$[Q,R]=0$ AND KOSTANT PARTITION FUNCTIONS

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1. The theorem

Let M be a compact manifold endowed with an almost complex structure $J \in \Gamma\mathrm{End}(\mathrm{T}M)$. We denote by $\mathrm{T}^J M$ the complex vector bundle of $+i$-eigenspaces of J acting on $\mathrm{T}M \otimes \mathbb{C}$, and by $\overline{\mathrm{T}}^J M$ the bundle of $-i$-eigenspaces of J. Then we have the splitting $\mathrm{T}M \otimes \mathbb{C} = \mathrm{T}^J M \oplus \overline{\mathrm{T}}^J M$. If M is a complex manifold endowed with an Hermitian metric, then $\mathrm{T}^J M$ may be identified with the *complex tangent bundle*, while $\overline{\mathrm{T}}^J M$ with the *complex cotangent bundle* of M. Informally, we will use this terminology even when M has only an almost complex structure.

To every complex vector bundle $\mathcal{E} \to M$ over M one can associate an integer as follows (cf. (2)). Set the notation $\Omega_J^\bullet(M,\mathcal{E}) = \Gamma(\Lambda^\bullet(\overline{\mathrm{T}}^J M)^* \otimes \mathcal{E})$ for the anti-holomorphic differential forms with values in $\mathcal{E}$, and consider the twisted Dolbeault-Dirac operator [5]

$$D_{\mathcal{E}} : \Omega_J^{\mathrm{even}}(M,\mathcal{E}) \to \Omega_J^{\mathrm{odd}}(M,\mathcal{E}).$$

This is a first-order elliptic differential operator, and its "virtual space of solutions" is well-defined as a formal difference of two finite-dimensional vector spaces:

$$Q(M,\mathcal{E}) = \mathrm{Ker}(D_{\mathcal{E}}) \ominus \mathrm{Coker}(D_{\mathcal{E}}). \tag{1}$$

The dimension of this virtual vector space is defined to be the integer

$$\dim Q(M,\mathcal{E}) = \dim \mathrm{Ker}(D_{\mathcal{E}}) - \dim \mathrm{Coker}(D_{\mathcal{E}}). \tag{2}$$

This number may be computed by the Atiyah-Singer index formula:

$$\dim Q(M,\mathcal{E}) = \int_M \mathrm{ch}(\mathcal{E})\,\mathrm{Todd}(\mathrm{T}^J M); \tag{3}$$

here $\mathrm{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ and $\mathrm{Todd}(\mathrm{T}^J M)$ is the Todd class of M.

Now assume that a compact, connected Lie group G acts compatibly on the manifold M and the bundle $\mathcal{E}$, and preserves the almost complex structure J. Then $Q(M,\mathcal{E})$ becomes a virtual representation of G, whose character we denote by $\chi_{\mathcal{E}}$.

To make this more explicit, we introduce the following notation for the Lie data:

- Denote by T the maximal torus of G, and

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- by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T, respectively;
- we will identify $\mathfrak{t}^*$ with the T-invariant subspace of $\mathfrak{g}^*$ under the coadjoint action.
- Let Λ stand for the weight lattice of T thought of as a subspace of $\mathfrak{t}^*$.
- We will use the notation e_λ for the character $T \to \mathbb{C}^*$ corresponding to $\lambda \in \Lambda$, and write t^λ for the value of this character on t in T. Thus we have $e_\lambda(t) = t^\lambda$ for $t \in T$, and also $t^\lambda = e^{i\langle\lambda,X\rangle}$ if $X \in \mathfrak{t}$ and $t = \exp(X)$.
- Denote the set of roots of G by $\mathfrak{R}$; these split into a positive and a negative part: $\mathfrak{R} = \mathfrak{R}^+ \cup \mathfrak{R}^-$.
- Write dt for the Haar measure on T satisfying $\int_T dt = 1$.

Further, for $X \in \mathfrak{g}$, we denote by VX the vector field

$$VX : M \to \mathrm{T}M, \quad VX : q \mapsto \frac{d}{dt}e^{-tX}q|_{t=0}$$

on M induced by the G-action.

Atiyah-Bott-Segal-Singer [2, 3, 4] give a fixed point formula for $\chi_{\mathcal{E}}(t)$. The Fourier transform $\mathcal{F}\chi_{\mathcal{E}}$ of $\chi_{\mathcal{E}}$ is then a function with finite support on Λ; its value

$$\mathcal{F}\chi_{\mathcal{E}}(\lambda) = \int_T e_{-\lambda}\chi_{\mathcal{E}}\,dt$$

is an integer, called the *multiplicity* of the weight λ in $\chi_{\mathcal{E}}$. It is then possible to express $\mathcal{F}\chi_{\mathcal{E}}(\lambda)$ in terms of partition functions, the first example of such an expression being Kostant's formula for the multiplicity of a weight λ in a finite dimensional representation of a compact Lie group in terms of the number of ways a weight can be expressed as a sum of positive roots.

Our focus will be the calculation of the dimension of the G-invariant part $Q(M,\mathcal{E})^G$ of $Q(M,\mathcal{E})$, obtained by taking G-invariants on the right hand side of (1):

$$Q(M,\mathcal{E})^G = \mathrm{Ker}(D_{\mathcal{E}})^G \ominus \mathrm{Coker}(D_{\mathcal{E}})^G.$$

According to the Weyl character formula, this integer may be expressed via the multiplicities as follows

$$\dim Q(M,\mathcal{E})^G = \int_T \prod_{\alpha\in\mathfrak{R}^-}(1-e_\alpha)\chi_{\mathcal{E}}\,dt. \tag{4}$$

Consider an equivariant line bundle $\mathcal{L}$ over M, endowed with a G-invariant Hermitian structure and an Hermitian connection ∇. Then the curvature ∇^2 will be equal to $-i\Omega$, where Ω is a closed real 2-form on M. The G-invariant connection ∇ determines a G-equivariant map, the *moment map* $\mu_G : M \to \mathfrak{g}^*$ by the formula:

$$i\langle\mu_G, X\rangle = L_X - \nabla_{VX}, \tag{5}$$

where L_X is the Lie derivative acting on the sections of $\mathcal{L}$. Observe that if $p \in M$ is a fixed point of the T-action, then $\mu_G(p)$ is in $\mathfrak{t}^* \subset \mathfrak{g}^*$, moreover,

$\mu_G(p)$ is exactly the T-weight of the fiber $\mathcal{L}_p$. Differentiating (5), we obtain the key equality

$$\langle d\mu_G, X\rangle + \Omega(VX,\cdot) = 0. \tag{6}$$

The space $\mu_G^{-1}(0)/G$ is called the *reduced space* of M with respect to G. The philosophy of *quantization commutes with reduction* (or [Q,R]=0 for short) is that the virtual space $Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$ may be identified with the virtual space of solutions of a Dirac operator associated to a vector bundle of the form $\mathcal{E}_0\otimes\mathcal{L}_0^k$ on the reduced space $\mu_G^{-1}(0)/G$. This idea was introduced in [10] in the form of a precise conjecture (cf. Theorem 6, and the discussion below). If $\mu_G^{-1}(0)/G$ is smooth, then, using the conjecture and applying the Atiyah-Singer formula (3) to the bundle $\mathcal{E}_0\otimes\mathcal{L}_0^k$, we can conclude that $\dim Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$ depends polynomially on k. This polynomiality thus is a key manifestation of the [Q,R]=0 principle, and it will be our main object of study in this article.

This reduction principle comes from considering the special case when M is a complex projective G-manifold, $\mathcal{L}$ is the ample bundle, and $\mathcal{E}$ is a holomorphic vector bundle on M. Then the G-action on M may be extended to a holomorphic action $G_{\mathbb{C}}\times M\to M$ of the complexification of the compact Lie group G, and [Q,R]=0 follows from the fact that (cf. [16]) the orbit of $\mu_G^{-1}(0)$ by this complexified action of $G_{\mathbb{C}}$ is dense in M.

Returning to the almost complex setting, where no complexified action exists, consider the case where $G=T$ is abelian. In this case, we will write $\mu: M\to \mathfrak{t}^*$ for the moment map, omitting the index T. A special case of the *quantization commutes with reduction* principle then reads as follows.

Theorem 1. *Let $\mathcal{E}^+$ and $\mathcal{E}^-$ be T-equivariant vector bundles over the almost complex manifold M. Let $\mathcal{L}$ be an equivariant line bundle with associated moment map $\mu: M\to\mathfrak{t}^*$. Suppose that the bundle $\mathcal{E}^+$ is equivalent to $\mathcal{E}^-$ over $\mu^{-1}(0)$. Then, for k large, the multiplicities $\mathcal{F}\chi_{\mathcal{E}^+\otimes\mathcal{L}^k}(0)$ and $\mathcal{F}\chi_{\mathcal{E}^-\otimes\mathcal{L}^k}(0)$ are equal.*

Following Meinrenken, we give a proof of this theorem in §5 based on the stationary phase principle applied to the integral formula of [7] for $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$.

Example 2. *Let us consider the simplest example: $M=\mathbb{P}^1(\mathbb{C})$, $\mathcal{L}$ is the dual of the tautological bundle, endowed with an action of the group* SU(2)*. The maximal torus T of this group corresponds to the set of diagonal matrices in* SU(2)*. Then we have the following table:*

k	...	-4	-3	-2	-1	0	1	2	3	...
$\dim Q(\mathcal{L}^k)$	...	-3	-2	-1	0	1	2	3	4	...
$\dim Q(\mathcal{L}^k)^T$	...	-1	0	-1	0	1	0	1	0	...
$\dim Q(\mathcal{L}^k)^{\mathrm{SU}(2)}$	...	0	0	-1	0	1	0	0	0	...

Thus we see that

- $\dim Q(\mathcal{L}^k)=k+1$*; it is thus a polynomial for all $k\in\mathbb{Z}$.*

- $\dim Q(\mathcal{L}^k)^T = \begin{cases} 1, & \text{if } 0 \le k \text{ is even},\\ -1, & \text{if } 0 > k \text{ is even},\\ 0, & \text{if } k \text{ odd}.\end{cases}$
  In particular, this is a quasi-polynomial for all $k \geq 0$.
- $\dim Q(\mathcal{L}^k)^{\mathrm{SU}(2)}$ *is, however, only polynomial for $k \geq 1$. Note that* $\dim Q(\mathcal{L}^{-k})^{\mathrm{SU}(2)}$ *is* not *polynomial for $k \geq 1$.*

This example shows, that, in general, $\dim Q(M, \mathcal{E} \otimes \mathcal{L}^k)^G$ is not polynomial for small k. To make a stronger statement, we set $\mathcal{E}$ to be the trivial line bundle, and introduce a key condition on $\mathcal{L}$ as follows.

Definition 3. *Given an almost complex manifold (M, J), we say that a line bundle $\mathcal{L}$ over M is* positive *if for an Hermitian structure, and a compatible connection ∇ on $\mathcal{L}$, the corresponding curvature $-i\Omega$ satisfies*

$$\Omega_q(V, JV) > 0 \quad \text{for all } 0 \neq V \in T_qM \tag{7}$$

at every point $q \in M$.

Note that in this case Ω is a symplectic form on M.

Remark 4. *The data of a positive line bundle is the same as that of a prequantizable symplectic manifold endowed with a Kostant line bundle (*[12]*); this latter is a G-equivariant line bundle with first Chern class equal to the class of $\Omega/2\pi$. Indeed, if (M, Ω) is a symplectic manifold, then we can choose a unique (up to homotopy) positive almost complex structure J, and then the Kostant property of $\mathcal{L}$ is equivalent to the existence of a G-invariant connection with property* (7)*.*

Definition 5. *Let Ξ be a free $\mathbb{Z}$-module. A function $P : \Xi \to \mathbb{C}$ is* quasi-polynomial *if there exists a sublattice $\Xi_0 \subset \Xi$ of finite index such that for every $\lambda \in \Xi$ the function P restricted to $\lambda + \Xi_0$ coincides with the restriction of a polynomial function from Ξ to Ξ_0.*

In particular, a function $P : \mathbb{Z} \to \mathbb{C}$ is quasi-polynomial if, for some nonzero $d \in \mathbb{Z}$, the function $l \mapsto P(ld + r)$ is polynomial for every $r \in \mathbb{Z}$.

Now we are ready to formulate the polynomiality statement for which we give a new proof. As we explain below, this is a corollary of results of [15].

Theorem 6. *Let (M, J) be a compact, connected almost complex manifold, endowed with the action of a connected compact Lie group G, and let $\mathcal{L}$ be a positive G-equivariant line bundle on M. Then*

- *the integer function*
$$k \to \dim Q(M, \mathcal{L}^k)^G$$
  is quasi-polynomial for $k \geq 1$, and
- *this quasi-polynomial is identically zero if $0 \notin \mu_G(M)$.*

Let us further comment on the relation between this theorem and the original Guillemin-Sternberg conjecture [10]. One may also consult [19] and [23] for more details and references.

Let M be a symplectic manifold equipped with a Kostant line bundle $\mathcal{L}$. If 0 is a regular value of μ_G, then the reduced space $\mu_G^{-1}(0)/G$ is a symplectic orbifold equipped with a Kostant line bundle $\mathcal{L}_0$. Guillemin-Sternberg formulated the conjecture that $Q(M,\mathcal{L})^G$ may be identified to $Q(\mu_G^{-1}(0)/G,\mathcal{L}_0)$.

Meinrenken, in his first approach to the Guillemin-Sternberg conjecture [13], determined the asymptotic behavior of $\dim Q(M,\mathcal{L}^k)^G$ for k large, under the assumption that 0 is a regular value. By a "stationary phase" method (that we borrowed in part for our proof of Theorem 1), he showed that $\dim Q(M,\mathcal{L}^k)^G$ is indeed equal to $\dim Q(\mu_G^{-1}(0)/G,\mathcal{L}_0^k)$ for k sufficiently large, and that the equality holds for all $k\geq 0$ if G is abelian. He has thus shown that the Guillemin-Sternberg conjecture for general compact connected Lie group G is equivalent to the fact that $\dim Q(M,\mathcal{L}^k)^G$ is quasi-polynomial in k for $k\geq 1$.

Meinrenken-Sjamaar in [15] formulated the Guillemin-Sternberg conjecture for the case when 0 is not necessarily a regular value of the moment map, and, using techniques of symplectic cutting, proved this more general statement. There is also an analytic proof of this generalized Guillemin-Sternberg conjecture by Tian and Zhang [21], and another proof by Paradan [17] using transversally elliptic operators. Theorem 6 is a consequence of this statement.

