, the product of the generalized functions θ_{ξ_k} is well defined.

Definition 7 *Let $S = \{\xi_1, \xi_2, ..., \xi_\ell\}$ be a finite subset of elements of $\mathfrak{t}^*$ contained in a half-space. Define*

$$\theta(S) = \theta_{\xi_1}\theta_{\xi_2}\cdots\theta_{\xi_\ell}.$$

We thus have

$$\theta(S)(X) = \int_{\mathbb{R}^\ell_+} e^{i(\sum y_j\xi_j, X)} dy_1 dy_2 \cdots dy_\ell.$$

The function $\theta(S)$ is homogeneous of degree $-\ell$: if t is a positive real number, $\theta(S)(tX) = t^{-\ell}\theta(S)(X)$ for $X \in \mathfrak{t}$. If $X_0 \in i\mathfrak{t}$ is such that $i(\xi, X_0) < 0$ for all $\xi \in S$, then we have in the space of generalized functions on $\mathfrak{t}$ the equality: $\theta(S)(X) = \lim_{\epsilon\to 0^+} \prod_{\xi\in S} i(\xi, X + \epsilon X_0)^{-1}$.

Assume $\Delta^+ = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$ is a subset of weights of T contained in a half space. We can then define the generalized function $\Theta(\Delta^+)$ as well as the generalized function $\theta(-i\Delta^+)$ on $\mathfrak{t}$. Let $t > 0$. We can consider the generalized function $X \mapsto \Theta(\Delta^+)(tX)$.

Lemma 8 *Let Δ^+ be a set of weights of T contained in a half space with cardinal ℓ. Let t be a real positive number. Then, in the space of generalized functions on $\mathfrak{t}$, we have the equality*

$$\lim_{t\to 0} t^\ell \Theta(\Delta^+)(\exp tX) = \theta(-i\Delta^+)(X).$$

Proof. Let

$$j_{\Delta^+}(X) = \prod_{\alpha\in\Delta^+} \frac{1 - e^{-\alpha(X)}}{\alpha(X)}.$$

Then $j_{\Delta^+}(X)$ is analytic and we have $\Theta(\Delta^+)(X)j_{\Delta^+}(X) = \theta(-i\Delta^+)(X)$. The function $\theta(-i\Delta^+)$ is homogeneous of degree $-\ell$, while $j_{\Delta^+}(tX)$ tends to 1 when t tends to 0. Thus

$$\lim_{t\to 0} t^\ell \Theta(\Delta^+)(\exp tX) = \lim_{t\to 0} t^\ell \Theta(\Delta^+)(\exp tX) j_{\Delta^+}(tX).$$

But we have

$$t^\ell \Theta(\Delta^+)(\exp tX) j_{\Delta^+}(tX) = t^\ell \theta(-i\Delta^+)(tX) = \theta(-i\Delta^+)(X).$$

∎
The same proof shows that the map $t \to \Theta(\Delta^+)(\exp tX)$ has an asymptotic expansion when t tends to 0 in the space of generalized functions on $\mathfrak{t}$.

Let T be acting on a complex vector space V of dimension ℓ.

Proposition 9 *Let M be an admissible $(T, S(V^*))$-module. Let r be the degree of Joseph polynomial $J(M)$. Then*

$$\Theta(M)(X) = \lim_{t \mapsto 0} t^{\ell - r} \operatorname{Tr}(M)(\exp tX)$$

exists in the space of generalized functions on $\mathfrak{t}$ and we have

$$\Theta(M)(X) = i^{-r} J(M)(X) \theta(i\Delta^+)(X).$$

Proof. We use the formula $\operatorname{Tr}(M)(\exp X) = U(\exp X) \prod_{\alpha \in \Delta^+} \Theta_{-\alpha}(\exp X)$ of Lemma 5. It is clear that, if r is the degree of Joseph polynomial, $t^{-r} U(\exp tX)$ has a limit when t tends to 0 equal to $i^{-r} J(M)(X)$. Thus $t^{\ell - r} U(\exp tX) \prod_{\alpha \in \Delta^+} \Theta_{-\alpha}(\exp tX)$ has a limit when t tends to 0 given by the formula above. ∎

Let C be a T-invariant complex cone in V of complex dimension d. The degree of the Joseph polynomial $J(C)$ is equal to $\ell - d$. We write $\Theta(C)$ for $\Theta(R(C))$. Thus we obtain the following proposition.

Proposition 10 *Let C be a T-invariant complex cone in V of complex dimension d. Then*

$$\Theta(C)(X) = \lim_{t \to 0} t^d \operatorname{Tr}(R(C))(\exp tX) = i^{-(\ell - d)} \theta(i\Delta^+)(X) J(C)(X).$$

Using the integral formula for $J(C)$, we may rewrite this as

$$\Theta(C)(X) = (-1)^d (-2i\pi)^{(\ell - d)} \theta(i\Delta^+)(X) \int_C a_V(X).$$

It will be more elegant to rewrite this formula in terms of the equivariant symplectic form of V . We will do this in Proposition 16 in the next section.

3 Asymptotic formulas for multiplicities

Consider a complex vector space V of complex dimension ℓ with an action of a reductive complex Lie group G. Let C be a closed complex algebraic cone invariant under the action of G. We fix a maximal compact subgroup K of G and choose a K-invariant hermitian form h on V. Thus K is a subgroup of $U(V)$. We identify $\hat{K}$ with dominant weights. We denote by $\mathfrak{k}$ the Lie algebra of K. We denote by $R(C)$ the space of regular functions on C and write the isotypic decomposition of $R(C)$

$$R(C) = \sum_{\lambda \in \hat{K}} R(C)_\lambda.$$

If all the spaces $R(C)_\lambda$ are finite dimensional and satisfy the growth condition: there exists a positive constant v and an integer N such that $\dim(R(C)_\lambda) \leq v(1 + \|\lambda\|^2)^N$, then the space $R(C)$ is a trace class representation of K. In particular the series $\sum_\lambda \mathrm{Tr}_{R(C)_\lambda}(\exp X)$ defines a generalized function of $X \in \mathfrak{k}$. We denote this function simply as $\mathrm{Tr}(R(C))(\exp X)$.

