Morris identities and Total residues

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1 Introduction

The purpose of this paper is to find explicit formulae for the total residue of some interesting rational functions with poles on hyperplanes determined by roots of type $A_n = \{(e^i - e^j) | 1 \leq i, j \leq (n+1), i \neq j\}$. As pointed out by Zeilberger [Z], these calculations are mere reformulations of Morris identities [M], where the total residue function replaces here the iterated constant term. The proof we give of these identities follows closely (as suggested in [Z]) Aomoto's computation [Ao] of generalized Selberg integrals. Recall that Selberg [Se] proved that the following integral:

$$S_n(k_1, k_2, k_3) = \int_{[0,1]^n} \prod_{i=1}^{n} x_i^{k_1} (1 - x_i)^{k_2} \prod_{1 \leq i < j \leq n} |(x_i - x_j)|^{k_3} dx$$

is a product of Γ functions. In this setting, k_1, k_2, k_3 are non negative integers. Here we will be interested in the Fourier transform of the function

$$\mathcal{F}_n(k_1, k_2, k_3)(x_0, x) = \frac{1}{\prod_{i=1}^{n} x_i^{k_1} (x_0 - x_i)^{k_2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{k_3}},$$

more particularly on the value

$$s_n(k_1, k_2, k_3)(\xi) = \int_{\mathbb{R}^{n+1}} e^{i x_0 \xi} \mathcal{F}_n(k_1, k_2, k_3)(x_0, x) dx_0 dx$$

where $\mathcal{F}_n(k_1, k_2, k_3)(x_0, x)$ is interpreted as a boundary value of an holomorphic function. As shown by Jeffrey-Kirwan [JK], the value of the function $s_n(k_1, k_2, k_3)(\xi)$ is easily deduced from the knowledge of the total residue of the integrand. This reduces the problem to purely algebraic consideration: "integration " means that we will explicitly compute the function $\mathcal{F}_n(k_1, k_2, k_3)(x_0, x)$ modulo derivatives in $x_1, x_2, \ldots, x_n$ according to the decomposition appearing in Equation 1.

Let us give some of our motivations for calculating these total residues. Let $C(A_n^+)$ be the cone spanned by the positive roots of A_n. Consider the locally polynomial function v_n on $C(A_n^+)$ obtained by convolution of all the Heaviside functions $H(\alpha)$ supported on $\mathbb{R}^+\alpha$, when α varies in the set of positive roots. The value of v_n at a point $\xi \in C(A_n^+)$ is the volume of the polytope

$$\mathcal{P}(A_n, \xi) = \{x = (x_\alpha) \in \mathbb{R}_+^{n(n+1)/2} | \sum_{\alpha \in A_n^+} x_\alpha \alpha = \xi\}.$$

The Laplace transform of the function v_n is the rational function $\frac{1}{\prod_{\alpha \in A_n^+} \alpha}$. Thus, following Jeffrey-Kirwan, the value of the function v_n at the point ξ follows from the knowledge of the total residue of the rational function $\frac{\xi^{n(n-1)/2}}{\prod_{\alpha \in A_n^+} \alpha}$. For $\xi = (e^1 - e^{n+1})$, the highest root, the determination of this residue follows from our calculation of $\mathcal{F}_n(1, 1, 1)$ modulo derivatives. We refer to [BSV] and [BLV] for more details. More precisely, the general purpose of the notes [BSV] is to explain results of Brion-Vergne [BV] on total residues and to give examples (due to several authors), where volume or number of integral points of flow polytopes can indeed be determined explicitly (a very few cases) via residue calculus. These notes will be hopefully expanded in a monograph in the future. We refer to [BLV] for description of a program to compute number of integral points in general flow polytopes, where no closed formula is known (in particular, a Maple program for the calculation of Kostant partition function at any point is available on TO BE ADDED). Particular examples are beautiful in themselves, and also of great value on testing programs.

For this article, we have extracted from [BSV] our own contribution on these explicit calculations. As byproduct of the identities proven here, we give in the notes [BSV] a new proof of a conjecture of Chan, Robbins and Yuen [CRY], for which the volume of the polytope $\mathcal{P}(A_n^+, \theta)$, where $\theta = (e^1 - e^{n+1})$ is the highest root, can be expressed as product of the first $n-1$ Catalan numbers. This number, as shown by Postnikov-Stanley [**?**, **?**] is also the value of Kostant partition function $K(A_{n-2}^+)$ at the point $e^1 + 2e^2 + 3e^3 + \cdots + (n-2)e^{n-2} - \frac{(n-1)(n-2)}{2} e^{n-1}$ of $C(A_{n-2}^+)$. Thus, as conjectured by Stanley and proved by Zeilberger, we reobtain

$$K(A_n^+)[1,2,3,4,\ldots,n,-\frac{n(n+1)}{2}] = \prod_{i=1}^{n} \frac{2i!}{i!(i+1)!}.$$

2 Total residues

We recall the basic results about the total residue function. The definition is given in [BV] and it formalizes notions introduced in Jeffrey-Kirwan [JK]. Let V be a r-dimensional real vector space and let V^* be the dual vector space. If e^i is a basis of V^*, we denote by e_i the dual basis. Let $\Delta \subset V^*$ be a finite subset of non-zero linear forms. Each $\alpha \in \Delta$ determines an hyperplane $\{x \in V : \alpha(x) = 0\}$ in V. Consider the hyperplane arrangement

$$\mathcal{H} = \bigcup_{\alpha \in \Delta} \{\alpha = 0\}.$$

The ring R_Δ of rational functions with poles on $\mathcal{H}$ is the ring $\Delta^{-1}S(V^*)$ generated by the ring $S(V^*)$ of polynomial functions on V, together with inverses of the linear functions $\alpha \in \Delta$. Thus a function in R_Δ can be written as $P(x)/\prod_{\alpha\in\Delta} \alpha(x)^{n_\alpha}$ where P is a polynomial function on V and n_α are non negative integers. The ring R_Δ has a $\mathbb{Z}$-gradation by the homogeneous degree which can be positive or negative.