In the present paper, we prove that $\dim Q(M,\mathcal{L}^k)^G$ is quasi-polynomial in k for $k\geq 1$ directly, without making the assumption that 0 is a regular value of the moment map. However, we will not pursue here the task of identifying geometrically the quasi-polynomial $\dim Q(M,\mathcal{L}^k)^G$.

Our main purpose is to show that this result may be obtained from the Atiyah-Bott fixed point formula for $\chi_{\mathcal{L}^k}$, using Theorem 1 as the only analytic input. The rest of the argument is based on combinatorial manipulations of Kostant partition functions and some simple geometric arguments "localizing positivity". The ideas underlying our paper originated in the works of Paradan [17, 18].

The paper is structured as follows: in §2 we describe the calculus of the expansions of the terms of the Atiyah-Bott fixed point formula for $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$ using partition functions (Corollary 10), and then introduce a quasi-polynomial character $\Delta_\mu[\mathcal{E},\mathfrak{a}]$ encoding the asymptotic behavior of this expansion. We begin §3 by Paradan's combinatorial formula decomposing a partition function in terms of convolution products of partitions functions in lower dimensions, then we apply this formula to our geometric setup (Proposition 35), which results in a decomposition of $\chi_\mathcal{E}$ in terms of certain characters of the type Δ_μ enumerated by fixed-point sets of subtori of T. This combinatorial decomposition is analogous at the level of characters to the stratification of the manifold M using the Morse function $\|\mu\|^2$ used by Witten [24] to compute intersection numbers on reduced spaces. We finish the proof of Theorem 6 in §4, by studying the terms of this expansion. The crucial part of the argument is a "localization of positivity" result: Proposition 44. Finally, we give a quick proof of Theorem 1 in §5. A list of notations given in §6 helps the reader to navigate the paper.

2. Fixed point formula and a formal character.

2.1. **The Fixed Point Formula.** As in the previous section, let us start with a connected, compact, almost complex T-manifold M, and a pair $(\mathcal{E}, \mathcal{L})$, consisting of a complex equivariant vector bundle and a line bundle on M. In this section, we embark on the study of the characters $\chi_{\mathcal{E}}$ and $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$. Our starting point is the Atiyah-Bott fixed point formula [2]. To simplify our situation, we assume that the set F of T-fixed points in M is finite. Let us introduce some notation for the fixed point data.

We will use the term *list* for set with multiplicities. A list Φ thus consists of a set $\{\Phi\}$, and a multiplicity function $m_\Phi : \{\Phi\} \to \mathbb{Z}_{>0}$. We will write

- $\psi \in \Phi$ if $\psi \in \{\Phi\}$;
- if $\psi \in \Phi$ and $m_\Phi(\psi) > 1$, then $\Phi - \{\psi\}$ will denote the list Φ with the multiplicity of ψ decreased by 1; if $m_\Phi(\psi) = 1$, then $\Phi - \{\psi\}$ will denote the list Φ with ψ removed;
- for a list Φ and a set S, we will write $\Phi \cap S$ for the list with underlying set $\{\Phi\} \cap S$ and multiplicity function coinciding with that of Φ on this set; we will write $\Phi \setminus S$ for the list with underlying set $\{\Phi\} \setminus S$ and multiplicity function coinciding with that of Φ on this set;
- the product $\prod_{\psi\in\Phi} \psi$ will stand for the product $\prod_{\psi\in\{\Phi\}} \psi^{m_\Phi(\psi)}$.

For example, for each fixed point $p \in F$, the weights of the T-action on $\mathcal{E}_p$ form a list, which we will denote by Ψ_p. Let ε_p be the function $T \to \mathbb{C}$ obtained by taking the trace of the T-action on the fiber $\mathcal{E}_p$. Thus we have $\varepsilon_p = \sum_{\eta\in\Psi_p} e_\eta$. Similarly, we denote by Φ_p the list of T-weights of the complex vector space $\overline{\mathrm{T}}{}^J_p M$.

Without loss of generality, we can make the additional simplifying assumption that the generic stabilizer of the T-action on M is finite; this is equivalent to the condition that Φ_p spans $\mathfrak{t}^*$ for all $p \in F$. With these preparations we can state the Atiyah-Bott fixed point formula for our case: it is an equality between two functions defined on an open and dense subset of T.

$$\chi_{\mathcal{E}} = \sum_{p\in F} \frac{\varepsilon_p}{\prod_{\phi\in\Phi_p}(1 - e_\phi)}. \tag{8}$$

Indeed, the right hand side is meaningful on the set

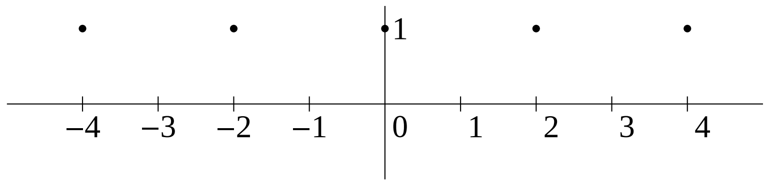
$$\{t \in T |\, t^\phi \neq 1\ \forall p \in F \text{ and } \phi \in \Phi_p\},$$

while the left hand side is regular on T. Let us see a few simple examples.

Example 7. *Let $M = P^1(\mathbb{C})$ with the action of $\mathrm{U}(1)$ given by $t\cdot(x : y) = (tx : t^{-1}y)$, and let $\mathcal{E} = \mathcal{L}^k$ be the kth tensor power of the dual of the tautological* *line bundle. There are 2 fixed points $p^+ = (1 : 0)$ and $p^- = (0 : 1)$, and we have*

$$\chi_{\mathcal{E}}(t) = \frac{t^k}{(1 - t^{-2})} + \frac{t^{-k}}{(1 - t^2)} = \sum_{j=0}^{k} t^{k-2j}.$$

The graph of the function $\mathcal{F}\chi_{\mathcal{E}}$ is pictured below for $k = 4$.



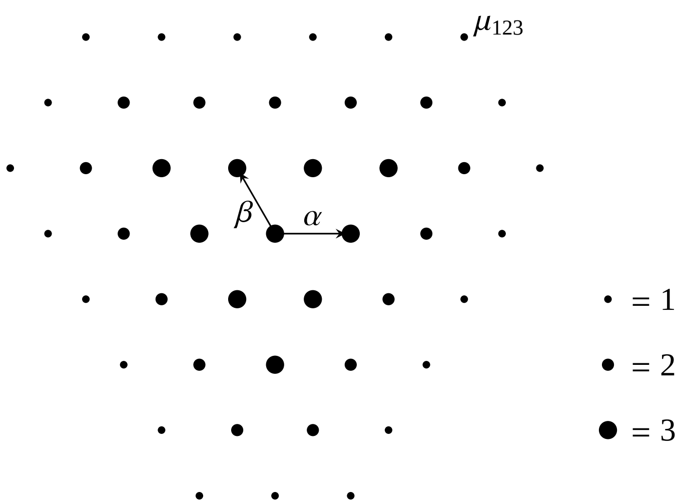
Example 8. *Let M be the flag variety of $\mathbb{C}^3$ endowed with the action of the group $\mathrm{U}(3)$. The subgroup $T = \{(t_1, t_2, t_3);\ t_1, t_2, t_3 \in \mathrm{U}(1)\} \subset \mathrm{U}(3)$ of diagonal matrices serves as the maximal torus in this case, and the weight lattice has a canonical diagonal bases: $\Lambda = \mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3$. Then the coordinate flag $\{\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3\}$ is invariant under T, and the rest of the fixed points in M^T may be obtained by applying to this flag the elements of the permutations group Σ_3 in a natural manner. We will use the notation $w \in \Sigma_3 \mapsto p_w \in M^T$ for this correspondence; in particular, the coordinate flag will be denoted by p_{123}.*

Consider the line bundle $\mathcal{L}$ induced from the character $t_1^4 t_2^{-1} t_3^3$ of T. Then

$$\chi_{\mathcal{L}^k} = \sum_{w \in \Sigma_3} w * \frac{t_1^{4k} t_2^{-k} t_3^{3k}}{(1 - t_2/t_1)(1 - t_3/t_2)(1 - t_3/t_1)},$$

where, again, $w*$ stands for the natural action of Σ_3 on the indices.

The function $\mathcal{F}\chi_{\mathcal{L}^k}$ (for $k = 1$) is depicted below on the root lattice generated by the simple roots $\alpha = \theta_1 - \theta_2$ and $\beta = \theta_2 - \theta_3$. The weight μ_{123} of the bundle $\mathcal{L}$ at p_{123} is $4\alpha + 3\beta$.



2.2. **The partition function.** Recall that $\chi_{\mathcal{E}}$ is determined by its Fourier coefficients $\mathcal{F}\chi_{\mathcal{E}} : \Lambda \to \mathbb{Z}$, and that this latter function has finite support in Λ. Our immediate goal is to write down (8) as an equality of two functions in the Fourier dual space $\{\Lambda \to \mathbb{Z}\}$.

Before we proceed, we need to introduce a few basic notions.

- We denote by $R(T)$ the set of finite integral linear combinations of the characters e_λ, $\lambda \in \Lambda$, and
- by $\hat{R}(T)$ the space of formal, possibly infinite, integral linear combinations of these characters. Thus the elements of $\hat{R}(T)$ are in one-to-one correspondence with the functions $m(\lambda) : \Lambda \to \mathbb{Z}$ via $\theta := \sum_{\lambda\in\Lambda} m(\lambda)e_\lambda \in \hat{R}(T)$. We will write $\mathcal{F}\theta$ for the function m in this case. Conversely, given a function m, we will call the corresponding series θ its character. If we extend the weights $\lambda \in \Lambda$ to linear functions on $\mathfrak{t}_{\mathbb{C}}$, then we can also think of the elements of $\hat{R}(T)$ as formal series of holomorphic exponential functions on $\mathfrak{t}_{\mathbb{C}}$.
- Informally, we will call $\delta \in \hat{R}(T)$ a *quasi-polynomial character* if its Fourier transform $\mathcal{F}\delta : \Lambda \to \mathbb{C}$ is quasi-polynomial (cf. Definition 5).

Observe that $\hat{R}(T)$ is a module over $R(T)$ and the set of quasi-polynomial characters forms a linear subspace in $\hat{R}(T)$ which is stable under multiplication by $R(T)$. In addition, elements of $\hat{R}(T)$ whose Fourier transforms are supported on a fixed acute cone may be multiplied.

With these preparations, we are ready to introduce the basic building block of our constructions. Let Φ be a list of nonzero elements of Λ. We will call $X \in \mathfrak{t}$ *polarizing* for Φ if $\langle\phi, X\rangle \neq 0$ for every $\phi \in \Phi$. For nonempty Φ and polarizing X, define the partition $\Phi = \Phi_+ \cup \Phi_-$ of Φ into

$$\Phi_+ = \{\phi \in \Phi \,|\, \langle\phi, X\rangle > 0\} \quad \text{and} \quad \Phi_- = \{\phi \in \Phi \,|\, \langle\phi, X\rangle < 0\},$$

and introduce the formal character

$$\Theta[\Phi\uparrow X] = (-1)^{|\Phi_-|} \prod_{\phi\in\Phi_-} e_{-\phi} \times \prod_{\phi\in\Phi_-} \sum_{k=0}^{\infty} e_{-k\phi} \times \prod_{\phi\in\Phi_+} \sum_{k=0}^{\infty} e_{k\phi}. \tag{9}$$

It is easy to verify that the products in this formula are meaningful, and hence the series $\Theta[\Phi\uparrow X]$ defines an element of $\hat{R}(T)$. We also set $\Theta[\emptyset\uparrow X] = 1$ for any $X \in \mathfrak{t}$.

We record a few basic properties of $\Theta[\Phi\uparrow X]$.

Lemma 9. (1) $\Theta[\Phi\uparrow X]$ *satisfies* $\prod_{\phi\in\Phi}(1 - e_\phi)\,\Theta[\Phi\uparrow X] = 1$.

(2) *Considered as a series of holomorphic functions of the form* $e^{i\lambda}$ *with* $\lambda \in \Lambda$, *the series* (9) *converges absolutely in a neighborhood of* $iX \in \mathfrak{t}_{\mathbb{C}}$.

(3) $\mathcal{F}\Theta[\Phi\uparrow X]$ *is supported in the pointed cone generated in* $\mathfrak{t}^*$ *by the set* $\Phi^+ \cup (-\Phi^-)$.

Note that if two formal characters in $\hat{R}(T)$ absolutely converge, and coincide on an open set of $\mathfrak{t}_{\mathbb{C}}$, then they also coincide as elements of $\hat{R}(T)$. In particular, it follows from Lemma 9 that in their domain of convergence the formal characters $\Theta[\Phi \uparrow X]$ are equal to the function $\prod_{\phi\in\Phi}(1 - e_\phi)^{-1}$ for any X.

This allows us to rewrite (8) as follows:

Corollary 10. *We have the following equality in* $\hat{R}(T)$*:*

$$\chi_{\mathcal{E}} = \sum_{p\in F} \varepsilon_p \Theta[\Phi_p \uparrow X]. \tag{10}$$

Remark 11. *The function* $\mathcal{F}\Theta[\Phi \uparrow X] : \Lambda \to \mathbb{Z}$*, traditionally, has been called the* partition function*, since, assuming* $\Phi = \Phi^+$*, its value at* μ *equals the number of ways one can write* μ *as a nonnegative integral linear combinations of vectors from* Φ*. In particular, the equality* (10) *applied to Weyl's formula for the characters leads to Kostants's formula for the multiplicity of a weight in an irreducible representation of a reductive Lie group.*

A key fact is that the Fourier transform $\mathcal{F}\Theta[\Phi \uparrow X]$, as a function on Λ, is piecewise quasi-polynomial. Let us explain this in more detail:

Definition 12. *Given a list* Φ *spanning* $\mathfrak{t}^*$*, we will call an element* $\gamma \in \mathfrak{t}^*$ Φ-regular *if it is not the linear combination of fewer than* $\dim(\mathfrak{t})$ *elements of* Φ*.*

The set of Φ-regular elements form the complement of a hyperplane arrangement in $\mathfrak{t}^*$, and we will use the term Φ-*tope* for the connected components of this set. It will be convenient to use the notation $\mathcal{T}(\gamma)$ for the tope containing the Φ-regular element γ. Note that topes are open convex cones, which are invariant under rescaling.