Recall that we can consider V as a symplectic space with symplectic form $B = -2\Im h$. We can thus consider the equivariant symplectic form $\Omega(X)$ on V given by formula 1.

Theorem 11 *Let V be a Hermitian vector space and let K be a closed subgroup of $U(V)$. Let $C \subset V$ be a closed K-invariant complex cone of pure dimension d. Assume that the restriction to C of the moment map $f : C \to \mathfrak{k}^*$ is proper. Then $R(C)$ is a trace class representation of K. Furthermore, we have the equality of generalized functions on $\mathfrak{k}$:*

$$\lim_{t \to 0} t^d \,\mathrm{Tr}(R(C))(\exp tX) = (-2i\pi)^{-d} \int_C e^{-i\Omega(X)}.$$

In this formula, the orientation on the regular part of C is given by its complex structure.

Before proving this theorem, let us start to explain the meaning of the generalized function $\int_C e^{-i\Omega(X)}$.

Lemma 12 *Suppose the moment map f induces a proper map $f : C \to \mathfrak{k}^*$. Then, if ϕ is a test function on $\mathfrak{k}$, the differential form $\int_{\mathfrak{k}} e^{-i\Omega(X)}\phi(X)dX$ is integrable on C.*

Proof. Consider the maximum $c > 0$ of $\|v\|^2$ on the compact set $\{v; \|f(v)\| = 1\} \cap C$. By homogeneity, we deduce then that for $v \in C$

$$c\|f(v)\| \geq \|v\|^2. \qquad (4)$$

If ϕ is a compactly supported C^∞ function on $\mathfrak{k}$, we denote by $\hat{\phi}$ the function on $\mathfrak{k}^*$ given by $\hat{\phi}(f) = \int_{\mathfrak{k}} e^{-i(f,X)}\phi(X)dX$. Then $\int_{\mathfrak{k}} e^{-i\Omega(X)}\phi(X)dX$ is a form on V given by

$$\int_{\mathfrak{k}} e^{-i\Omega(X)}\phi(X)dX = (\int_V e^{-i(f(v),X)}\phi(X)dX)e^{-i\Omega} = \hat{\phi}(f(v))\sum_{k=0}^{\ell}(-i)^k\frac{\Omega^k}{k!}$$

and the function $v \mapsto \hat{\phi}(f(v))$ is rapidly decreasing on C by the above estimate (4). By Lelong's theorem on integrability on algebraic varieties, the restriction to the regular part of C of $\hat{\phi}(f(v))\frac{\Omega^d}{d!}$ is integrable. We thus can define a generalized function $\int_C e^{-i\Omega(X)}$ by the formula

$$\int_{\mathfrak{k}}(\int_C e^{-i\Omega(X)})\phi(X)dX = \int_C(\int_{\mathfrak{k}} e^{-i(f(v),X)}\phi(X)dX)e^{-i\Omega} = \int_C \hat{\phi}(f(v))(-i)^d\frac{\Omega^d}{d!}.$$

∎

Proposition 13 *The restriction of the moment map $f : C \to \mathfrak{k}^*$ is proper if and only if the representation $R(C)$ is trace class. Furthermore, if $f : C \to \mathfrak{k}^*$ is proper, the map $t \mapsto \mathrm{Tr}(R(C))(\exp tX)$ has an asymptotic expansion when t tends to* 0.

Proof. We prove first that if f is proper, the representation $R(C)$ is trace class. As f is homogeneous, the fact that f is proper on C is equivalent to the condition that $C \cap f^{-1}(0)$ is 0. By Mumford's lemma (Lemma 3), we first see that all homogeneous invariant polynomials $P \in R_n(C)^G$ with strictly positive homogeneous degree n vanishes on C. Let T be a maximal torus of K and let B be a Borel subgroup of G containing T. Let U be the unipotent subgroup of B. Consider $R(C)^U$. It is a representation space for T. If $n(\lambda)$ is the multiplicity of the representation of K with dominant weight λ in $R(C)$, then $n(\lambda)$ is also the multiplicity of the weight λ in $R(C)^U$. Furthermore $R(C)^U$ is finitely generated. Indeed, the algebra $R(C)^U$ is isomorphic to the ring of G-invariant regular functions of the G-variety $G \times_U C$. Denote by $P(R(C)^U) \subset i\mathfrak{t}^*$ the set of weights occurring in the representation of T in $R(C)^U$. By the previous discussion, all weights $\mu \in P(R(C)^U)$ are non zero. This implies that there exists $X_0 \in \mathfrak{t}$ such that $i(\mu, X_0) > 0$ for all $\mu \in P(R(C)^U)$: if $P(R(C)^U)$ was not contained in a half space, there would be positive rational numbers r_j such that $0 = \sum_j r_j\mu_j$ where $\mu_j \in P(R(C)^U)$. By finding a common denominator, we see that 0 would necessarily belong to $P(R(C)^U)$. We now choose a finite set I of generators f_a for $R(C)^U$ with $a \in I$. The weight μ_a of f_a satisfies $i\mu_a(X_0) > 0$. The space $R(C)^U$ is a module over the polynomial ring $\mathbb{C}[X_a]$ where X_a acts by multiplication by f_a. The representation of T in the ring $\mathbb{C}[X_a]$ is trace class. Thus the representation in the quotient $R(C)^U$ is also trace class. It follows that multiplicities $n(\lambda)$ of representations with dominant weight λ occurring in $R(C)$ are of at most polynomial growth. Conversely, if $R(C)$ is trace class, this implies necessarily that $R(C)^G = \mathbb{C}$ as otherwise the trivial representation will occur with infinite multiplicity. This in turns by Mumford's lemma implies that $f^{-1}(0) \cap C = \{0\}$.

