Equivariant relative Thom forms and Chern characters

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December 2007

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1 Introduction

These notes are the first chapter of a monograph, dedicated to a detailed proof of the equivariant index theorem for transversally elliptic operators.

In this preliminary chapter, we prove a certain number of natural relations in equivariant cohomology. These relations include the Thom isomorphism in equivariant cohomology, the multiplicativity of the relative Chern characters, and the Riemann-Roch relation between the relative Chern character of the Bott symbol and of the relative Thom class. In the spirit of Mathai-Quillen, we give "explicit" representatives of a certain number of relative classes. We believe that this construction has its interest in the non equivariant case as well. As remarked by Cartan (see [9]) and emphasized in Mathai-Quillen [11], computations in the ordinary de Rham cohomology of vector bundles are deduced easily from computations in the equivariant cohomology of vector spaces. In particular, we give here an explicit formula for the relative Thom form $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) \in \mathcal{H}^*(\mathcal{V}, \mathcal{V}\backslash M)$ of a Euclidean vector bundle $p : \mathcal{V} \to M$ provided with a Euclidean connection: $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) := [p^* \mathrm{Eul}(\mathcal{V}), \beta_{\mathcal{V}}]$, where $\mathrm{Eul}(\mathcal{V})$ is the Euler class and $\beta_{\mathcal{V}}$ an explicit form depending of the connection, defined outside the zero section of $\mathcal{V}$, such that $p^* \mathrm{Eul}(\mathcal{V}) = d\beta_{\mathcal{V}}$. We similarly give an explicit formula for the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b) \in \mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$ of the Bott morphism on the vector bundle $\mathcal{V}$, if $\mathcal{V}$ is provided with a complex structure. The Riemann-Roch relation

$$p^* \left(\mathrm{Todd}(\mathcal{V})\right) \mathrm{Ch}_{\mathrm{rel}}(\sigma_b) = \mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$$

holds in relative cohomology, and follows from the formulae.

Our constructions in the de Rham model for equivariant cohomology are strongly influenced by Quillen's construction of characteristic classes via super-connections and super-traces. The articles of Quillen [13] and Mathai-Quillen [11] are our main background. However, Quillen did not use relative cohomology while our constructions are systemically performed in **relative cohomology**, therefore are more precise. This relative construction was certainly present in the mind of Quillen, and we do not pretend to a great originality. Indeed, if a morphism $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ between two vector bundles over a manifold M is invertible outside a closed subset F, the construction of Quillen of the Chern character $\mathrm{Ch}_{\mathrm{Q}}(\sigma) = \mathrm{Str}(e^{\mathbb{A}_\sigma^2})$ is defined using a super-connection $\mathbb{A}_\sigma$ with zero degree term the odd endomorphism $i(\sigma \oplus \sigma^*)$ and this construction provides also a form β defined outside F such that the equality $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-) = d\beta$ holds on $M \setminus F$. Thus the couple (α, β) of differential forms, with $\alpha := \mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$, defines naturally a class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in the de Rham relative group $\mathcal{H}^*(M, M \setminus F)$ that we call the Quillen's relative Chern character. Now, if $a = (\alpha, \beta)$ is a closed element in relative cohomology, e.g. α is a closed form on M and $\alpha = d\beta$ outside F, the couple (α, β) of differential forms leads naturally to usual de Rham closed differential forms on M with support as close as we want from F. Indeed, using a function χ identically equal to 1 on a neighborhood of F, the closed differential form $\mathrm{p}(a) := \chi\alpha + d\chi\beta$ is supported as close as we want of F. Thus Quillen's super-connection construction gives us three representations of the Chern character: the Quillen Chern character $\mathrm{Ch}_{\mathrm{Q}}(\sigma)$, the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ and the Chern character $\mathrm{Ch}_{\mathrm{sup}}(\sigma) = \mathrm{p}(\mathrm{Ch}_{\mathrm{rel}}(\sigma))$ supported near F. We study the relations between these classes and prove some basic relations. Our previous article [12] explained the construction of the relative Chern character in ordinary cohomology. Here these constructions are done in equivariant cohomology and are very similar.

As an important example, we consider σ_b the Bott morphism on a complex vector bundle $\sigma_b : \wedge^+\mathcal{V} \to \wedge^-\mathcal{V}$ over $\mathcal{V}$, given by the exterior product by $v \in \mathcal{V}$. This morphism has support the zero section M of $\mathcal{V}$. Quillen's Chern character $\mathrm{Ch}_{\mathrm{Q}}(\sigma_b)$ is particularly pleasant as it is represented by a differential form with "Gaussian look" on each fiber of $\mathcal{V}$. However, for many purposes, it is important to construct the Chern character of σ_b, as a differential form supported near the zero section of $\mathcal{V}$. More precisely, we here consider systematically the relative class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)$ in $\mathcal{H}^*(\mathcal{V}, \mathcal{V} \setminus M)$ which contains all information.

A similar construction of the Thom form, using the Berezin integral instead of a super-trace, leads to explicit formulae for the relative Thom class. Again here, we have three representatives of the Thom classes, the Mathai-Quillen Thom form which has a "Gaussian look", the relative Thom form, and the Thom form with support near the zero section. Our main result is Theorem 4.7 where these three representatives are given in the equivariant cohomology of vector spaces.

These explicit formulae allows us to derive the well known relations between Thom classes in cohomology and K-theory (Theorem 6.4). Here again, following Mathai-Quillen construction, we perform all calculations on the equivariant cohomology groups of an Euclidean vector space, and we apply Chern-Weil morphism to deduce relations in any vector bundle.

We also include in this chapter proofs of Thom isomorphisms in various equivariant cohomologies spaces, using Atiyah's "rotation" construction. For the case of relative cohomology, we need to define the product in de Rham relative cohomology and some of its properties. This is the topic of Section 3.

In the second chapter of this monograph, we will generalize our results to equivariant cohomology classes with $C^{-\infty}$-coefficients. This will be an essential ingredient of our new index formula for transversally elliptic operators (see [14]).

2 Equivariant cohomologies

2.1 Definitions and notations

If f is a map on a space M, the notation $f(x)$ means, depending of the context, either the value of f at the point x of M, or the function $x \mapsto f(x)$ where x is a running variable in M.

When a compact Lie group K acts linearly on a vector space E, we denote E^K the sub-space of K-invariant vectors.

Let N be a manifold, and let $\mathcal{A}(N)$ be the algebra of differential forms on N. We denote by $\mathcal{A}_c(N)$ the sub-algebra of compactly supported differential forms. We will consider on $\mathcal{A}(N)$ and $\mathcal{A}_c(N)$ the $\mathbb{Z}$-grading by the exterior degree. It induces a $\mathbb{Z}_2$-grading on $\mathcal{A}(N)$ in even or odd forms. We denote by d the de Rham differential. If α is a closed differential form, we sometimes denote also by α its de Rham cohomology class.

If S is a vector field on N, we denote by $\mathcal{L}(S) : \mathcal{A}^k(N) \to \mathcal{A}^k(N)$ the Lie derivative, and by $\iota(S) : \mathcal{A}^k(N) \to \mathcal{A}^{k-1}(N)$ the contraction of a differential form by the vector field S.

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. We denote $\mathcal{C}^{\mathrm{pol}}(\mathfrak{k})$ the space of polynomial functions on $\mathfrak{k}$ and $\mathcal{C}^\infty(\mathfrak{k})$ the space of C^∞-functions on $\mathfrak{k}$. The algebra $\mathcal{C}^{\mathrm{pol}}(\mathfrak{k})$ is isomorphic to the symmetric algebra $S(\mathfrak{k}^*)$ of $\mathfrak{k}^*$.

We suppose that the manifold N is provided with an action of K. We denote $X \mapsto VX$ the corresponding morphism from $\mathfrak{k}$ into the Lie algebra of vector fields on N: for $n \in N$,

$$V_nX := \frac{d}{d\epsilon} \exp(-\epsilon X) \cdot n|_{\epsilon=0}, \quad X \in \mathfrak{k}.$$

Let $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N) = (\mathcal{C}^{\mathrm{pol}}(\mathfrak{k}) \otimes \mathcal{A}(N))^K$ be the $\mathbb{Z}$-graded algebra of equivariant polynomial functions $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. Its $\mathbb{Z}$-grading is the grading induced by the exterior degree and where elements of $\mathfrak{k}^*$ have degree two. Let $D = d - \iota(VX)$ be the equivariant differential:

$$(D\alpha)(X) = d(\alpha(X)) - \iota(VX)\alpha(X).$$

Let $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N) := \mathrm{Ker}D/\mathrm{Im}D$ be the equivariant cohomology algebra with polynomial coefficients. It is a module over $\mathcal{C}^{\mathrm{pol}}(\mathfrak{k})^K$.

Remark 2.1 *If K is not connected, $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ depends of K, and not only of the Lie algebra of K. However, for notational simplicity, we do not include K in the notation.*

If $g : M \to N$ is a K-equivariant map from the K-manifold M to the K-manifold N, then we obtain a map $g^* : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, M)$, which induces a map g^* in cohomology. When U is an open invariant subset of N, we denote by $\alpha \mapsto \alpha|_U$ the restriction of $\alpha \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ to U.

If S is a K-invariant vector field on N, the operators $\mathcal{L}(S)$ and $\iota(S)$ are extended from $\mathcal{A}(N)$ to $\mathcal{C}^{\mathrm{pol}}(\mathfrak{k}) \otimes \mathcal{A}(N)$: they commute with the K-action, so $\mathcal{L}(S)$ and $\iota(S)$ act on $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$. Cartan's relation holds:

$$\mathcal{L}(S) = D \circ \iota(S) + \iota(S) \circ D. \tag{1}$$

If N is non-compact, we can also consider the space $\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{k}, N) := (\mathcal{C}^{\mathrm{pol}}(\mathfrak{k}) \otimes \mathcal{A}_c(N))^K$ of equivariant polynomial forms $\alpha(X)$ which are compactly supported on N. We denote by $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N)$ the corresponding cohomology algebra. If N is an oriented manifold, integration over N defines a map $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N) \to \mathcal{C}^{\mathrm{pol}}(\mathfrak{k})^K$. If $\pi : N \to B$ is a K-equivariant fibration with oriented fibers, then the integral over the fiber defines a map $\pi_* : \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N) \to \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, B)$.

Finally, we give more definitions in the case of a K-equivariant real vector bundle $p : \mathcal{V} \to M$. We may define two sub-algebras of $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V})$ which are stable under the derivative D. The sub-algebra $\mathcal{A}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V})$ consists of polynomial equivariant forms on $\mathcal{V}$ such that all partial derivatives are rapidly decreasing along the fibers. We may also consider the sub-algebra $\mathcal{A}_{\text{fiber cpt}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V})$ of K-equivariant forms on $\mathcal{V}$ which have a compact support in the fibers of $p : \mathcal{V} \to M$. The inclusions $\mathcal{A}_{\text{fiber cpt}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \subset \mathcal{A}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \subset \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V})$ give rise to the natural maps $\mathcal{H}_{\text{fiber cpt}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V})$. If the fibers of $\mathcal{V}$ are oriented, integration over the fiber defines a map $p_* : \mathcal{H}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M)$.

Let $\mathcal{A}^\infty(\mathfrak{k}, N)$ be the $\mathbb{Z}_2$-graded algebra of equivariant *smooth* maps $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. Its $\mathbb{Z}_2$-grading is the grading induced by the exterior degree. The equivariant differential D is well defined on $\mathcal{A}^\infty(\mathfrak{k}, N)$ and respects the $\mathbb{Z}_2$-grading. Let $\mathcal{H}^\infty(\mathfrak{k}, N) := \mathrm{Ker} D / \mathrm{Im} D$ be the corresponding cohomology algebra with C^∞-coefficients. We denote by $\mathcal{A}_c^\infty(\mathfrak{k}, N)$ the sub-algebra of equivariant differential forms with compact support and by $\mathcal{H}_c^\infty(\mathfrak{k}, N)$ the corresponding algebra cohomology algebra . Then $\mathcal{H}^\infty(\mathfrak{k}, N)$ and $\mathcal{H}_c^\infty(\mathfrak{k}, N)$ are $\mathbb{Z}_2$-graded algebras. If N is oriented, integration over N defines a map $\mathcal{H}_c^\infty(\mathfrak{k}, N) \to \mathcal{C}^\infty(\mathfrak{k})^K$.

Let $\mathcal{V}$ be a K-equivariant real vector bundle over a manifold M. We define similarly $\mathcal{A}_{\text{dec-rap}}^\infty(\mathfrak{k}, \mathcal{V})$ and $\mathcal{H}_{\text{dec-rap}}^\infty(\mathfrak{k}, \mathcal{V})$ as well as $\mathcal{A}_{\text{fiber cpt}}^\infty(\mathfrak{k}, \mathcal{V})$ and $\mathcal{H}_{\text{fiber cpt}}^\infty(\mathfrak{k}, \mathcal{V})$. There are natural maps $\mathcal{H}_c^\infty(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}_{\text{fiber cpt}}^\infty(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}_{\text{dec-rap}}^\infty(\mathfrak{k}, \mathcal{V})$ and an integration map $\mathcal{H}_{\text{dec-rap}}^\infty(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}^\infty(\mathfrak{k}, M)$ if the fibers of $\mathcal{V} \to M$ are oriented.

After these lengthy definitions, we hope that at this point the reader is still with us.

If K is the identity (we say the non-equivariant case), then the operator D is the usual de Rham differential d. We systemically skip the letter $\mathfrak{k} = \{0\}$ in the corresponding notations. Thus the equivariant cohomology group $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$ coincide with the usual de Rham cohomology group $\mathcal{H}(N)$. The compactly supported cohomology space is denoted by $\mathcal{H}_c(N)$ and the rapidly decreasing cohomology space by $\mathcal{H}_{\text{dec-rap}}(\mathcal{V})$. (In this article, we will only work with cohomology groups, so the notation $\mathcal{H}$ refers always to cohomology).

2.2 Equivariant cohomology of vector bundles

It is well known that the cohomology of a vector bundle is the cohomology of the basis. The same equivariant Poincaré lemma holds in equivariant cohomology (see for example [8]). We review the proof.

Let $p : \mathcal{V} \to M$ be a K-equivariant (real) vector bundle. Let $i : M \to \mathcal{V}$ be the inclusion of the zero section.

Theorem 2.2

- *The map* $i^* : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M)$ *is an isomorphism with inverse* p^**.*
- *The map* $i^* : \mathcal{H}^{\infty}(\mathfrak{k}, \mathcal{V}) \to \mathcal{H}^{\infty}(\mathfrak{k}, M)$ *is an isomorphism with inverse* p^**.*

Proof. We prove the statement in the $\mathcal{C}^\infty$ case. As $p \circ i = \mathrm{Id}_M$, we have $i^* \circ p^* = \mathrm{Id}_{\mathcal{H}^\infty(\mathfrak{k},M)}$.

Let us prove $p^* \circ i^* = \mathrm{Id}_{\mathcal{H}^\infty(\mathfrak{k},\mathcal{V})}$. We denote by (m, v) a point of $\mathcal{V}$ with $m \in M$ and $v \in \mathcal{V}_m$. For $t \geq 0$, let $h(t)(m, v) = (m, tv)$ be the homothety on the fiber. The transformations $h(t)$ verify $h(t_1)h(t_2) = h(t_1t_2)$ and commute with the action of K.

Let $\alpha \in \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V})$ be a closed element and let $\alpha(t) = h(t)^*\alpha$. Thus $\alpha(0) = p^* \circ i^*(\alpha)$, while $\alpha(1) = \alpha$. From Formula (1), we obtain for $t > 0$:

$$\frac{d}{dt}\alpha(t) = \frac{1}{t}\mathcal{L}(S)\cdot(\alpha(t)) = D\Big(\frac{1}{t}\iota(S)(\alpha(t))\Big). \tag{2}$$

Here S is the Euler vector field on $\mathcal{V}$: at each point (m, v) of $\mathcal{V}$, $S_{(m,v)} = v$.

It is easily checked that $\frac{1}{t}(\iota(S)\alpha(t))$ is continuous at $t = 0$. Indeed, locally if $\alpha = \sum_{I,J}\nu_{I,J}(X, m, v)dm_I dv_J$, $\alpha(t) = \sum_{I,J}\nu_{I,J}(X, m, tv)dm_I t^{|J|}dv_J$, and $\iota(S)$ kills all components with $|J| = 0$. Integrating Equation (2) from 0 to 1, we obtain:

$$\alpha - p^* \circ i^*(\alpha) = D\Big(\int_0^1 \frac{1}{t}\iota(S)\alpha(t)dt\Big).$$

Thus we obtain the relation $p^* \circ i^* = \mathrm{Id}_{\mathcal{H}^\infty(\mathfrak{k},\mathcal{V})}$.