We say that a subset σ of Δ is a **basic subset** of Δ if the elements $\alpha \in \sigma$ form a basis of V^*. We denote by $\mathcal{B}(\Delta)$ the set of basic subsets of Δ. For σ a basic subset of Δ, the rational function f_σ defined by

$$f_\sigma(x) := \frac{1}{\prod_{\alpha\in\sigma} \alpha(x)}.$$

is called a **simple fraction**.

In appropriate coordinates $x_1, \ldots, x_r$, the function f_σ is simply $\frac{1}{x_1 x_2 \cdots x_r}$. We denote by S_Δ the subspace of R_Δ spanned by such "simple" elements f_σ .

Via the results of Orlik-Solomon, one can obtain a description of S_Δ as isomorphic to the r-cohomology group of the complement of the hyperplanes $\{\alpha = 0\}$.

The space S_Δ is contained in the homogeneous component of degree $-r$ of R_Δ. The elements v of V act on R_Δ by differentiation:

$$(\partial(v)f)(x) := \frac{d}{d\epsilon} f(x + \epsilon v)|_{\epsilon=0}$$

and we have the direct sum decomposition [BV]

$$R_\Delta = S_\Delta \oplus (\sum_{v\in V} \partial(v) R_\Delta). \tag{1}$$

Thus we see that only simple fractions cannot be obtained as derivatives.

As a corollary of this decomposition, we can define the projection map

$$\mathrm{Tres}_\Delta : R_\Delta \to S_\Delta.$$

Definition 1 *If $\phi \in R_\Delta$, the function $\mathrm{Tres}_\Delta\phi$ is called the total residue of the function ϕ.*

Let us note the following obvious property.

Lemma 2 *Assume $\Gamma \subset \Delta$ is a subset of Δ. Then*

$$R_\Gamma \subset R_\Delta,$$

$$S_\Gamma \subset S_\Delta.$$

Furthermore, if $f \in R_\Gamma$, then $\mathrm{Tres}_\Delta(f)$ belongs to S_Γ and

$$\mathrm{Tres}_\Gamma(f) = \mathrm{Tres}_\Delta(f).$$

Due to the preceding lemma, if ϕ is a rational function with poles on the collection of hyperplanes $\mathcal{B}$ determined by Γ, we can compute the total residue of ϕ with respect to any subset Δ containing Γ. It will be a sum of simple elements with denominators products of elements of Γ. Thus we simply write $\mathrm{Tres}(\phi)$ for the total residue of a rational function ϕ with poles on a collection of hyperplanes.

We conclude this section with some remarks and examples.

Remark 3

- *The total residue of a function is again a function (and not a number).*

  By definition, this function can be expressed as a linear combination of the simple fractions $f_\sigma(x)$.

  Example: If $V = \mathbb{R}e_1$ is one dimensional, and $\Delta = \{e^1\}$, then R_Δ is the ring of Laurent series

$$L = \{f(x) = \sum_{k\geq -q} a_k x^k\}$$

(the linear form e^1 is simply denoted by x, as $e^1(xe_1) = x$).

The total residue of a function $f(x) \in L$ is the function $\frac{a_{-1}}{x}$ where the constant a_{-1} is the usual residue denoted $\mathrm{Res}_{x=0} f$.

- Tres *vanishes on homogeneous rational functions in R_Δ of degree m, whenever $m \neq -r$.*

- Tres *vanishes on derivatives.*

  If $f = \frac{P}{\prod_k \alpha_k}$ ($P \in S(V^*), \alpha_k \in \Delta$) has a denominator product of linear forms $\alpha_k \in \Delta$ which do not generate V^*, then it is easy to see that f is a derivative and the total residue of f is equal to 0.

- *If Δ does not span V^*, then $S_\Delta = 0$.*

- *Elements f_σ are in general not linearly independent.*

  Example: Let V be a 2-dimensional vector space with basis e_1, e_2. Let Δ be the set
  $$\Delta = \{e^1, e^2, (e^1 - e^2)\}.$$
  Then we have the linear relation
  $$\frac{1}{e^1 e^2} = \frac{1}{e^2(e^1 - e^2)} - \frac{1}{e^1(e^1 - e^2)}$$
  between elements f_{σ_1},f_{σ_2}, f_{σ_3} with $\sigma_1 = \{e^1, e^2\}$, $\sigma_2 = \{e^1, (e^1 - e^2)\}$ and $\sigma_3 = \{e^2, (e^1 - e^2)\}$ basic subsets of Δ.

Example 4 Let us do a computation. Let V be a vector space with basis e_0, e_1, e_2 and let

$$\Delta = \{e^1, e^2, (e^0 - e^1), (e^0 - e^2), (e^1 - e^2)\}.$$

We write $x \in V$ as $x = x_0 e_0 + x_1 e_1 + x_2 e_2$. Consider the following function W_3 of R_Δ:

$$W_3(x_0, x_1, x_2) = \frac{x_0^2}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2) x_1 x_2}.$$

Then W_3 is homogeneous of degree -3. To compute the total residue of $W_3(x_0, x_1, x_2)$, we write x_0 as a linear combination of linear forms in the denominator of W_3, in order to reduce the degree of the denominator. For example, writing $x_0^2 = ((x_0 - x_1) + x_1)((x_0 - x_2) + x_2)$, we obtain

$$W_3(x_0, x_1, x_2) = \frac{1}{(x_1 - x_2)x_1x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1}$$
$$+\frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)}.$$

The first and last fractions have denominators with linearly dependent forms, so that their total residue is zero and we obtain:

$$\mathrm{Tres}(W_3(x)) = \frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1}.$$

More precisely, in the direct sum decomposition,

$$R_\Delta = S_\Delta \oplus (\partial_{x_0} R_\Delta + \partial_{x_1} R_\Delta + \partial_{x_2} R_\Delta),$$

we have,

$$W_3(x_0, x_1, x_2) - \mathrm{Tres}(W_3(x)) = U_3(x_0, x_1, x_2),$$

with

$$U_3(x_0, x_1, x_2) = -\partial_{x_1} \frac{x_0 - 2x_2}{x_2(x_0 - x_2)(x_1 - x_2)} - \partial_{x_2} \frac{x_0 - 2x_1}{x_1(x_0 - x_1)(x_1 - x_2)}.$$

We conclude this section with the following obvious property.