Let us see that $t \mapsto \mathrm{Tr}(R(C))(\exp tX)$ has an asymptotic expansion in t, when t tends to 0. We see by Lemma 5 that the character of the representation of T in $R(C)^U$ is of the form $B\Theta(-S)$ where $B \in R(T)$ and S is the finite set μ_a. Thus we see as in the proof of Lemma 8 that the generalized function of $Y \in \mathfrak{t}$ given by $\mathrm{Tr}\, R(C)^U(\exp tY)$ has an asymptotic expansion in the space of generalized function on $\mathfrak{t}$ when t tends to 0.

Let W be the Weyl group and let R^+ be a positive root system. Weyl integration formula implies that there is an isomorphism A between K-invariant generalized functions on $\mathfrak{k}$ and W-anti-invariant generalized functions on $\mathfrak{t}$. If θ is a K-invariant generalized function on $\mathfrak{k}$ and ϕ a K-invariant test function on $\mathfrak{k}$:

$$\int_{\mathfrak{k}} \theta(X)\phi(X)dX = \mathrm{vol}(K/T)|W|^{-1} \int_{\mathfrak{t}} A(\theta)(Y)(\prod_{\alpha \in R^+} \alpha(Y))\phi(Y)dY$$

where $\mathrm{vol}(K/T)$ is the volume of K/T for the K-invariant invariant measure compatible with dX/dY. Let $(\delta(t)\theta)(X) = \theta(tX)$. If N is the cardinal of R^+, then $A\delta(t) = t^{-N}\delta(t)A$. Consider the generalized function $r(C)(X) = \mathrm{Tr}\, R(C)(\exp X)$ in a neighborhood of 0 in $\mathfrak{k}$. Thus it is sufficient to prove that $\delta(t)A(r(C))$ has an asymptotic expansion when t tends to 0. Weyl's character formula implies the equality of generalized functions of $Y \in \mathfrak{t}$

$$(-1)^N \prod_{\alpha \in R^+} \frac{e^{(\alpha,Y)/2} - e^{-(\alpha,Y)/2}}{\alpha(Y)} A(r(C))(Y) = \sum_{w \in W} \epsilon(w) e^{(w\rho,Y)}\, \mathrm{Tr}\, R(C)^U(\exp wY).$$

Thus $A(r(C))(tY)$ has an asymptotic expansion when t tends to 0. ∎

Consider on V the Euclidean scalar product given by the real part of the Hermitian form h and the orientation given by its complex structure.

Thus V is an oriented Euclidean space, and we can construct a representative $a_V(X)$ of the $S0(V)$-equivariant Thom class $[a_V]$ of V. The equivariant form $e^{-i\Omega(X)}$ is a $\mathrm{Sp}(V)$-equivariant closed form on V. The intersection of the groups $SO(V)$ and $\mathrm{Sp}(V)$ is the unitary group $U(V)$. Thus both forms $a_V(X)$ and $e^{-i\Omega(X)}$ can be considered as closed $U(V)$-equivariant differential forms on V. The main tool in proving Theorem 11 is to compare these two $U(V)$-equivariant forms. The Lie algebra of $U(V)$ is denoted by $\mathfrak{u}$. Let us write Mathai-Quillen [7] representative a_V of $[a_V]$. It is a rapidly decreasing representative instead of a compactly supported representative. We use notations of [3], Formulae (13)-(21).

The space V is of real dimension 2ℓ. Define, for $X \in \mathfrak{so}(V)$, the following even element of the $\mathbb{Z}/2\mathbb{Z}$ graded algebra $\mathcal{A}(V) \otimes \Lambda V$

$$f_V(X) = -\|x\|^2 + d\mathbf{x} + \tau(X), \tag{5}$$

In orthonormal coordinates $x_1, x_2, ..., x_{2\ell}$,

$$f_V(X) = -\sum_i x_i^2 + \sum_i dx_i e_i + \frac{1}{2}\sum_{i<j}(Xe_i, e_j)e_i \wedge e_j.$$

Consider the Berezin integral $T : \mathcal{A}(V) \otimes \Lambda V \to \mathcal{A}(V)$: For $\alpha \in \mathcal{A}(V) \otimes \Lambda V$, the element $T(\alpha) \in \mathcal{A}(V)$ is such that $T(\alpha)e_1 \wedge e_2 \wedge \cdots \wedge e_{2\ell}$ is the projection of α on $\mathcal{A}(V) \otimes \Lambda^{2\ell}V$. Define $a_V(X) = (-\pi)^{-\ell}T(e^{f_V(X)})$. (In the notations of [3] , Formula (16)), this is the form $u_{\psi,V}$, with $\psi(t) = (-\pi)^{-\ell}e^t$). Then $a_V(X)$ is a representative of the equivariant Thom class of V. We have

$$\begin{aligned} a_V(X) &= (-\pi)^{-\ell}e^{-\|x\|^2}T(\exp\sum_i dx_i e_i + \frac{1}{2}\sum_{i<j}(Xe_i, e_j)e_i \wedge e_j) \\ &= (-\pi)^{-\ell}e^{-\|x\|^2}\sum_{I;\|I\| even} P_I(X/2)dx_{I'} \end{aligned} \tag{6}$$

where, for a subset I of cardinal $|I|$ even, P_I is an homogeneous polynomial on $\mathfrak{so}(V)$ of degree $(|I|/2)$ which coincides up to sign with the Pfaffian of the matrix $X_I = \{(Xe_i, e_j)_{ij\in I}\}$ and I' is the set of complementary indices.

Let us consider $t \in \mathbb{R}$ and $t > 0$. We consider $f_V(t, X) = -t^2\|x\|^2 + td\mathbf{x} + \tau(X)$. Consider the closed equivariant form $a_{t,V}$ defined by $a_{t,V}(X) = (-\pi)^{-\ell}T(e^{f_V(t,X)})$. This is still an equivariant closed form on V. When $t \to 0$, then $a_{t,V}(X)$ tends to the constant function $(-2\pi)^{-\ell}\det_{V,o}^{1/2}(X)$. Here

$\det_{V,o}^{1/2}(X)$ is a square root of the determinant of the transformation $X \in \mathfrak{so}(V)$ determined by the orientation of V.