2.3 The Chern-Weil construction

Let $\pi : P \to B$ be a principal bundle with structure group G. For any G-manifold Z, we define the manifold $\mathcal{Z} = P \times_G Z$ which is fibred over B with typical fiber Z. Let $\mathcal{A}(P \times Z)_{\mathrm{hor}} \subset \mathcal{A}(P \times Z)$ be the sub-algebra formed by the differential forms on $P \times Z$ which are *horizontal*: $\gamma \in \mathcal{A}(P \times Z)_{\mathrm{hor}}$ if $\iota(VX)\alpha = 0$ on $P \times Z$ for every $X \in \mathfrak{g}$. The algebra $\mathcal{A}(\mathcal{Z})$ admits a natural identification with the basic subalgebra

$$\mathcal{A}(P \times Z)_{\mathrm{bas}} := (\mathcal{A}(P \times Z)_{\mathrm{hor}})^G.$$

Let $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ be a connection one form on P, with curvature form $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in (\mathcal{A}^2(P)_{\mathrm{hor}} \otimes \mathfrak{g})^G$. The connection one form ω defines, for any G-manifold Z, a projection from $\mathcal{A}(P \times Z)^G$ onto $\mathcal{A}(P \times Z)_{\mathrm{bas}}$.

The Chern-Weil homomorphism

$$\phi_\omega^Z : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z) \longrightarrow \mathcal{A}(\mathcal{Z}). \tag{3}$$

is defined as follows (see [4], [8]). For a G-equivariant form $\alpha(X), X \in \mathfrak{g}$ on Z, the form $\phi_\omega^Z(\alpha) \in \mathcal{A}(\mathcal{Z})$ is equal to the projection of $\alpha(\Omega) \in \mathcal{A}(P \times Z)^G$ on the basic subspace $\mathcal{A}(P \times Z)_{\mathrm{bas}} \simeq \mathcal{A}(\mathcal{Z})$.

In the case where Z is the $\{\mathrm{pt}\}$, ϕ_ω^Z is the usual Chern-Weil homomorphism which associates to a G-invariant polynomial Q the characteristic form $Q(\Omega)$.

The main property of the equivariant cohomology differential D proved by Cartan (see [4], [9]) is the following proposition.

Proposition 2.3
$$\phi_\omega^Z \circ D = d \circ \phi_\omega^Z.$$
Thus a closed equivariant form on Z gives rise to a closed de Rham form on $\mathcal{Z}$.

We can repeat the construction above in the equivariant case.

Let K and G be two compact Lie groups. Assume that P is provided with an action of $K \times G$: $(k, g)(y) = kyg^{-1}$, for $k \in K, g \in G, y \in P$. We assume that G acts freely. Thus the manifold $P/G = B$ is provided with a left action of K. Let ω be a K-invariant connection one form on P, with curvature form Ω. For $Y \in \mathfrak{k}$, we denote by $\mu(Y) = -\iota(VY)\omega \in \mathcal{C}^\infty(P) \otimes \mathfrak{g}$ the moment of Y. The equivariant curvature form is

$$\Omega(Y) = \Omega + \mu(Y),\ X \in \mathfrak{k}.$$

Let Z be a G-manifold. We consider the Chern-Weil homomorphism

$$\phi_\omega^Z : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z) \longrightarrow \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{Z}). \tag{4}$$

It is defined as follows (see [4],[8]). For a G-equivariant form α on Z, the value of the equivariant form $\phi_\omega^Z(\alpha)$ at $Y \in \mathfrak{k}$ is equal to the projection of $\alpha(\Omega(Y)) \in \mathcal{A}(P \times Z)^G$ onto the the basic subspace $\mathcal{A}(P \times Z)_{\mathrm{bas}} \simeq \mathcal{A}(\mathcal{Z})$. For $Z = \{\mathrm{pt}\}$, and Q a G-invariant polynomial on $\mathfrak{g}$, the form $\phi_\omega^Z(Q)(Y) = Q(\Omega(Y))$ is the Chern-Weil characteristic class constructed in [3], see also [7].

Proposition 2.4 *The map $\phi_\omega^Z : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{Z})$ satisfies*

$$\phi_\omega^Z \circ D = D \circ \phi_\omega^Z.$$

Here, on the left side the equivariant differential is with respect to the action of G while, on the right side, the equivariant differential is with respect to the group K.

3 Relative equivariant cohomology

Let N be a manifold provided with an action of a compact Lie group K.

3.1 Definition and basic properties

Let F be a closed K-invariant subset of N. To an equivariant cohomology class on N vanishing on $N \setminus F$, we associate a relative equivariant cohomology class. Let us explain the construction (see [7],[12] for the non-equivariant case).

Consider the complex $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ with

$$\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) := \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N) \oplus \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)$$

and differential $D_{\mathrm{rel}}\,(\alpha, \beta) = \big(D\alpha, \alpha|_{N\setminus F} - D\beta\big)$.

Definition 3.1 *The cohomology of the complex* $(\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F), D_{\mathrm{rel}})$ *is the relative equivariant cohomology space* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$*.*

In the case where K is the identity, we skip the letter $\mathfrak{k}$ in the notation. Then D_{rel} is the usual relative de Rham differential and $\mathcal{H}(N, N \setminus F)$ is the usual de Rham relative cohomology group [6].

The complex $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ is $\mathbb{Z}$-graded : for $k \in \mathbb{Z}$, we take

$$\big[\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)\big]^k = \big[\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)\big]^k \oplus \big[\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)\big]^{k-1}.$$

Since D_{rel} sends $\big[\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)\big]^k$ into $\big[\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)\big]^{k+1}$, the $\mathbb{Z}$-grading descends to the relative cohomology spaces $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$. The class defined by a D_{rel}-closed element $(\alpha, \beta) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ will be denoted $[\alpha, \beta]$.

Remark that $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ is a module over $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$. Indeed the multiplication by a closed equivariant form $\eta \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$,

$$\eta \cdot (\alpha, \beta) = (\eta \wedge \alpha, \eta|_{N\setminus F} \wedge \beta),$$

on $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ commutes with D_{rel}.

If S is a K-invariant vector field on N, we define on $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ the operations $\mathcal{L}(S)(\alpha, \beta) := (\mathcal{L}(S)\alpha, \mathcal{L}(S)\beta)$ and $\iota(S)(\alpha, \beta) := (\iota(S)\alpha, -\iota(S)\beta)$. It is immediate to check that Cartan' relation (1) holds

$$\mathcal{L}(S) = \iota(S) \circ D_{\mathrm{rel}} + D_{\mathrm{rel}} \circ \iota(S). \tag{5}$$

We consider now the following maps.

- The projection $j : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ is the degree 0 map defined by $j(\alpha, \beta) = \alpha$.
- The inclusion $i : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ is the degree $+1$ map defined by $i(\beta) = (0, \beta)$.
- The restriction $r : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)$ is the degree 0 map defined by $r(\alpha) = \alpha|_{N\setminus F}$.

It is easy to see that i, j, r induce maps in cohomology that we still denote by i, j, r.

Proposition 3.2 • *We have an exact triangle*

$$\begin{tikzcd} & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \arrow[dr, "j"] & \\ \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F) \arrow[ur, "i"] & & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N). \arrow[ll, "r"] \end{tikzcd}$$

• *If $F \subset F'$ are closed K-invariant subsets of N, the restriction map $(\alpha, \beta) \mapsto (\alpha, \beta|_{N \setminus F'})$ induces a map*

$$\mathbf{r}_{F',F} : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F'). \tag{6}$$

• *If g is a diffeomorphism of N which preserves F and commutes with the action of K, then $g^*(\alpha, \beta) = (g^*\alpha, g^*\beta)$ induces a transformation g^* of $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$.*

Proof. This proof is the same than in the non equivariant-case [6] and we skip it.

The same statements hold in the $\mathcal{C}^\infty$-case. Here we consider the complex $\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F)$ with $\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F) := \mathcal{A}^\infty(\mathfrak{k}, N) \oplus \mathcal{A}^\infty(\mathfrak{k}, N \setminus F)$ and differential $D_{\mathrm{rel}}(\alpha, \beta) = \big(D\alpha, \alpha|_{N \setminus F} - D\beta\big)$.

Definition 3.3 *The cohomology of the complex $(\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F), D_{\mathrm{rel}})$ is the relative equivariant cohomology spaces $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$.*

The complex $\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F)$ is $\mathbb{Z}_2$-graded by taking $[\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F)]^\epsilon = [\mathcal{A}^\infty(\mathfrak{k}, N)]^\epsilon \oplus [\mathcal{A}^\infty(\mathfrak{k}, N \setminus F)]^{\epsilon+1}$. Since D_{rel} send $[\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F)]^\epsilon$ into $[\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus F)]^{\epsilon+1}$, the $\mathbb{Z}_2$-grading descends to the relative cohomology spaces $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$.

Here the space $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$ is a module over $\mathcal{H}^\infty(\mathfrak{k}, N)$.

Lemma 3.4 • *We have an exact triangle*

$$\begin{tikzcd} & \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \arrow[dr, "j"] & \\ \mathcal{H}^\infty(\mathfrak{k}, N \setminus F) \arrow[ur, "i"] & & \mathcal{H}^\infty(\mathfrak{k}, N). \arrow[ll, "r"] \end{tikzcd}$$

• *If $F \subset F'$ are closed K-invariant subsets of N, the restriction $(\alpha, \beta) \mapsto (\alpha, \beta|_{N \setminus F'})$ induces a map $\mathbf{r}_{F',F} : \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F')$.*

• *If g is a diffeomorphism of N which preserves F and commutes with the action of K, then $g^*(\alpha, \beta) = (g^*\alpha, g^*\beta)$ induces a transformation g^* of $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$.*

3.2 Excision

Let $\chi \in \mathcal{C}^\infty(N)^K$ be a K-invariant function such that χ is identically equal to 1 on a neighborhood of F. If $\beta \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)$, note that $d\chi \wedge \beta$ defines an equivariant form on N, since $d\chi$ is equal to 0 in a neighborhood of F. We define

$$I^\chi : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \tag{7}$$

by $I^\chi(\alpha, \beta) = (\chi\alpha + d\chi \wedge \beta, \chi\beta)$. Then

$$I^\chi \circ D_{\mathrm{rel}} = D_{\mathrm{rel}} \circ I^\chi, \tag{8}$$

so that I^χ defines a map $I^\chi : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$.

Lemma 3.5 *The map I^χ is independent of χ. In particular, I^χ is the identity in relative cohomology.*

Proof. If (α, β) is D_{rel}-closed, then $I^{\chi_1}(\alpha, \beta) - I^{\chi_2}(\alpha, \beta) = D_{\mathrm{rel}}((\chi_1 - \chi_2)\beta, 0)$. This shows that I^χ is independent of χ. Choosing $\chi = 1$, we see that $I^\chi = \mathrm{Id}$ in cohomology.

It follows from the above proposition that we can always choose a representative (α, β) of a relative cohomology class, with α and β supported in a neighborhood of F as small as we want. This will be important to define the integral over the fiber of a relative cohomology class with support intersecting the fibers in compact subsets. The integration will be defined in Section 3.6

In particular, if F is compact, we define a map

$$\mathrm{p}_c : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N). \tag{9}$$

by setting $\mathrm{p}_c(\alpha, \beta) = \chi\alpha + d\chi \wedge \beta$, where $\chi \in \mathcal{C}^\infty(N)^K$ is a K-invariant function with compact support such that χ is identically equal to 1 on a neighborhood of F.

An important property of the relative cohomology group is the *excision property.* Let U be a K-invariant neighborhood of F. The restriction $(\alpha, \beta) \mapsto (\alpha|_U, \beta|_{U \setminus F})$ induces a map

$$\mathbf{r}^U : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, U, U \setminus F).$$

Proposition 3.6 *The map $\mathbf{r}^U$ is an isomorphism.*

Proof. Let us choose $\chi \in \mathcal{C}^\infty(N)^K$ supported in U and equal to 1 in a neighborhood of F. The map (7) defines in this context three maps : $I^\chi_N : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$, $I^\chi_U : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, U, U \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, U, U \setminus F)$ and $I^\chi_{N,U} : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, U, U \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$.

We check easily that $I^\chi_{N,U} \circ \mathbf{r}^U = I^\chi_N$ on $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$, and that $\mathbf{r}^U \circ I^\chi_{N,U} = I^\chi_U$ on $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, U, U \setminus F)$. From Lemma 3.5, we know that I^χ_N and I^χ_U are the identity maps in cohomology. This proves that $\mathbf{r}^U$ is an isomorphism in cohomology.

The same statements holds in the $\mathcal{C}^\infty$-case.

Proposition 3.7 • *Let U be a K-invariant neighborhood of F. The map $\mathbf{r}^U$: $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^\infty(\mathfrak{k}, U, U \setminus F)$ is an isomorphism.*

• *If F is compact, there is a natural map $\mathrm{p}_c : \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^\infty_c(\mathfrak{k}, N)$.*

3.3 Product in relative equivariant cohomology

Let F_1 and F_2 be two closed K-invariant subsets of N. We will now define a graded product

$$\begin{aligned} \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) &\longrightarrow \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)) \\ (\quad a \quad , \quad b \quad) &\longmapsto a \diamond b \,. \end{aligned} \tag{10}$$

Let $U_1 := N \setminus F_1$, $U_2 := N \setminus F_2$ so that $U := N \setminus (F_1 \cap F_2) = U_1 \cup U_2$. Let $\Phi := (\Phi_1, \Phi_2)$ be a partition of unity subordinate to the covering $U_1 \cup U_2$ of U. By averaging by K, we may suppose that the functions Φ_k are invariant.

Since $\Phi_i \in \mathcal{C}^\infty(U)^K$ is supported in U_i, the product $\gamma \mapsto \Phi_i \gamma$ defines a map $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F_i) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus (F_1 \cap F_2))$. Since $d\Phi_1 = -d\Phi_2 \in \mathcal{A}(U)^K$ is supported in $U_1 \cap U_2 = N \setminus (F_1 \cup F_2)$, the product $\gamma \mapsto d\Phi_1 \wedge \gamma$ defines a map $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus (F_1 \cup F_2)) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus (F_1 \cap F_2))$.

With the help of Φ, we define a bilinear map $\diamond_\Phi$: $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$ as follows. For $a_i := (\alpha_i, \beta_i) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$, $i = 1, 2$, we define

$$a_1 \diamond_\Phi a_2 := \Big(\alpha_1 \wedge \alpha_2, \beta_\Phi(a_1, a_2)\Big)$$

with

$$\beta_\Phi(a_1, a_2) = \Phi_1 \beta_1 \wedge \alpha_2 + (-1)^{|a_1|} \alpha_1 \wedge \Phi_2 \beta_2 - (-1)^{|a_1|} d\Phi_1 \wedge \beta_1 \wedge \beta_2.$$

Remark that all equivariant forms $\Phi_1\beta_1 \wedge \alpha_2$, $\alpha_1 \wedge \Phi_2 \beta_2$ and $d\Phi_1 \wedge \beta_1 \wedge \beta_2$ are well defined on $U_1 \cup U_2$. So $a_1 \diamond_\Phi a_2 \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$. It is immediate to verify that

$$D_{\mathrm{rel}}(a_1 \diamond_\Phi a_2) = (D_{\mathrm{rel}} a_1) \diamond_\Phi a_2 + (-1)^{|a_1|} a_1 \diamond_\Phi (D_{\mathrm{rel}} a_2).$$

Thus $\diamond_\Phi$ defines a bilinear map $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$. Let us see that this product do not depend of the choice of the partition of unity. If we have another partition $\Phi' = (\Phi'_1, \Phi'_2)$, then $\Phi_1 - \Phi'_1 = -(\Phi_2 - \Phi'_2)$. It is immediate to verify that, if $D_{\mathrm{rel}}(a_1) = 0$ and $D_{\mathrm{rel}}(a_2) = 0$, one has

$$a_1 \diamond_\Phi a_2 - a_1 \diamond_{\Phi'} a_2 = D_{\mathrm{rel}}\Big(0, (-1)^{|a_1|}(\Phi_1 - \Phi'_1)\beta_1 \wedge \beta_2\Big).$$

in $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$. So the product on the relative cohomology spaces will be denoted by $\diamond$.

Remark 3.8 *The same formulae defines a $\mathbb{Z}_2$-graded product*

$$\begin{array}{rcl} \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2) & \longrightarrow & \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)) \\ (\quad a \quad , \quad b \quad) & \longmapsto & a \diamond b \,. \end{array} \tag{11}$$

We note the following properties, which are well known in the non-equivariant case.

Proposition 3.9

- *The relative product is compatible with restrictions: if $F_1 \subset F_1'$ and $F_2 \subset F_2'$ are closed invariant subsets of N, then the diagram* (12)

$$\begin{array}{ccccc} \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) & \times & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) & \xrightarrow{\diamond} & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)) \\ \downarrow \mathbf{r}_1 & & \downarrow \mathbf{r}_2 & & \downarrow \mathbf{r}_{12} \\ \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1') & \times & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2') & \xrightarrow{\diamond} & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1' \cap F_2')) \end{array}$$

  is commutative. Here the $\mathbf{r}_i$ are the restrictions maps defined in (6).