Lemma 13. *Given* Φ *and* X *as above, and a* Φ*-tope* $\mathcal{T}$*, there exists a quasi-polynomial character* $\delta[\Phi \uparrow X, \mathcal{T}]$ *such that* $\mathcal{F}\Theta[\Phi \uparrow X]$ *coincides with* $\mathcal{F}\delta[\Phi \uparrow X, \mathcal{T}]$ *on* $\Lambda \cap \mathcal{T}$*.*

This lemma is proved in [9] (see also [8]).

Example 14. *Let* $\mathfrak{t}^* = \mathbb{R}\alpha$*,* $\Lambda = \mathbb{Z}\alpha$*,* $\Phi = [\alpha]$ *and let* $X \in \mathfrak{t}$ *to be the vector satisfying* $\langle \alpha, X\rangle = 1$*. Then*

$$\Theta[\Phi \uparrow X] = \sum_{k=0}^{\infty} e_{k\alpha}.$$

Then $\mathcal{T} := \{t\alpha, t > 0\}$ *is a tope and* $\delta[\Phi \uparrow X, \mathcal{T}] = \sum_{k\in\mathbb{Z}} e_{k\alpha}$*. The character* $\delta[\Phi \uparrow X, \mathcal{T}]$ *is quasi-polynomial as the multiplicity* $\mathcal{F}\delta[\Phi \uparrow X, \mathcal{T}]$ *is the constant function* 1 *on* $\mathbb{Z}\alpha$*.*

2.3. **The asymptotics of the character.** Now we return to our geometric setup. We continue to assume that the torus T acts on the compact almost complex manifold M with a finite set of fixed points. Also, recall our notation $\mu : M \to \mathfrak{t}^*$ for the moment map associated to the Hermitian T-equivariant line bundle $\mathcal{L}$ as in §1, and the fact that for $p \in F$, $\mu(p)$ is the weight of the T-action on the fiber $\mathcal{L}_p$.

Definition 15. *For $p \in F$, and a subset $S \subset \mathfrak{t}^*$ denote by S^p the shifted subset $S - \mu(p)$. In particular, we have $\gamma^p = \gamma - \mu(p)$ for $\gamma \in \mathfrak{t}^*$.*

The moment map μ gives rise to a real affine hyperplane arrangement whose complement is the open set

$$\bigcap_{p\in F}\left\{\gamma \in \mathfrak{t}^* | \, \gamma^p \text{ is } \Phi_p\text{-regular}\right\} \subset \mathfrak{t}^*. \tag{11}$$

We will use the term *alcove* for the connected components of the set (11). The alcoves are thus minimal nonempty intersections of the translated polyhedral cones $\mathcal{T} + \mu(p)$, where $p \in F$, and $\mathcal{T}$ is a tope of Φ_p. Just as in the case of topes, we will use the notation $\mathfrak{a}(C)$ for the alcove containing the connected subset C of the set (11).

Remark 16. *If $\mathcal{L}$ is a positive line bundle as in Definition 3, then $\mu(M)$ is the convex hull of the points $\mu(p)$, where $p \in F$, and the set (11) is contained in the set of regular values of μ.*

Next, we define a quasi-polynomial character by formally replacing the generating function for the partition function $\Theta[\Phi_p \uparrow X]$ in (10) by appropriately chosen corresponding quasi-polynomials $\delta[\Phi_p \uparrow X, \mathcal{T}]$ (cf. Lemma 13).

Definition 17. *Given a T-equivariant vector bundle $\mathcal{E}$ over M, and an alcove $\mathfrak{a} \subset \mathfrak{t}^*$, we define the formal character*

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}] = \sum_{p\in F} \varepsilon_p \cdot \delta[\Phi_p \uparrow X, \mathcal{T}(\mathfrak{a}^p)], \tag{12}$$

where $\mathfrak{a}^p = \mathfrak{a} - \mu(p)$.

Remark 18. *Note that we omitted the dependence on X in the notation (cf. Corollary 23).*

This character acquires a geometric meaning for the sequence of bundles $\mathcal{E} \otimes \mathcal{L}^k$, $k = 1, 2, \dots$ (cf. Lemma 21 and Proposition 22).

The relation of the character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ to symplectic reduction may be described as follows. In the case when the moment map $\mu : M \to \mathfrak{t}^*$ is associated to a positive line bundle (and hence Ω is a nondegenerate 2-form), any element γ in an alcove $\mathfrak{a}$ is a regular value of μ, and the torus T acts with finite stabilizers on $\mu^{-1}(\gamma)$. Then the set $\mu^{-1}(\gamma)/T$ is an orbifold, which does not change if γ remains in $\mathfrak{a}$, and thus we can denote it by $M_\mathfrak{a}$. The bundle $\mathcal{E}$ descends to an orbifold bundle $\mathcal{E}_\mathfrak{a}$ on $M_\mathfrak{a}$, and each character λ allows us to twist $\mathcal{E}_\mathfrak{a}$ by the associated line bundle $L_\lambda = \mu^{-1}(\gamma) \times_T \mathbb{C}_\lambda$ over $M_\mathfrak{a}$. We then consider the integer $\dim(Q(M_\mathfrak{a}, \mathcal{E}_\mathfrak{a} \otimes L_\lambda))$, which, according to the Atiyah-Singer index formula, is a polynomial function of λ if $M_\mathfrak{a}$ is smooth. In the general case, when $M_\mathfrak{a}$ is an orbifold, the index formula for orbifolds ([1], see also [22]) implies that the function $\lambda \to \dim(Q(M_\mathfrak{a}, \mathcal{E}_\mathfrak{a} \otimes$

L_λ)) is a quasi-polynomial. In this setup, the character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ appears as the generating function of this quasi-polynomial:

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}] = \sum_\lambda \dim Q(M, \mathcal{E} \otimes L_\lambda)\, e_\lambda.$$

We will not use this interpretation in what follows.

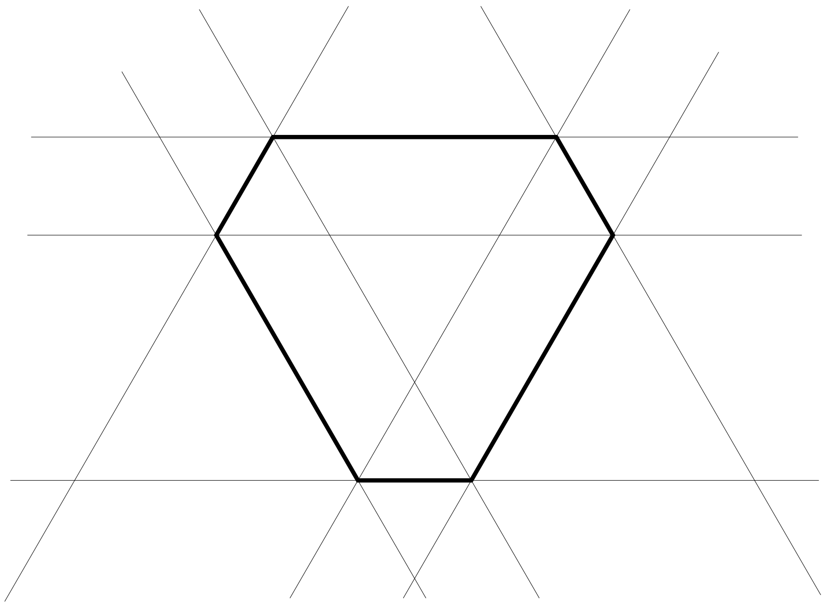
Example 19. *We return to Example 7, with μ is associated to the dual of the tautological bundle over $P^1(\mathbb{C})$. Then there are* 3 *alcoves*

$$\mathfrak{a}_1 :=]-\infty, -1[; \quad \mathfrak{a}_2 :=]-1, 1[; \quad \mathfrak{a}_3 :=]1, \infty[$$

For $\mathcal{E}$ the trivial bundle, we have

$$\Delta_\mu[\mathcal{E}, \mathfrak{a}_1] = 0; \quad \Delta_\mu[\mathcal{E}, \mathfrak{a}_2](n) = 1; \quad \Delta_\mu[\mathcal{E}, \mathfrak{a}_3] = 0.$$

Example 20. *We return to Example 8, with μ associated to the line bundle $\mathcal{L}$. The straight lines cut the plane into alcoves. The support of the multiplicity function $\mathcal{F}\chi_{\mathcal{L}}$ is the highlighted hexagon.*



Since $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ is a linear combination of quasi-polynomial characters, it is itself quasi-polynomial. The following extension of this fact holds.

Lemma 21. *The function* $(\lambda, k) \mapsto \mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](\lambda)$ *is quasi-polynomial on the lattice* $\Lambda \times \mathbb{Z}$.

Proof. Indeed, we have

$$\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}] = \sum_{p \in F} \varepsilon_p e_{k\mu(p)} \delta[\Phi_p \uparrow X, \mathfrak{a}^p].$$

The statement now follows from the fact that for a formal character $\theta \in \hat{R}(T)$, and $\lambda, \nu \in \Lambda$, we have $\mathcal{F}e_{k\mu}\theta(\lambda) = \mathcal{F}\theta(\lambda - k\mu)$. ◻

For small k, in particular for $k = 0$, $\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}]$ does not have any direct relationship with $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$. Since ε_p for $p \in F$ is the trace of a finite-dimensional representation, its multiplicity function $\mathcal{F}\varepsilon_p$, $p \in F$ has finite support in Λ. For large k, we have the following statement.

Proposition 22. *Let $\mathfrak{b}$ be a compact subset of an alcove $\mathfrak{a}$. Then there exist a positive integer K such that for every $k > K$ and $\lambda \in k\mathfrak{b} \cap \Lambda$, the equality*

$$\mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](\lambda) = \mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) \tag{13}$$

holds.

Proof. Recall the notation Ψ_p introduced for the list of weights of $\mathcal{E}_p$ at $p \in F$. According to (10), we have

$$\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) = \sum_{p\in F} \sum_{\eta\in\Psi_p} \mathcal{F}\Theta[\Phi_p \uparrow X](\lambda - \eta - k\mu(p)),$$

while, by Lemma 21,

$$\mathcal{F}\Delta_\mu[\mathcal{E} \otimes \mathcal{L}^k, \mathfrak{a}](\lambda) = \sum_{p\in F} \sum_{\eta\in\Psi_p} \mathcal{F}\delta[\Phi_p \uparrow X, \mathcal{T}(\mathfrak{a}^p)](\lambda - \eta - k\mu(p)).$$

Hence, by the definition of the quasi-polynomial character δ given in Lemma 13, these two expressions coincide as long as for each $p \in F$ and $\eta \in \Psi_p$, we have $\lambda - \eta - k\mu(p) \in \mathcal{T}(\mathfrak{a}^p)$. Since topes are invariant under rescaling, and $\mathfrak{a}^p = \mathfrak{a} - \mu(p)$, we can conclude that (13) holds if

$$\frac{\lambda}{k} - \frac{\eta}{k} \in \mathfrak{a} \quad \text{for each } \eta \in \cup_{p\in F}\Psi_p. \tag{14}$$

As the set $\cup_{p\in F}\Psi_p$ is finite, for large enough k, we will have $\mathfrak{b} - \eta/k \in \mathfrak{a}$ for every η from this set. Hence (14) holds for large enough k, uniformly in $\lambda \in k\mathfrak{b} \cap \Lambda$. This completes the proof. □

Corollary 23. *The quasi-polynomial character $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ does not depend on the choice of the polarizing vector X.*

Indeed, assume that the subset $\mathfrak{b} \subset \mathfrak{a}$ in Proposition 22 contains an open set U, and observe that the set of pairs (ξ, t) satisfying the condition $t > K, \xi \in t\mathfrak{b}$ with K as in Proposition 22 contains a translate of the open cone $\{(r\gamma, r)| \, r > 0, \gamma \in U\} \subset \mathfrak{t}^* \times \mathbb{R}$. According to Lemma 21, $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ is quasi-polynomial on $\Lambda \times \mathbb{Z}$, and hence the right hand side of (13) completely determines the left hand side. Since $\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}$ does not depend on X, we can conclude that neither does $\mathcal{F}\Delta_\mu[\mathcal{E}, \mathfrak{a}]$.

In what follows, we will need the extension of the definition of $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ to the case when the generic stabilizer of the T action is not finite.

Definition 24. *Suppose the Lie group G with Lie algebra $\mathfrak{g}$ acts on a manifold M. Then for a subset $C \subset M$ we denote by*

$$\mathfrak{g}_C = \{X \in \mathfrak{g};\ VX \text{ vanishes on } C\}$$

the stabilizer subalgebra of $\mathfrak{g}$, *and by* G_C *the connected subgroup of* G *with Lie algebra* $\mathfrak{g}_C$.

In particular, T_M is the *connected component* of the generic stabilizer of M containing the identity element, and $\mathfrak{t}_M \subset \mathfrak{t}$ is the Lie algebra of T_M. Then for every $p \in F$, the weights Φ_p span the annihilator $\mathfrak{t}_M^\perp \subset \mathfrak{t}^*$.

Clearly, the group T_M acts on each of the fibers $\mathcal{E}_q$, $q \in M$, and since M is connected, this representation does not depend on q. In particular, for two fixed points p and q in F, the weights $\mu(p)$ and $\mu(q)$ of T differ by an element of $\mathfrak{t}_M^\perp$. Thus the affine-linear subspace

$$A_M = \mu(p) + \mathfrak{t}_M^\perp \tag{15}$$

of $\mathfrak{t}^*$ does not depend on $p \in F$. Note that according to equation (6), the image $\mu(M)$ is contained in A_M.

Now we can repeat the definitions (11) and (12) with $\mathfrak{t}^*$ replaced by $\mathfrak{t}_M^\perp$. More precisely, we consider the open set in A_M consisting of those elements γ for which $\gamma^p \in \mathfrak{t}_M^\perp$ is Φ_p-regular for any $p \in F$. An alcove $\mathfrak{a}$ is a connected component of this open set. For an alcove $\mathfrak{a}$, we denote by $\mathcal{T}(\mathfrak{a}^p)$ the Φ_p-tope in $\mathfrak{t}_M^\perp$ containing $\mathfrak{a}^p$. The formal character $\Delta_\mu[\mathcal{E},\mathfrak{a}]$ may be defined by equation (12). The multiplicity function $\mathcal{F}\Delta_\mu[\mathcal{E},\mathfrak{a}]$ is then supported on a finite number of translates (by vectors from the set $\cup_{p\in F}\{\Psi_p\}$) of $\mathfrak{t}_M^\perp \cap \Lambda$, and it is quasi-polynomial on each translate.