We have ([3], Equation (20))

$$\frac{d}{dt}T(e^{f_V(t,X)}) = d_X(T(\mathbf{x}e^{f_V(t,X)}). \tag{7}$$

We thus have the transgression formula for $X \in \mathfrak{so}(V)$

$$a_V(X) - (-2\pi)^{-\ell} \overset{1/2}{\underset{V,o}{\det}}(X) = d_X\beta(X) \tag{8}$$

with

$$\beta(X) = (-\pi)^{-\ell}\int_0^1 e^{-t^2\|x\|^2}T((\sum_i x_ie_i)(\exp\sum_i tdx_ie_i+\frac{1}{2}\sum_{i<j}(Xe_i,e_j)e_i\wedge e_j)).$$

Thus $\beta(X)$ is of the form

$$(-\pi)^{-\ell}\sum_{I,j,|I|even,j\notin I}\phi_{I,j}(x)P_I(X/2)x_jdx_K$$

where K is the complementary subset to $I\cup\{j\}$ and $\phi_{I,j}(x) = \int_0^1 e^{-t^2\|x\|^2}t^{|K|}dt$. Note in particular that $\phi_{I,j}$ is a C^∞-function on V which is uniformly bounded on V and that its partial derivatives are of at most polynomial growth.

Lemma 14 *For $X \in \mathfrak{u}$, we have*

$$a_V(X) = (-2\pi)^{-\ell}(\overset{1/2}{\underset{V,o}{\det}} X)e^{-i\Omega(X)} + (d_X\nu)(X).$$

Furthermore, if ϕ is a compactly supported C^∞-function on $\mathfrak{u}$, then $\int_{\mathfrak{u}}\nu(X)\phi(X)dX$ is a rapidly decreasing form on V.

Proof. Consider Formula 8 . Thus we have for $X \in \mathfrak{u} \subset \mathfrak{so}(V)$,

$$a_V(X)e^{-i\Omega(X)} - (-2\pi)^{-\ell}\overset{1/2}{\underset{V,o}{\det}}(X)e^{-i\Omega(X)} = d_X(\beta(X)e^{-i\Omega(X)}).$$

Now, by Formula 2, we have also $a_V(X)(e^{-i\Omega(X)}-1) = d_X(a_V(X)\omega(\frac{e^{-id_X\omega}-1}{d_X\omega}))$. Finally we have $a_V(X) - (-2\pi)^{-\ell}\det_{V,o}^{1/2} Xe^{-i\Omega(X)} = d_X\nu$ with

$$\nu(X) = \beta(X)e^{-i\Omega(X)} - a_V(X)\omega(\frac{e^{-id_X\omega}-1}{d_X\omega}). \tag{9}$$

The function $\frac{e^{-ix}-1}{x}$ is uniformly bounded on $\mathbb{R}$. It is then easy to see that the form $a_V(X)\omega(\frac{e^{-id_X\omega}-1}{d_X\omega})$ is rapidly decreasing for all $X \in \mathfrak{u}$ while $\beta(X)e^{-i\Omega(X)}$ is rapidly decreasing when integrated against a test function on $\mathfrak{u}$.

Example

Let $T = \{e^{i\theta}; \theta \in \mathbb{R}\}$ be the circle group. We denote by J the basis of the Lie algebra $\mathfrak{t}$ of T such that $\exp(\theta J) = e^{i\theta}$. Let $V = \mathbb{R}^2$ with coordinates (x_1, x_2). Let T acting on $V = \mathbb{R}^2$ by rotations so that $J_V = x_2\partial_{x_1} - x_1\partial_{x_2}$. We note $\|x\|^2 = x_1^2 + x_2^2$. We have

$$a_V(\theta J) = \pi^{-1}e^{-\|x\|^2}(-\theta/2 + dx_1 \wedge dx_2)$$

while

$$e^{-i\Omega(\theta J)} = e^{-i\theta\|x\|^2/2}(1 - idx_1 \wedge dx_2).$$

Remark that on $V - \{0\}$, we have $a_V(\theta J) = (d_{\mathfrak{t}}\nu_1)(\theta J)$ with

$$\nu_1(\theta J) = (2\pi)^{-1}\frac{e^{-\|x\|^2}}{\|x\|^2}(x_2dx_1 - x_1dx_2)$$

Similarly $(2\pi)^{-1}\theta e^{-i\Omega(\theta J)} = -(d_{\mathfrak{t}}\nu_2)(\theta J)$ with

$$\nu_2(\theta J) = (2\pi)^{-1}\frac{e^{-i\theta\|x\|^2/2}}{\|x\|^2}(x_2dx_1 - x_1dx_2).$$

The above transgression formula reads

$$a_V(\theta J) + (2\pi)^{-1}\theta e^{-i\Omega(\theta J)} = (d_{\mathfrak{t}}\nu)(\theta J)$$

with

$$\nu(\theta J) = \nu_1(\theta J) - \nu_2(\theta J) = (2\pi)^{-1}\frac{e^{-\|x\|^2} - e^{-i\theta\|x\|^2/2}}{\|x\|^2}(x_2dx_1 - x_1dx_2)$$

well defined on V and which obviously satisfies the property stated in Lemma 14.

Lemma 15 *Let $C \subset V$ be a K-invariant closed complex cone. Assume the restriction $f : C \to \mathfrak{k}^*$ is a proper map. We have the equality:*

$$(-2\pi)^{-\ell}\det_{V,o}^{1/2}(X)\int_C e^{-i\Omega(X)} = \int_C a_V(X).$$

Proof. In the notations of the preceding lemma, the difference

$$\int_C a_V(X) - (-2\pi)^{-\ell} \det_{V,o}^{1/2}(X) \int_C e^{-i\Omega(X)}$$

is equal to $\int_C (d_{\mathfrak{g}}\nu)(X)\phi(X)dX$. As C is a cycle by Lelong's theorem, this last integral is equal to 0 and we obtain the equality in the lemma. ∎

We first prove Theorem 11 in the case of a torus.