- *The relative product is graded commutative : $a_1 \diamond a_2 = (-1)^{|a_1|\cdot|a_2|} a_2 \diamond a_1$.*

- *The relative product is associative.*

The same statements holds in the $\mathcal{C}^{\infty}$-case.

Proof. The first point follows from the definition. Let $U_i = N \setminus F_i$, $U_i' = N \setminus F_i'$. Let $\Phi_1 + \Phi_2 = 1$ be a partition of unity on $U_1 \cup U_2$. Then $\Phi' = (\Phi_1', \Phi_2')$ with $\Phi_i' := \Phi_i|_{U_i'}$ is a partition of unity on $U_1' \cup U_2'$. Then, at the level of equivariant forms, we have $\beta_{\Phi}(a_1, a_2)|_{N \setminus (F_1' \cap F_2')} = \beta_{\Phi'}(\mathbf{r}_1(a_1), \mathbf{r}_2(a_2))$. The commutative diagram (12) follows.

The second point is immediate from the definition.

We now prove the third point. Let F_1, F_2 and F_3 be three closed invariant subsets of N. Let $a_i \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$ for $i = 1, 2, 3$. In order to prove that $(a_1 \diamond a_2) \diamond a_3 = a_1 \diamond (a_2 \diamond a_3)$ in $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$ we introduce a multi-linear map $\top : E_1 \times E_2 \times E_3 \longrightarrow \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$ where $E_i = \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$.

Let $U_i = N \setminus F_i$ and $U = N \setminus (F_1 \cap F_2 \cap F_3)$. Let $\mathcal{D}$ be the data formed by :

- A partition of unity $\Phi_1 + \Phi_2 + \Phi_3 = 1$ on $U_1 \cup U_2 \cup U_3 = U$, where the functions Φ_i are K-invariant.

- Invariant one forms Λ_1, Λ_2 and Λ_3 on U supported respectively in $U_2 \cap U_3$, $U_1 \cap U_3$ and $U_1 \cap U_2$.

We suppose that the data $\mathcal{D}$ satisfies the following conditions

$$d\Phi_1 = \Lambda_2 - \Lambda_3,\ d\Phi_2 = \Lambda_3 - \Lambda_1,\ d\Phi_3 = \Lambda_1 - \Lambda_2. \tag{13}$$

Then we have

$$D\Lambda_1(X) = D\Lambda_2(X) = D\Lambda_3(X). \tag{14}$$

We denote $\Theta(X)$ the equivariant 2-form equal to $D\Lambda_1(X)$: (14) shows that $\Theta(X)$ is supported in $U_1 \cap U_2 \cap U_3$.

With the help of $\mathcal{D}$, we define a three-linear map $\top_{\mathcal{D}}$ from $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) \times \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_3)$ into $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$ as follows. For $a_i := (\alpha_i, \beta_i) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$, $i = 1, 2, 3$, we define $\top_{\mathcal{D}}(a_1, a_2, a_3) := (\alpha_1 \wedge \alpha_2 \wedge \alpha_3, \beta_{\mathcal{D}}(a_1, a_2, a_3))$ with

$$\begin{aligned} \beta_{\mathcal{D}}(a_1, a_2, a_3) = {} & \Phi_1 \beta_1 \alpha_2 \alpha_3 + (-1)^{|a_1|} \Phi_2 \alpha_1 \beta_2 \alpha_3 + (-1)^{|a_1|+|a_2|} \Phi_3 \alpha_1 \alpha_2 \beta_3 \\ & + (-1)^{|a_2|} \Lambda_1 \alpha_1 \beta_2 \beta_3 - (-1)^{|a_1|+|a_2|} \Lambda_2 \beta_1 \alpha_2 \beta_3 + (-1)^{|a_1|} \Lambda_3 \beta_1 \beta_2 \alpha_3 \\ & - (-1)^{|a_2|} \Theta \beta_1 \beta_2 \beta_3. \end{aligned} \tag{15}$$

Remark that all equivariant forms which appears in the right hand side of (15) are well defined on $U_1 \cup U_2 \cup U_3$. So $\top_{\mathcal{D}}(a_1, a_2, a_3) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$.

The following relation is "immediate" to verify:

$$D_{\mathrm{rel}}(\top_{\mathcal{D}}(a_1, a_2, a_3)) =$$
$$\top_{\mathcal{D}}(D_{\mathrm{rel}} a_1, a_2, a_3) + (-1)^{|a_1|} \top_{\mathcal{D}}(a_1, D_{\mathrm{rel}} a_2, a_3) + (-1)^{|a_1|+|a_2|} \top_{\mathcal{D}}(a_1, a_2, D_{\mathrm{rel}} a_3).$$

Thus $\top_{\mathcal{D}}$ defines a three-linear map from from $E_1 \times E_2 \times E_3$ into $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$. Let us see that this map do not depend of the choice of the data $\mathcal{D}$. Let $\mathcal{D}' = \{\Phi'_i, \Lambda'_i \text{ for } i = 1, 2, 3\}$ be another data which satisfies conditions (13).

We consider the functions $f_i = \Phi_i - \Phi'_i$ on U. If $\{i, j, k\} = \{1, 2, 3\}$ the function f_i is supported in $U_i \cap (U_j \cup U_k) = (U_i \cap U_j) \cup (U_i \cap U_k)$. The relations $f_1 + f_2 + f_3 = 0$ on U shows that there exists K-invariant functions θ_i on U such that θ_i is supported in $U_j \cap U_k$ and

$$f_1 = \theta_2 - \theta_3,\ f_2 = \theta_3 - \theta_1,\ f_3 = \theta_1 - \theta_2.$$

We see then that

$$\Lambda_1 - \Lambda'_1 - d\theta_1 = \Lambda_2 - \Lambda'_2 - d\theta_2 = \Lambda_3 - \Lambda'_3 - d\theta_3$$

is an invariant one form on U supported on $U_1 \cap U_2 \cap U_3$: let us denote it by Δ. We have $\Theta(X) - \Theta'(X) = D\Delta(X)$.

Then for D_{rel}-closed elements a_i, one checks that $\top_{\mathcal{D}}(a_1, a_2, a_3) - \top_{\mathcal{D}'}(a_1, a_2, a_3)$ is equal to $D_{\mathrm{rel}}(0, -\delta) = (0, D\delta)$ with

$$\begin{aligned} \delta = {} & (-1)^{|a_2|} \theta_1 \alpha_1 \beta_2 \beta_3 - (-1)^{|a_1|+|a_2|} \theta_2 \beta_1 \alpha_2 \beta_3 + (-1)^{|a_1|} \theta_3 \beta_1 \beta_2 \alpha_3 \\ & - (-1)^{|a_2|} \Delta \beta_1 \beta_2 \beta_3. \end{aligned}$$

Let us denote $\top$ the three-linear map induced by $\top_{\mathcal{D}}$ in relative equivariant cohomology.

Now we will see that the map $(a_1, a_2, a_3) \mapsto (a_1 \diamond a_2) \diamond a_3$ coincides with $\top$. Let $\phi_1 + \phi_2 = 1$ be a partition of unity on $U_{12} := U_1 \cup U_2$, and let $\varphi_{12} + \varphi_3 = 1$ be a partition of unity on $U_{12} \cup U_3$: all the functions are supposed K-invariant. Then we take the data $\mathcal{D} = \{\Phi_i, \Lambda_i \text{ for } i = 1, 2, 3\}$ defined by the relations:

- $\Phi_1 = \varphi_{12}\phi_1$, $\Phi_2 = \varphi_{12}\phi_2$, $\Phi_3 = \varphi_3$,
- $\Lambda_1 = -d(\varphi_{12})\phi_2$, $\Lambda_2 = d(\varphi_{12})\phi_1$, $\Lambda_3 = -\varphi_{12} d\phi_1$.

One checks that $\mathcal{D}$ satisfies conditions (13), and that for $a_i \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$ the following equality

$$\top_{\mathcal{D}}(a_1, a_2, a_3) = (a_1 \diamond_\phi a_2) \diamond_\varphi a_3$$

holds in $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2 \cap F_3))$.

One proves in the same way that the map $(a_1, a_2, a_3) \mapsto a_1 \diamond (a_2 \diamond a_3)$ coincides with $\top$. We have then proved the associativity of the relative product $\diamond$.

3.4 Inverse limit of equivariant cohomology with support

Let F be a closed K-invariant subset of N. We consider the set $\mathcal{F}_F$ of all open invariant neighborhoods U of F which is ordered by the relation $U \leq V$ if and only if $V \subset U$. For any $U \in \mathcal{F}_F$, we consider the algebra $\mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ of equivariant differential forms on N with support contained in U: $\alpha \in \mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ if there exists a closed set $\mathcal{C}_\alpha \subset U$ such that $\alpha(X)|_n = 0$ for all $X \in \mathfrak{k}$ and all $n \in U \setminus \mathcal{C}_\alpha$. Note that the vector space $\mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ is naturally a module over $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$.

The algebra $\mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ is stable under the differential D, and we denote by $\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ the corresponding cohomology algebra. Note that $\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ is naturally a module over $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$. If $U \leq V$, we have then an inclusion map $\mathcal{A}^{\mathrm{pol}}_V(\mathfrak{k}, N) \hookrightarrow \mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ which gives rise to a map $r_{U,V} : \mathcal{H}^{\mathrm{pol}}_V(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ of $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$-modules.

Definition 3.10 *We denote by $\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$ the inverse limit of the inverse system $(\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N), r_{U,V}; U, V \in \mathcal{F}_F)$. We will call $\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$ the equivariant cohomology of N supported on F.*

Note that the vector space $\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$ is naturally a module over $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$. Let us give the following basic properties of the equivariant cohomology spaces with support.

Lemma 3.11 • *$\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) = \{0\}$ if $F = \emptyset$.*

• *There is a natural map $\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$. If F is compact, this map factors through $\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k}, N)$.*

• *If $F \subset F'$ are closed K-invariant subsets, there is a restriction morphism*

$$\mathbf{r}^{F',F} : \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}_{F'}(\mathfrak{k}, N). \tag{16}$$

• *If F_1 and F_2 are two closed K-invariant subsets of N, the wedge product of forms defines a natural product*

$$\mathcal{H}^{\mathrm{pol}}_{F_1}(\mathfrak{k}, N) \times \mathcal{H}^{\mathrm{pol}}_{F_2}(\mathfrak{k}, N) \xrightarrow{\wedge} \mathcal{H}^{\mathrm{pol}}_{F_1 \cap F_2}(\mathfrak{k}, N). \tag{17}$$

• *If $F_1 \subset F_1'$ and $F_2 \subset F_2'$ are closed K-invariant subsets, then the diagram*

$$\begin{array}{ccccc}
\mathcal{H}^{\mathrm{pol}}_{F_1}(\mathfrak{k}, N) & \times & \mathcal{H}^{\mathrm{pol}}_{F_2}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^{\mathrm{pol}}_{F_1 \cap F_2}(\mathfrak{k}, N) \\
\downarrow{\scriptstyle \mathbf{r}^1} & & \downarrow{\scriptstyle \mathbf{r}^2} & & \downarrow{\scriptstyle \mathbf{r}^{12}} \\
\mathcal{H}^{\mathrm{pol}}_{F_1'}(\mathfrak{k}, N) & \times & \mathcal{H}^{\mathrm{pol}}_{F_2'}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^{\mathrm{pol}}_{F_1' \cap F_2'}(\mathfrak{k}, N)
\end{array} \tag{18}$$

is commutative. Here the $\mathbf{r}^i$ are the restriction morphisms defined in (16).

All the maps in the previous lemma preserve the structures of $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$-module. The same definition and properties hold in the $\mathcal{C}^\infty$-case.

Definition 3.12 *We denote by $\mathcal{H}^\infty_F(\mathfrak{k}, N)$ the inverse limit of the inverse system $(\mathcal{H}^\infty_U(\mathfrak{k}, N), r_{U,V}; U, V \in \mathcal{F}_F)$. We will call $\mathcal{H}^\infty_F(\mathfrak{k}, N)$ the equivariant cohomology of N supported on F.*

Lemma 3.13 • *$\mathcal{H}^\infty_F(\mathfrak{k}, N) = \{0\}$ if $F = \emptyset$.*

• *There is a natural map $\mathcal{H}^\infty_F(\mathfrak{k}, N) \to \mathcal{H}^\infty(\mathfrak{k}, N)$. If F is compact, this map factors through $\mathcal{H}^\infty_F(\mathfrak{k}, N) \to \mathcal{H}^\infty_c(\mathfrak{k}, N)$.*

• *If $F \subset F'$ are closed K-invariant subsets, there is a restriction morphism $\mathbf{r}^{F',F} : \mathcal{H}^\infty_F(\mathfrak{k}, N) \to \mathcal{H}^\infty_{F'}(\mathfrak{k}, N)$.*

• *If F and R are two closed K-invariant subsets of N, the wedge product of forms defines a natural product*

$$\mathcal{H}^\infty_{F_1}(\mathfrak{k}, N) \times \mathcal{H}^\infty_{F_2}(\mathfrak{k}, N) \xrightarrow{\wedge} \mathcal{H}^\infty_{F_1 \cap F_2}(\mathfrak{k}, N). \tag{19}$$

• *If $F_1 \subset F_1'$ and $F_2 \subset F_2'$ are closed K-invariant subsets, then the diagram*

$$\begin{array}{ccccc}
\mathcal{H}^\infty_{F_1}(\mathfrak{k}, N) & \times & \mathcal{H}^\infty_{F_2}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^\infty_{F_1 \cap F_2}(\mathfrak{k}, N) \\
\downarrow{\scriptstyle \mathbf{r}^1} & & \downarrow{\scriptstyle \mathbf{r}^2} & & \downarrow{\scriptstyle \mathbf{r}^{12}} \\
\mathcal{H}^\infty_{F_1'}(\mathfrak{k}, N) & \times & \mathcal{H}^\infty_{F_2'}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^\infty_{F_1' \cap F_2'}(\mathfrak{k}, N)
\end{array} \tag{20}$$

is commutative.

3.5 Morphism p_F

If F is any closed invariant subset of N, we define a morphism

$$\mathrm{p}_F : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \longrightarrow \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \tag{21}$$

of $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$-modules.

Proposition 3.14 *For any open invariant neighborhood U of F, we choose $\chi \in \mathcal{C}^\infty(N)^K$ with support in U and equal to 1 in a neighborhood of F.*

- *The map*

$$\mathrm{p}^\chi(\alpha, \beta) = \chi\alpha + d\chi\beta \tag{22}$$

defines a morphism $\mathrm{p}^\chi : \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ of $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$-modules.

In consequence, let $\alpha \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ be a closed *equivariant form and $\beta \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)$ such that $\alpha|_{N\setminus F} = D\beta$, then $\mathrm{p}^\chi(\alpha, \beta)$ is a* closed *equivariant form supported in U.*

- *The cohomology class of $\mathrm{p}^\chi(\alpha, \beta)$ in $\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$ does not depend of χ. We denote this class by $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$.*
- *For any neighborhoods $V \subset U$ of F, we have $r_{U,V} \circ \mathrm{p}_V = \mathrm{p}_U$.*

Proof. The proof is similar to the proof of Proposition 2.3 in [12]. We repeat the main arguments.

The equation $\mathrm{p}^\chi \circ D_{\mathrm{rel}} = D \circ \mathrm{p}^\chi$ is immediate to check. In particular $\mathrm{p}^\chi(\alpha, \beta)$ is closed, if $D_{\mathrm{rel}}(\alpha, \beta) = 0$. Directly: $D(\chi\alpha + d\chi\beta) = d\chi\alpha - d\chi D\beta = 0$. This proves the first point. For two different choices χ and χ', we have

$$\begin{aligned}
\mathrm{p}^\chi(\alpha, \beta) - \mathrm{p}^{\chi'}(\alpha, \beta) &= (\chi - \chi')\alpha + d(\chi - \chi')\beta \\
&= D\left((\chi - \chi')\beta\right).
\end{aligned}$$

Since $\chi - \chi' = 0$ in a neighborhood of F, the right hand side of the last equation is well defined, and is an element of $\mathcal{A}^{\mathrm{pol}}_U(\mathfrak{k}, N)$. This proves the second point. Finally, the last point is immediate, since $\mathrm{p}_U(\alpha, \beta) = \mathrm{p}_V(\alpha, \beta) = \mathrm{p}^\chi(\alpha, \beta)$ for $\chi \in \mathcal{C}^\infty(N)^K$ with support in $V \subset U$.

Definition 3.15 *Let $\alpha \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ be a* closed *equivariant form and $\beta \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N \setminus F)$ such that $\alpha|_{N\setminus F} = D\beta$. We denote by $\mathrm{p}_F(\alpha, \beta) \in \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$ the element defined by the sequence $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}^{\mathrm{pol}}_U(\mathfrak{k}, N)$, $U \in \mathcal{F}_F$. We have then a morphism $\mathrm{p}_F : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$.*

The following proposition summarizes the functorial properties of p.