Lemma 5 *Assume that $V = V_1 \oplus V_2$ and $\Delta = \Delta_1 \cup \Delta_2$, with $\Delta_i \subset V_i$, then*

$$R_\Delta = R_{\Delta_1} \otimes R_{\Delta_2},$$

$$S_\Delta = S_{\Delta_1} \otimes S_{\Delta_2}.$$

If $\phi = \phi_1\phi_2$ with $\phi_1 \in R_{\Delta_1}, \phi_2 \in R_{\Delta_2}$, then

$$\mathrm{Tres}(\phi) = \mathrm{Tres}(\phi_1)\mathrm{Tres}(\phi_2).$$

3 Morris identities for A_{r+1}

Consider a $(r+2)$ dimensional real vector space, with basis $e^0, e^1, e^2, \ldots, e^r, e^{r+1}$. We denote by A_{r+1} the root system of $SL(r+2)$. We realize it as the collection of elements $(e^i - e^j)$ with $0 \leq i \leq r+1, 0 \leq j \leq r+1$, and $i \neq j$. In particular the vector space E_{r+1} spanned by the elements $(e^i - e^j)$ has dimension $r+1$. We write

$$\Pi = \{(e^0 - e^1), (e^1 - e^2), \ldots, (e^r - e^{r+1})\}$$

for the system of simple roots defined by A_{r+1}. Then

$$f_\Pi = \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^r - e^{r+1})}$$

is an element of $S_{A_{r+1}}$ (in particular is homogeneous of degree $-(r+1)$).

In fact the following proposition shows that the dimension of $S_{A_{r+1}}$ is $(r+1)!$ and a particularly nice basis of $S_{A_{r+1}}$ is given by the elements

$$f_w = w \cdot f_\Pi = w \cdot \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)(e^r - e^{r+1})}$$

where w is a permutation of $\{0, 1, 2, \ldots, r\}$ (the permutation w leaves $(r+1)$ fixed).

Proposition 6 *The map from*

$$\oplus_{i=0}^r S_{A_r} \to S_{A_{r+1}}$$

given by

$$\sum_{i=0}^r f_i \mapsto \sum_{i=0}^r f_i \frac{1}{(e^i - e^{r+1})}$$

is a bijection.

Proof. If f is in S_{A_r}, then for any $0 \leq i \leq r$, the element $f \frac{1}{(e^i - e^{r+1})}$ is in $S_{A_{r+1}}$, and it is easy to see that the map above is injective.

To prove that it is surjective, observe first that for any set of elements $K \subset \{0, \ldots, r\}$:

$$\prod_{j \in K} \frac{1}{(e^{r+1} - e^j)} = \sum_{i \in K} \frac{1}{\prod_{j \in K; j \neq i}(e^i - e^j)} \frac{1}{(e^{r+1} - e^i)}.$$

If ν is a subset of A_r, we denote by $f_\nu = \frac{1}{\prod_{\alpha\in\nu}\alpha}$. Then

$$\prod_{j\in K}\frac{1}{(e^{r+1}-e^j)}f_\nu = \sum_{i\in K}\frac{1}{\prod_{j\in K; j\neq i}(e^i-e^j)}f_\nu\frac{1}{(e^{r+1}-e^i)}.$$

If $\nu\cup\{(e^{r+1}-e^j), j\in K\}$ is a basic subset of A_{r+1}, then for every $i\in K$,

$$\nu\cup\{(e^i-e^j), j\in K, j\neq i\}\cup\{(e^{r+1}-e^i)\}$$

is a basic subset of A_{r+1}. The result follows.

We continue to set up the notations we need to formulate our main result. Consider the group Σ_r of permutations of $\{1,\ldots,r\}$ and denote by $\epsilon(w)$ the sign of an element $w\in\Sigma_r$.

Let $0\leq\ell\leq r$ and denote by $P_{\ell,r}$ the Σ_r-invariant polynomial

$$P_{\ell,r} = \sum_{w\in\Sigma_r} w\cdot[(e^1-e^{r+1})(e^2-e^{r+1})\cdots(e^\ell-e^{r+1})].$$

In particular

$$P_{0,r} = r! \qquad\qquad P_{r,r} = r!\prod_{j=1}^{r}(e^j-e^{r+1}).$$

When r is fixed, we will write P_ℓ for $P_{\ell,r}$, when $\ell>0$.

We consider the rational function given by

$$\phi_{r+1}(\ell,k_1,k_2,k_3)$$
$$= \frac{P_\ell}{(\prod_{j=1}^r(e^j-e^{r+1}))^{k_1}(\prod_{j=1}^r(e^0-e^j))^{k_2}(\prod_{1\leq i<j\leq r}(e^i-e^j))^{k_3}}.$$

In particular

$$\phi_{r+1}(0,k_1,k_2,k_3)$$
$$= r!\frac{1}{(\prod_{j=1}^r(e^j-e^{r+1}))^{k_1}(\prod_{j=1}^r(e^0-e^j))^{k_2}(\prod_{1\leq i<j\leq r}(e^i-e^j))^{k_3}}.$$

Here, k_1, k_2 and k_3 are non negative integers, so that $\phi_{r+1}(\ell,k_1,k_2,k_3)$ is an element of $R_{A_{r+1}}$ of homogeneity degree $\ell-(k_1+k_2)r-k_3\frac{r(r-1)}{2}$. If k_3 is odd, this function is anti-invariant under the group Σ_r of permutations of $\{1,\ldots,r\}$. If k_3 is even, this function is invariant.