Proposition 16 *Let T be a torus acting on V such that all weights of T are contained in a half-space. Let C be a T-invariant closed complex cone in V of complex dimension d. Then*

$$\lim_{t\to 0} t^d \operatorname{Tr} R(C)(\exp tX) = \Theta(C)(X) = (-2i\pi)^{-d} \int_C e^{-i\Omega(X)}.$$

Proof. Let Δ^+ be the set of weights of T in V. Let $X_0 \in i\mathfrak{t}$ such that $(\alpha, X_0) > 0$ for all $\alpha \in \Delta^+$. Let ϵ a small positive number. By Proposition 10, we have

$$\Theta(C)(X) = i^{-(\ell-d)} \lim_{\epsilon\to 0^+} \prod_{\alpha\in\Delta^+} (\alpha, X+\epsilon X_0)^{-1} J(C)(X+\epsilon X_0).$$

We use now Proposition 6 and we obtain

$$J(C)(X+\epsilon X_0) = (-1)^d (2\pi)^{l-d} \int_C a_V(X+\epsilon X_0).$$

If $v = \sum_{\alpha\in\Delta^+} v_\alpha$ is the decomposition of $v \in V$ in eigenvectors of weights α, we have $f(v)(X+\epsilon X_0) = i < v, (X+\epsilon X_0)v >= -i\sum_{\alpha\in\Delta^+} \alpha(X+\epsilon X_0)\|v_\alpha\|^2$. As $\alpha(X)$ is imaginary and $\alpha(X_0) > 0$, the function $e^{if(v)(X+\epsilon X_0)}$ is rapidly decreasing on V and we have

$$\int_C e^{-i\Omega(X)} = \int_C e^{-if(v)(X)}(-i)^d \Omega^d/d! = \lim_{\epsilon\to 0^+} \int_C e^{-i\Omega(X+\epsilon X_0)}.$$

Using notations of Lemma 14, we have

$$a_V(X+\epsilon X_0) - (-2\pi)^{-\ell} \det_{V,o}^{1/2}(X+\epsilon X_0) e^{-i\Omega(X+\epsilon X_0)} = (d_{\mathfrak{g}}\nu)(X+\epsilon X_0)$$

where ν is given by Formula 9. The function $\frac{e^{iz}-1}{z}$ is uniformly bounded when z varies in the upper-half plane. Thus $\nu(X+\epsilon X_0)$ is rapidly decreasing on V. Thus we obtain

$$\overset{-1/2}{\det_{V,o}}(X+\epsilon X_0)\int_C a_V(X+\epsilon X_0) = (-2\pi)^{-\ell}\int_C e^{-i\Omega(X+\epsilon X_0)}.$$

We have

$$\overset{1/2}{\det_{V,o}}(X+\epsilon X_0) = (-i)^{\ell}\prod_{\alpha\in\Delta^+}(\alpha, X+\epsilon X_0). \tag{10}$$

Thus, taking limits when ϵ tends to 0, we obtain the formula above. ∎

We now prove Theorem 11.

Proof. Assume first that the compact group K contains the group S^1, where $S^1=\{e^{i\theta}I\}$ is the center of $U(V)$. Let T be a maximal torus of K. Then the action of T on C is such all weights are contained in a half space. We then know that $R(C)$ is a trace class representation of T. Thus the formula

$$\lim_{t\to 0} t^d\,\mathrm{Tr}(R(C))(\exp tX) = (-2i\pi)^{-d}\int_C e^{-i\Omega(X)}$$

is valid in the space of generalized functions on $\mathfrak{t}$. A fortiori it is valid in the space of generalized functions on $\mathfrak{k}$. If K does not contain S^1, we can consider the group $\tilde{K}$ generated by K and S^1. We obtain the equality

$$\lim_{t\to 0} t^d\,\mathrm{Tr}(R(C))(\exp tX) = (-2i\pi)^{-d}\int_C e^{-i\Omega(X)}$$

as a generalized functions on $\tilde{\mathfrak{k}}$.

Under our hypothesis that f is proper on C, the generalized function $\int_C e^{-i\Omega(X)}$ restricts to $\mathfrak{k}$ as well as $\mathrm{Tr}(R(C))(\exp X)$. Furthermore the function $\mathrm{Tr}(R(C))(\exp tX)$ has an asymptotic expansion as a generalized function on $\mathfrak{k}$. Calculation of Proposition 13 shows that this asymptotic expansion is the restriction to $\mathfrak{k}$ of the asymptotic expansion of $\mathrm{Tr}(R(C))(\exp tX)$ for $X\in\tilde{\mathfrak{k}}$. By restricting to $\mathfrak{k}$, we obtain our theorem. ∎

4 Applications to Nilpotent Orbits

Let G be a real semi-simple connected Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G. Let K be a maximal compact subgroup of G. Let

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Consider a nilpotent orbit O of G in $\mathfrak{g}^*$ of dimension $2d$. Consider the Kirillov symplectic form σ_O on O with associated Liouville measure $d\beta_O$. The action of G on (O, σ_O) is Hamiltonian and the moment map is the canonical injection. We denote by $\sigma_O(X)$ the equivariant symplectic form of the orbit. It is homogeneous of degree 1 with respect to homotheties. Define the G-invariant generalized function F_O of $X \in \mathfrak{g}$ given by

$$F_O(X) = \int_O e^{-i(f,X)} d\beta_O(f) = (-2i\pi)^{-d} \int_O e^{-i\sigma_0(X)}.$$

On the other hand, consider the closure $C(O)$ of the nilpotent orbit $c(O)$ of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$ associated to O by Kostant-Sekiguchi correspondence [10].

Vogan's conjecture is the equality of generalized functions of $X \in \mathfrak{k}$:

$$F_O(X) = \lim_{t \mapsto 0} t^d \operatorname{Tr} R(C(O))(\exp tX).$$

Let us sketch first an heuristic argument to explain why it is natural to expect this equality in the context of equivariant cohomology.