Proposition 3.16 • *If $F \subset F'$ are closed invariant subsets of N, then the diagram*

$$\begin{array}{ccc}
\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) & \xrightarrow{\mathrm{p}_F} & \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \\
\downarrow{\scriptstyle \mathbf{r}_1} & & \downarrow{\scriptstyle \mathbf{r}^1} \\
\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F') & \xrightarrow{\mathrm{p}_{F'}} & \mathcal{H}^{\mathrm{pol}}_{F'}(\mathfrak{k}, N)
\end{array} \tag{23}$$

is commutative. Here $\mathbf{r}_1$ *and* $\mathbf{r}^1$ *are the restriction morphisms (see (6) and (16)).*

• *If* F_1, F_2 *are closed invariant subsets of* N*, then the diagram*

(24)

$$\begin{tikzcd} \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2) \arrow[r, "\diamond"] \arrow[d, "\mathrm{p}_{F_1} \times \mathrm{p}_{F_2}"] & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)) \arrow[d, "\mathrm{p}_{F_1 \cap F_2}"] \\ \mathcal{H}^{\mathrm{pol}}_{F_1}(\mathfrak{k}, N) \times \mathcal{H}^{\mathrm{pol}}_{F_2}(\mathfrak{k}, N) \arrow[r, "\wedge"] & \mathcal{H}^{\mathrm{pol}}_{F_1 \cap F_2}(\mathfrak{k}, N) \end{tikzcd}$$

is commutative.

Proof. The proof of the first point is left to the reader. Let us prove the second point. Let W be an invariant open neighborhood of $F_1 \cap F_2$. Let V_1, V_2 be invariant open neighborhoods respectively of F_1 and F_2 such that $V_1 \cap V_2 \subset W$. Choose $\chi_i \in \mathcal{C}^\infty(N)^K$ supported in V_i and equal to 1 in a neighborhood of F_i. Then $\chi_1\chi_2$ is supported in W and equal to 1 in a neighborhood of $F_1 \cap F_2$. Let $\Phi_1 + \Phi_2 = 1_{N \setminus (F_1 \cap F_2)}$ be a partition of unity relative to the decomposition $N \setminus (F_1 \cap F_2) = (N \setminus F_1) \cup (N \setminus F_2)$, and where the function Φ_i are K-invariant.

Then one checks easily that

$$\begin{aligned} \mathrm{p}^{\chi_1}(a_1) \wedge \mathrm{p}^{\chi_2}(a_2) - \mathrm{p}^{\chi_1\chi_2}(a_1 \diamond_\Phi a_2) = \\ D\Big((-1)^{|a_1|}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2) - (-1)^{|a_1|}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2)\Big) \end{aligned} \tag{25}$$

for any D_{rel}-closed forms $a_i = (\alpha_i, \beta_i) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_i)$. Remark that $\Phi_1\beta_1\beta_2$ is defined on $N \setminus F_2$, so that $d\chi_2(\Phi_1\beta_1\beta_2)$ is well defined on N and supported in V_2. Thus the equivariant form $(-1)^{|a_1|}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2) - (-1)^{|a_1|}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2)$ is well defined on N and supported in $V_1 \cap V_2 \subset W$. Thus $\mathrm{p}^{\chi_1}(a_1) \wedge \mathrm{p}^{\chi_2}(a_2)$ and $\mathrm{p}^{\chi_1\chi_2}(a_1 \diamond_\Phi a_2)$ are equal in $\mathcal{H}^{\mathrm{pol}}_W(\mathfrak{k}, N)$. As this holds for any neighborhood W of $F_1 \cap F_2$, this proves that $\mathrm{p}_{F_1}(a_1) \wedge \mathrm{p}_{F_2}(a_2) = \mathrm{p}_{F_1 \cap F_2}(a_1 \diamond a_2)$.

If we take $F' = N$ in (23), we see that the map $\mathrm{p}_F : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N)$ factors the natural map $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$.

By the same formulae, we define the morphism of $\mathbb{Z}_2$-graded algebras:

$$\mathrm{p}_F : \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^\infty_F(\mathfrak{k}, N), \tag{26}$$

which enjoys the same properties:

Proposition 3.17 • *If* $F \subset F'$ *are closed invariant subsets of* N*, then the diagram*

$$\begin{tikzcd} \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \arrow[r, "\mathrm{p}_F"] \arrow[d, "\mathbf{r}_1"] & \mathcal{H}^\infty_F(\mathfrak{k}, N) \arrow[d, "\mathbf{r}^1"] \\ \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F') \arrow[r, "\mathrm{p}_{F'}"] & \mathcal{H}^\infty_{F'}(\mathfrak{k}, N) \end{tikzcd} \tag{27}$$

is commutative. Here $\mathbf{r}_1$ *and* $\mathbf{r}^1$ *are the restriction morphisms (see (6) and (16)).*

• *If* F_1, F_2 *are closed invariant subsets of* N*, then the diagram*

(28)

$$\begin{array}{ccccc}
\mathcal{H}^\infty(\mathfrak{k}, N, N\setminus F_1) & \times & \mathcal{H}^\infty(\mathfrak{k}, N, N\setminus F_2) & \xrightarrow{\diamond} & \mathcal{H}^\infty(\mathfrak{k}, N, N\setminus (F_1\cap F_2)) \\
\downarrow{\scriptstyle \mathrm{p}_{F_1}} & & \downarrow{\scriptstyle \mathrm{p}_{F_2}} & & \downarrow{\scriptstyle \mathrm{p}_{F_1\cap F_2}} \\
\mathcal{H}^\infty_{F_1}(\mathfrak{k}, N) & \times & \mathcal{H}^\infty_{F_2}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^\infty_{F_1\cap F_2}(\mathfrak{k}, N)
\end{array}$$

is commutative.

If F is a compact K-invariant subset of N, we have a natural morphism

$$\mathcal{H}^{\mathrm{pol}}_F(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k}, N). \tag{29}$$

The composition of p_F with this morphism is the morphism p_c defined in (9) : $\mathrm{p}_c(\alpha,\beta)$ is the class of $\mathrm{p}^\chi(\alpha,\beta) = \chi\alpha + d\chi\beta$, where $\chi \in \mathcal{C}^\infty(N)^K$ has a compact support and is equal to 1 in a neighborhood of F.

3.6 Integral over the fiber in relative cohomology

Let $\pi : N \to B$ be a K-invariant fibration, with oriented fibers : the orientation is assumed to be invariant relatively to the action of K. Recall the definition of $\mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, N)$, the sub-algebra of $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N)$ formed by the equivariant forms which have a support that intersects the fibers of π in compact subsets, and of $\mathcal{H}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, N)$ the corresponding cohomology space.

We have an integration morphism $\pi_* : \mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, N) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, B)$ satisfying the following rules:

$$\pi_*(D\alpha) = D(\pi_*(\alpha)), \tag{30}$$

$$\pi_*(\pi^*(\gamma)\wedge\alpha) = \gamma\wedge\pi_*(\alpha), \tag{31}$$

for $\alpha \in \mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, N)$ and $\gamma \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, B)$. Thanks to (30) the integration morphism descends to the cohomology :

$$\pi_* : \mathcal{H}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, N) \to \mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k}, B).$$

Note that π_* sends $\mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k}, N)$ to $\mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k}, B)$ and that (31) is still valid in cohomology.

Remark 3.18 *If* U *is an invariant open subset of* B*, we have an integration morphism* $\pi_* : \mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, \pi^{-1}(U)) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, U)$.

Let F be a K-invariant closed subset of N which is **compact**. We will define an integration morphism $\pi_* : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus \pi(F))$ which makes the following diagram

$$\begin{tikzcd} \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{p}_c"] & \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus \pi(F)) \arrow[d, "\mathrm{p}_c"] \\ \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N) \arrow[r, "\pi_*"] & \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, B) \end{tikzcd} \tag{32}$$

commutative.

To perform the integration, it is natural to choose a representative $[\alpha, \beta]$ of a relative cohomology class a where α and β are supported near F. This can be done via the Lemma 3.5: let us choose a K-invariant function χ which is compactly supported and is equal to 1 on a neighborhood of F. Let $(\alpha, \beta) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$. Then the equivariant form $\chi\alpha + d\chi\beta$ is compactly supported and can be integrated over the fiber. Similarly, the form $\chi\beta$ belongs to $\mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, \pi^{-1}(B \setminus \pi(F)))$ and can be integrated over the fiber. The expression

$$\pi_*^\chi(\alpha, \beta) := (\pi_*(\chi\alpha + d\chi\beta), \pi_*(\chi\beta))$$

defines an element in $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus \pi(F))$. Since we have the relation $\pi_*^\chi \circ D_{\mathrm{rel}} = D_{\mathrm{rel}} \circ \pi_*^\chi$ the map π_*^χ descends to cohomology.

Furthermore, if (α, β) is D_{rel}-closed, and χ_1, χ_2 are two different choices of functions χ, we verify that

$$\pi_*^{\chi_1}(\alpha, \beta) - \pi_*^{\chi_2}(\alpha, \beta) = D_{\mathrm{rel}}\Big(\pi_*((\chi_1 - \chi_2)\beta), 0\Big)$$

so that the map π_*^χ is independent of the choice of χ.

This allows us to make the following definition.

Definition 3.19 *Let us choose $\chi \in \mathcal{C}^\infty(N)$ a K-invariant function identically equal to* 1 *on a neighborhood of F, and with compact support. Then we define*

$$\pi_* : [\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)]^k \longrightarrow [\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus \pi(F))]^{k - \dim N + \dim B}$$

by the formula : $\pi_*([\alpha, \beta]) = [\pi_*(\chi\alpha + d\chi\beta), \pi_*(\chi\beta)]$.

If B is a point, then the integral of $a \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$ over the fiber is just the integral over N of the class $\mathrm{p}_c(a) \in \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{k}, N)$.

The same definition makes sense for equivariant forms with $\mathcal{C}^\infty$-coefficients and defines a map $\pi_* : \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^\infty(\mathfrak{k}, B, B \setminus \pi(F))$.

We now prove

Proposition 3.20
- *The diagram (32) is commutative*
- *Let F_2 be a* compact *K-invariant set of N. Let F_1 be a closed K-invariant set in B. Then, for any $a \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus F_1)$ and $b \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2)$, we have*
$$\pi_*(\pi^* a \diamond b) = a \diamond \pi_*(b)$$
in $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus (F_1 \cap \pi(F_2)))$.

Proof. Let $\chi \in \mathcal{C}^\infty(N)^K$ and $\chi' \in \mathcal{C}^\infty(B)^K$ be two compactly supported functions : χ is identically equal to 1 on a neighborhood of F and χ' is identically equal to 1 on a neighborhood of $\pi(F)$. For $[\alpha, \beta] \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F)$, the equivariant class $\mathrm{p}_c \circ \pi_*[\alpha, \beta]$ is represented by

$$\mathrm{p}^{\chi'}(\pi_*^\chi(\alpha, \beta)) = \chi' \pi_*(\chi\alpha + d\chi\beta) + d\chi' \pi_*(\chi\beta).$$

On the other hand, the equivariant class $\pi_* \circ \mathrm{p}_c[\alpha, \beta]$ is represented by $\pi_*(\chi\alpha + d\chi\beta)$. We check that

$$\mathrm{p}^{\chi'}(\pi_*^\chi(\alpha, \beta)) - \pi_*(\chi\alpha + d\chi\beta) = D\Big((\chi' - 1)\pi_*(\chi\beta)\Big),$$

where $(\chi' - 1)\pi_*(\chi\beta)$ is an equivariant form on B with compact support. Then, the first point is proved.

Let us prove the second point. We work with the invariant open subsets $U = N \setminus (\pi^{-1}(F_1) \cap F_2), U_1 = N \setminus \pi^{-1}(F_1), U_2 = N \setminus F_2$ of N, and the invariant open subsets $U' = B \setminus (F_1 \cap \pi(F_2)), U_1' = B \setminus F_1, U_2' = B \setminus \pi(F_2)$ of B. Note that $\pi^{-1}U_1' = U_1$ and $\pi^{-1}U_2' \subset U_2$: hence $\pi^{-1}U' \subset U$.

Let $\Phi_1 + \Phi_2 = 1$ be a partition of unity on $U = U_1 \cup U_2$, and let $\Phi_1' + \Phi_2' = 1$ a partition of unity on $U' = U_1' \cup U_2'$: all the functions are supposed invariant. Let (α_1, β_1) and (α_2, β_2) be respectively the representatives of $a \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus F_1)$ and $b \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F_2)$. The equivariant forms α_2, β_2 are chosen so that their supports belong to a compact neighborhood of F_2.

Then $\pi_*(\pi^*(a) \diamond b)$ is represented by $(\alpha_1\pi_*(\alpha_2), \beta)$ with

$$\beta = \underbrace{\beta_1\pi_*(\Phi_1\alpha_2)}_{\beta(1)} + \underbrace{(-1)^{|a|}\alpha_1\pi_*(\Phi_2\beta_2)}_{\beta(2)} + \underbrace{\beta_1\pi_*(d\Phi_1\beta_2)}_{\beta(3)}.$$

On the other hand $a \diamond \pi_*(b)$ is represented by $(\alpha_1\pi_*(\alpha_2), \beta')$ with

$$\beta' = \underbrace{\Phi_1'\beta_1\pi_*(\alpha_2)}_{\beta'(1)} + \underbrace{(-1)^{|a|}\Phi_2'\alpha_1\pi_*(\beta_2)}_{\beta'(2)} + \underbrace{\beta_1 d\Phi_1'\pi_*(\beta_2)}_{\beta'(3)}.$$

Note that the equivariant forms $\beta(i), \beta'(i)$ are well defined on $B \setminus (F_1 \cap \pi(F_2))$.

Lemma 3.21 *The equivariant form $\delta = \Phi_2'\pi_*(\beta_2) - \pi_*(\Phi_2\beta_2)$ is defined on $B \setminus (F_1 \cap \pi(F_2))$ and supported on $B \setminus (F_1 \cup \pi(F_2))$. We have*

$$\beta - \beta' = D\left(\beta_1 \wedge \delta\right),$$

where the equivariant form $\beta_1 \wedge \delta$ is defined on $B \setminus (F_1 \cap \pi(F_2))$. It gives the following relation in $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus (F_1 \cap \pi(F_2)))$: $(\alpha_1\pi_*(\alpha_2), \beta') - (\alpha_1\pi_*(\alpha_2), \beta) = D_{\mathrm{rel}}(0, \beta_1\delta)$.

Proof. The invariant function Φ_2 is defined on U, supported on U_2, and equal to 1 on $U_2 \setminus (U_1 \cap U_2)$. Then its restriction $\Phi_2|_{\pi^{-1}(U')}$ is supported on $\pi^{-1}(U_2')$ and equal to 1 on $\pi^{-1}(U_2') \setminus (U_1 \cap \pi^{-1}(U_2'))$. Similarly the function

$\pi^*\Phi_2'$ is defined on $\pi^{-1}(U')$, supported on $\pi^{-1}(U_2')$ and equal to 1 on $\pi^{-1}(U_2') \setminus (U_1 \cap \pi^{-1}(U_2'))$. Hence the difference $\pi^*\Phi_2' - \Phi_2|_{\pi^{-1}(U')}$ is defined on $\pi^{-1}(U')$ and supported on $U_1 \cap \pi^{-1}(U_2')$. This shows that $\delta = \pi_*((\pi^*\Phi_2' - \Phi_2|_{\pi^{-1}(U')})\beta_2)$ is defined on U' and supported on $U_1' \cap U_2'$.

We have

$$\begin{aligned}\beta_1 \wedge D(\delta) &= \beta_1 d\Phi_2' \pi_*(\beta_2) - \beta_1 \pi_*(d\Phi_2\beta_2) + \beta_1 \wedge \Big(\Phi_2'\pi_*(\alpha_2) - \pi_*(\Phi_2\alpha_2)\Big) \\ &= -\beta'(3) + \beta(3) + \beta_1 \wedge \Big((1-\Phi_1')\pi_*(\alpha_2) - \pi_*((1-\Phi_1)\alpha_2)\Big) \\ &= -\beta'(3) + \beta(3) - \beta'(1) + \beta(1).\end{aligned}$$

We see also that

$$\begin{aligned}(-1)^{|a|-1} D(\beta_1) \wedge \delta &= -(-1)^{|a|}\Phi_2'\alpha_1\pi_*(\beta_2) + (-1)^{|a|}\alpha_1\pi_*(\Phi_2\beta_2) \\ &= -\beta'(2) + \beta(2).\end{aligned}$$

Finally we have proved that $D(\beta_1 \wedge \delta) = \beta(1) + \beta(2) + \beta(3) - \beta'(1) - \beta'(2) - \beta'(3)$.