Remark 7 *If* $k_1 \geq 1$ *then*

$$\phi_{r+1}(r, k_1, k_2, k_3) = \phi_{r+1}(0, k_1 - 1, k_2, k_3).$$

Let $k_1, k_2, k_3 \geq 0$, $0 \leq \ell \leq r$. Let $D = (k_1 + k_2)r + k_3\frac{r(r-1)}{2} - \ell$.
Then the function

$$\mathcal{F}(r, \ell, k_1, k_2, k_3) = (e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$$

is of homogeneity degree equal to $-(r + 1)$.

The functions $\mathcal{F}$ are the ones of which we are interested in computing the total residue.

In particular the function

$$W_{r+1} = \frac{1}{r!}(e^0 - e^{r+1})^{r(r+1)/2}\phi_{r+1}(0, 1, 1, 1) = \frac{(e^0 - e^{r+1})^{r(r+1)/2}}{\prod_{0\leq i<j\leq r+1}(e^i - e^j)} \tag{2}$$

is the one that is needed to compute the volume of the Chan Robbins Yuen polytope, that is the volume of the polytope

$$\mathcal{P}(A_{r+1}, (e^0 - e^{r+1})) = \{x = (x_\alpha) \in \mathbb{R}_+^{(r+1)(r+2)/2} | \sum_{\alpha \in A_{r+1}^+} x_\alpha \alpha = (e^0 - e^{r+1})\}.$$

As in the calculations of generalized Selberg integrals, total residues of the functions $\mathcal{F}(r, \ell, k_1, k_2, k_3)$ satisfy inductive formulae. The intermediate functions $\mathcal{F}(r, \ell, k_1, k_2, k_3)$ allows us to reduce the calculation of $\mathcal{F}(r, 0, k_1, k_2, k_3)$, modulo derivatives, to the calculation of $\mathcal{F}(r - 1, 0, k_1, k_2, k_3)$.

Let us explain our strategy, which follows closely Aomoto [**?**]. In the following, we use x_i instead of e^i, etc... as it is a more familiar notation for computing derivatives. Write $Denom = (\prod_{j=1}^r (x_j - x_{r+1}))^{k_1}(\prod_{j=1}^r (x_0 - x_j))^{k_2}(\prod_{1\leq i<j\leq r}(x_i - x_j))^{k_3}$ so that $\frac{P_\ell}{Denom} = \phi_{r+1}(\ell, k_1, k_2, k_3)$. We want to compute $P_\ell/Denom$ modulo derivatives. Thus we introduce derivatives of functions of the form $Q/Denom$ where Q is a polynomial of degree $\ell + 1$. By Leibniz rule, we need to apply derivations to Q and $1/Denom$. Applying derivations to $1/Denom$ would a priori increase the degree of the denominator. But, luckily, we will see that the additional factors of the denominator will be cancelled by **antisymmetrisation** of the numerator. Finally under differentiation and antisymmetrisation, we will see that a function $Q/Denom$ leads to a function $Q'/Denom$, where Q' is a polynomial of degree ℓ. Carefully choosing Q in function of P_ℓ allows us to obtain recurrence formulae.

The function W_3 is the function considered in Example 4 of Section 2 (where we have set $e^{r+1} = 0$). In particular, we see that, in general, the function W_{r+1} is not in the space $S_{A_{r+1}}$. However, its projection on $S_{A_{r+1}}$ is particularly nice. As a consequence of the more general identities proved in Theorem 8 the following holds

$$\text{Tres}\left(\frac{(e^0 - e^{r+1})^{r(r+1)/2}}{\prod_{0\le i<j\le r+1}(e^i - e^j)}\right) = \prod_{i=1}^{r-1} \frac{(2i)!}{i!(i+1)!} \sum_{w\in\Sigma_r} \epsilon(w)w \cdot f_\Pi \tag{3}$$

The reformulation of Morris identities we were speaking of in the introduction are the content of the following theorem.

Theorem 8 *Let $k_1, k_2, k_3 \geq 0$, $0 \leq \ell \leq r$. Let $D = (k_1 + k_2)r + k_3\frac{r(r-1)}{2} - \ell$.*

Then the function $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ is of homogeneity degree equal to $-(r + 1)$, and we have

- *If k_3 is odd,*

$$\text{Tres}((e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))$$
$$= C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w\in\Sigma_r} \epsilon(w)w.f_\Pi].$$

- *If k_3 is even,*

$$\text{Tres}((e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))$$
$$= C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w\in\Sigma_r} w.f_\Pi].$$

The constants $C_{r+1}(\ell, k_1, k_2, k_3)$ are determined uniquely by the relations:

- *for $1 \leq \ell \leq r$,*

$$(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))C_{r+1}(\ell, k_1, k_2, k_3)$$
$$= (k_1 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3).$$

- $$C_{r+1}(r,k_1,k_2,k_3)=C_{r+1}(0,k_1-1,k_2,k_3).$$
- *If* $r>1$,
$$C_{r+1}(r-1,1,k_2,k_3)=C_r(0,k_3,k_2,k_3).$$
- $$C_{r+1}(0,k_1,k_2,k_3)=C_{r+1}(0,k_2,k_1,k_3).$$
- $$C_{r+1}(0,1,1,0)=r!.$$
- *If* k_1 *or* $k_2=0$,
$$C_{r+1}(\ell,k_1,k_2,k_3)=0.$$

Constants $C_{r+1}(\ell,k_1,k_2,k_3)$ will be explicitly determined as a product of Γ factors in corollaries 10 and 11.