We denote by κ the Killing form on $\mathfrak{g}$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by κ. Consider on $\mathfrak{k}$ and $\mathfrak{p}$ the Euclidean scalar products such that $\kappa(\xi_0 + \xi_1, \xi_0 + \xi_1) = \|\xi_1\|^2 - \|\xi_0\|^2$ for $\xi_0 \in \mathfrak{k}$ and $\xi_1 \in \mathfrak{p}$. The space $\mathfrak{p}_{\mathbb{C}}$ is a Hermitian vector space where K acts unitarily. Let us denote by $\Omega_{\mathfrak{p}}$ the corresponding symplectic form. If $\xi = \xi_1 + i\xi_2 \in \mathfrak{p}_{\mathbb{C}}$, then $\Omega_{\mathfrak{p}} = \kappa(d\xi_1, d\xi_2)$. It is easy to see that the moment map $f : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{k}^*$ is given by $f(\xi_1 + i\xi_2) = -[\xi_1, \xi_2]$ for $\xi_1, \xi_2 \in \mathfrak{p}$. Let $\mathcal{N}_{\mathbb{C}}$ be the nilpotent cone in $\mathfrak{g}_{\mathbb{C}}$. Remark that when $\xi = \xi_1 + i\xi_2$ is in the nilpotent cone $\mathcal{N}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}_{\mathbb{C}}$ and not equal to 0, then $f(\xi)$ is not zero. Indeed ξ_1 and ξ_2 are semi-simple elements of $\mathfrak{g}$ and if $[\xi_1, \xi_2]$ was equal to 0, then $\xi_1 + i\xi_2$ would be semi-simple. Thus f restricts to a proper map from $C(O)$ to $\mathfrak{k}^*$. By Theorem 11 $\lim_{t \mapsto 0} t^d \operatorname{Tr} R(C(O))(\exp tX)$ has a limit when t tends to 0 and we have

$$\lim_{t \mapsto 0} t^d \operatorname{Tr} R(C(O))(\exp tX) = (2i\pi)^{-d} \int_{C(O)} e^{-i\Omega_{\mathfrak{p}}(X)}.$$

We know ([12]) that there is a K-invariant diffeomorphism $R : c(O) \to O$ which commutes with the homotheties. By change of coordinates

$$F_O(X) = (-2i\pi)^{-d} \int_{c(O)} e^{-iR^*\sigma_O(X)}.$$

Consider the 1-form ω on $\mathfrak{p}_{\mathbb{C}}$ given by $\omega = \frac{1}{2}(\kappa(\xi_1, d\xi_2) - \kappa(\xi_2, d\xi_1))$ so that $\Omega_{\mathfrak{p}}(X) = d_X\omega$. We may consider for s a positive number and $X \in \mathfrak{k}$

$$F(s)(X) = (-2i\pi)^{-d} \int_{c(O)} e^{-iR^*\sigma_O(X) - isd_X\omega}.$$

As the K- equivariant cohomology class of $R^*\sigma_O(X) + sd_X\omega$ remains constant, this integral should be independent of s. Consider now the homothety $\delta(s)(\xi) = s^{-1/2}\xi$. We obtain

$$F(s)(X) = (-2i\pi)^{-d} \int_{c(O)} e^{-i\delta(s)R^*\sigma_O(X) - is\delta(s)d_X\omega}.$$

We have $\delta(s)(R^*\sigma_O(X)) = s^{-1/2}(R^*\sigma_O(X))$ while $s\delta(s)d_X\omega = d_X\omega$ as ω is homogeneous of degree 2. Thus

$$F(s)(X) = (-2i\pi)^{-d} \int_{c(O)} e^{-is^{-1/2}R^*\sigma_O(X) - id_X\omega}.$$

Assume that, as we wish, $F(s)$ is independent of s. Taking the limit when s tends to ∞ we obtain $F_O(X) = F(0)(X) = F(\infty)(X) = (-2i\pi)^{-d} \int_{c(O)} e^{-i\Omega_{\mathfrak{p}}(X)}$. Vogan's conjecture now follows from Theorem 11.

The main difficulty is thus to prove that indeed $F(s)(X)$ is independent of s. This is equivalent to prove a Stokes formula on $c(O)$. This obviously requires some care as $c(O)$ is an open subset of $C(O)$ and I do not know how to prove this fact in general.

Proposition 17 *If G is a complex Lie group, then Vogan's conjecture is true.*

Proof. In this case we have $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. The space $\mathfrak{p}$ is equal to $i\mathfrak{k}$. We identify $\mathfrak{g}$ to the Hermitian space $\mathfrak{p}_{\mathbb{C}}$. Thus O is equipped with two K-invariant symplectic forms: the Kirillov symplectic form σ_O and the restriction to O of the symplectic form $\Omega_{\mathfrak{p}}$ of the Hermitian space $\mathfrak{p}_{\mathbb{C}}$. Let $\omega = \frac{1}{2}(\kappa(\xi_1, d\xi_2) - \kappa(\xi_2, d\xi_1))$. If $\xi = \xi_1 + i\xi_2 \in O$, the moment map for the symplectic form σ_O is $\xi \mapsto \xi_1$, while the moment map for the symplectic form $\Omega_{\mathfrak{p}}$ is $\xi \mapsto -[\xi_1, \xi_2]$. If $f(s)(\xi_1 + i\xi_2) = \xi_1 - s[\xi_1, \xi_2]$, remark that $\|f(s)(\xi)\|^2 = \|\xi_1\|^2 + s^2\|[\xi_1, \xi_2]\|^2$, so that for all $s \in \mathbb{R}$

$$\|f(s)(\xi)\|^2 \geq \|\xi_1\|^2 \tag{11}$$

Introduce

(12)
$$F(s)(X) = (-2i\pi)^{-d} \int_O e^{-i\sigma_O(X) - is\Omega_{\mathfrak{p}}(X)} = (-2i\pi)^{-d} \int_O e^{-i\sigma_O(X) - isd_X\omega}.$$

We will soon see that this is indeed well defined as a generalized function for all $s \in \mathbb{R}$. We have

$$\int_{\mathfrak{k}} F(s)(X)\phi(X)dX = (2\pi)^{-d}\frac{1}{d!}\int_O \hat{\phi}(f(s))(\sigma_O + s\Omega_{\mathfrak{p}})^d.$$