Remark 3.22 *If F is a* closed but non-compact *subset of N, the morphism $\pi_* : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, B, B \setminus \pi(F))$ is still defined in the case where the image $\pi(F)$ is closed and the intersection of F with each fiber is compact. The second point of Proposition (3.20) still holds in this case.*

3.7 The Chern-Weil construction

Let $\pi : P \to B$ be a principal bundle with structure group G. Let $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ be a connection one form on P, with curvature form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$.

For any G-manifold Z, we define $\mathcal{Z} = P \times_G Z$. If F is a G invariant closed subset of Z, then $\mathcal{F} := P \times_G F$ is a closed subset of $\mathcal{Z}$. We consider the Chern-Weil homomorphism

$$\phi_\omega : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z, Z \setminus F)) \longrightarrow \mathcal{A}(\mathcal{Z}, \mathcal{Z} \setminus \mathcal{F})$$

defined by $\phi_\omega(\alpha, \beta) = (\phi_\omega^Z(\alpha), \phi_\omega^{Z\setminus F}\beta)$. We have the relation:

Proposition 3.23 *We have $d_{\mathrm{rel}} \circ \phi_\omega = \phi_\omega \circ D_{\mathrm{rel}}$.*

We can repeat the construction above in the equivariant case. If P is a G-principal bundle with left action of K, and F is a $G \times K$-invariant closed subset of P, we define

$$\phi_\omega : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z, Z \setminus F) \longrightarrow \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{Z}, \mathcal{Z} \setminus \mathcal{F})$$

by $\phi_\omega(\alpha, \beta) = (\phi_\omega^Z \alpha, \phi_\omega^{Z\setminus F}\beta)$.

Proposition 3.24 *The map $\phi_\omega : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z, Z \setminus F) \to \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{Z}, \mathcal{Z} \setminus \mathcal{F})$ satisfies*

$$\phi_\omega \circ D_{\mathrm{rel}} = D_{\mathrm{rel}} \circ \phi_\omega.$$

On the left side the equivariant differential is with respect to the action of G while on the right side, this is with respect to the group K.

4 Explicit formulae for Thom Classes in relative cohomology

4.1 The equivariant Thom forms of a vector space

Let V be an Euclidean oriented vector space of dimension d. Consider the group $\mathrm{SO}(V)$ of orthogonal transformations of V preserving the orientation. Let $\mathfrak{so}(V)$ be its Lie algebra. Consider the projection $\pi : V \to \{\mathrm{pt}\}$ and the closed subset $F = \{0\} \subset V$. We denote $\int_V$ the integration morphism $\pi_* : \mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\}) \to \mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$ defined in Section 3.6.

In this section, we will describe a generator over $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$ of the equivariant relative cohomology space of the pair $(V, V \setminus \{0\})$. This explicit representative is basically due to Mathai-Quillen [11]. As a consequence, we obtain the following theorem.

Theorem 4.1 *There exists a unique class* $\mathrm{Th}_{\mathrm{rel}}(V)$ *in* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$ *such that* $\int_V \mathrm{Th}_{\mathrm{rel}}(V) = 1$. *This class is called the relative Thom class.*

Before establishing the unicity, a *closed* form $\alpha \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$, or in $\mathcal{A}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$, or in $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$, of integral 1 will be called a Thom form. A Thom class will be the class defined by a Thom form.

We start by constructing Thom forms in the spaces $\mathcal{A}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$, $\mathcal{A}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$ and $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$. The Lie algebra $\mathfrak{so}(V)$ is identified with the Lie algebra of antisymmetric endomorphisms of V. Let $(e_1, \ldots, e_d)$ be an oriented orthonormal basis of V. Denote by $\mathrm{T} : \wedge V \to \mathbb{R}$ the Berezin integral normalized by $\mathrm{T}(e_1 \wedge \cdots \wedge e_d) = 1$.

Definition 4.2 *The pfaffian of* $X \in \mathfrak{so}(V)$ *is defined by*

$$\mathrm{Pf}(X) = \mathrm{T}(\mathrm{e}^{\sum_{k<l} \langle X e_k, e_\ell \rangle e_k \wedge e_\ell}).$$

Here the exponential is computed in the algebra $\wedge V$.

Recall that $\mathrm{Pf}(X)$ is an $\mathrm{SO}(V)$-invariant polynomial on $\mathfrak{so}(V)$ such that $(\mathrm{Pf}(X))^2 = \det_V(X)$. In particular this polynomial is identically equal to 0, if d is odd. We also denote by $\mathrm{Pf}(X) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{so}(V), V)$ the function on V identically equal to $\mathrm{Pf}(X)$.

Let $x_k = (e_k, x)$ be the coordinates on V. We consider the equivariant map $f_t : \mathfrak{so}(V) \to \mathcal{A}(V) \otimes \wedge V$ given by the formula

$$f_t(X) = -t^2 \|x\|^2 + t \sum_k dx_k e_k + \frac{1}{2} \sum_{k<l} \langle X e_k, e_\ell \rangle e_k \wedge e_\ell. \tag{33}$$

For any (real or complex) vector space $\mathcal{A}$, the Berezin integral is extended to a map $\mathrm{T} : \mathcal{A} \otimes \wedge V \to \mathcal{A}$ by $\mathrm{T}(\alpha \otimes \xi) = \alpha\, \mathrm{T}(\xi)$ for $\alpha \in \mathcal{A}$ and $\xi \in \wedge V$. If $\mathrm{d} : \mathcal{A} \to \mathcal{A}$ is a linear map, we extend it on $\mathcal{A} \otimes \wedge V$ by $\mathrm{d}(\alpha \otimes \xi) = \mathrm{d}(\alpha) \otimes \xi$. Note that this extension satisfies $\mathrm{T} \circ \mathrm{d} = \mathrm{d} \circ \mathrm{T}$.

The Berezin integral is the "super-commutative" analog of the super-trace for endomorphisms of a super-space. We now give a construction for the relative Thom form, analogous to Quillen's construction of the Chern character. We will discuss Quillen's construction of the Chern character in the next chapter (the formulae we give here for the "super-commutative case" are strongly inspired by the formulae for the curvature of the super-connection attached to the Bott symbol: see Section 6.1).

We consider the $\mathrm{SO}(V)$-equivariant forms on V defined by

$$\mathrm{C}^t_\wedge(X) \quad := \quad \mathrm{T}\left(\mathrm{e}^{f_t(X)}\right), \tag{34}$$

$$\eta^t_\wedge(X) \quad := \quad -\,\mathrm{T}\left((\sum_k x_k e_k)\,\mathrm{e}^{f_t(X)}\right), \tag{35}$$

for $X \in \mathfrak{so}(V)$. Here the exponential is computed in the super-algebra $\mathcal{A}(V) \otimes \wedge V$.

Lemma 4.3 *The equivariant form $\mathrm{C}^t_\wedge(X)$ is closed. Furthermore,*

$$\frac{d}{dt}\mathrm{C}^t_\wedge = -D(\eta^t_\wedge). \tag{36}$$

Proof. The proof of the first point is given in [4] (Chapter 7, Theorem 7.41). We recall the proof. If $e \in V$, we denote by $\iota_\wedge(e)$ the derivation of $\wedge V$ such that $\iota_\wedge(e)v = \langle e, v\rangle$ when $v \in \wedge^1 V = V$. We extend it to a derivation of $\mathcal{A}(V) \otimes \wedge V$. We denote by $\iota_\wedge(\mathbf{x})$ the operator $\sum_k x_k \iota_\wedge(e_k)$. The equivariant derivative D which is defined on $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V)) \otimes \mathcal{A}(V)$ is extended to a derivation of $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V)) \otimes \mathcal{A}(V) \otimes \wedge V$. We have $\mathrm{T} \circ D = D \circ \mathrm{T}$.

It is easy to verify that

$$(D - 2t\iota_\wedge(\mathbf{x}))f_t(X) = 0. \tag{37}$$

The exponential $\mathrm{e}^{f_t(X)}$ satisfies also $(D - 2t\iota_\wedge(\mathbf{x}))(\mathrm{e}^{f_t(X)}) = 0$, since D and $\iota_\wedge(\mathbf{x})$ are derivations. The Berezin integral is such that $\mathrm{T}(\iota_\wedge(\mathbf{x})\alpha) = 0$ for any $\alpha \in \mathcal{A}(V) \otimes \wedge V$. This shows that $D\left(\mathrm{T}(\mathrm{e}^{f_t(X)})\right) = 0$.

Let us prove the second point. As $(D - 2t\iota_\wedge(\mathbf{x}))f_t(X) = 0$, we have

$$\begin{aligned}
D \circ \mathrm{T}\left((\sum x_k e_k)\,\mathrm{e}^{f_t(X)}\right) &= \mathrm{T} \circ (D - 2t\iota_\wedge(\mathbf{x}))\left((\sum x_k e_k)\,\mathrm{e}^{f_t(X)}\right) \\
&= \mathrm{T}\left(((D - 2t\iota_\wedge(\mathbf{x})) \cdot (\sum x_k e_k))\,\mathrm{e}^{f_t(X)}\right) \\
&= \mathrm{T}\left((\sum dx_k e_k - 2t\|x\|^2)\,\mathrm{e}^{f_t(X)}\right) \\
&= \mathrm{T}\left(\frac{d}{dt}\,\mathrm{e}^{f_t(X)}\right).
\end{aligned}$$

When $t = 0$, then $\mathrm{C}^0_\wedge(X)$ is just equal to $\mathrm{Pf}(\frac{X}{2}) = \frac{1}{2^{d/2}}\,\mathrm{Pf}(X)$. When $t = 1$, then $\mathrm{C}^1_\wedge(X) = \mathrm{T}(\mathrm{e}^{f_1(X)}) = e^{-\|x\|^2}Q(dx, X)$ is a closed equivariant class with a

Gaussian look on V (with $Q(dx,X)$ a polynomial in dx,X that we will write more explicitly in a short while). This form was considered by Mathai-Quillen in [11].

Definition 4.4 *The Mathai-Quillen form is the closed equivariant form on V defined by*

$$\mathrm{C}^1_\wedge(X) := \mathrm{T}\left(\mathrm{e}^{f_1(X)}\right), \quad X \in \mathfrak{so}(V).$$

We have $\eta^t_\wedge = \mathrm{e}^{-t^2\|x\|^2}\, Q(t,X,x,dx)$ where $Q(t,X,x,dx)$ depends polynomially of (t,X,x,dx) where $t \in \mathbb{R}, X \in \mathfrak{so}(V), x \in V, dx \in \mathcal{A}^1(V)$. Thus, if $x \neq 0$, when t goes to infinity, $\eta^t_\wedge$ is an exponentially decreasing function of t. We can thus define the following equivariant form on $V \setminus \{0\}$:

$$\beta_\wedge(X) = \int_0^\infty \eta^t_\wedge(X)dt, \quad X \in \mathfrak{so}(V). \tag{38}$$

If we integrate (36) between 0 and ∞, we get

$$\mathrm{C}^0_\wedge = D(\beta_\wedge) \quad \text{on} \quad V \setminus \{0\}.$$

Thus the couple $(\mathrm{C}^0_\wedge, \beta_\wedge)$ defines a canonical relative class

$$\left[\mathrm{Pf}(\tfrac{X}{2}), \beta_\wedge(X)\right] \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\}) \tag{39}$$

of degree equal to $\dim(V)$.

Consider now the equivariant cohomology with compact support of V. Following (9) we have a map $\mathrm{p}_c : \mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\}) \to \mathcal{H}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$. Thus $\left[\mathrm{Pf}(\frac{X}{2}), \beta_\wedge(X)\right]$ provides us a class

$$\mathrm{C}_V := \mathrm{p}_c\left(\left[\mathrm{Pf}(\tfrac{X}{2}), \beta_\wedge(X)\right]\right) \in \mathcal{H}^{\mathrm{pol}}_c(\mathfrak{so}(V), V). \tag{40}$$

As the map p_c commutes with the integration over the fiber (see (32)), it is the same thing to compute the integral of (39) or of (40).

The orientation on V is given by $dx_1 \wedge \cdots \wedge dx_d$.

Proposition 4.5 *Let $\chi \in \mathcal{C}^\infty(V)$ be an $\mathrm{SO}(V)$-invariant function with compact support and equal to 1 in a neighborhood of 0. The form*

$$\mathrm{C}^\chi_V(X) = \chi\, \mathrm{Pf}(\tfrac{X}{2}) + d\chi \int_0^\infty \eta^t_\wedge(X)dt$$

is a closed equivariant form with compact support on V. Its cohomology class in $\mathcal{H}^\infty_c(\mathfrak{so}(V), V)$ coincides with C_V : in particular, it does not depend of the choice of χ. For every $X \in \mathfrak{so}(V)$,

$$\frac{1}{\epsilon_d}\int_V \mathrm{C}^\chi_V(X) = 1,$$

with $\epsilon_d = (-1)^{\frac{d(d-1)}{2}}\pi^{d/2}$. Thus $\frac{1}{\epsilon_d}\mathrm{C}^\chi_V(X)$ is a Thom form in $\mathcal{A}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$.

Proof. The first assertions are immediate to prove, since $\mathrm{C}_V = \mathrm{p}_c\left(\left[\mathrm{Pf}(\frac{X}{2}), \beta_\wedge(X)\right]\right)$. The class of C_V^χ in compactly supported cohomology do not depend of χ. To compute its integral, we may choose $\chi = f(\|x\|^2)$ where $f \in \mathcal{C}^\infty(\mathbb{R})$ has a compact support and is equal to 1 in a neighborhood of 0. The component of maximal degree of the differential form $d\chi \wedge \eta_\wedge^t(X)$ is

$$-2(-1)^{\frac{d(d-1)}{2}} t^{d-1} f'(\|x\|^2)\|x\|^2 e^{-t^2\|x\|^2} dx_1 \wedge \cdots \wedge dx_d.$$

Hence, using the change of variables $x \to \frac{1}{t}x$,

$$\begin{aligned}\int_V \mathrm{C}_V^\chi(X) &= -2(-1)^{\frac{d(d-1)}{2}} \int_0^\infty t^{d-1} \left(\int_V f'(\|x\|^2)\|x\|^2 e^{-t^2\|x\|^2} dx\right) dt \\ &= (-1)^{\frac{d(d-1)}{2}} \int_0^\infty \underbrace{\left(\int_V f'(\frac{\|x\|^2}{t^2})(\frac{-2\|x\|^2}{t^3}) e^{-\|x\|^2} dx\right)}_{I(t)} dt.\end{aligned}$$

Since for $t > 0$, $I(t) = \frac{d}{dt}\left(\int_V f(\frac{\|x\|^2}{t^2}) e^{-\|x\|^2} dx\right)$, we have $\int_V \mathrm{C}_V^\chi(X) = (-1)^{\frac{d(d-1)}{2}} \int_V e^{-\|x\|^2} dx = (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$.

Using Mathai-Quillen form, it is possible to construct representatives of a Thom form with Gaussian look.

Proposition 4.6 *The Mathai-Quillen form $\mathrm{C}_\wedge^1(X)$ is a closed equivariant form which belongs to $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$. For every $X \in \mathfrak{so}(V)$,*

$$\frac{1}{\epsilon_d} \int_V \mathrm{C}_\wedge^1(X) = 1,$$

with $\epsilon_d = (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. Thus $\frac{1}{\epsilon_d}\mathrm{C}_\wedge^1(X)$ is a Thom form in $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$.

Proof. Indeed, only the term in $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_d$ will contribute to the integral. This highest term is $(-1)^{\frac{d(d-1)}{2}} e^{-\|x\|^2} dx$.

We summarize Propositions 4.5 and 4.6 in the following theorem.

Theorem 4.7 *Let V be an oriented Euclidean vector space with oriented orthonormal basis $(e_1, \ldots, e_d)$. Let $\epsilon_d := (-1)^{\frac{d(d-1)}{2}} \pi^{d/2}$. Let $\mathrm{T} : \wedge V \to \mathbb{R}$ be the Berezin integral. Let, for $X \in \mathfrak{so}(V)$,*

$$\begin{aligned} f_t(X) &= -t^2\|x\|^2 + t\sum_k dx_k e_k + \frac{1}{2}\sum_{k<l} \langle X e_k, e_\ell\rangle e_k \wedge e_\ell, \\ \eta_\wedge^t(X) &= -\mathrm{T}\left((\sum_k x_k e_k)\, \mathrm{e}^{f_t(X)} \right), \\ \beta_\wedge(X) &= \int_0^\infty \eta_\wedge^t(X) dt. \end{aligned}$$

• *Let*
$$\mathrm{Th}_{\mathrm{rel}}(V)=\frac{1}{\epsilon_d}\left(\mathrm{Pf}(\tfrac{X}{2}),\beta_\wedge(X)\right),\quad X\in\mathfrak{so}(V).$$
Then $\mathrm{Th}_{\mathrm{rel}}(V)\in\mathcal{A}^{\mathrm{pol}}(\mathfrak{so}(V),V,V\setminus\{0\})$ *is a Thom form. It defines a Thom class, still denoted* $\mathrm{Th}_{\mathrm{rel}}(V)$*, in* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V),V,V\setminus\{0\})$.