Denote by $\mathbb{C}_\lambda$ the trivial line bundle over M endowed with the action e_λ of T. As an immediate consequence of the decomposition

$$\mathcal{E} = \bigoplus_{\lambda\in\Lambda/\Lambda\cap\mathfrak{t}_M^\perp} \mathbb{C}_\lambda \otimes (\mathcal{E}\otimes\mathbb{C}_{-\lambda})^{T_M}, \tag{16}$$

one has the explicit formula

$$\Delta_\mu[\mathcal{E},\mathfrak{a}] = \sum_{\lambda\in\Lambda/\Lambda\cap\mathfrak{t}_M^\perp} e_\lambda \Delta_\mu[(\mathcal{E}\otimes\mathbb{C}_{-\lambda})^{T_M},\mathfrak{a}], \tag{17}$$

where the sum is understood as taken over any system of representatives of the quotient. Note that this formula expresses the formal T-character $\Delta_\mu[\mathcal{E},\mathfrak{a}]$ through quasi-polynomial characters for the torus $T' = T/T_M$, which acts on M with generic finite stabilizer.

We have the following simple corollary of (17):

Lemma 25. *If for some* $\lambda \in \Lambda$, *the multiplicity* $\mathcal{F}\Delta_\mu[\mathcal{E},\mathfrak{a}](\lambda)$ *is not zero, then the restriction of* λ *to* $\mathfrak{t}_M$ *is a weight of the representation of* T_M *on a fiber of* $\mathcal{E}$.

We end this section with a comment on the situation, where the affine space A_M given by equation (15) is linear, i.e. passes through the origin. This is equivalent to the condition that T_M acts trivially on the fibers of $\mathcal{L}$, i.e. $\mathcal{L}$ is a T/T_M-line bundle.

Lemma 26. *Let* $\mathcal{E}$ *be a* T*-bundle, and* $\mathcal{L}$ *be a* T/T_M*-line bundle on* M. *Then* $k \mapsto \mathcal{F}\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k,\mathfrak{a}](0)$ *is a quasi-polynomial function of* k.

Proof. Applying (17) to the bundle $\mathcal{E}\otimes\mathcal{L}^k$, and using the condition on $\mathcal{L}$, we obtain the equality

$$\Delta_\mu[\mathcal{E}\otimes\mathcal{L}^k,\mathfrak{a}](0)=\Delta_\mu[\mathcal{E}^{T_M}\otimes\mathcal{L}^k,\mathfrak{a}](0).$$

Since $\mathcal{E}^{T_M}$ is a T/T_M-equivariant vector bundle, we can replace T by T/T_M. According to Lemma 21, $\mathcal{F}\Delta_\mu[\mathcal{E}^{T_M}\otimes\mathcal{L}^k,\mathfrak{a}](\lambda)$ is quasi-polynomial in $(\lambda,k)\in(\mathfrak{t}_M^\perp\cap\Lambda)\times\mathbb{Z}$, and hence $\mathcal{F}\Delta_\mu[\mathcal{E}^{T_M}\otimes\mathcal{L}^k,\mathfrak{a}](0)$ is a quasi-polynomial function of k. □

3. Decomposition of partition functions

3.1. The decomposition formula. In this section, we prove a decomposition formula for the generating function $\Theta[\Phi\uparrow X]$ of the partition function introduced in (9). This formula is due to Paradan [18], and it will serve as the combinatorial engine of our proof of Theorem 6.

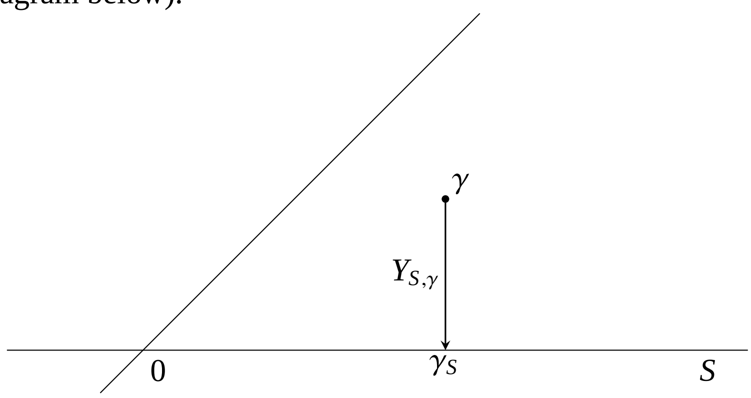
Definition 27. *Given a list Φ of weights in $\Lambda\subset\mathfrak{t}^*$, introduce the set of* **Φ-rational subspaces**

$$\mathcal{RS}(\Phi)=\{S\subset\mathfrak{t}^*\ \textit{linear};\ \Phi\cap S\ \textit{spans}\ S\}.$$

This is the set of linear subspaces of $\mathfrak{t}^*$ spanned by some subset of Φ:

Remark 28. *1. Note that $\{0\}\in\mathcal{RS}(\Phi)$, and $\mathfrak{t}^*\in\mathcal{RS}(\Phi)$ if Φ spans $\mathfrak{t}^*$.*
2. Comparing this definition to Definition 12, we see that all subspaces $S\in\mathcal{RS}(\Phi)$, except for $S=\mathfrak{t}^*$, consist of nonregular elements.

Now, fix a positive definite scalar product $(\cdot,\cdot)$ on $\mathfrak{t}^*$, and denote by $\gamma\mapsto\gamma^*$ the induced linear bijection between $\mathfrak{t}^*$ and $\mathfrak{t}$. For each rational subspace $S\in\mathcal{RS}(\Phi)$ and $\gamma\in\mathfrak{t}^*$, introduce the notation γ_S for the orthogonal projection of γ onto S, and $Y_{S,\gamma}$ for the vector $(\gamma_S-\gamma)^*\in\mathfrak{t}$ (see the diagram below).



Finally, recall from Lemma 13 that, on a Φ-tope $\mathcal{T}$, the partition function $\mathcal{F}\Theta[\Phi\uparrow X]$ coincides with a quasi-polynomial $\mathcal{F}\delta[\Phi\uparrow X,\mathcal{T}]:\Lambda\to\mathbb{Z}$. It is thus natural to compare the two functions at all points of Λ. As we will see, the difference may be expressed as a sum of (convolution) products of partition functions and quasi-polynomials coming from lower-dimensional systems.

Proposition 29. *(Paradan) Let Φ be a list of vectors in Λ, and let X be a polarizing vector for Φ (see the definition of Θ in* (9)*). Assume that $\gamma \in \mathfrak{t}^*$ is such that for every $S \in \mathcal{RS}(\Phi)$, the projection $\gamma_S \in S$ is $(\Phi \cap S)$-regular, while the orthogonal component $Y_{S,\gamma}$ is polarizing for $\Phi \backslash S$. Then*

$$\Theta[\Phi \uparrow X] = \sum_{S \in \mathcal{RS}(\Phi)} \Theta[\Phi \backslash S \uparrow Y_{S,\gamma}] \cdot \delta[\Phi \cap S \uparrow X, \mathcal{T}(\gamma_S)]. \tag{18}$$

Observe that if $-\gamma$ is in the dual cone to the cone generated by $\Phi^+ \cup -\Phi^-$, then all the terms but the one corresponding to $S = \{0\}$ vanish, and hence, in this case, the identity (18) is tautological.

Example 30. *Let $\mathfrak{t}^* = \mathbb{R}\alpha$, $\Lambda = \mathbb{Z}\alpha$, $\Phi := [\alpha]$ and set $X \in \mathfrak{t}$ to be the vector satisfying $\langle \alpha, X\rangle = 1$. Then*

$$\Theta[\Phi \uparrow X] = \sum_{k=0}^{\infty} e_{k\alpha}.$$

The set $\mathcal{RS}(\Phi)$ has two elements: $S = \{0\}$ and $S = \mathfrak{t}^*$.

If we let $\gamma = t\alpha$ for some $t > 0$, then on the right hand side of (18) *we have*

- $\delta[\Phi \uparrow X, \mathcal{T}(\gamma_S)] = \sum_{k \in \mathbb{Z}} e_{k\alpha}$ *for $S = \mathfrak{t}^*$, and*
- $\Theta[\Phi \uparrow Y_{S,\gamma}] = -\sum_{k>0} e_{-k\alpha}$*, for $S = \{0\}$.*

Then formula (18) *reads:*

$$\Theta[\Phi \uparrow X] = \delta[\Phi \uparrow X, \mathcal{T}(\gamma_S)] + \Theta[\Phi \uparrow Y_{S,\gamma}].$$

Proof of Proposition 29. Replacing γ by its orthogonal projection on the subspace generated by Φ, we may assume that V is spanned by Φ.

Observe that for each term on the right hand side of (18), the Fourier transform restricted to a tope of Φ is quasi-polynomial. We show that the Fourier transforms of the two sides of (18) coincide.

We start by showing that the Fourier coefficients of the two sides coincide on the tope $\mathcal{T}(\gamma)$. The term corresponding to $S = \mathfrak{t}^*$ is $\delta[\Phi \uparrow X, \mathcal{T}(\gamma)]$, whose Fourier coefficients coincide with those of $\Theta[\Phi \uparrow X]$ on the tope $\mathcal{T}(\gamma)$. For any $S \in \mathcal{RS}(\Phi)$ different from $\mathfrak{t}^*$, by construction, the Fourier transform of the corresponding term $\Theta[\Phi \backslash S \uparrow Y_{S,\gamma}] \cdot \delta[\Phi \cap S \uparrow X, \mathcal{T}(\gamma_S)]$ is a function on Λ supported on the subset $\{\lambda;\ \langle \lambda, Y_{S,\gamma}\rangle \geq 0\}$ (cf. Lemma 9). Since $\langle \gamma, Y_{S,\gamma}\rangle = -|\gamma_S - \gamma|^2 < 0$, we see that this function vanishes on a conic neighborhood of the half line $\mathbb{R}^+\gamma$. This implies that the Fourier coefficients of all these terms vanish on all of $\mathcal{T}(\gamma)$, and thus indeed, the Fourier coefficients of the two sides of (18) coincide on $\mathcal{T}(\gamma)$.

To extend the equality of Fourier coefficients to the rest of Λ, we use induction on the number of elements in Φ. If Φ is empty, then both sides are equal to 1. Now pick an element $\phi \in \Phi$, and consider $\Phi' = \Phi - \{\phi\}$ (cf. the beginning of §2 for our conventions). Clearly $(1-e_\phi)\cdot\Theta[\Phi \uparrow X] = \Theta[\Phi' \uparrow X]$. If we restrict the Fourier transform of this equation to a tope $\mathcal{T}$, we obtain

$$(1 - e_\phi)\, \delta[\Phi \uparrow X, \mathcal{T})] = \delta[\Phi' \uparrow X, \mathcal{T}']$$

if Φ' generates V and $\mathcal{T}'$ is the tope of Φ containing $\mathcal{T}$, while

$$(1 - e_\phi)\,\delta[\Phi \uparrow X, \mathcal{T})] = 0$$

if Φ' does not generate V.

We multiply both sides of (18) by $(1 - e_\phi)$, and compare the results. On the left hand side, we end up with $\Theta[\Phi' \uparrow X]$. For a term on the right hand side corresponding to $S \in \mathcal{RS}(\Phi)$, we separate 3 cases:

1. $\phi \notin S$**:** In this case, $S \in \mathcal{RS}(\Phi')$, $\Phi \cap S = \Phi' \cap S$ and

$$(1 - e_\phi) \cdot \Theta[\Phi \backslash S \uparrow Y_{S,\gamma}] = \Theta[\Phi' \backslash S \uparrow Y_{S,\gamma}].$$

Thus, after multiplication by $(1 - e_\phi)$, we end up with the term

$$\Theta[\Phi' \backslash S \uparrow Y_{S,\gamma}] \cdot \delta[\Phi' \cap S \uparrow X, \mathcal{T}(\gamma_S)]. \tag{19}$$

2. $\phi \in S$**, and** $S \in \mathcal{RS}(\Phi')$**:** In this case $\Phi \backslash S = \Phi' \backslash S$ while $(\Phi \cap S) - \{\phi\} = \Phi' \cap S$, which implies that

$$(1 - e_\phi)\,\delta[\Phi \cap S \uparrow X, \mathcal{T}(\gamma_S)] = \delta[\Phi' \cap S \uparrow X, \mathcal{T}'(\gamma_S)].$$

Thus we end up with the term (19) again.

3. $\phi \in S$**, and** $S \notin \mathcal{RS}(\Phi')$**:** In this case,

$$(1 - e_\phi)\,\delta[\Phi \cap S \uparrow X, \mathcal{T}(\gamma_S)] = 0.$$

Thus multiplying the right hand side of (18) by $(1 - e_\phi)$ has the effect of replacing Φ by Φ'. Using the inductive assumption, we can conclude that after multiplying both sides of (18) by $(1 - e_\phi)$ for any $\phi \in \Phi$, we obtain an identity. As Φ spans $\mathfrak{t}^*$, this implies that the Fourier coefficients of the difference of the two sides of (18) form a periodic function with respect to the sublattice of finite index in Λ generated by Φ. Since we also know that these coefficients vanish on $\mathcal{T}(\gamma)$, they must vanish on all of Λ. This completes the proof. □

3.2. **Paradan's decomposition of a character.** In this paragraph, we substitute the decomposition (18) into formula (10), and then find a geometric interpretation of the resulting expression.

Note that while performing this substitution, we have the freedom of varying the vector γ in (18) depending on the fixed point p. We take advantage of this possibility, by shifting γ by $\mu(p)$ for $p \in F$, and thus replacing γ by the vector $\gamma^p = \gamma - \mu(p)$.

With this choice, we obtain

$$\chi_\mathcal{E} = \sum_{p \in F} \sum_{S \in \mathcal{RS}(\Phi_p)} \varepsilon_p \cdot \Theta[\Phi_p \backslash S \uparrow Y_{S,\gamma^p}] \cdot \delta[\Phi_p \cap S \uparrow X, \mathcal{T}(\gamma_S^p)]. \tag{20}$$

Now we turn to the geometric meaning of this sum. Recall that each $X \in \mathfrak{t}$ defines a vector field VX on M; this vector field vanishes on the fixed point set F.