Remark 9 The function $(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3)$ is invariant or anti-invariant under the group Σ_r depending on the parity of k_3, then its total residue must be an element of $S_{A_{r+1}}$ which is invariant or anti-invariant by Σ_r. There are $(r+1)$ linearly independent such functions. Let us consider the basis $w\cdot f_\Pi$ of $S_{A_{r+1}}$ with w a permutation of $\{0,1,2,\ldots,r\}$. For homogeneity reasons, it is easy to see that the component of the total residue of $\mathcal{F}(r,\ell,k_1,k_2,k_3)$ on $w\cdot f_\Pi$ is equal to 0 unless the permutation w of $\{0,1,\ldots,r\}$ fixes 0, thus the total residue of $\mathcal{F}(r,\ell,k_1,k_2,k_3)$ belongs to the vector space spanned by elements f_w with $w\in\Sigma_r$, and the total residue of the function
$$(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3))$$
is proportional to either $[\sum_{w\in\Sigma_r}\epsilon(w)w.f_\Pi]$ or $[\sum_{w\in\Sigma_r}w.f_\Pi]$. The calculation of the constant of proportionality is thus equivalent to the Morris iterated constant term identity. However, we will give here a direct proof.

The recurrence formula above determines entirely the constants
$$C_{r+1}(\ell,k_1,k_2,k_3).$$

Indeed, we first check that constants $C_{r+1}(\ell, k_1, k_2, 0)$ are uniquely determined by the recurrence relations above. The first one reads, for $1 \leq \ell \leq r$,

$$(k_1 + k_2 - 2)C_{r+1}(\ell, k_1, k_2, 0) = (k_1 - 1)C_{r+1}(\ell - 1, k_1, k_2, 0).$$

The second reads

$$C_{r+1}(r, k_1, k_2, 0) = C_{r+1}(0, k_1 - 1, k_2, 0).$$

So if $k_1 > 1$, we can increase ℓ to $\ell = r$, using the first relation. Using the second relation, we decrease k_1 to $k_1 - 1$. Thus, using alternatively one and two, we compute $C_{r+1}(\ell, k_1, k_2, 0)$ from the value of $C_{r+1}(0, 1, k, 0)$. By the symmetry relation (relation 4), we can also assume $k_2 = 1$. We are finally reduced to $C_{r+1}(0, 1, 1, 0)$. Its value is given by the relation 5.

Assume now that $k_3 > 0$. Similarly in the first relation, if $k_1 > 1$, the constant $(k_1 - 1) + \frac{k_3}{2}(r - \ell)$ is strictly positive, so if $k_1 > 1$, we can increase ℓ to $\ell = r$, then using the second relation, we decrease k_1 to $k_1 - 1$. We thus may determine the constant $C_{r+1}(\ell, k_1, k_2, k_3)$ from $C_{r+1}(0, 1, k_2, k_3)$. If $k_1 = 1$, the first relation reads:

$$(k_2 - 1 + \frac{k_3}{2}(2r - \ell - 1))C_{r+1}(\ell, k_1, k_2, k_3)$$

$$= (\frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3).$$

Thus, if $r > 1$, we can increase ℓ up to $(r-1)$, using this relation. Then using the third, we decrease $r+1$ to r. In conclusion we determine $C_{r+1}(\ell, k_1, k_2, k_3)$ from the value of the constants $C_2(\ell, k_1, k_2, k_3)$. But if $r = 1$, there is no factor corresponding to k_3, so that $C_2(\ell, k_1, k_2, k_3) = C_2(\ell, k_1, k_2, 0)$, value that we have determined previously.

Corollary 10 *Assume $r > 1$.*

- *If $k_3 > 0$, or if $k_1 + k_2 > 2$, then for $1 \leq \ell \leq r$,*

$$C_{r+1}(\ell, k_1, k_2, k_3) = \prod_{j=1}^{\ell} \frac{k_1 - 1 + (r - j)\frac{k_3}{2}}{k_1 + k_2 - 2 + (2r - j - 1)\frac{k_3}{2}} C_{r+1}(0, k_1, k_2, k_3).$$

- *If* $k_1 + k_2 \geq 2$,

$$C_{r+1}(0, k_1, k_2, k_3) = r! \prod_{j=0}^{r-1} \frac{\Gamma(1 + \frac{k_3}{2})\Gamma(k_1 + k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j+1)\frac{k_3}{2})\Gamma(k_1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})}.$$

Corollary 11 *We have*

- $$C_{r+1}(\ell, 1, 1, 1) = \prod_{j=1}^{\ell} \frac{(r-j)}{(2r-j-1)} C_{r+1}(0, 1, 1, 1).$$

- $$C_{r+1}(0, 1, 1, 1) = r! \prod_{i=1}^{r-1} C_i,$$

  where $C_i = \frac{(2i)!}{i!(i+1)!}$ *is the i-th Catalan number.*

- $$C_{r+1}(0, k, 1, 1) = r! \prod_{i=k-1}^{r+k-3} \frac{1}{2i+1} \binom{r+k+i-1}{2i}.$$

- $$C_{r+1}(\ell, k, 1, 1) = \prod_{j=1}^{\ell} \frac{2(k-1) + (r-j)}{2(k-1) + (2r-j-1)} C_{r+1}(0, k, 1, 1).$$

The second corollary is of course a consequence of the first, using several times the duplication formula for the Gamma function, but it is somewhat easier to use directly the recurrence formulas in k_1, k_2, k_3, with $k_3 = 1, k_2 = 1$ as the value of k_2, k_3 remains constant and equal to 1, through the recurrence.

Let us first verify the corollaries , assuming Theorem 8.