• *Let*
$$\mathrm{Th}_{\mathrm{c}}(V)=\frac{1}{\epsilon_d}\mathrm{C}_V^\chi(X)=\frac{1}{\epsilon_d}\left(\chi\,\mathrm{Pf}(\tfrac{X}{2})+d\chi\beta_\wedge(X)\right),\quad X\in\mathfrak{so}(V).$$
Then $\mathrm{Th}_{\mathrm{c}}(V)\in\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{so}(V),V)$ *is a Thom form. It defines a Thom class, still denoted* $\mathrm{Th}_{\mathrm{c}}(V)$*, in* $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V),V)$.

Here $\chi\in\mathcal{C}^\infty(V)^{\mathrm{SO}(V)}$ *is an invariant function with compact support and equal to* 1 *in a neighborhood of* 0.

• *Let*
$$\mathrm{Th}_{\mathrm{MQ}}(V)=\frac{1}{\epsilon_d}\mathrm{C}_\wedge^1(X)=\frac{1}{\epsilon_d}\,\mathrm{T}\left(\mathrm{e}^{f_1(X)}\right),\quad X\in\mathfrak{so}(V)$$
be the Mathai-Quillen form. Then $\mathrm{Th}_{\mathrm{MQ}}(V)\in\mathcal{A}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{so}(V),V)$ *is a Thom form. It defines a Thom class, still denoted* $\mathrm{Th}_{\mathrm{MQ}}(V)$*, in* $\mathcal{H}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{so}(V),V)$.

Thus the use of the Berezin integral allowed us to give slim formulae for Thom forms in relative cohomology, as well as in compactly supported cohomology or in rapidly decreasing cohomology.

Let us explain the relation between the Thom forms $\frac{1}{\epsilon_d}\mathrm{C}_V^\chi$ and $\frac{1}{\epsilon_d}\mathrm{C}_\wedge^1$. For $t>0$, it is easy to see that the forms $\mathrm{C}_V^\chi(X)$ and $\mathrm{C}_\wedge^t(X)$ differ by the differential of an equivariant form with Gaussian decay. We could deduce this fact as a corollary of the unicity theorem, that we will prove soon, but we prove it here directly by giving explicit transgression formulae.

Proposition 4.8 *For any* $t>0$, $\mathrm{C}_V^\chi=\mathrm{C}_\wedge^t$ *in* $\mathcal{H}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{so}(V),V)$.

Proof. Fix $t>0$. The form $\mathrm{C}_\wedge^t=\mathrm{e}^{-t^2\|x\|^2}P(t,dx,X)$ is rapidly decreasing on V, thus $\mathrm{C}_\wedge^t\in\mathcal{A}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{so}(V),V)$. Define on $V\setminus\{0\}$ the equivariant form $\beta_\wedge^t=\int_t^\infty\eta_\wedge^s ds$. We have $\eta_\wedge^s=\mathrm{e}^{-s\|x\|^2}P(s,X,x,dx)$ where $P(s,X,x,dx)$ depends polynomially of (s,X,x,dx). For $s=t+u$, $e^{-s\|x\|^2}=\mathrm{e}^{-t\|x\|^2}\mathrm{e}^{-u\|x\|^2}$, thus $\beta_\wedge^t$ is rapidly decreasing when $\|x\|$ tends to ∞. The transgression formula (36) integrated between t and ∞ shows that $\mathrm{C}_\wedge^t=D(\beta_\wedge^t)$ on $V\setminus\{0\}$. Thus $\mathrm{C}_V^{\chi,t}:=\chi\mathrm{C}_\wedge^t+d\chi\beta_\wedge^t$ is a closed equivariant form belonging to $\mathcal{A}_{\text{dec-rap}}^{\mathrm{pol}}(\mathfrak{so}(V),V)$. The two following transgression formulae are evident to prove:
$$\mathrm{C}_V^\chi-\mathrm{C}_V^{\chi,t}=D\left(\chi\int_0^t\eta_\wedge^s ds\right)\quad\text{and}\quad\mathrm{C}_V^{\chi,t}-\mathrm{C}_\wedge^t=D\left((\chi-1)\beta_\wedge^t\right).$$

Thus we obtain $\mathrm{C}_V^\chi - \mathrm{C}_\wedge^t = D(\delta)$, where

$$\delta := \chi \int_0^t \eta_\wedge^s ds + (\chi - 1)\beta_\wedge^t$$

is an equivariant form defined on V which is rapidly decreasing. So we have $\mathrm{C}_V^\chi = \mathrm{C}_\wedge^t$ in $\mathcal{H}_{\text{dec-rap}}^{\text{pol}}(\mathfrak{so}(V), V)$.

Remark 4.9 *In the next sections, we will keep the same notations for the Thom classes and for their representatives defined in Theorem 4.7.*

Before going on in proving the unicity, let us give more explicit formulae for the Thom forms we have constructed.

4.2 Explicit formulae for the Thom forms of a vector space

If $I = [i_1, i_2, \dots, i_p]$ (with $i_1 < i_2 < \cdots < i_p$) is a subset of $[1, 2, \dots, d]$, we denote by $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$. If X is an antisymmetric matrix, we denote by X_I the sub-matrix $(\langle Xe_i, e_j\rangle)_{i,j\in I}$, which is viewed as an antisymmetric endomorphism of the vector space V_I generated by $e_i, i \in I$: let $\mathrm{Pf}(X_I)$ be its pfaffian, where V_I is oriented by e_I. One sees easily that

$$\mathrm{e}^{\sum_{k<l}\langle Xe_k, e_\ell\rangle e_k \wedge e_\ell} = \sum_I \mathrm{Pf}(X_I)\, e_I \quad \text{in} \quad \wedge V. \tag{41}$$

Only those I with $|I|$ even will contribute to the sum (41), as otherwise the pfaffian of X_I vanishes.

If I and J are two disjoint subsets of $\{1, 2, \dots, d\}$, we denote by $\epsilon(I, J)$ the sign such that $e_I \wedge e_J = \epsilon(I, J)\, e_{I\cup J}$.

Proposition 4.10 • *We have* $\mathrm{Th}_{\mathrm{rel}}(V)(X) = \frac{1}{\epsilon_d}\left(\mathrm{Pf}(\frac{X}{2}), \beta_\wedge(X)\right)$ *with*

$$\beta_\wedge(X) = \sum_{k,I,J} \gamma_{(k,I,J)}\, \mathrm{Pf}\left(\tfrac{X_I}{2}\right) \frac{x_k dx_J}{\|x\|^{|J|+1}},$$

with

$$\gamma_{(k,I,J)} = -\frac{1}{2}(-1)^{\frac{|J|(|J|+1)}{2}}\Gamma\left(\tfrac{|J|+1}{2}\right)\epsilon(I, J)\epsilon(\{k\}, I \cup J).$$

Here for $1 \le k \le d$, the sets I, J vary over the subsets of $\{1, 2, \dots, d\}$ such that $\{k\} \cup I \cup J$ is a partition of $\{1, 2, \dots, d\}$. Only those I with $|I|$ even will contribute to the sum.

• *We have*

$$\mathrm{Th}_{\mathrm{c}}(V)(X) = \frac{1}{\epsilon_d}\Big(f(\|x\|^2)\,\mathrm{Pf}(\tfrac{X}{2}) + 2f'(\|x\|^2)(\sum x_i dx_i)\beta_\wedge(X)\Big)$$

where f is a compactly supported function on $\mathbb{R}$, identically equal to 1 in a neighborhood of 0.

- *We have*

$$\mathrm{Th}_{\mathrm{MQ}}(V)(X) = \frac{1}{(\pi)^{d/2}} e^{-\|x\|^2} \sum_I (-1)^{\frac{|I|}{2}} \epsilon(I, I') \,\mathrm{Pf}(\tfrac{X_I}{2}) dx_{I'}.$$

Here I runs over the subset of $\{1, 2, \dots, d\}$ with an even number of elements, and I' denotes the complement of I.

Proof. Since $dx_k e_k$ and $dx_l e_l$ commute we have

$$\begin{aligned} \mathrm{e}^{t \sum_k dx_k e_k} &= \prod_k (1 + t dx_k e_k) \\ &= \sum_J (-1)^{\frac{|J|(|J|-1)}{2}} dx_J \, e_J \, t^{|J|}. \end{aligned}$$

If we use (41), we get

$$\begin{aligned} \mathrm{e}^{f_t(X)} &= \mathrm{e}^{-t^2\|x\|^2} \left(\sum_I \mathrm{Pf}(\tfrac{X_I}{2}) \, e_I \right) \left(\sum_J (-1)^{\frac{|J|(|J|-1)}{2}} dx_J \, e_J \, t^{|J|} \right) \\ &= \mathrm{e}^{-t^2\|x\|^2} \sum_{I,J} \epsilon(I, J)(-1)^{\frac{|J|(|J|-1)}{2}} \,\mathrm{Pf}(\tfrac{X_I}{2}) \, dx_J \, e_{I\cup J} \, t^{|J|}, \end{aligned} \tag{42}$$

where the I, J run over the subsets of $\{1, 2, \dots, d\}$ which are disjoint. If we take $t = 1$ in (42), we see that

$$\mathrm{T}\left(\mathrm{e}^{f_1(X)}\right) = \mathrm{e}^{-\|x\|^2} \sum_I \epsilon(I, I')(-1)^{\frac{|I'|(|I'|-1)}{2}} \,\mathrm{Pf}(\tfrac{X_I}{2}) \, dx_{I'}.$$

The third point then follows since $(-1)^{\frac{|I'|(|I'|-1)}{2}} = (-1)^{\frac{d(d-1)}{2}} (-1)^{\frac{|I|}{2}}$.

Equation (42) gives also

$$\begin{aligned} \eta^t_\wedge(X) &= -\,\mathrm{e}^{-t^2\|x\|^2} \,\mathrm{T}\left((\sum_k x_k e_k) \Big(\sum_{I,J} \epsilon(I, J)(-1)^{\frac{|J|(|J|-1)}{2}} \,\mathrm{Pf}(\tfrac{X_I}{2}) \, dx_J \, e_{I\cup J} \, t^{|J|} \Big) \right) \\ &= -\,\mathrm{e}^{-t^2\|x\|^2} \sum_{k,I,J} \epsilon(I, J)\epsilon(\{k\}, I \cup J)(-1)^{\frac{|J|(|J|+1)}{2}} \,\mathrm{Pf}(\tfrac{X_I}{2}) \, x_k \, dx_J \, t^{|J|}, \end{aligned}$$

where the sum runs over the partitions $\{k\} \cup I \cup J = \{1, \dots, d\}$. If we integrate the last equality between 0 and infinity, and use the formula $\int_0^\infty e^{-t^2} t^a dt = \frac{1}{2}\Gamma(\frac{a+1}{2})$, we get the first point.

The second point follows from the definition.

Let us give the formulae for $\mathrm{Th}_{\mathrm{rel}}, \mathrm{Th}_{\mathrm{c}}, \mathrm{Th}_{\mathrm{MQ}}$ for small dimensions. Here f is a compactly supported function on $\mathbb{R}$, identically equal to 1 in a neighborhood of 0.

Example 4.11 *If* $\dim V = 1$, *then* $\mathrm{SO}(V)$ *is reduced to the identity. We have*

$$\beta_\wedge = -\frac{1}{2}\pi^{1/2}\frac{x}{|x|} = -\frac{1}{2}\pi^{1/2}\,\mathrm{sign}(x),$$

so that

$$\begin{aligned}
\mathrm{Th}_{\mathrm{rel}}(V) &= \Big(0, -\frac{1}{2}\,\mathrm{sign}(x)\Big),\\
\mathrm{Th}_{\mathrm{c}}(V) &= -f'(x^2)|x|dx,\\
\mathrm{Th}_{\mathrm{MQ}}(V) &= \frac{1}{\pi^{1/2}}\,\mathrm{e}^{-\|x\|^2}\,dx.
\end{aligned}$$

Example 4.12 *Let* $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ *of dimension* 2. *We have*

$$\beta_\wedge = \frac{x_1 dx_2 - x_2 dx_1}{2\|x\|^2}$$

so that

$$\begin{aligned}
\mathrm{Th}_{\mathrm{rel}}(V)(X) &= \frac{-1}{\pi}\Big(\mathrm{Pf}(\tfrac{X}{2}), \beta_\wedge\Big),\\
\mathrm{Th}_{\mathrm{c}}(V)(X) &= -\frac{1}{\pi}\Big(f(\|x\|^2)\,\mathrm{Pf}(\tfrac{X}{2}) + f'(\|x\|^2)dx_1\wedge dx_2\Big),\\
\mathrm{Th}_{\mathrm{MQ}}(V)(X) &= \frac{1}{\pi}\,\mathrm{e}^{-\|x\|^2}(-\,\mathrm{Pf}(\tfrac{X}{2}) + dx_1\wedge dx_2).
\end{aligned}$$

Example 4.13 *Let* $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ *of dimension* 3. *We have*

$$\begin{aligned}
\beta_\wedge(X) = &-\frac{\pi^{1/2}}{4\|x\|}\Big(x_1\langle Xe_2, e_3\rangle + x_2\langle Xe_3, e_1\rangle + x_3\langle Xe_1, e_2\rangle\Big)\\
&+\frac{\pi^{1/2}}{4\|x\|^3}\Big(x_1 dx_2\wedge dx_3 + x_2 dx_3\wedge dx_1 + x_3 dx_1\wedge dx_2\Big)
\end{aligned}$$

so that

$$\mathrm{Th}_{\mathrm{rel}}(V)(X) = \frac{-1}{\pi^{3/2}}(0, \beta_\wedge(X)).$$

The equivariant form $\mathrm{Th}_{\mathrm{c}}(V)(X)$ *is equal to*

$$\frac{1}{4\pi}\frac{f'(\|x\|^2)}{\|x\|}\Big((x_1\langle Xe_2, e_3\rangle + x_2\langle Xe_3, e_1\rangle + x_3\langle Xe_1, e_2\rangle)(d\|x\|^2) - 2dx_1\wedge dx_2\wedge dx_3\Big).$$

The equivariant form $\mathrm{Th}_{\mathrm{MQ}}(V)(X)$ *is equal to*

$$-\frac{1}{2\pi^{3/2}}\,\mathrm{e}^{-\|x\|^2}\Big(\langle Xe_2, e_3\rangle dx_1 + \langle Xe_3, e_1\rangle dx_2 + \langle Xe_1, e_2\rangle dx_3 - 2dx_1\wedge dx_2\wedge dx_3\Big).$$

4.3 Unicity of the Thom forms of a vector space

The following theorem is well known in the non equivariant case.

Theorem 4.14 • *The relative class* $\mathrm{Th}_{\mathrm{rel}}(V)$ *is a free generator of* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$ *over* $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$*. Thus* $\mathrm{Th}_{\mathrm{rel}}(V)$ *is the unique class in* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$ *of integral* 1.

• *The equivariant class* $\mathrm{Th}_{\mathrm{c}}(V)$ *is a free generator of* $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$ *over* $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$*. Thus* $\mathrm{Th}_{\mathrm{c}}(V)$ *is the unique class in* $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$ *of integral* 1.

• *The Mathai-Quillen class* $\mathrm{Th}_{\mathrm{MQ}}(V)$ *is a free generator of* $\mathcal{H}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$ *over* $\mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$*. Thus* $\mathrm{Th}_{\mathrm{MQ}}(V)$ *is the unique class in* $\mathcal{H}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{so}(V), V)$ *of integral* 1.

The same theorem holds in the $\mathcal{C}^\infty$-category.

Proof. We give the proof of the second point first. Let $t_V(X)$ be a closed equivariant form with compact support, so that $\int_V t_V = 1$. We want to prove that any closed equivariant form $\alpha(X)$ in $\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$ is proportional to $t_V(X)$ in cohomology : $\alpha = Q\, t_V$ in $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$ with $Q \in \mathcal{C}^{\mathrm{pol}}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$.

Let $\mathrm{Ex}(v) = -v$ be the transformation $-\mathrm{Id}_V$. We consider the space $V \times V$. The transformation $g(t)(v_1, v_2) = (\cos(t)v_1 + \sin(t)v_2, -\sin(t)v_1 + \cos(t)v_2)$ is a one-parameter transformation of $V \times V$ which commutes with the diagonal action of $\mathrm{SO}(V)$ and preserves the pair $(V\times V, V\times V\setminus\{(0,0)\})$. The transformation $g(0)$ is the identity, while $g(\frac{\pi}{2})(v_1, v_2) = (v_2, -v_1)$. Let $S : V \times V \to V \times V$ be the vector field on $V \times V$ associated to the action of the one-parameter subgroup $g(t)$. Thus $S_{v_1,v_2} = (-v_2, v_1)$.