Definition 31. *For $p \in F$ and $S \in \mathcal{RS}(\Phi_p)$, denote by $C(p,S)$ the connected component of the set*

$$M^{S^\perp} = \{m \in M |\, VX(m) = 0 \text{ for every } X \in S^\perp\}$$

which contains p. Let $\mathrm{FPC}(M)$ *stand for the set of all connected* **fixed point components** *$C(p,S)$ obtained this way:*

$$\mathrm{FPC}(M) = \{C(p,S) |\, p \in F,\ S \in \mathcal{RS}(\Phi_p)\}.$$

Observe that the set $M^{S^\perp}$ is also the fixed point set of the subtorus of T with Lie algebra $S^\perp$.

Lemma 32. *For $p \in F$ and $S \in \mathcal{RS}(\Phi_p)$, consider $C(p,S) \in \mathrm{FPC}(M)$. Then*

(1) *the set $C(p,S)$ is a smooth, T-invariant submanifold of M,*
(2) *the Lie algebra of the generic stabilizer $\mathfrak{t}_{C(p,S)}$ of $C(p,S)$ in $\mathfrak{t}$ is exactly $S^\perp$, and*
(3) *for $q \in C(p,S) \cap F$, we have $S \in \mathcal{RS}(\Phi_q)$ and $C(q,S) = C(p,S)$.*

Proof. Recall the basic fact that a torus action on a manifold can be linearized near a fixed point. It is sufficient then to verify these statements in a linear model. Let V be a vector space with a linear T-action, with a list of weights Ψ, and let $S \in \mathcal{RS}(\Psi)$. Then the set of points stabilized by the torus $\exp(S^\perp)$ is the direct sum of all eigenspaces of V whose weight is in S. This easily implies all three statements.

□

It follows then that there is a one-to-one correspondence

$$\{(p,S) |\, p \in F,\ S \in \mathcal{RS}(\Phi_p)\} \leftrightarrow \{(p,C) |\, C \in \mathrm{FPC}(M),\ p \in C \cap F\},$$

and hence we can regroup the terms of the sum in (20) according to the fixed point component C to which it corresponds. To write down the resulting formula, we need to introduce new notation for the vectors γ_S and $Y_{S,\gamma}$ in terms of the component C.

The manifold C inherits a T-invariant almost complex structure, and the set of weights of the fiber of the complex vector bundle $\mathrm{T}^J C$ at $p \in C \cap F$ is $\Phi_p \cap \mathfrak{t}_C^\perp$. Consider the T-equivariant line bundle $\mathcal{L}_C$ on C obtained as the restriction of $\mathcal{L}$ to C, and recall from (15) the definition of the affine space A_C.

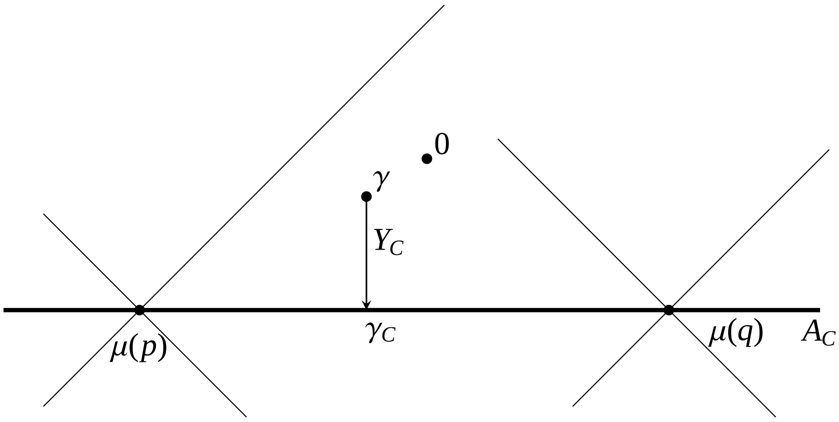
Using our scalar product, we decompose $\mathfrak{t}^* = \mathfrak{t}_C \oplus \mathfrak{t}_C^\perp$. If p and $q \in C \cap F$, then $\mu(p) - \mu(q)$ belongs to $\mathfrak{t}_C^\perp$, thus the projection $(\mu(p) - \gamma)_{\mathfrak{t}(C)}$ of $\mu(p) - \gamma$ to $\mathfrak{t}(C)$ does not depend on $p \in C \cap F$. This is the polarizing vector in any term of the sum in (20) associated to the component C.

Using this observation, we introduce the following notations.

Definition 33. *From now on, we will write γ_C for the orthogonal projection of γ on the affine space A_C. We introduce the notation*

$$Y_C \stackrel{\text{def}}{=} (\mu(p) - \gamma)_{\mathfrak{t}(C)} = \gamma_C - \gamma$$

for the polarizing vector in $\mathfrak{t}(C)$, *omitting the dependence on* γ.



Regrouping the terms of (20), and using these notations, we obtain the formula

$$(21)\quad \chi_{\mathcal{E}} = \sum_{C\in \mathrm{FPC}(M)} \sum_{p\in C\cap F} \varepsilon_p \cdot \Theta[\Phi_p \backslash \mathfrak{t}_C^{\perp} \uparrow Y_C] \cdot \delta[\Phi_p \cap \mathfrak{t}_C^{\perp} \uparrow X, \mathcal{T}(\gamma_C - \mu(p))].$$

Recall that $\mathfrak{a}(\gamma_C)$ stands for the alcove containing γ_C. We claim that the term

$$\Delta_C := \sum_{p\in C\cap F} \varepsilon_p \cdot \Theta[\Phi_p \backslash \mathfrak{t}_C^{\perp} \uparrow Y_C] \cdot \delta[\Phi_p \cap \mathfrak{t}_C^{\perp} \uparrow X, \mathcal{T}(\gamma_C - \mu(p))]$$

has the form $\Delta_\mu[\tilde{\mathcal{E}}_C, \mathfrak{a}(\gamma_C)]$ for an infinite-dimensional bundle $\tilde{\mathcal{E}}_C$ over C, whose fiber at the fixed point p has character $\varepsilon_p \cdot \Theta[\Phi_p \backslash \mathfrak{t}_C^{\perp} \uparrow Y_C]$.

This bundle may be constructed as follows. The bundle $KC = \overline{\mathrm{T}}^J M/\overline{\mathrm{T}}^J C$ is a T-equivariant complex bundle on C, whose T_C-weights are constant along C; denote the set of these T_C-weights by Φ_C. These weights thus may be obtained by restricting $\Phi_p \backslash \mathfrak{t}_C^{\perp}$ to $\mathfrak{t}_C$ for any $p \in C \cap F$. Now, we can split Φ_C into the disjoint union $\Phi_C^+ \cup \Phi_C^-$ according to the sign of the value of the weights on the polarizing vector $Y_C \in \mathfrak{t}_C$, and we obtain a direct sum decomposition of KC in $KC = KC_+ \oplus KC_-$, where KC_+ and KC_- are the subspaces generated by eigenvectors of T_C with weights from Φ_C^+ and Φ_C^-, respectively.

Then, recalling the definition of the formal character Θ from (9), we obtain the following statement.

Lemma 34. *Define the infinite-dimensional* T*-equivariant virtual bundle*

$$(22)\qquad \mathcal{S}^{\bullet}(KC\uparrow Y_C) = (-1)^{\operatorname{rank} KC_-} \det(KC_-^*) \otimes \bigoplus_{m=0}^{\infty} \mathcal{S}^{[m]}(KC_-^* \oplus KC_+)$$

over C, *where* $\mathcal{S}^{[m]}(V)$ *stands for the mth symmetric tensor product of the vector space* V, *and* $\det(V)$ *for its top exterior product. The fibers of this bundle over points of* C *form a* T_C*-representation with finite multiplicities, moreover, for* $p \in C \cap F$, *the* T*-character of the fiber* $\mathcal{S}^{\bullet}(KC \uparrow Y_C)_p$ *is* $\Theta[\Phi_p \backslash \mathfrak{t}_C^{\perp} \uparrow Y_C]$.

Denote by $\mathcal{E}_C$ the restriction of $\mathcal{E}$ to C. Then the combination of the fixed point formula with Paradan's decomposition leads to the following statement.

Proposition 35 (Paradan). *Let γ be a generic point in $\mathfrak{t}^*$. Then we have the following equality in $\hat{R}(T)$:*

$$\chi_{\mathcal{E}} = \sum_{C \in \mathrm{FPC}(M)} \Delta_\mu[\mathcal{E}_C \otimes \mathcal{S}^\bullet(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)]. \tag{23}$$

If C consists of a single fixed point $p \in F$, then the corresponding term is $\varepsilon_p \cdot \Theta[\Phi_p \uparrow (\mu(p) - \gamma)^*]$. It is reassuring to compare this to (10), which contains a similar Θ-term functions but oriented differently. According to Lemma 9 (1), the two terms, interpreted as generalized functions, coincide on the set $\{t \in T;\ t^\phi \neq 1\ \forall \phi \in \Phi_p\}$.

We also observe that these are the only terms which correspond to smooth functions on some open sets of T; all the other terms correspond to generalized functions supported in positive-codimensional subtori of T. One can think of this formula then as a refinement of the Atiyah- Bott formula (8).

Remark 36.

(1) *The alcove $\mathfrak{a}(\gamma_C)$ is an open set in $A_C = \mu(p) + \mathfrak{t}_C^\perp$ when $p \in C$.*
(2) *The moment map $\mu : M \to \mathfrak{t}^*$ of the line bundle μ satisfies equation (6), and this implies that $\mu(C)$ is contained in the affine space A_C. Moreover, if $\mathcal{L}$ is a positive line bundle as in Definition 3, then $\mu(C)$ is the convex hull of the finite set $\{\mu(p);\ p \in C \cap F\}$.*
(3) *Proposition 52 implies the vanishing of those the terms of the sum in* (23) *which corresponding to a fixed point component C satisfying $\mathfrak{a}(\gamma_C) \not\subset \mu(C)$.*
(4) *We note that the terms of the sum* (23) *are not quasi-polynomial. Rather, the term corresponding to the fixed point component C is quasi-polynomial along each element an infinite set of linear translates of $\mathfrak{t}_C^\perp \subset \mathfrak{t}^*$.*
(5) *When $C = M$, then the term of the sum* (23) *reduces to $\Delta_\mu[\mathcal{E}, \mathfrak{a}(\gamma)]$, which is an actual quasi-polynomial character (cf. Lemma 21 and Proposition 22).*

Example 37. *In Example 7, the elements of* $\mathrm{FPC}(M)$ *are easy to list:*

$$\mathrm{FPC}(M) := \{M, p^+, p^-\}.$$

If we consider $\gamma = 0$, the corresponding decomposition of $\chi_{\mathcal{E}}$ reads as follows:

$$t^{-4} + t^{-2} + 1 + t^2 + t^4 = \sum_{k \in \mathbb{Z}} t^{2k} - t^6 \sum_{k=0}^{\infty} t^{2k} - t^{-6} \sum_{k=0}^{\infty} t^{-2k}.$$

Example 38. *In Example 8 (see also Example 20), the set of fixed point components* $\mathrm{FPC}(M)$ *consists of the following elements:*

- *the complex 3-dimensional manifold M itself,*

- *the* 6 *fixed points* p_w, $w \in \Sigma_3$, *corresponding to the vertices of the highlighted hexagon. The corresponding values of the moment map, are as follows:*

$$\mu_{123} = 4\alpha + 3\beta,\ \mu_{213} = -\alpha + 3\beta,\ \mu_{132} = 4\alpha + \beta,$$

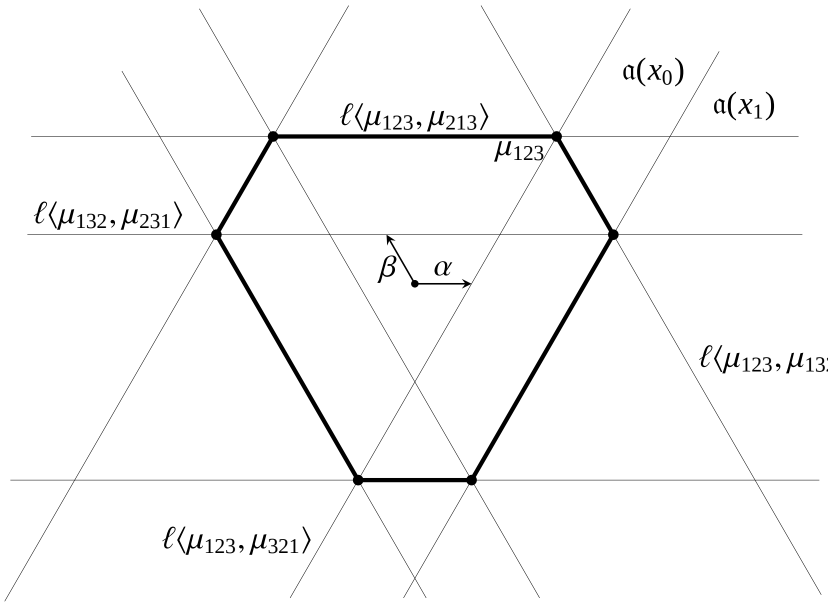
$$\mu_{321} = -3\alpha - 4\beta,\ \mu_{231} = -3\alpha + \beta,\ \mu_{312} = -\alpha - 4\beta$$

- 9 *components isomorphic to* $\mathbb{P}^1(\mathbb{C})$, *whose images are intervals which span the 9 lines on the picture below. Each of these components contains precisely two fixed points; we will use the notation* $C[p_v, p_w]$ *for the component containing the fixed points* p_v *and* p_w, *and* $\ell\langle\mu_v, \mu_w\rangle$ *for the corresponding line.*

  For example, the fixed point component $C[p_{123}, p_{213}]$ *may be described as the set of flags of the form*

$$\mathbb{C}v \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3.$$

  The stabilizer group of this submanifold is $\{(t, t, u);\ t, u \in \mathrm{U}(1)\}$.



Thus decomposition (23) *of the character* $\chi_{\mathcal{L}}$ *involves* 16 *formal characters. On each of the alcoves, each of these terms gives us a polynomial function. Clearly, these polynomials need to add up to 0 in an alcove which is not in the support of* $\mathcal{F}\chi_{\mathcal{E}}$. *The support of* $\mathcal{F}\chi_{\mathcal{E}}$, *which is the convex hull of the points* $\mu(p_w)$, $w \in \Sigma_3$, *is the highlighted hexagon on the diagram.*

We will consider two such cases: the alcove $\mathfrak{a}(x_0)$ *containing the point* $x_0 = 7\alpha + 5\beta$, *and the alcove* $\mathfrak{a}(x_1)$ *containing the point* $x_1 = 11\alpha + 5\beta$. *We will express an element* $\lambda \in \Lambda$ *in the basis of simple roots:* $\lambda = k_1\alpha + k_2\beta$. *The multiplicity function* $\mathcal{F}\chi_{\mathcal{L}}$ *thus is a function of* (k_1, k_2), *with* $k_1, k_2 \in \mathbb{Z}$.