To verify the first corollary, we verify the recurrence relations. The first being obvious, we check the second:

$$C_{r+1}(r, k_1, k_2, k_3) = C_{r+1}(0, k_1 - 1, k_2, k_3).$$

We write

$$C_{r+1}(r, k_1, k_2, k_3)$$

$$= r!\prod_{j=1}^{r} \frac{k_1 - 1 + (r-j)\frac{k_3}{2}}{k_1 + k_2 - 2 + (2r-j-1)\frac{k_3}{2}}$$

$$\times \prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_1+k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}.$$

In the first product, we change j in $(r-j)$, in the second we use $\Gamma(z+1) = z\Gamma(z)$ and we obtain:

$$r!\prod_{j=0}^{r-1} \frac{k_1 - 1 + j\frac{k_3}{2}}{k_1 + k_2 - 2 + (r+j-1)\frac{k_3}{2}} \frac{\Gamma(1+\frac{k_3}{2})(k_1+k_2-2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})(k_1-1+j\frac{k_3}{2})}$$

$$\times \prod_{j=0}^{r-1} \frac{\Gamma(k_1-1+k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(k_1-1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$= C_{r+1}(0, k_1-1, k_2, k_3).$$

We verify the third condition.
We write

$$C_{r+1}(r-1, 1, k_2, k_3)$$

$$= r!\prod_{j=1}^{r-1} \frac{(r-j)\frac{k_3}{2}}{k_2 - 1 + (2r-j-1)\frac{k_3}{2}} \prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$= r!\prod_{j=1}^{r-1} \frac{(r-j)\frac{k_3}{2}}{k_2 - 1 + (2r-j-1)\frac{k_3}{2}} \prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}$$

$$\times\Gamma(k_2+(r-1)\frac{k_3}{2}) \prod_{j=1}^{r-1} \frac{\Gamma(k_2+(r+j-1)\frac{k_3}{2})}{\Gamma(1+j\frac{k_3}{2})}$$

In the first product, we change j in $(r-j)$, in the last we use $\Gamma(z+1) = z\Gamma(z)$, and we obtain after simplification that $C_{r+1}(r-1, 1, k_2, k_3)$ is equal to

$$r!\prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}\Gamma(k_2+(r-1)\frac{k_3}{2})$$

$$\times \prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

$$= r! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+r\frac{k_3}{2})} \prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

$$= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})} \frac{\Gamma(\frac{k_3}{2})}{\Gamma(r\frac{k_3}{2})} \prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}$$

while

$$C_r(0,k_3,k_2,k_3) = (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_3+k_2-1+(r-1+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_3+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}.$$

$$= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1+\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})} \prod_{j=0}^{r-2} \frac{\Gamma(k_2-1+(r+j)\frac{k_3}{2})}{\Gamma((j+2)\frac{k_3}{2})}$$

It remains to verify

$$\frac{\Gamma(\frac{k_3}{2})}{\Gamma(r\frac{k_3}{2})} \prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r+j-1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})} = \prod_{j=0}^{r-2} \frac{\Gamma(k_2-1+(r+j)\frac{k_3}{2})}{\Gamma((j+2)\frac{k_3}{2})}.$$

which is true.

The remaining properties are obvious.

We now prove Theorem 8 by induction on r.

Proof. If $k_1 = 0$, the remaining roots $(e^i - e^j)$ occurring in the denominator of $\phi_{r+1}(\ell, 0, k_2, k_3)$ are contained in the hyperplane $\sum_{i=0}^r e^i = 0$. So the total residue of $\phi_{r+1}(\ell, 0, k_2, k_3)$ is 0. The same argument shows that $\phi_{r+1}(\ell, k_1, 0, k_3)$ is 0 .

We thus may assume that $k_1, k_2 > 0$. We first show that the function $(e^0 - e^{r+1})\phi_{r+1}(\ell-1, k_1, k_2, k_3)$ is proportional to the function $\phi_{r+1}(\ell, k_1, k_2, k_3)$ modulo $\sum_{i=1}^r \partial_i R_{A_{r+1}}$. Thus the functions $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell-1, k_1, k_2, k_3)$

and $(e^0 - e^{r+1})^{D-r}\phi_{r+1}(\ell-1, k_1, k_2, k_3)$ will be proportional too, modulo the vector space $\sum_{i=1}^r \partial_i R_{A_{r+1}}$, their total residues will be proportional and we will get the first recursive relations for the constant C_{r+1}.

We write $U = (\prod_{j=1}^r (x_j - x_{r+1}))^{-k_1} (\prod_{j=1}^r (x_0 - x_j))^{-k_2} (\prod_{1\leq i<j\leq r} (x_i - x_j))^{-k_3}$ so that $P_\ell U = \phi_{r+1}(\ell, k_1, k_2, k_3)$.

Consider $Q_{\ell+1} = (x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})$. This is a polynomial of degree $\ell+1$. As in Aomoto, the strategy consists in computing derivatives of $Q_{\ell+1}U$ followed by antisymetrization. We will obtain a function of the form $Q'_\ell U$, where Q'_ℓ is a polynomial of degree ℓ. The function Q'_ℓ will be expressed in function of P_ℓ and $P_{\ell-1}$. This will imply the wanted recurrence relation, as by definition $Q'_\ell U$ is zero modulo derivatives.

We compute:

$$\partial_1[(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U]$$

This is equal to

$$\begin{gathered} -(1-k_2)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U \\ +(1-k_1)(x_0 - x_1)(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U \\ -k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \sum_{j=2}^r \frac{1}{x_1 - x_j} U. \end{gathered}$$

Using $(x_0 - x_1) = (x_0 - x_{r+1}) + (x_{r+1} - x_1)$, this is also equal to

$$\begin{gathered} = (k_1 + k_2 - 2)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U \\ +(1-k_1)(x_0 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U \\ -k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \sum_{j=2}^r \frac{1}{x_1 - x_j} U. \end{gathered}$$

Assume first that k_3 is odd, so that U is anti-invariant by the group Σ_r.
Let us antisymmetrize over permutations. We obtain

$$\sum_{w\in\Sigma_r} \epsilon(w) w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U)) =$$

$$(k_1 + k_2 - 2)P_\ell U + (1-k_1)(x_0 - x_{r+1})P_{\ell-1} U$$

$$-k_3 \sum_{w\in\Sigma_r} w\cdot\left((x_0-x_1)(x_1-x_{r+1})(x_2-x_{r+1})\cdots(x_\ell-x_{r+1})\sum_{j=2}^{r}\frac{1}{x_1-x_j}\right)U.$$

To compute $\sum_{w\in\Sigma_r} w\cdot((x_0-x_1)(x_1-x_{r+1})(x_2-x_{r+1})\cdots(x_\ell-x_{r+1})\frac{1}{x_1-x_j})$, we first sum over the transposition $(j,1)$.