We denote by π_1, π_2 the first and second projections of $V \times V$ on V. Let α_1, α_2 be closed equivariant forms in $\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$. The exterior product $A_{12} = \pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2$ belongs to $\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{so}(V), V \times V)$.

Let us apply the transformation $g(t)$ to A_{12}. Define

$$A(t) = g(t)^*(A_{12}).$$

Then $A(0) = \pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2$ while $A(\frac{\pi}{2}) = \pi_2^*\alpha_1 \wedge \pi_1^* \mathrm{Ex}^* \alpha_2$. If the equivariant form α_i are supported in the balls $B(0, r_i) \subset V$, we see that the equivariant $A(t)$ is supported in the ball $B(0, r_1 + r_2) \subset V \times V$ for any $t \in \mathbb{R}$.

We have $\frac{d}{dt}A(t) = \mathcal{L}(S)A(t) = D(\iota(S)A(t))$ from Cartan's relation (1). Thus the cohomology class of $A(t)$ remains constant. We obtain that, for homogeneous elements α_1, α_2,

$$\pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2 = \pi_2^*\alpha_1 \wedge \pi_1^* \mathrm{Ex}^* \alpha_2 = (-1)^{|\alpha_1||\alpha_2|}\pi_1^* \mathrm{Ex}^* \alpha_2 \wedge \pi_2^*\alpha_1$$

in $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V \times V)$.

We now consider the first projection $\pi_1 : V \times V \to V$ and the corresponding integral over the fiber $(\pi_1)_*$ (see Section 3.6). Note that the map $(\pi_1)_* \circ \pi_2^*$:

$\mathcal{H}_c^{\rm pol}(\mathfrak{so}(V), V) \rightarrow \mathcal{C}^{\rm pol}(\mathfrak{so}(V))^{\mathrm{SO}(V)}$ corresponds to the integration map $\int_V$. Thus we obtain the relation

$$\left(\int_V \alpha_2\right)\alpha_1 = (-1)^{|\alpha_1||\alpha_2|}\left(\int_V \alpha_1\right)\mathrm{Ex}^*\,\alpha_2$$

in $\mathcal{H}_c^{\rm pol}(\mathfrak{so}(V), V)$.

Let us apply this relation to the couple $(\alpha, \mathrm{Ex}^*\, t_V)$. As $\int_V \mathrm{Ex}^*\, t_V = (-1)^{\dim V}$, we obtain that $\alpha = Q(X) t_V$, with $Q(X) = \epsilon \int_V \alpha(X)$. Thus, if α is not zero, we can assume that its degree is of same parity that $\dim V$, and we obtain the relation

$$\alpha = \left(\int_V \alpha\right) t_V \tag{43}$$

in $\mathcal{H}_c^{\rm pol}(\mathfrak{so}(V), V)$. This proves that t_V is unique in cohomology, and is a generator of $\mathcal{H}_c^{\rm pol}(\mathfrak{so}(V), V)$.

The third point is proved in exactly the same way.

Let us prove the first point. To prove the fact that $\mathrm{Th}_{\rm rel}(V)$ is a generator of $\mathcal{H}^{\rm pol}(\mathfrak{so}(V), V, V\setminus\{0\})$, we apply the same one parameter transformation $g(t)$ which acts on $V \times V$ and preserves the subset $\{(0,0)\} \subset V\times V$. Here we use the product $\diamond$ from

$$\mathcal{H}^{\rm pol}(\mathfrak{so}(V), V\times V, (V\setminus\{0\})\times V) \times \mathcal{H}^{\rm pol}(\mathfrak{so}(V), V\times V, V\times(V\setminus\{0\}))$$

into $\mathcal{H}^{\rm pol}(\mathfrak{so}(V), V\times V, V\times V\setminus\{(0,0)\})$. For a couple (a_1,a_2) of $D_{\rm rel}$-closed elements in $\mathcal{A}^{\rm pol}(\mathfrak{so}(V), V, V\setminus\{0\})$, we consider the product

$$\pi_1^*(a_1)\diamond\pi_2^*(a_2) \in \mathcal{A}^{\rm pol}(\mathfrak{so}(V), V\times V, V\times V\setminus\{(0,0)\})$$

and their transformations $g(t)^*(\pi_1^* a_1 \diamond \pi_2^* a_2)$ which are in the same cohomology class. We need the

Lemma 4.15 *We have the following equality*

$$g(\tfrac{\pi}{2})^*\left(\pi_1^*\alpha_1\diamond\pi_2^*(\alpha_2)\right) = (-1)^{|\alpha_1||\alpha_2|}\pi_1^*\,\mathrm{Ex}^*\,\alpha_2\diamond\pi_2^*\alpha_1$$

in $\mathcal{H}^{\rm pol}(\mathfrak{so}(V), V\times V, V\times V\setminus\{(0,0)\})$.

Proof. We consider the covering $V\times V\setminus\{(0,0)\} = U_1\cup U_2$ where $U_1 := (V\setminus\{0\})\times V$ and $U_2 = V\times(V\setminus\{0\})$. Let $\Phi = (\Phi_1,\Phi_2)$ a partition of unity subordinate to this covering: the functions Φ_k are supposed $\mathrm{SO}(V)$-invariant. We have also the group of symmetry generated by $\theta := g(\frac{\pi}{2})$. We have $\theta(U_1) = U_2$, $\theta(U_2) = U_1$ and $\theta^2(x,y) = (-x,-y)$. We can suppose that the functions Φ_k are invariant under θ^2, and that $\theta^*(\Phi_1) = \Phi_2$.

Let $a_k := (\alpha_k, \beta_k) \in \mathcal{A}^{\rm pol}(\mathfrak{so}(V), V, V\setminus\{0\})$, $k = 1,2$. Recall that $\pi_1^* a_1 \diamond_\Phi \pi_2^* a_2$ is equal to $(\pi_1^*\alpha_1\wedge\pi_2^*\alpha_2, \beta_\Phi(\pi_1^*a_1, \pi_2^*a_2))$ with

$$\begin{aligned}&\beta_\Phi(\pi_1^*a_1,\pi_2^*a_2) =\\ &\qquad \Phi_1\pi_1^*\beta_1\wedge\pi_2^*\alpha_2 + (-1)^{|a_1|}\pi_1^*\alpha_1\wedge\Phi_2\pi_2^*\beta_2 - (-1)^{|a_1|}d\Phi_1\wedge\pi_1^*\beta_1\wedge\pi_2^*\beta_2.\end{aligned}$$

Then $\theta^*(\pi_1^* a_1 \diamond_\Phi \pi_2^* a_2)$ is equal to $(\theta^*(\pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2), \theta^*(\beta_\Phi(a_1, a_2)))$. We know already that $\theta^*(\pi_1^*\alpha_1 \wedge \pi_2^*\alpha_2)$ is equal to $(-1)^{|a_1||a_2|}\pi_1^* \operatorname{Ex}^* \alpha_2 \wedge \pi_2^*\alpha_1$. Let us compute

$$
\begin{aligned}
&\theta^*(\beta_\Phi(\pi_1^* a_1, \pi_2^* a_2))\\
&\quad = \quad \theta^* \Big(\Phi_1 \pi_1^*\beta_1 \wedge \pi_2^*\alpha_2 + (-1)^{|a_1|}\pi_1^*\alpha_1 \wedge \Phi_2\pi_2^*\beta_2 - (-1)^{|a_1|} d\Phi_1 \wedge \pi_1^*\beta_1 \wedge \pi_1^*\beta_2\Big)\\
&\quad = \quad \Phi_2 \pi_2^*\beta_1 \wedge \pi_1^* \operatorname{Ex}^* \alpha_2 + (-1)^{|a_1|}\pi_2^*\alpha_1 \wedge \Phi_1 \pi_1^* \operatorname{Ex}^* \beta_2\\
&\qquad\qquad -(-1)^{|a_1|} d\Phi_2 \wedge \pi_2^*\beta_1 \wedge \pi_1^* \operatorname{Ex}^* \beta_2\\
&\quad = \quad (-1)^{|a_1||a_2|}\beta_\Phi(\pi_1^* \operatorname{Ex}^* a_2, \pi_2^* a_1).
\end{aligned}
$$

So the elements $\theta^*(\pi_1^* a_1 \diamond_\Phi \pi_2^* a_2)$ and $(-1)^{|a_1||a_2|}\pi_1^* \operatorname{Ex}^* a_2 \diamond_\Phi \pi_2^* a_1$ coincide in $\mathcal{A}^{\mathrm{pol}}(\mathfrak{so}(V), V \times V, V \times V \setminus \{(0,0)\})$.

We have proved that

$$\pi_1^* a_1 \diamond \pi_2^* a_2 = (-1)^{|a_1||a_2|}\pi_1^* \operatorname{Ex}^* a_2 \diamond \pi_2^* a_1$$

holds in $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V \times V, V \times V \setminus \{(0,0)\})$ for any couple (a_1, a_2) in $\mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\})$. Then we consider the integral over the fiber of π_1 and we apply Proposition 3.20. The rest of the proof is the same.

Remark 4.16 *At the level of equivariant forms, Equality (43) can be made more specific as follows. We have the following equality in* $\mathcal{A}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$*:*

$$\alpha(X) = \left(\int_V \alpha(X)\right) t_V(X) + D(\delta)(X) \tag{44}$$

where $\delta = (-1)^{\dim(V)+1}(\pi_1)_* \left(\int_0^1 \iota(S)A(t)\right)$*, and* $A(t) = g(t)^*(\pi_1^*\alpha \wedge \pi_2^* \operatorname{Ex}^* t_V)$*. We have also a control on the support of the equivariant form* δ*. If* α *and* t_V *are supported respectively in the balls* $B(0,r)$ *and* $B(0,\epsilon)$ *of* V*, the form* δ *is supported in the ball* $B(0, r+\epsilon)$*.*

Since the map $\mathrm{p}_c : \mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\}) \to \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$ sends the class $\mathrm{Th}_{\mathrm{rel}}(V)$ to the class $\mathrm{Th_c}(V)$, Theorem 4.14 shows that p_c is an *isomorphism*.

We can specify this property as follows. We have $\mathrm{p}_c = j \circ \mathrm{p}_{\{0\}}$ where $\mathrm{p}_{\{0\}} : \mathcal{H}^{\mathrm{pol}}(\mathfrak{so}(V), V, V \setminus \{0\}) \to \mathcal{H}_{\{0\}}^{\mathrm{pol}}(\mathfrak{so}(V), V)$ and

$$j : \mathcal{H}_{\{0\}}^{\mathrm{pol}}(\mathfrak{so}(V), V) \to \mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V).$$

is a natural map (see Lemma 3.11): let us recall its definition. A class in $\mathcal{H}_{\{0\}}^{\mathrm{pol}}(\mathfrak{so}(V), V)$ is defined by a collection $[\gamma_U] \in \mathcal{H}_U^{\mathrm{pol}}(\mathfrak{so}(V), V)$, where U runs over the open neighborhood of $\{0\}$, and such that $r_{U,U'}[\gamma_{U'}] = [\gamma_U]$ for $U' \subset U$. The image of $([\gamma_U])_U$ by j is the class defined by the closed the equivariant form γ_U in $\mathcal{H}_c^{\mathrm{pol}}(\mathfrak{so}(V), V)$, for any *relatively compact* open neighborhood U.

Theorem 4.17 • *The maps j and $\mathrm{p}_{\{0\}}$ are isomorphisms.*

• *Similarly, the maps* $\mathrm{p}_{\{0\}} : \mathcal{H}^\infty(\mathfrak{so}(V), V, V \setminus \{0\}) \to \mathcal{H}^\infty_{\{0\}}(\mathfrak{so}(V), V)$ *and* $j : \mathcal{H}^\infty_{\{0\}}(\mathfrak{so}(V), V) \to \mathcal{H}^\infty_c(\mathfrak{so}(V), V)$ *are isomorphisms.*

Proof. Since $\mathrm{p}_c = j \circ \mathrm{p}_{\{0\}}$ is an isomorphism, it is enough to prove that j is one to one. Let $([\gamma_U])_U$ an element in the kernel of j : for any *relatively compact* open neighborhood U of $\{0\}$, we have $\gamma_U = 0$ in $\mathcal{H}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$. If we show that $\gamma_U = 0$ in $\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{so}(V), V)$, it gives that $([\gamma_U])_U = 0$. Let $U' = B(0, r)$ be a ball and $0 < \epsilon << 1$ such that $\overline{B(0, r + \epsilon)}$ is included in U.

We use (44) with the closed equivariant form $\gamma_{U'}$ supported in the ball $B(0, r)$, and a Thom form t_V supported in the ball $B(0, \epsilon)$. We have the following relation in $\mathcal{A}^{\mathrm{pol}}_U(\mathfrak{so}(V), V)$

$$\begin{aligned} \gamma_{U'}(X) &= \left(\int_V \gamma_{U'}(X)\right) t_V(X) + D(\delta)(X) \\ &= D(\delta)(X) \end{aligned}$$

since $\gamma_{U'} = \gamma_U = 0$ in $\mathcal{H}^{\mathrm{pol}}_c(\mathfrak{so}(V), V)$. This proves that $\gamma_U = r_{U,U'}(\gamma_{U'}) = 0$ in $\mathcal{H}^{\mathrm{pol}}_U(\mathfrak{so}(V), V)$.

4.4 Explicit relative Thom form of a vector bundle and Thom isomorphism

Let M be a manifold. Let $p := \mathcal{V} \to M$ be a real oriented Euclidean vector bundle over M of rank d. In this section, we will describe a generator over $\mathcal{H}(M)$ of the relative cohomology space of the pair $(\mathcal{V}, \mathcal{V} \setminus M)$. We will use Chern-Weil construction.

Recall the sub-space $\mathcal{A}_{\text{fiber cpt}}(\mathcal{V}) \subset \mathcal{A}(\mathcal{V})$ of differential forms on $\mathcal{V}$ which have a compact support in the fibers of $p : \mathcal{V} \to M$. We have also defined the sub-space $\mathcal{A}_{\text{dec-rap}}(\mathcal{V})$. The integration over the fiber is well defined on the three spaces $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$, $\mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V})$ and $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V})$ and take values in $\mathcal{A}(M)$. A Thom form on $\mathcal{V}$ will be a closed element which integrates to the constant function 1 on M.

The bundle $\mathcal{V}$ is associated to a principal bundle $P \to M$ with structure group $G = \mathrm{SO}(V)$, where $V = \mathbb{R}^d$. We denote $\mathfrak{so}(V)$ by $\mathfrak{g}$. An element $y \in P$ above $x \in M$ is by definition a map $y : V \to \mathcal{V}_x$ conserving the inner product and the orientation. Thus P is equipped with an action of $\mathrm{SO}(V)$: $g \cdot y = yg^{-1} : V \to \mathcal{V}_x$.

Definition 4.18 *Let ω be a connection one form on P, with curvature form Ω. The Euler form $\mathrm{Eul}(\mathcal{V}, \omega) \in \mathcal{A}(M)$ of $\mathcal{V} \to M$ is the closed differential form on M defined by*

$$\mathrm{Eul}(\mathcal{V}, \omega) := \mathrm{Pf}\left(-\frac{\Omega}{2\pi}\right).$$

The class of $\mathrm{Eul}(\mathcal{V}, \omega)$, which does not depend of ω, is denoted $\mathrm{Eul}(\mathcal{V}) \in \mathcal{H}^d(M)$.

Remark 4.19 *Since the pfaffian vanishes on $\mathfrak{so}(V)$ when $\dim V$ is odd, the Euler class $\mathrm{Eul}(\mathcal{V}) \in \mathcal{H}^d(M)$ is identically equal to 0 when the rank of $\mathcal{V}$ is odd.*

Consider the Chern-Weil map

$$\phi^Z_\omega : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, Z) \longrightarrow \mathcal{A}(\mathcal{Z})$$

where the manifold Z is the $\{\mathrm{pt}\}$, V, or $Z = V \setminus \{0\}$: hence the quotient manifold $\mathcal{Z} = P \times_G Z$ are respectively M, $\mathcal{V}$ and $\mathcal{V} \setminus M$. In order to simplify the notations we denote all these maps by ϕ_ω. The map ϕ_ω maps $\mathcal{A}^{\mathrm{pol}}_c(\mathfrak{g}, V)$ to $\mathcal{A}_{\text{fiber cpt}}(\mathcal{V})$ and $\mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{g}, V)$ to $\mathcal{A}_{\text{dec-rap}}(\mathcal{V})$.

Recall the equivariant differential forms defined in Theorem 4.7 :

$$\begin{aligned}
\mathrm{Th}_{\mathrm{rel}}(V) &\in \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, V, V \setminus \{0\}),\\
\mathrm{Th}_{\mathrm{c}}(V) &\in \mathcal{A}^{\mathrm{pol}}_c(\mathfrak{g}, V),\\
\mathrm{Th}_{\mathrm{MQ}}(V) &\in \mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{g}, V).
\end{aligned}$$

We can take the image of these forms via the Chern-Weil map. We consider the transgression form $\beta_{\mathcal{V},\omega} \in \mathcal{A}^{d-1}(\mathcal{V} \setminus M)$ defined by $\beta_{\mathcal{V},\omega} = \phi_\omega(\beta_\wedge)$.