We begin with $\mathfrak{a}(x_0)$. *In fact, because of the support conditions, only the following 6 fixed point components contribute:*

- $C = M$ *contributes the constant function* $f_1(k_1, k_2) = 3$.

- *The term corresponding to* $C = p_{123}$*is*

$$\Delta_{p_{123}} = -e_{\mu_{123}} e_{(\alpha+\beta+(\alpha+\beta))} \cdot \sum_{k=0}^{\infty} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)}.$$

  The Fourier transform of this character restricted to $\mathfrak{a}(x_0) \cap \Lambda$ *is the function* $f_2(k_1, k_2) = -(k_2 - 4)$.
- *The term corresponding to* $C = C[p_{123}, p_{132}]$ *is*

$$\Delta_{C[p_{123}, p_{132}]} := e_{\mu_{123}} e_{(\alpha+(\alpha+\beta))} \cdot \sum_{k\in\mathbb{Z}} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)}.$$

  The Fourier transform of this character restricted to $\mathfrak{a}(x_0) \cap \Lambda$ *is* $f_3(k_1, k_2) = (k_1 - 5)$.
- *The term corresponding to* $C = C[p_{123}, p_{213}]$ *is*

$$\Delta_{C[p_{123}, p_{213}]} := e_{\mu_{123}} e_{(\beta+(\alpha+\beta))} \cdot \sum_{k\in\mathbb{Z}} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)}.$$

  The Fourier transform of this character restricted to $\mathfrak{a}(x_0) \cap \Lambda$ *is* $f_4(k_1, k_2) = (k_2 - 4)$.
- *The term corresponding to* $C = C[p_{132}, p_{231}]$ *is*

$$\Delta_C := -e_{\mu_{132}} e_{(\alpha+\beta)} \cdot \sum_{k\in\mathbb{Z}} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{k\beta} \cdot \sum_{k=0}^{\infty} e_{k(\alpha+\beta)}.$$

  The Fourier transform of this character restricted to on $\mathfrak{a}(x_0) \cap \Lambda$ $f_5(k_1, k_2) = -(k_2 - 1)$.
- *The term corresponding to* $C = C[p_{123}, p_{321}]$

$$\Delta_C := -e_{\mu_{123}} e_{\beta} \cdot \sum_{k\in\mathbb{Z}} e_{k(\alpha+\beta)}) \cdot \sum_{k=0}^{\infty} e_{k\alpha} \cdot \sum_{k=0}^{\infty} e_{-k\beta}.$$

  The Fourier transform of this character restricted to $\mathfrak{a} \cap \Lambda$ *is* $f_6(k_1, k_2) = -(k_1 - k_2 - 1)$.

Now, it remains to observe that the sum of all these contributions vanishes:

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 3 - (k_2 - 4) + (k_1 - 5) + (k_2 - 4) - (k_2 - 1) - (k_1 - k_2 - 1) = 0.$$

On the alcove $\mathfrak{a}(x_1)$*, the calculation is quite similar. One still has the contributions* f_1*,* f_2*,* f_3*,* f_4*,* f_5*,* f_6*, and there are 2 additional terms: one coming from the fixed point* p_{132}*, and the other from* $C[p_{132}, p_{321}]$. *The first contributions restricted to this alcove is* $(k_1 - k_2) - 6$*, while the second is the same expression with opposite sign. Thus we have* 8 *contributions adding up to* 0.

4. Quasi-polynomial behavior of multiplicities

4.1. **Decomposition of a G-character.** Returning to the setup of §1, we consider a compact connected Lie group G acting compatibly on an almost complex manifold M, bundles $\mathcal{E}$ and $\mathcal{L}$ and the connection ∇ on $\mathcal{L}$.

Let T be the maximal torus of G. Our first goal is to understand what formula (23) tells us about $\dim Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$. Let W_G be the Weyl group of G, and choose a triangular decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{n}^+\oplus\mathfrak{n}^-$ of the complexification of the Lie algebra $\mathfrak{g}$ of G. This choice induces a splitting of the roots of G into positive and negative ones: $\mathfrak{R} = \mathfrak{R}^+\cup\mathfrak{R}^-$, where $\mathfrak{R}^\pm$ is the list of weights of T on $\mathfrak{n}^\pm$; it also provides us with the subset $\Lambda_{\mathrm{dom}}\subset\Lambda$ of dominant weights, which serve as a fundamental domain of the W_G-action on $\Lambda\subset\mathfrak{t}^*$. Then the irreducible characters $\chi_\lambda : T\to\mathbb{C}$ of G are parametrized by $\lambda\in\Lambda_{\mathrm{dom}}$.

Remark 39. *As observed by Atiyah-Bott, the Weyl character formula*

$$\chi_\lambda := \sum_{w\in W_G}\frac{e_{w\lambda}}{\prod_{\alpha\in\mathfrak{R}^-}(1-e_{w\alpha})}$$

is the Atiyah-Bott formula for $\dim Q(G\cdot\lambda,\mathcal{L}_\lambda)$ *associated to the line bundle* $\mathcal{L}_\lambda = G\times_{G_\lambda}\mathbb{C}_\lambda$ *on the coadjoint orbit* $G\cdot\lambda$.

We consider the G-equivariant moment map $\mu_G : M\to\mathfrak{g}^*$ satisfying equation (5). Then the map μ, obtained as the composition of μ_G with the restriction $\mathfrak{g}^*\to\mathfrak{t}^*$, serves as a moment map for the T-action.

Now our character $\chi_{\mathcal{E}\otimes\mathcal{L}^k}\in R(T)$ may be expressed in a unique way as a finite linear combination of irreducible characters χ_λ, $\lambda\in\Lambda_{\mathrm{dom}}$. In particular, the sought after quantity $\int_G\chi_{\mathcal{E}\otimes\mathcal{L}^k}\,dg = \dim Q(M,\mathcal{E}\otimes\mathcal{L}^k)^G$ is exactly the coefficient of the trivial character in the decomposition of $\chi_{\mathcal{E}\otimes\mathcal{L}^k}$ as a linear combination of irreducible characters. To obtain an explicit formula for this multiplicity, we observe that the Weyl character formula for χ_λ may be stated in the following way:

Lemma 40. *Introduce the element*

$$\omega_G = \prod_{\alpha\in\mathfrak{R}^-}(1-e_\alpha)\in R(T).$$

Then for $\lambda_1,\lambda_2\in\Lambda_{\mathrm{dom}}$, *we have*

$$\mathcal{F}\omega_G\chi_{\lambda_1}(\lambda_2) = \begin{cases}1 \text{ if } \lambda_1=\lambda_2,\\ 0 \text{ otherwise.}\end{cases}$$

Applying this to our situation with $\lambda = 0$, we obtain the formula

$$\dim Q(M,\mathcal{E})^G = \mathcal{F}\omega_G\chi_{\mathcal{E}}(0).$$

Now we make the formal observation that multiplying $\chi_{\mathcal{E}}$ by ω_G amounts to tensoring $\mathcal{E}$ by the trivial super-bundle over M with fiber $\wedge\mathfrak{n}^- = \wedge^{\mathrm{even}}\mathfrak{n}^-\ominus$

$\wedge^{\mathrm{odd}}\mathfrak{n}^-$ endowed with the standard (adjoint) T-action. This leads to the following formula:

$$\dim Q(M, \mathcal{E} \otimes \mathcal{L}^k)^G = \chi_{\mathcal{E}\otimes\wedge\mathfrak{n}^-}(0). \tag{24}$$

Now Proposition 35 combined with Remark 36 (3) implies

Corollary 41. *For any integer k, we have*

(25)
$$\dim Q(M, \mathcal{E}\otimes\mathcal{L}^k)^G = \sum_{C\in\mathrm{FPC}(M)} \mathcal{F}\Delta_\mu \left[\mathcal{E}_C \otimes \mathcal{L}_C^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\right](0),$$

and those terms for which $\gamma_C \notin \mu(C)$ vanish.

It turns out that one can significantly strengthen the condition on C under which the corresponding term in (25) vanishes. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{q}$ be the T-invariant decomposition of $\mathfrak{g}$ with $\mathfrak{q} = [\mathfrak{t}, \mathfrak{g}]$. This induces a decomposition $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{q}^*$ which, in turn, gives us a map $\mu_\perp : M \to \mathfrak{q}^*$ satisfying

$$\mu_G(m) = \mu(m) \oplus \mu_\perp(m).$$

Now introduce the notation $\mathrm{Slice}(M) = \mu_\perp^{-1}(0)$. The set $\mathrm{Slice}(M)$ is a T-space, and on $\mathrm{Slice}(M)$, the map μ_G coincides with μ. Also, note that any G-orbit in M intersects $\mathrm{Slice}(M)$.

Theorem 42 (Paradan)**.** *Let γ be generic in $\mathfrak{t}^*$, and $C \in \mathrm{FPC}(M)$ be a fixed point component. Then the term*

$$\Delta_\mu \left[\mathcal{E}_C \otimes \mathcal{L}_C^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC \uparrow Y_C), \mathfrak{a}(\gamma_C)\right] \tag{26}$$

in (25) *vanishes unless*

$$\gamma_C \in \mu(C \cap \mathrm{Slice}(M)).$$

Proof. Recall that for a Hermitian vector space H the formula

$$c(v) : \omega \mapsto v \wedge \omega - v^* \lfloor \omega, \quad v \in H,\ \omega \in \wedge H, \tag{27}$$

defines the Clifford action of H on the exterior algebra of H, i.e. a linear map $H \to \mathrm{End}(\wedge H)$. A simple computation shows that $c(v)^2 = -\|v\|^2 \cdot \mathrm{id}$, and hence $c(v)$ is an isomorphism whenever $v \neq 0$.

Now assume that the inverse image $\mu^{-1}(\gamma_C)$ does not intersect $\mathrm{Slice}(M) \cap C$. Then, for any $q \in C$ such that $\mu(q) = \gamma_C$, the component $\mu_\perp(q)$ does not vanish, and hence the map $[q, \omega] \mapsto [q, c(\mu_\perp(q))\omega]$ is a T-equivariant bundle-map $C \times \Lambda^{\mathrm{even}}\mathfrak{n}^- \to C \times \Lambda^{\mathrm{odd}}\mathfrak{n}^-$, which is an isomorphism over $\mu^{-1}(\gamma_C) \cap C$. Now, using Proposition 52, we can conclude that the term (26) vanishes. □

4.2. **The main result.** At this point, we impose the condition of positivity on our line bundle $\mathcal{L}$. Recall that this means that the curvature of the connection ∇ on $\mathcal{L}$ is of the form $-i\Omega$, where the closed 2-form Ω is such that the quadratic form $V \mapsto \Omega_q(V, JV)$ is positive definite at each point $q \in M$. Note that this condition, in particular, implies that Ω is symplectic.

Now we turn to the proof of Theorem 6, which we repeat here for reference.

Theorem 43. *Let (M, J) be a compact, connected almost complex manifold, endowed with the action of a connected compact Lie group G, and let $\mathcal{L}$ be a positive G-equivariant line bundle on M. Then*

- *the integer function*
$$k \to \dim Q(M, \mathcal{L}^k)^G$$
is quasi-polynomial for $k \geq 1$, and
- *this quasi-polynomial is identically zero if $0 \notin \mu_G(M)$.*

Proof. Let γ be a regular element of $\mathfrak{t}^*$. For $C \in \mathrm{FPC}(M)$, we retain the notation γ_C and Y_C from Definition 33. Then, according to Theorem 42, we have

$$\dim Q(M, \mathcal{L}^k)^G = \sum_C \mathcal{F}\Delta_\mu \left[\mathcal{L}^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC\uparrow Y_C), \mathfrak{a}(\gamma_C)\right](0), \tag{28}$$
$$\text{where } \{C \in \mathrm{FPC}(M);\ \gamma_C \in \mu(C \cap \mathrm{Slice}(M))\}.$$

First, consider the terms of this sum corresponding to $C \in \mathrm{FPC}(M)$ for which the affine-linear subspace A_C defined in (15) passes through the origin: $0 \in A_C$. Lemma 26 shows that in this case, the function

$$k \to \mathcal{F}\Delta_\mu \left[\mathcal{L}^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC\uparrow Y_C), \mathfrak{a}(\gamma_C)\right](0)$$

is a quasi-polynomial function of k. Furthermore, if $0 \in A_C$ but $0 \notin \mu_G(M)$, then for γ sufficiently close to 0, the orthogonal projection γ_C of γ to A_C is also close to 0, and thus $\gamma_C \notin \mu(C \cap \mathrm{Slice}(M)) \subset \mu_G(M)$. Hence, according to Theorem 42, the term of (28) corresponding to such $C \in \mathrm{FPC}(M)$ vanishes.

Now, both assertions of Theorem 43 will follows if we show that, for γ chosen sufficiently near 0 and $k \geq 1$, the term on the right hand side of (28) corresponding to $C \in \mathrm{FPC}(M)$ with $0 \notin A_C$ vanishes.

Indeed, consider a fixed point component $C \in \mathrm{FPC}(M)$ for which $0 \notin A_C$, i.e. for which the stabilizer $\mathfrak{t}_C$ acts nontrivially on the fibers of the line bundle $\mathcal{L}_C$, and let $\gamma \in \mathfrak{t}^*$ be a generic element, for which there is a $q \in C$ satisfying $\mu_\perp(q) = 0$ and $\mu(q) = \gamma_C$.