In particular we need to prove that $(\sigma_O + s\Omega_{\mathfrak{p}})^d$ defines a tempered measure on O. This is easy to see using Rao's explicit description [9] of σ_O that we recall. We choose a Kostant three dimensional algebra (H, X, Y) such that $Y \in O$, $H \in \mathfrak{p}$ and $(X - Y) \in \mathfrak{k}$. We consider the stabilizer $G(H)$ of H. Let $\mathfrak{g}_j$ be the subspace of $\mathfrak{g}$ where $\operatorname{ad} H$ acts with eigenvalue j. The Lie algebra of $G(H)$ is $\mathfrak{g}_0$. Let $\mathfrak{n}^+ = \sum_{j>0} \mathfrak{g}_j$ and $\mathfrak{n}^- = \sum_{j<0} \mathfrak{g}_j$. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}^+$ is a parabolic decomposition of $\mathfrak{g}$. Let N be the subgroup of G with Lie algebra $\mathfrak{n}^+$. Let $Q = G(H)N$ be the parabolic subgroup of G with Lie algebra $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{n}^+$. We have $\mathfrak{g}(Y) \subset \mathfrak{q}$. The cotangent bundle $T^*(G/Q)$ can be identified with $G \times_Q \mathfrak{n}^+$ via the Killing form. We write $V_k = \sum_{i \geq k} \mathfrak{g}_i$. The space V_2 is a representation space for Q. We can then form the complex vector bundle $\mathcal{V} = G \times_Q V_2$ over the compact manifold G/Q and $\mathcal{V}$ is a sub-vector bundle of $T^*(G/Q)$. The closure of the orbit O is equal to $G(V_2)$. Furthermore the map $p : G \times_Q V_2 \to \overline{O}$ is an isomorphism over O. As O is a nilpotent orbit, Kirillov form is exact on O. Indeed, consider the invariant 1-form ω_O on O such that $\omega_O((\exp \epsilon J) \cdot f) = \kappa(J, f)$ for all $J \in \mathfrak{g}$ and $f \in O$. This is well defined as $\kappa(J, f)$ vanishes if $J \in \mathfrak{g}(f)$. The form ω_O is such that $\sigma_O = d(\omega_O)$. Consider now on the subspace $\mathcal{V} = G \times_Q V_2$ of $T^*(G/Q)$ the restriction $\tilde{\omega}^{G/Q}$ of the canonical 1-form $\omega^{G/Q}$ of $T^*(G/Q)$. It is easy to see that $p^*\omega_O = \tilde{\omega}^{G/Q}|_{p^{-1}(O)}$. Indeed we need only to verify this equality at the point $[e, Y]$ and for vector fields $J_{T^*(G/Q)}$ generated by the G-action of $J \in \mathfrak{g}$. We have $p^*\omega_O(J_{T^*(G/Q)})_{(e,Y)} = \kappa(Y, J)$. We have also $\tilde{\omega}^{G/Q}(J_{T^*(G/Q)})_{(e,Y)} = \kappa(Y, J)$. Thus we see that the inverse image of σ_O under the map p^* coincides with the restriction $\tilde{\sigma}^{G/Q}$ to $p^{-1}(O)$ of the canonical symplectic form $\sigma^{G/Q} = d\omega^{G/Q}$ of $T^*(G/Q)$.

Consider $\mathcal{V} = G \times_Q V_2$. Thus $\mathcal{V}$ is a desingularisation of the closure of O. If h is a function on $\mathfrak{g}^*$, we have

$$\int_O h\sigma_O^d = \int_{\mathcal{V}} (p^*h)(\tilde{\sigma}^{G/Q})^d.$$

In local coordinates of $\mathcal{V} = U \times V_2$, with U an open subset of G/Q, we can write $\tilde{\omega}^{G/Q} = \sum_i \theta^i y_i$ where θ^i are 1-forms on U and y_i linear coordinates on the vector space V_2. Then $(\tilde{\sigma}^{G/Q})^d = (\sum_i d\theta^i y_i + \sum_i \theta^i dy_i)^d$ is of polynomial behaviour in y_i, dy_i. The base of the vector bundle $\mathcal{V}$ being compact, if h is a function on $\mathfrak{g}^*$ such that the function p^*h on $\mathcal{V}$ is rapidly decreasing in the fiber directions, the integral $\int_{\mathcal{V}} (p^*h)(\tilde{\sigma}^{G/Q})^d$ is convergent. The pull-back $p^*\omega$ of the 1-form ω on $\mathfrak{g}^*$ is also of polynomial behaviour along the fibers of $\mathcal{V}$, thus so is $p^*\Omega_{\mathfrak{p}} = p^*\omega$. It follows that for any $s \in \mathbb{R}$, the integral $\int_{\mathcal{V}} (p^*h)(\tilde{\sigma}^{G/Q} + s\Omega_{\mathfrak{p}})^d$ is convergent. Let ϕ be a test function on $\mathfrak{k}$. Consider the function $h(s)(\xi) = \hat{\phi}(f(s)(\xi))$ on $\mathfrak{g}^*$. We have

$$\int_{\mathfrak{k}} F(s)(X)\phi(X)dX = (2i\pi)^{-d}\frac{1}{d!}\int_{\mathcal{V}} p^*(h(s))(\tilde{\sigma}^{G/Q} + s\Omega_{\mathfrak{p}})^d.$$

Let us show that the pull-back $p^*(h(s))$ on $\mathcal{V}$ is rapidly decreasing on fibers. Consider for example the fiber V_2 of $\mathcal{V}$. We need to analyze the behaviour of the function $\xi \mapsto \hat{\phi}(f(s)(\xi))$ restricted to V_2. We write $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in \mathfrak{k}$. Then as $\xi \in V_2$ is nilpotent, we have $\kappa(\xi, \xi) = 0$. This gives $\|\xi_1\|^2 = \|\xi_2\|^2$ so that $\|\xi_1\|^2 = \frac{1}{2}v\|\xi\|^2$. The function $\hat{\phi}$ is a rapidly decreasing function on $\mathfrak{k}^*$. Inequality 11 shows that the function $\xi \to \hat{\phi}(f(s)(\xi))$ is rapidly decreasing on V_2. Thus $F(s)$ is a tempered generalized function on $\mathfrak{k}$.