If $2\le j\le \ell$, we use

$$\frac{(x_0-x_1)(x_1-x_{r+1})(x_j-x_{r+1})}{x_1-x_j}+\frac{(x_0-x_j)(x_j-x_{r+1})(x_1-x_{r+1})}{x_j-x_1}$$

$$=-(x_1-x_{r+1})(x_j-x_{r+1}).$$

If $j>\ell$, we use

$$\frac{(x_1-x_{r+1})(x_0-x_1)}{x_1-x_j}+\frac{(x_j-x_{r+1})(x_0-x_j)}{x_j-x_1}$$

$$=(x_0-x_1)+(x_{r+1}-x_j)=(x_0-x_{r+1})+(x_{r+1}-x_1)+(x_{r+1}-x_j).$$

We obtain that

$$2\sum_{w\in\Sigma_r} w\cdot((x_0-x_1)(x_1-x_{r+1})(x_2-x_{r+1})\cdots(x_\ell-x_{r+1})\sum_{j=2}^{r}\frac{1}{x_1-x_j})$$

is equal to

$$(-(\ell-1)-2(r-\ell))P_\ell+(x_0-x_{r+1})(r-\ell)P_{\ell-1}.$$

Thus finally, we obtain

$$\sum_{w\in\Sigma_r}\epsilon(w)w\cdot(\partial_1((x_0-x_1)(x_1-x_{r+1})(x_2-x_{r+1})\cdots(x_\ell-x_{r+1})\phi_{r+1}(0,k_1,k_2,k_3))=$$

$$(k_1+k_2-2+\frac{k_3}{2}(2r-\ell-1))\phi_{r+1}(\ell,k_1,k_2,k_3)$$
$$+(x_0-x_{r+1})(-k_1+1-\frac{k_3}{2}(r-\ell))\phi_{r+1}(\ell-1,k_1,k_2,k_3).$$

If k_3 is even, we also obtain

$$\sum_{w\in\Sigma_r} w\cdot(\partial_1((x_0-x_1)(x_1-x_{r+1})(x_2-x_{r+1})\cdots(x_\ell-x_{r+1}))\phi_{r+1}(0,k_1,k_2,k_3))=$$

$$(k_1+k_2-2+\frac{k_3}{2}(2r-\ell-1))\phi_{r+1}(\ell,k_1,k_2,k_3)$$

$$+(x_0-x_{r+1})(-k_1+1-\frac{k_3}{2}(r-\ell))\phi_{r+1}(\ell-1,k_1,k_2,k_3).$$

Thus we see that

$$(x_0-x_{r+1})\phi_{r+1}(\ell-1,k_1,k_2,k_3) \text{ is proportional to } \phi_{r+1}(\ell,k_1,k_2,k_3),$$

modulo derivatives with respect to x_1, x_2,..., x_r. In particular the total residue of the function $(x_0-x_{r+1})^{D-(r+1)}\phi_{r+1}(\ell,k_1,k_2,k_3)$ is proportional to the total residue of the function $(x_0-x_{r+1})^{D-(r+1)+1}\phi_{r+1}(\ell-1,k_1,k_2,k_3)$. This proves the first property.

We proceed to the proof of the second property. We return to the notation $x_i=e^i$.

To avoid confusion in the following argument we will write explicitly the dependence by the parameters of D, that is we will write, whenever necessary,

$$D=D_{A_{r+1}}(\ell,k_1,k_2,k_3)=(k_1+k_2)r+k_3\frac{r(r-1)}{2}-\ell.$$

We now compute the total residue of

$$(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(r-1,1,k_2,k_3)$$

with $D=D_{A_{r+1}}(r-1,1,k_2,k_3)=(1+k_2)r+k_3\frac{r(r-1)}{2}-(r-1)$.

We have:

$$r!(e^0-e^{r+1})^{D-(r+1)}\phi_{r+1}(r-1,1,k_2,k_3)$$

$$=(e^0-e^{r+1})^{D-(r+1)}\frac{P_{r-1}}{(e^1-e^{r+1})(e^2-e^{r+1})\cdots(e^r-e^{r+1})}\phi_{r+1}(0,0,k_2,k_3).$$

with

$$\frac{P_{r-1}}{(e^1-e^{r+1})(e^2-e^{r+1})\cdots(e^r-e^{r+1})}$$

$$\frac{\sum_{w\in\Sigma_r} w\cdot[(e^1-e^{r+1})(e^2-e^{r+1})\cdots(e^{r-1}-e^{r+1})]}{(e^1-e^{r+1})(e^2-e^{r+1})\cdots(e^r-e^{r+1})}$$

$$=(r-1)!\sum_{j=1}^{r}\frac{1}{(e^j-e^{r+1})}.$$

Consider the subgroup Cyc_r generated by the circular permutation of $1, \ldots, r$. Then

$$\sum_{j=1}^{r} \frac{1}{(e^j - e^{r+1})} = \sum_{w \in \mathrm{Cyc}_r} w \cdot \frac{1}{(e^r - e^{r+1})}.$$

Assume k_3 odd. Thus

$$\frac{r!}{(r-1)!} \phi_{r+1}(r-1, 1, k_2, k_3) = [\sum_{w \in \mathrm{Cyc}_r} w \cdot \frac{1}{(e^r - e^{r+1})}] \phi_{r+1}(0, 0, k_2, k_3)$$

$$= \sum_{w \in \mathrm{Cyc}_r} \epsilon(w) w \cdot [\frac{1}{(e^r - e^{r+1})} \phi_{r+1}(0, 0, k_2, k_3)]$$

as $\phi_{r+1}(0, 0, k_2, k_3)$ is anti-invariant under Σ_r.