Assume, *ad absurdum*, that the zero weight occurs with nonzero multiplicity in the T-module

$$\Delta_\mu \left[\mathcal{L}^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC\uparrow Y_C), \mathfrak{a}(\gamma_C)\right].$$

According to Lemma 25, this implies that the representation of T_C on the fiber of the bundle $\mathcal{L}^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC \uparrow Y_C)$ at some point of C contains the trivial weight. This implies that at every point $q \in C$ the Lie algebra element $Y_C \in \mathfrak{t}_C$ annihilates a nonzero vector in the fiber

$$(\mathcal{L}^k \otimes \wedge\mathfrak{n}^- \otimes \mathcal{S}^\bullet(KC\uparrow Y_C))_q. \tag{29}$$

To find a contradiction, we will give a positive lower bound on the eigenvalues of Y_C on this space for an appropriately chosen γ. Let us consider the eigenvalues of Y_C on each of the 3 tensor factors:

- The eigenvalue $y(\mathcal{L}^k)$ of Y_C acting on $\mathcal{L}^k_q$ is equal to $k\langle\mu(q), Y_C\rangle = k\langle\gamma_C, Y_C\rangle$.
- The eigenvalues of Y_C on $\wedge\mathfrak{n}^-$ are parametrized by subsets $I \subset \mathfrak{R}^-$ of the negative roots: we have the expression $y(\wedge\mathfrak{n}^-, I) = \sum_{\alpha\in I}\langle\alpha, Y_C\rangle$ for the corresponding eigenvalues.
- Finally, recall from the discussion before Lemma 34 the splitting $\Phi_C = \Phi_C^+ \cup \Phi_C^-$ of the $\mathfrak{t}_C$-weights of the conormal bundle KC according to the sign of their value on Y_C. It follows then that all eigenvalues of Y_C on $\mathcal{S}^{[m]}(KC_-^* \oplus KC_+)_q$ are nonnegative, and hence, according to (22), all eigenvalues of Y_C on $\mathcal{S}^\bullet(KC \uparrow Y_C)$ are bounded from below by the eigenvalue of Y_C on $\wedge KC_-^*$:

$$y(\wedge KC_-^*) = -\sum_{\eta\in\Phi_C^-}\langle\eta, Y_C\rangle. \tag{30}$$

Hence, to prove that Y_C does not annihilate vectors in the vector space (29), it suffices to show that for some regular $\gamma \in \mathfrak{t}^*$,

$$y(\mathcal{L}^k) + y(\wedge\mathfrak{n}^-, I) + y(\wedge KC_-^*) > 0 \text{ for every } I \subset \mathfrak{R}^-. \tag{31}$$

Denote by β_C the projection of the origin onto A_C. Naturally, for γ near the origin, the point γ_C will be close to β_C. Also, for such γ, under the identification of $\mathfrak{t}$ with $\mathfrak{t}^*$, the vector Y_C will be close to β_C. This implies that for γ near the origin

$$y(\mathcal{L}^k) = k\langle\gamma_C, Y_C\rangle \sim k(\beta_C, \beta_C) = k\|\beta_C\|^2 > 0. \tag{32}$$

By the definition of Φ_C^-, all terms of the sum in (30) are also positive. Thus our worry is the set of negative terms which could appear in $y(\wedge\mathfrak{n}^-, I)$: these correspond to $\alpha \in \mathfrak{R}^-$ for which $\langle\alpha, Y_C\rangle < 0$. Again, by continuity, for γ in a small neighborhood of the origin, for such a root α, there are two possibilities:

- either $(\beta_C, \alpha) = 0$,
- or the three numbers $\langle\alpha, Y_C\rangle$, (β_C, α), and (γ_C, α) are all negative.

When $(\beta_C, \alpha) = 0$, then for γ sufficiently close to the origin in $\mathfrak{t}^*$, the sum

$$y(\mathcal{L}^k) + \sum\{\langle\alpha, Y_C\rangle;\ \alpha \in \mathfrak{R}^-,\ (\beta_C, \alpha) = 0\}$$

remains positive.

If $\langle\alpha, Y_C\rangle, (\beta_C, \alpha), (\gamma_C, \alpha) < 0$, then $y(\wedge\mathfrak{n}^-, I)$ needs to be neutralized using the summands of $y(\wedge KC_-^*)$ in (30). To prove that indeed, $y(\wedge\mathfrak{n}^-, I) + y(\wedge KC_-^*) \geq 0$, it will be sufficient to show that all such roots appear as weights of KC. More precisely, it is sufficient to show, that whenever $\alpha \in \mathfrak{R}^-$ and $(\gamma_C, \alpha) < 0$, then α restricted to $\mathfrak{t}_C$ is a weight of KC.

Proposition 44. *Let (M, Ω, J) be a positive symplectic G-manifold. Let $q \in M$ be a point such that $\mu_G(q) = \mu(q)$, i.e. $\mu_\perp(q) = 0$. Then the list of complex weights of the stabilizer group T_q in T_qM with respect to the almost complex structure J contains the the list of restricted weights*

$$[\alpha|\mathfrak{t}_q;\ \alpha \in \mathfrak{R},\ (\mu(q), \alpha) > 0]. \tag{33}$$

Proof. As usual, we use the notation $VX(q)$ for the tangent vector in T_qM corresponding to $X \in \mathfrak{g}$ under the G-action on M. Since μ_G is G-equivariant, we have

$$\mathfrak{g}_q = \{X \in \mathfrak{g};\ VX(q) = 0\} \subset \mathfrak{g}_{\mu(q)} = \{X \in \mathfrak{g};\ \mathrm{ad}^* X(\mu(q)) = 0\}$$

for the stabilizer Lie subalgebras. Thus under the correspondence $X \mapsto VX(q)$, the $G_{\mu(q)}$-invariant complement $\mathfrak{g}^{\perp}_{\mu(q)}$ of $\mathfrak{g}_{\mu(q)}$ in $\mathfrak{g}$ maps injectively into T_qM. We will look for our weights (33) among the weights of this image.

Our first observation is that by contracting the equation $\langle d\mu_G, X\rangle + \iota(VX)\Omega = 0$ by VZ, and using the G-equivariance of μ_G, we can identify the symplectic form on this image. We obtain

$$\Omega_q(VX, VZ) = \langle \mu_G(q), [X, Z]\rangle \tag{34}$$

for $X, Z \in \mathfrak{g}$. Since $\mu_G(q) = \mu(q)$, using the structure of reductive Lie algebras, we can conclude that this antisymmetric form restricts to a non-degenerate form on $V\mathfrak{g}^{\perp}_{\mu(q)} \subset \mathrm{T}_qM$.

Let us summarize our situation: we have a real vector space T_qM endowed with an action of a torus T_q, a T_q-invariant symplectic form Ω and a T_q-invariant positive complex structure J. We have a T_q-invariant symplectic subspace $V\mathfrak{g}^{\perp}_{\mu(q)} \subset \mathrm{T}_qM$, which is, however, not necessarily invariant under the complex structure J. Finally, we know the T_q-weights of the space $V\mathfrak{g}^{\perp}_{\mu(q)} \otimes \mathbb{C}$: in our case this is the set of restricted weights $\{\alpha|\mathfrak{t}_q;\ \alpha \in \mathfrak{R},\ (\mu(q), \alpha) \neq 0\}$. Our goal is to find out which of these weights appear as weights of the complex vector space (T_qM, J).

The following lemma, which is a simple exercise in linear algebra, gives us the answer:

Lemma 45. *Let W be a real vector space on which a torus group T_W acts, and assume that W is endowed with an invariant symplectic form Ω_W and an invariant complex structure J_W, satisfying*

$$\Omega_W(X, J_W X) > 0 \text{ for all } 0 \neq X \in W. \tag{35}$$

Let α be a T_W-weight, and let $Z \in \mathfrak{t}_W$ be such that $\langle \alpha, Z\rangle \neq 0$. Then the multiplicity of α in the complex vector space (W, J_W) equals to half of the number of positive eigenvalues of the quadratic form

$$X \mapsto \frac{1}{\langle \alpha, Z\rangle}\Omega(X, Z \cdot X) \text{ on } (W_\alpha \oplus W_{-\alpha}) \cap W,$$

where W_α is the α weight-space of $W \otimes \mathbb{C}$.

This lemma has the following corollaries:

Corollary 46. (1) *Given Ω_W on W, the complex eigenvalues of T_W with respect to those complex structures J_W satisfying* (35) *do not depend on J_W. Thus we can speak of the* symplectic T_W-weights *of W.*

(2) *If $W' \subset W$ is a T_W-invariant symplectic subspace, then the symplectic weights of W' form a sublist of the list of symplectic weights of W.*
(3) *In the special case when the quadratic form $X \mapsto \Omega(X, Z \cdot X)$ is positive definite, all symplectic weights α of W satisfy $\langle \alpha, Z \rangle > 0$.*

Now we apply Lemma 45 and its corollaries to our situation: $W = \mathrm{T}_q M$, $W' = V\mathfrak{g}^{\perp}_{\mu(q)}$ and $T_W = T_q$, and we can conclude that the symplectic weights of $V\mathfrak{g}^{\perp}_{\mu(q)}$ form a subset of the symplectic weights of $\mathrm{T}_q M$. Now we observe that the natural $\mathfrak{t}_q$-action on $V\mathfrak{g}^{\perp}_{\mu(q)}$ extends to an action of $\mathfrak{t}$ via the formula $Z \cdot VX = V[Z, X]$, and hence, using this action, we can try to verify the condition in Corollary 46 (3) for a $Z \in \mathfrak{t}$ not necessarily in $\mathfrak{t}_q$.

Denote by Z_q the element of $\mathfrak{t}$ which corresponds to $\mu(q)$ under our chosen inner product; thus we have

$$\langle \alpha, Z_q \rangle = (\mu(q), \alpha) \text{ and } \langle \mu(q), Y \rangle = (Z_q, Y) \quad \text{for } \alpha \in \mathfrak{t}^*,\ Y \in \mathfrak{t}.$$

The weights of the space $\mathfrak{g}^{\perp}_{\mu(q)} \otimes \mathbb{C}$ then may be listed as the subset $\{\alpha \in \mathfrak{R};\ \langle \alpha, Z_q \rangle \neq 0\}$, and hence, according to Corollary 46, Proposition 44 will follow if we show that the quadratic form

$$X \mapsto \Omega_q(VX, Z_q \cdot VX)$$

is positive definite on $\mathfrak{g}^{\perp}_{\mu(q)} = \mathfrak{g}^{\perp}_{Z_q}$. This is indeed the case: using (34), we see that for $X \in \mathfrak{g}^{\perp}_{Z_q}$ we have

$$\Omega_q(VX, Z_q \cdot VX) = \langle \mu(q), [X, [Z_q, X]] \rangle = (Z_q, [X, [Z_q, X]]) = ([Z_q, X], [Z_q, X]).$$

This last expression is positive for $0 \neq X \in \mathfrak{g}^{\perp}_{Z_q}$ and this completes the proof. □

This ends our proof of Theorem 43. To recapitulate, we have chosen a generic $\gamma \in \mathfrak{t}^*$ near the origin, and considered a fixed point component $C \in \mathrm{FPC}(M)$ such that $0 \neq A_C$; we needed to show that the corresponding contribution to (28) vanishes. In view of Theorem 42, we could assume that there is a $q \in C$ satisfying $\mu_{\perp}(q) = 0$ and $\mu(q) = \gamma_C$, where γ_C is the projection of γ on to the affine subspace $A_C \subset \mathfrak{t}_C$. Using simple estimates we reduced this vanishing to the inequality (31) on the eigenvalues of an element $Y_C \in \mathfrak{t}$ on a certain vector space associated to C. The first term on the left hand side of this inequality is positive and bounded away from zero according to (32). A quick calculation then shows that the sum of the last two terms is nonnegative if $\{\alpha|\mathfrak{t}_C \in \mathfrak{R}^-;\ (\gamma_C, \alpha) < 0\}$ is a subset of the $\mathfrak{t}_C$-weight of KC; this last statement is the content of Proposition 44. □

5. The asymptotic result in the torus case

The purpose of this last section is to give a proof of a version of Theorem 1. This result was also first proved by Meinrenken.

Theorem 47. *Let M be a compact almost complex T manifold, $\mathcal{L}$ a T-equivariant line bundle over M, with moment map μ. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a super-bundle. If for some $\gamma \in \mathfrak{t}^*$, the T-equivariant bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are isomorphic on $\mu^{-1}(\gamma)$, then there is a neighborhood $\mathfrak{b}$ of γ and $K > 0$ such that $\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) = 0$ for $k > K$ and $\lambda \in k\mathfrak{b} \cap \Lambda$.*

Proof. As $\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda)$ is an integer, it will be sufficient to show that there is a neighborhood $\mathfrak{b}$ of γ and a constant C such that $k|\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda)| < C$ when $k \geq 1$ and $\lambda \in k\mathfrak{b}$.

We choose a neighborhood $\mathfrak{a}$ of γ such that the vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are isomorphic over $\mu^{-1}(\mathfrak{a})$ and choose $\mathfrak{b}$ a compact neighborhood of γ contained in $\mathfrak{a}$.

Let us begin with the Atiyah-Bott fixed point formula (8);

$$\chi_{\mathcal{E}\otimes\mathcal{L}^k} = \sum_{p\in F} \frac{e_{k\mu(p)}\varepsilon_p}{\prod_{\phi\in\Phi_p}(1-e_\phi)}. \tag{36}$$

According to our hypothesis that the T-equivariant vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are isomorphic on $\mu^{-1}(\mathfrak{a})$, we have $\varepsilon_p = \varepsilon_p^+ - \varepsilon_p^- = 0$ if $\mu(p) \in \mathfrak{a}$. Denote by F' the set of $p \in F$ such that $\mu(p) \notin \mathfrak{a}$. Then we have

$$\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) = \int_T e_{-\lambda}(t) \sum_{p\in F'} \frac{e_{k\mu(p)}(t)\varepsilon_p(t)}{\prod_{\phi\in\Phi_p}(1-e_\phi(t))}\, dt. \tag{37}$$

To estimate this integral, we would like to exchange the summation and the integration in this formula. However, the terms of the sum are singular expressions, and thus we can only estimate the part of this integral where the terms of the sum are bounded.

To find this partial estimate, we proceed as follows. Consider the open set

$$T_{\mathrm{reg}} = \{g \in T \mid e_\phi(g) \neq 1\ \forall\phi \in \Phi_p,\ p \in F'\},$$

of those elements $g \in T$ for which the terms of our sum are regular, and for each $g \in T_{\mathrm{reg}}$ pick a ball $U_g \subset \mathfrak{t}$ centered at $0 \in \mathfrak{t}$ such that for $X \in U_g$, we have $g\exp(U_g) \subset T_{\mathrm{reg}}$. Now, let $\rho_g : T \to [0,1]$ be an auxiliary smooth function with compact support on $g\exp(U_g)$, and consider the piece

$$\int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}\otimes\mathcal{L}^k}(t)\, dt \tag{38}$$

of the integral in (37) supported in $g\exp(U_g)$. Pulling this integral back to $\mathfrak{t}$ via the map $g\exp : \mathfrak{t} \to T$, we can estimate the absolute value of (38) as being less or equal than

$$\sum_{p\in F'} \left| \int_{\mathfrak{t}} e_{k\langle\mu(p)-\lambda/k, X\rangle} \frac{\rho_g(g\exp(X))\,\varepsilon_p(g\exp(X))}{\prod_{\phi\in\Phi_p}(1-e^{i\langle\phi,X\rangle}e^{i\phi}(g))}\, dX \right|. \tag{39}$$

Note that we omitted the constant factor $e^{ik\mu(p)-i\lambda}(g)$, since it is of absolute value 1.