Let us now show that $F(s)$ is independent of s. We denote by $\sigma^{G/Q}(X)$ the equivariant symplectic form of $T^*(G/Q)$. We still denote by ω the pull back to $\mathcal{V}$ of the form ω on $\mathfrak{g}^*$. We have

$$F(s)(X) = (-2i\pi)^{-d}\int_{\mathcal{V}} e^{-i\tilde{\sigma}^{G/Q}(X) - isd_X\omega}.$$

The vector bundle $\mathcal{V}$ is a K-equivariant vector bundle over a compact base. Using the same argument as in Proposition 35 of [3], we can conclude that $F(s)$ is independent of s. Let us recall the proof. Let $\alpha(s)(X) = e^{-i\tilde{\sigma}^{G/Q}(X) - isd_X\omega}$. Thus $\alpha(s)$ is a K-equivariant closed form on $\mathcal{V}$ and its cohomology class is independent of s. More precisely, we have

$$\frac{d}{ds}\alpha(s) = d_{\mathfrak{k}}\beta(s) \tag{13}$$

with

$$\beta(s)(X) = -i\omega e^{-i\tilde{\sigma}^{G/Q}(X) - isd_X\omega} = -i\omega\alpha(s)(X).$$

Let ϕ be a test function on $\mathfrak{k}$, let $\alpha(s,\phi) = \int_{\mathfrak{k}} \alpha(s,X)\phi(X)dX$ and $\beta(s,\phi) = \int_{\mathfrak{k}} \beta(s,X)\phi(X)dX$. The forms $\alpha(s,\phi)$ and $\beta(s,\phi)$ are forms on $\mathcal{V}$. We have $\beta(s,\phi) = -i\omega\alpha(s,\phi)$. We have

$$\alpha(s,\phi) = \int_{\mathfrak{k}} e^{-i\tilde{\sigma}^{G/Q}(X) - isd_X\omega}\phi(X)dX = \hat{\phi}(f(s))e^{-i\tilde{\sigma}^{G/Q} - isd\omega}$$

thus we have proved that $\alpha(s,\phi)$ is a rapidly decreasing form on $\mathcal{V}$. It follows that $\beta(s,\phi) = -i\omega\alpha(s,\phi)$ is also rapidly decreasing on $\mathcal{V}$. We have

$$\int_{\mathfrak{k}} F(s)(X)\phi(X)dX = (-2i\pi)^{-d}\int_{\mathcal{V}} \alpha(s,\phi).$$

Now,

$$\int_{\mathcal{V}} \frac{d}{ds}\alpha(s,\phi) = \int_{\mathcal{V}} \frac{d}{ds}(\alpha(s,\phi))_{[2d]} = \int_{\mathcal{V}} d(\beta(s,\phi))_{[2d-1]}$$

using Relation 13. By Stokes theorem, this is equal to 0 as $\beta(s,\phi)$ is rapidly decreasing on $\mathcal{V}$. This concludes the proof of the fact that $F(s)(X)$ is independent of s. ∎

Consider the case where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a real semi-simple Lie algebra. In some other instances where we know sufficiently well a diffeomorphism of O into $c(O)$, we can conclude similarly that Vogan's conjecture holds. Take for example the case of a minimal nilpotent orbit O. We have $O = G \cdot E_\theta$ where θ is the highest restricted root and E_θ a root vector for θ. Let $E_\theta, H_\theta, E_{-\theta}$ be a Kostant triple, with $H_\theta \in \mathfrak{p}$. Then $c(O)$ is the orbit of $\frac{1}{2}(E_\theta + E_{-\theta}) + \frac{1}{2}iH_\theta$.

Lemma 18 *Let $\xi = \xi_0 + \xi_1 \in O$, with $\xi_0 \in \mathfrak{k}$ and $\xi_1 \in \mathfrak{p}$. Let $a = \frac{1}{2}\|H_\theta\|$. Then the map $V(\xi) = a^{1/2}\|\xi_1\|^{-1/2}\xi_1 + ia^{3/2}\|\xi_1\|^{-3/2}[\xi_0,\xi_1])$ is a K-equivariant symplectic diffeomorphism from (O,σ_O) to $(c(O),\Omega_h)$.*

Proof. Writing $G = KAN$ we see that $O = K \cdot (\mathbb{R}^+ E_\theta)$. From this description, it is easy to see that if $\xi = \xi_0 + \xi_1 \in O$, the elements ξ_0 and ξ_1 generates a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and that $V(\xi)$ is in $c(O)$. Indeed this is true for E_θ, with $V(E_\theta) = \frac{1}{2}(E_\theta + E_{-\theta}) + \frac{1}{2}iH_\theta$. Furthermore for any $\xi = \xi_0 + \xi_1 \in O$, we have $[[\xi_0,\xi_1],\xi_1] = a^{-2}\|\xi_1\|^2\xi_0$. The moment map for $\Omega_{\mathfrak{p}}$ is $(v_1,v_2) \mapsto -[v_1,v_2]$. We see thus that the moment map for $V^*\Omega_{\mathfrak{p}}$ is equal to the moment map for σ_O. We thus have for all $X \in \mathfrak{k}$ the equality $\iota(X_O)(\sigma_0 - V^*\Omega_{\mathfrak{p}}) = 0$. As orbits of K are of codimension 1, this implies $\sigma_O - V^*\Omega_{\mathfrak{p}} = 0$. ∎

Corollary 19 *Vogan's conjecture is true for minimal orbits.*

Proof. Indeed we have $\int_O e^{i\sigma_O(X)} = \int_{c(O)} e^{i\Omega_{\mathfrak{p}}(X)}$. ∎

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Michèle Vergne: E.N.S. et UA 762 du CNRS.
DMI, Ecole Normale Supérieure.
45 rue d'Ulm, 75005 Paris. France
E-mail: vergne@dmi.ens.fr