Remark that:

$$\phi_{r+1}(0, 0, k_2, k_3) = r \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)$$

so that

$$(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \phi_{r+1}(0, 0, k_2, k_3)$$

$$= r(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3).$$

It follows that we have

$$(e^0 - e^{r+1})^{D-(r+1)} \phi_{r+1}(r-1, 1, k_2, k_3)$$

$$= \sum_{w \in \mathrm{Cyc}_r} \epsilon(w) w \cdot [(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)].$$

We now use Lemma 5 to compute the total residue of the last term in the equality.

We write the vector space E_{r+1} as $E_r \oplus \mathbb{R}(e^r - e^{r+1})$, and we consider $\Delta' = A_r \cup \{(e^r - e^{r+1})\}$

Using the decomposition $(e^0 - e^{r+1}) = (e^r - e^{r+1}) + (e^0 - e^r)$, we write

$$(e^0 - e^{r+1})^{D-(r+1)} = \sum_{i\geq 0, j\geq 0, i+j=D-(r+1)} c_{ij}(e^r - e^{r+1})^i (e^0 - e^r)^j.$$

Thus

$$(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)$$

$$= \sum_{i\geq 0, j\geq 0, i+j=D-(r+1)} c_{ij} \frac{(e^r - e^{r+1})^i}{(e^r - e^{r+1})} \frac{(e^0 - e^r)^j}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)$$

belongs to the vector space $R_{\{(e^r - e^{r+1})\}} \otimes R_{A_r}$ and we can easily compute the total residue using Lemma5, as well as the obvious calculation for a one dimensional space. Precisely

$$\mathrm{Tres}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)]$$

$$= \sum_{i\geq 0, j\geq 0, i+j=D-(r+1)} c_{ij} \mathrm{Tres}(\frac{(e^r - e^{r+1})^i}{(e^r - e^{r+1})})$$

$$\times \mathrm{Tres}[\frac{(e^0 - e^r)^j}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)].$$

Only the term $i = 0$ gives a non zero residue, so we obtain

$$\mathrm{Tres}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)^{k_2}} \phi_r(0, k_3, k_2, k_3)]$$

$$= \frac{1}{(e^r - e^{r+1})} \mathrm{Tres}[(e^0 - e^r)^{D-(r+1)-k_2} \phi_r(0, k_3, k_2, k_3)].$$

Now

$$D - (r+1) - k_2 = D_{A_{r+1}}(r-1, 1, k_2, k_3) - (r+1) - k_2 = D_{A_r}(0, k_3, k_2, k_3) - r.$$

So we obtain

$$\mathrm{Tres}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} (e^0 - e^r)^{k_2} \phi_r(0, k_3, k_2, k_3)]$$

$$= \frac{1}{(e^r - e^{r+1})} \mathrm{Tres}[(e^0 - e^r)^{D_{A_r}(0,k_3,k_2,k_3)-r} \phi_r(0, k_3, k_2, k_3)].$$

We apply induction hypothesis on r. We have

$$\mathrm{Tres}[(e^0 - e^r)^{D_{A_r}(0,k_3,k_2,k_3)-r} \phi_r(0, k_3, k_2, k_3)]$$
$$= C_r(0, k_3, k_2, k_3) \sum_{w' \in \Sigma_{r-1}} \epsilon(w') w' \cdot [\frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)}].$$

By Formula 3, as the total residue commutes with the action of W, we obtain:

$$\mathrm{Tres}(e^0 - e^{r+1})^{D-(r+1)} \phi_{r+1}(r - 1, 1, k_2, k_3)$$
$$= C_r(0, k_3, k_2, k_3) \sum_{w \in Cyc_r} \epsilon(w) w \cdot [\frac{1}{(e^r - e^{r+1})}]$$
$$\times \sum_{w' \in \Sigma_{r-1}} \epsilon(w') w' \cdot [\frac{1}{(e^0 - e^1) \cdots (e^{r-1} - e^r)}].$$

But

$$\sum_{w \in Cyc_r} \epsilon(w) w \cdot [\frac{1}{(e^r - e^{r+1})}] \sum_{w' \in \Sigma_{r-1}} \epsilon(w') w' \cdot [\frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)}]$$

$$= \sum_{ww' \in Cyc_r \Sigma_{r-1} = \Sigma_r} \epsilon(ww') ww' \cdot [\frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)}]$$

$$= \sum_{w \in \Sigma_r} \epsilon(w) w . f_\Pi.$$

Thus we obtain the second relation. The case k_3 even is completely analogous, so the proof of the first and second relation is complete. The symmetry property in k_1, k_2 is obvious.

Let us check

$$C_{r+1}(0, 1, 1, 0) = r!.$$

More precisely, we have the following exact formula (without projection on S_Δ).

Lemma 12
$$\frac{(e^0 - e^{r+1})^{(r-1)}}{\prod_{i=1}^r (e^0 - e^i)(e^i - e^{r+1})} = \sum_{w \in \Sigma_r} w \cdot f_\Pi.$$

Indeed, by reduction to the same denominator, the right hand side can be written as

$$\frac{P}{\prod_{i=1}^r (e^0 - e^i)(e^i - e^{r+1}) \prod_{1 \leq i < j \leq r} (e^i - e^j)}$$

From invariance consideration, P has to be anti-invariant under Σ_r , so is divisible by $\prod_{1 \leq i < j \leq r}(e^i - e^j)$. From degree consideration, we obtain the desired equality, and $C_{r+1}(0,1,1,0) = r!$.

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