Now we recall the following standard estimate from Fourier Analysis.

Lemma 48. *Let $0 \neq \eta \in \mathfrak{t}^*$, and $H : \mathfrak{t} \to \mathbb{C}$ be a smooth compactly supported function. Then for every positive integer d, the inequality*

$$\left| \int_{\mathfrak{t}} e^{i\langle \eta, X\rangle} H(X)\, dX \right| \leq \frac{C_d(H)}{||\eta||^{2d}}$$

holds, where one can take

$$C_d(H) = \max_{X\in\mathfrak{t}} \left| \left[\sum_i \partial_i^2 \right]^d (H(X)) \right|;$$

This constant depends only on a finite number of derivatives of H.

Now we return to (39), and consider expression $\mu(p) - \lambda/k$ in the exponent, for $\lambda/k \in \mathfrak{b}$. Since $\mu(p)$ is not in $\mathfrak{a}$, we have the bound $|\mu(p) - \lambda/k| \geq \delta > 0$, which depends only on the choice of the relatively compact set $\mathfrak{b} \subset \mathfrak{a}$. Applying Lemma 48 to $\eta = k(\mu(p) - \lambda/k)$, we obtain the following

Corollary 49. *For fixed compact $\mathfrak{b} \subset \mathfrak{a}$, $g \in T_{\mathrm{reg}}$, and smooth function $\rho_g : T \to [0, 1]$ with compact support in U_g, the integral* (38) *goes to zero faster than any power of k, uniformly in $\lambda \in k\mathfrak{b}$.*

5.1. **Equivariant forms.** In order to bound the rest of the integral (37), for each $g \in T \setminus T_{\mathrm{reg}}$, we will replace the Atiyah-Bott formula by an expression, which is regular at g. Such formulas were given in [7]; here we sketch the setup and the relevant notions. We start with the identity element of T: $g = \mathrm{id}$.

For a manifold M with a T-action, we define the algebra $\mathcal{A}_T(M)$ of *equivariant forms* as the space of smooth maps $\alpha : \mathfrak{t} \to \Gamma(\wedge^\bullet \mathrm{T}^* M)^T$, from $\mathfrak{t}$ to the set of invariant differential forms on M. As a matter of notation, we will write $\alpha(X)$ for the resulting differential form on M, and $\alpha(X, q)$ for the value of this differential form at $q \in M$.

The equivariant differential $D : \mathcal{A}_T(M) \to \mathcal{A}_T(M)$ is given by the formula

$$D\alpha(X) = d\alpha(X) - VX \lfloor \alpha(X);$$

we have $D^2 = 0$. Accordingly, $\alpha \in \mathcal{A}_T(M)$ is called *equivariantly closed* if $D\alpha = 0$. The formulas in [6] express the integral $\int_M \alpha : \mathfrak{t} \to \mathbb{C}$ of an equivariantly closed form α in terms of local data on M. We follow the exposition of ([5],chapters 7,8).

Returning to our setup of T-manifold M, endowed with a line bundle $\mathcal{L}$ with curvature $R^{\mathcal{L}} = -i\Omega$, we observe that we have already encountered such equivariantly closed forms: indeed, equation (6) may be interpreted as saying that the expression

(40) $$R_{\mathcal{L}}(X) = R^{\mathcal{L}} + L_X - \nabla_{VX} = i\langle \mu, X\rangle - i\Omega,$$

the *equivariant curvature* of the bundle $\mathcal{L}$, is equivariantly closed. The equivariant curvature may be constructed for any equivariant bundle $\mathcal{B}$ over M by choosing a T-invariant connection ∇ on $\mathcal{B}$ with curvature $R_{\mathcal{B}}$. Then, again, we can define $R_{\mathcal{B}}(X) = R_{\mathcal{B}} + L_X - \nabla_{VX}$ which is a smooth map from $\mathfrak{t}$

to the T-invariant sections of the bundle of algebras $\wedge^\bullet \mathrm{T}^*M \otimes \mathrm{End}(\mathcal{B})$. We can then define the equivariant forms

(41)
$$\mathrm{ch}_{\mathcal{B}}(X) = \mathrm{Tr}_{\mathcal{B}}\left[\exp(R_{\mathcal{B}}(X))\right], \quad \mathrm{Todd}_{\mathcal{B}}(X) = \det_{\mathcal{B}}\left[\frac{R_{\mathcal{B}}(X)}{1-\exp(-R_{\mathcal{B}}(X))}\right],$$

where the trace and the determinant are taken in $\mathrm{End}(\mathcal{B})$. These forms are called, respectively, the *equivariant Chern class* and the *equivariant Todd class* of the bundle $\mathcal{B}$. Note that the latter is only defined in a neighborhood of $0 \in \mathfrak{t}$.

Applying this construction to the bundles $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathrm{T}^J M$, we obtain the equivariant curvature forms $R_{\mathcal{E}^+}$, $R_{\mathcal{E}^-}$ and $R_{\mathrm{T}^J M}$, respectively. Now we make the crucial observation, that since $\mathcal{E}^+$ is isomorphic to $\mathcal{E}^-$ over $\mu^{-1}(\mathfrak{a})$, we can assume that the corresponding connections $\nabla_{\mathcal{E}^+}$ and $\nabla_{\mathcal{E}^-}$ are chosen to coincide over $\mu^{-1}(\mathfrak{a})$, and thus we have

$$R_{\mathcal{E}^+}(X,q) = R_{\mathcal{E}^-}(X,q) \quad \text{if } \mu(q) \in \mathfrak{a}. \tag{42}$$

Now we are ready to write down the relevant formula from [7] (see also [5],chapter 8):

$$\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\exp X) = \frac{1}{(2i\pi)^{\dim M/2}} \int_M \mathrm{ch}_{\mathcal{L}^k}(X)\left[\mathrm{ch}_{\mathcal{E}^+}(X) - \mathrm{ch}_{\mathcal{E}^-}(X)\right]\mathrm{Todd}_{\mathrm{T}^J M}(X);$$

this equality is valid for X from the neighborhood U_{id} of $0 \in \mathfrak{t}$ where $\mathrm{Todd}_{\mathrm{T}^J M}(X)$ is defined.

Writing $\mathrm{ch}_{\mathcal{E}}(X)$ for $\mathrm{ch}_{\mathcal{E}^+}(X) - \mathrm{ch}_{\mathcal{E}^-}(X)$ and using (40), we can rewrite this expression as

$$\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\exp X) = \frac{1}{(2i\pi)^{\dim M/2}} \int_M e^{ik\langle\mu,X\rangle - ik\Omega}\,\mathrm{ch}_{\mathcal{E}}(X)\,\mathrm{Todd}_{\mathrm{T}^J M}(X). \tag{43}$$

Now we are proceeding quite similarly to our analysis of the Atiyah-Bott formula above. We choose an auxiliary smooth function $\rho_{\mathrm{id}} : T \to [0,1]$ with compact support in $\exp(U_{\mathrm{id}})$ and we write

$$(2i\pi)^{\dim M/2} \int_T \rho_{\mathrm{id}}(t) e_{-\lambda}(t) \chi_{\mathcal{E}\otimes\mathcal{L}^k}(t)\,dt = \int_M \sum_{j=0}^{\dim M/2} (-i)^j \frac{k^j}{j!}\Omega(q)^j \tag{44}$$
$$\times \int_{\mathfrak{t}} \rho_{\mathrm{id}}(\exp(X)) e^{ik\langle\mu(q)-\lambda/k,X\rangle}\,\mathrm{ch}_{\mathcal{E}}(X,q)\,\mathrm{Todd}_{\mathrm{T}^J M}(X,q)\,dX.$$

Now we observe that due to (42), the factor $\mathrm{ch}_{\mathcal{E}}(X,q)$ vanishes whenever $\mu(q) \in \mathfrak{a}$, and hence we can again assume that $|\mu(q) - \lambda/k| > \delta$ for some positive δ depending on $\mathfrak{b}$ only. Since both M and the support of ρ_{id} are compact, we have bounds on the derivatives of the integrand in (44), which are uniform in q. Hence we can apply Lemma 48 again to conclude that for each d, there is a constant C_d, independent of q, such that the integral over $\mathfrak{t}$ in (44) is bound by $C_d k^{-2d}$. Integrating over M then gives us

Corollary 50. *The integral* (44) *goes to zero faster than any negative power of k.*

Finally, we extend these arguments to all $g \in T$, using the generalization of the above formula given in [7, Theorem 3.23]. We first introduce the twisted versions of our characteristic forms: if $s \in T$ acts trivially on M, then we can define the twisted Chern character

$$\mathrm{ch}_{\mathcal{B},s}(X) = \mathrm{Tr}\left[s \exp(R_{\mathcal{B}}(X))\right],$$

and

$$D_{\mathcal{B},s} = \det\left[1 - s^{-1}\exp(-R_{\mathcal{B}}(X))\right],$$

as s acts fiberwise in any T-equivariant vector bundle over M.

Now let $g \in T$ be an arbitrary element, denote by M^g the submanifold fixed by g (thus g acts trivially on M^g) and let NM^g be the normal bundle of M^g in M. Then the formula in [7] states that

$$\chi_{\mathcal{E}\otimes\mathcal{L}^k}(g \exp X) = \frac{1}{2i\pi^{\dim M^g/2}} \int_{M^g} \frac{\mathrm{ch}_{\mathcal{L}^k,g}(X)\,\mathrm{ch}_{\mathcal{E},g}(X)\,\mathrm{Todd}_{M^g}(X)}{D_{NM^g,g}(X)}. \tag{45}$$

Starting from here, the arguments are identical to those we gave in the case $g = \mathrm{id}$, and hence they will be omitted.

We can summarize what we proved as follows.

Lemma 51. *For $g \in T$, let U_g be a neighborhood of $0 \in \mathfrak{t}$ such that for $X \in U_g$ the characteristic classes $\mathrm{Todd}_{M^g}(X)$ and $\frac{1}{D_{NM^g,g}(X)}$ are defined on M^g. Then for any smooth function $\rho_g : T \to [0, 1]$ compactly supported in $g\exp(U_g)$, and any $\lambda \in k\mathfrak{b}$, the integral*

$$\int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}\otimes\mathcal{L}^k}(t)\, dt$$

goes to zero faster than any negative power of k.

Now we can easily finish the proof of the theorem. Indeed, the sets $\{g\exp(U_g) |\, g \in T\}$ form an open cover of the compact torus T. We can thus pick a finite subset $S \subset T$ such that $\cup_{g\in S}\, g\exp(U_g) = T$. Next, we choose a partition of unity subordinated to this cover, i.e functions $\rho_g : T \to [0, 1]$, $g \in S$ such that ρ_g is compactly supported in $g\exp(U_g)$ and $\sum_{g\in S} \rho_g = 1$. Then, for $\lambda \in k\mathfrak{b}$, we have

$$\int_T e_{-\lambda}(t)\chi_{\mathcal{E}\otimes\mathcal{L}^k}(t)\, dt = \sum_{g\in S} \int_T \rho_g(t) e_{-\lambda}(t) \chi_{\mathcal{E}\otimes\mathcal{L}^k}(t)\, dt.$$

Each term of the sum goes to zero as $k \to \infty$ uniformly in λ, and hence so does their sum, the expression on the left hand side, which equals $\mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}$. This completes the proof of Theorem 47.

□

Let us formulate the corollary of Theorem 47 that we used to conclude the vanishing of certain quasi-polynomial characters on some alcoves. Without loss of generality we can assume that $\mathfrak{t}_M$, the infinitesimal stabilizer of M in $\mathfrak{t}$, is trivial.

Proposition 52. *Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a super-bundle. Let $\mathfrak{a} \subset \mathfrak{t}^*$ be an alcove. If for some $\gamma \in \mathfrak{a}$, the T-equivariant bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are isomorphic on $\mu^{-1}(\gamma)$, then $\Delta_\mu[\mathcal{E}, \mathfrak{a}] = 0$.*

Proof. According to Lemma 13 and Proposition 22, this follows from the fact that for a compact $\mathfrak{b} \subset \mathfrak{a}$ and k sufficiently large

$$\lambda \in k\mathfrak{b} \quad \Rightarrow \quad \mathcal{F}\chi_{\mathcal{E}\otimes\mathcal{L}^k}(\lambda) = 0.$$

□

6. List of notations

- (M, ω) – compact symplectic manifold; $\mathcal{E}$ – vector, $\mathcal{L}$ – line bundle over M.
- $\mathrm{T}M$ – the tangent bundle of M, $J \in \mathrm{End}(\mathrm{T}M)$ stands for a complex structure, T^J and $\overline{\mathrm{T}}^J$ denote the $\pm i$ eigenspaces of J.
- T – compact torus group, $\mathfrak{t}$ – its Lie algebra, Λ – weight lattice of T, G – compact Lie group with maximal torus T and Lie algebra $\mathfrak{g}$.
- $\mu_G : M \to \mathfrak{g}$ and $\mu : M \to \mathfrak{t}$ – moment maps.
- $\mathcal{F}\eta$ – the Fourier transform/multiplicity function of the formal character η of T.
- $\Theta[\Phi\uparrow X]$ – formal character of the partition with a list of weights Φ and oriented by the vector X (cf. (9)).
- $\delta[\Phi\uparrow X, \mathcal{T}]$ – formal quasi-polynomial character, whose multiplicity function coincides with that of $\Theta[\Phi\uparrow X]$ on the tope $\mathcal{T}$ (cf. Lemma 13).
- $\Delta_\mu[\mathcal{E}, \mathfrak{a}]$ – the asymptotic character associated to $\mathcal{E}$ and μ (cf. Definition 17).
- $G_C, T_C, \mathfrak{g}_C, \mathfrak{t}_C$, generic stabilizer groups of the subset $C \subset M$, and their Lie algebras.
- $\mathrm{FPC}(M)$ – set of connected components of fixed point sets of M.
- $\mathcal{E}_C$ – vector bundle restricted to the submanifold C.

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