Definition 4.20 • *The relative form $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}, \omega) \in \mathcal{A}^d(\mathcal{V}, \mathcal{V} \setminus M)$ is the image of $\mathrm{Th}_{\mathrm{rel}}(V) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, V, V \setminus \{0\})$ by the Chern-Weil map ϕ_ω :*

$$\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}, \omega) = \frac{1}{\epsilon_d}\left(p^* \mathrm{Pf}(\tfrac{\Omega}{2}), \beta_{\mathcal{V},\omega}\right).$$

• *The form $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V}, \omega) \in \mathcal{A}^d_{\text{fiber cpt}}(\mathcal{V})$ is defined as the image of $\mathrm{Th}_{\mathrm{c}}(V) \in \mathcal{A}^{\mathrm{pol}}_c(\mathfrak{g}, V)$ by the Chern-Weil map ϕ_ω.*

• *The Mathai-Quillen form $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}, \omega) \in \mathcal{A}^d_{\text{dec-rap}}(\mathcal{V})$ is defined as the image of $\mathrm{Th}_{\mathrm{MQ}}(V) \in \mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{g}, V)$ by the Chern-Weil map ϕ_ω:*

The form $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}, \omega)$ is relatively closed, since $\beta_{\mathcal{V}}$ is defined on $\mathcal{V} \setminus M$ and satisfies $d\beta_{\mathcal{V},\omega} = p^* \mathrm{Pf}(\frac{\Omega}{2})$. The forms $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V}, \omega)$ and $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}, \omega)$ are closed de Rham differential forms. We denote $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$, $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ and $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ the corresponding cohomology classes. Since the map ϕ_ω commutes with the integration over the fiber, these cohomology classes images are Thom classes.

We obtain the analog of Theorem 4.14.

Theorem 4.21 .

• *The relative class $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ is a free generator of $\mathcal{H}(\mathcal{V}, \mathcal{V} \setminus M)$ over $\mathcal{H}(M)$. Thus $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ is the unique class in $\mathcal{H}(\mathcal{V}, \mathcal{V} \setminus M)$ with integral 1 along the fiber. We say that $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ is the Thom class in $\mathcal{H}(\mathcal{V}, \mathcal{V} \setminus M)$.*

• *The class $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ is a free generator of $\mathcal{H}_{\text{fiber cpt}}(\mathcal{V})$ over $\mathcal{H}(M)$. Thus $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ is the unique class in $\mathcal{H}_{\text{fiber cpt}}(\mathcal{V})$ with integral 1 along the fiber. We say that $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ is the Thom class in $\mathcal{H}_{\text{fiber cpt}}(\mathcal{V})$.*

• *The Mathai-Quillen class $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ is a free generator of $\mathcal{H}_{\text{dec-rap}}(\mathcal{V})$ over $\mathcal{H}(M)$. Thus $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ is the unique class in $\mathcal{H}_{\text{dec-rap}}(\mathcal{V})$ with integral 1 along the fiber. We say that $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ is the Thom class in $\mathcal{H}_{\text{dec-rap}}(\mathcal{V})$.*

Proof. The proof is the same than the proof of Theorem 4.14. We work on the sum of vector bundles $\mathcal{V}\oplus\mathcal{V}$ over M and we relate the identity on $\mathcal{V}\oplus\mathcal{V}$ to the exchange of the two copies of $\mathcal{V}$ via a one-parameter transformation group.

From the explicit formulae for $\mathrm{Th}_{\text{fiber cpt}}, \mathrm{Th}_{\mathrm{MQ}}$, we obtain the following proposition.

Proposition 4.22 *The restriction of the Thom classes* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ *or* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ *to* M *is the Euler class* $\mathrm{Eul}(\mathcal{V})$.

Let us give explicit expressions for the forms $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V},\omega), \mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)$ as well as $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)$. Let us fix an oriented orthonormal basis $(e_1,\dots,e_d)$ of V. We write $\omega=\sum_{k,\ell}\omega_{k\ell}E_\ell^k$ where E_ℓ^k is the transformation of V such that $E_\ell^k(e_i)=\delta_{i,k}e_\ell$. Here $\omega_{k\ell}$ are 1-forms on P and $\omega_{k\ell}=-\omega_{\ell k}$. The curvature Ω is $\Omega=\sum_{k,\ell}\Omega_{k\ell}E_\ell^k$ where $\Omega_{k\ell}=d\omega_{k\ell}+\sum_j\omega_{j\ell}\wedge\omega_{kj}$.

The connection ∇ on $\mathcal{V}\to M$ induced by the connection form ω is the operator $\nabla=d+\omega$ acting on $(\mathcal{C}^\infty(P)\otimes V)^G$ with values in $(\mathcal{A}^1(P)_{\mathrm{hor}}\otimes V)^G$. The Chern-Weil map $\phi_\omega:\mathcal{A}^\infty(\mathfrak{g},V)\to(\mathcal{A}(P\times V)_{\mathrm{hor}})^G$ admit a natural extension

$$\phi_\omega^\wedge:(\mathcal{C}^\infty(\mathfrak{g})\otimes\mathcal{A}(V)\otimes\wedge V)^G\longrightarrow(\mathcal{A}(P\times V)_{\mathrm{hor}}\otimes\wedge V)^G$$

such that $\mathrm{T}\circ\phi_\omega^\wedge=\phi_\omega\circ\mathrm{T}$.

Let $f_t\in(\mathcal{C}^{\mathrm{pol}}(\mathfrak{g})\otimes\mathcal{A}(V)\otimes\wedge V)^G$ be the map defined in (33). The element $f_t^\omega:=\phi_\omega^\wedge(f_t)$ is defined by the equation

$$f_t^\omega=-t^2\|x\|^2+t\sum_i\eta_ie_i+\frac{1}{2}\sum_{k<l}\Omega_{k\ell}e_k\wedge e_\ell. \tag{45}$$

where $\eta_i=dx_i+\sum_\ell\omega_{\ell i}x_\ell$ is a horizontal 1-form on $P\times V$. If $I=\{i_1<\cdots<i_k\}$ is a subset of $[1,2,\dots,d]$, we denote by η_I the product $\eta_{i_1}\wedge\cdots\wedge\eta_{i_k}$.

Let $\eta_\wedge^t\in(\mathcal{C}^{\mathrm{pol}}(\mathfrak{g})\otimes\mathcal{A}(V))^G$ be the transgression forms defined in (34). The element $\phi_\omega(\eta_\wedge^t)\in\mathcal{A}(P\times V)_{\mathrm{bas}}$ satisfies

$$\begin{aligned}\phi_\omega(\eta_\wedge^t) &= -\phi_\omega\circ\mathrm{T}\Big((\sum_k x_ke_k)\,\mathrm{e}^{f_t(X)}\Big)\\ &= -\mathrm{T}\circ\phi_\omega^\wedge\Big((\sum_k x_ke_k)\,\mathrm{e}^{f_t(X)}\Big)\\ &= -\mathrm{T}\Big((\sum_k x_ke_k)\,\mathrm{e}^{f_t^\omega}\Big).\end{aligned}$$

We use the same notations as in Proposition 4.10.

Proposition 4.23 • *The form* $\beta_{\mathcal{V},\omega}=\phi_\omega(\beta_\wedge)$ *is defined in* $\mathcal{A}(P\times(V\setminus\{0\}))_{\mathrm{bas}}$ *by the relation*

$$\beta_{\mathcal{V},\omega}=-\int_0^\infty\mathrm{T}\Big((\sum x_ke_k)\,\mathrm{e}^{-t^2\|x\|^2+t\sum_k\eta_ke_k+\frac{1}{2}\sum_{k<\ell}\Omega_{k\ell}e_k\wedge e_\ell}\Big).$$

More explicitly,

$$\beta_{\mathcal{V},\omega} = \sum_{k,I,J} \gamma_{(k,I,J)} \operatorname{Pf}\left(\tfrac{\Omega_I}{2}\right) \frac{x_k \eta_J}{\|x\|^{|J|+1}}.$$

- *The form* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)$ *is defined in* $\mathcal{A}(P\times V)_{\text{bas}}$ *by the relation*

$$\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega) = \frac{1}{\epsilon_d}(\chi \operatorname{Pf}(\tfrac{\Omega}{2}) + d\chi \beta_{\mathcal{V},\omega}).$$

where χ *is a* $\mathrm{SO}(V)$*-invariant function on* V*, identically equal to* 1 *in a neighborhood of* 0*.*

- *The form* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)$ *is defined in* $\mathcal{A}(P\times V)_{\text{bas}}$ *by the relation*

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega) = \frac{1}{\epsilon_d} \mathrm{T}\left(\mathrm{e}^{-\|x\|^2+\sum_k \eta_k e_k + \frac{1}{2}\sum_{k<\ell}\Omega_{k\ell} e_k\wedge e_\ell}\right).$$

More explicitly

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega) = \frac{1}{(\pi)^{d/2}} \mathrm{e}^{-\|x\|^2} \sum_{I \text{ even}} (-1)^{\frac{|I|}{2}} \epsilon(I,I') \operatorname{Pf}(\tfrac{\Omega_I}{2})\eta_{I'}.$$

This gives an explicit expression of $\beta_{\mathcal{V},\omega}$ and thus of $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V},\omega)$ in functions of the variables x_j, dx_j on V, the connection one forms $\omega_{k\ell}$ and the curvature forms $\Omega_{k\ell}$ of an Euclidean connection on M.

The form $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)$ is the representative with Gaussian look of the Thom form constructed by Mathai-Quillen [11].

We also obtain:

Theorem 4.24 • *The map* $\mathcal{H}(\mathcal{V},\mathcal{V}\setminus M)\to\mathcal{H}_M(\mathcal{V})$ *is an isomorphism.*
- *If* M *is compact, the map* $\mathcal{H}(\mathcal{V},\mathcal{V}\setminus M)\to\mathcal{H}_c(\mathcal{V})$ *is an isomorphism.*

4.5 Explicit equivariant relative Thom form of a vector bundle and Thom isomorphism

We can repeat the construction above in the equivariant case. Assume now that $\mathcal{V}$ is a K-equivariant vector bundle. Then $\mathcal{V}$ is associated to a K-equivariant principal bundle $P\to M$ with structure group $G=\mathrm{SO}(V)$. The principal bundle P is provided with an action of $K\times G$. If $y: V\to\mathcal{V}_x$ is an orthonormal frame, then $(g,k)\cdot y = kyg^{-1}$ is a frame over kx.

Let ω be a K-invariant connection one form on P, with curvature form Ω. For $Y\in\mathfrak{k}$, we denote by $\mu(Y) = -\iota(VY)\omega \in \mathcal{C}^\infty(P)\otimes\mathfrak{g}$ the moment of $Y\in\mathfrak{k}$. The equivariant curvature form is $\Omega(Y)=\Omega+\mu(Y)$, $Y\in\mathfrak{k}$.

Consider the Chern-Weil maps

$$\phi_\omega : \mathcal{A}^{\mathrm{pol}}(\mathfrak{g},Z)\longrightarrow \mathcal{A}^{\mathrm{pol}}(\mathfrak{k},\mathcal{Z})$$

where the manifold Z is the $\{\mathrm{pt}\}$, V, or $Z=V\setminus\{0\}$. See (4) for the definition.

Definition 4.25 *Let ω be a connection one form on P, with curvature form Ω. The equivariant Euler form* $\mathrm{Eul}(\mathcal{V},\omega) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, M)$ *of $\mathcal{V} \to M$ is the closed equivariant form on M defined by*

$$\mathrm{Eul}(\mathcal{V},\omega)(Y) := \mathrm{Pf}\left(-\frac{\Omega(Y)}{2\pi}\right), \ Y \in \mathfrak{k}.$$

In other words, $\mathrm{Eul}(\mathcal{V},\omega)$ *is the image by the Chern-Weil map ϕ_ω of the invariant polynomial $X \mapsto \mathrm{Pf}(-\frac{X}{2\pi})$. The class of* $\mathrm{Eul}(\mathcal{V},\omega)$*, which does not depend of ω, is denoted* $\mathrm{Eul}(\mathcal{V}) \in \mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M)$*.*

As integration over the fiber commutes with the Chern-Weil construction, the image by ϕ_ω of Thom classes are Thom classes. We define the transgression form $\beta_{\mathcal{V},\omega}(Y) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V} \setminus M)$ by

$$\beta_{\mathcal{V},\omega}(Y) = (\phi_\omega(\beta_\wedge))(Y).$$

Definition 4.26 • *The relative equivariant form* $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V},\omega) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, \mathcal{V}, \mathcal{V}\setminus M)$ *is the image of* $\mathrm{Th}_{\mathrm{rel}}(V) \in \mathcal{A}^{\mathrm{pol}}(\mathfrak{g}, V, V \setminus \{0\})$ *by the Chern-Weil map ϕ_ω. More explicitly, for $Y \in \mathfrak{k}$*

$$\mathrm{Th}_{\mathrm{rel}}(\mathcal{V},\omega)(Y) = \frac{1}{\epsilon_d}\left(p^* \,\mathrm{Pf}(\tfrac{\Omega(Y)}{2}), \beta_{\mathcal{V},\omega}(Y)\right).$$

• *We define the equivariant form* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)(Y) \in \mathcal{A}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V})$ *as the image of* $\mathrm{Th}_{\mathrm{c}}(V) \in \mathcal{A}^{\mathrm{pol}}_{c}(\mathfrak{g}, V)$ *by the Chern-Weil map ϕ_ω:*

$$\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)(Y) := (\phi_\omega \mathrm{Th}_{\mathrm{c}}(V))(Y).$$

• *We define the Mathai-Quillen form* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y) \in \mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V})$ *as the image of* $\mathrm{Th}_{\mathrm{MQ}}(V) \in \mathcal{A}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{g}, V)$ *by the Chern-Weil map ϕ_ω:*

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y) := (\phi_\omega \mathrm{Th}_{\mathrm{MQ}}(V))(Y).$$

The equivariant form $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V},\omega)$ is relatively closed, since $\beta_{\mathcal{V},\omega}$ is defined on $\mathcal{V}\setminus M$ and satisfies $D(\beta_{\mathcal{V},\omega}) = p^* \,\mathrm{Pf}(\frac{\Omega}{2})$. The equivariant forms $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)$ and $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)$ are closed equivariant differential forms. We denote $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$, $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ and $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ the corresponding equivariant cohomology classes. Since the map ϕ_ω commutes with the integration over the fiber, these cohomology classes images are equivariant Thom classes.

With the same proof as the non equivariant case, we obtain the following Theorem.

Theorem 4.27 • *The relative class* $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})(Y)$ *is a free generator of* $\mathcal{H}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$ *over* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M)$*. Thus* $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ *is the unique class in* $\mathcal{H}^{\mathrm{pol}}_{\mathrm{rel}}(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$ *with integral* 1 *along the fiber. We say that* $\mathrm{Th}_{\mathrm{rel}}(\mathcal{V})$ *is the equivariant Thom form in* $\mathcal{H}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$*.*

• *The equivariant class* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})(Y)$ *is a free generator of* $\mathcal{H}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k},\mathcal{V})$ *over* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k},M)$*. Thus* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ *is the unique class in* $\mathcal{H}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k},\mathcal{V})$ *with integral* 1 *along the fiber. We say that* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ *is the equivariant Thom form in* $\mathcal{H}^{\mathrm{pol}}_{\text{fiber cpt}}(\mathfrak{k},\mathcal{V})$*.*

• *The Mathai-Quillen class* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})(Y)$ *is a free generator of* $\mathcal{H}^{\mathrm{pol}}_{\text{dec-rap}}(\mathfrak{k},\mathcal{V})$ *over* $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k},M)$*. Thus* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ *is the unique class in* $\mathcal{H}_{\text{dec-rap}}(\mathfrak{k},\mathcal{V})$ *with integral* 1 *along the fiber. We say that* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$ *is the equivariant Thom form in* $\mathcal{H}_{\text{dec-rap}}(\mathfrak{k},\mathcal{V})$*.*

Proof. The proof is the same than the proof of Theorem 4.21.

From the explicit formulae for $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)(Y), \mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y)$, we obtain the following proposition.

Proposition 4.28 *The restriction of the equivariant Thom class* $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V})$ *(or* $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V})$*) to* M *is the equivariant Euler class* $\mathrm{Eul}(\mathcal{V})$*.*

Let us give explicit expressions for the equivariant forms $\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)(Y)$ as well as $\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y)$: we will express them as K-equivariant map from $\mathfrak{k}$ into $\mathcal{A}(P\times V)_{\mathrm{bas}}$. We use the same notations as in Proposition 4.23.

We denote by $\mu(Y)$ the moment of $Y\in\mathfrak{k}$ with respect to the connection $\nabla = d+\omega$. By definition $\mu(Y)=\mathcal{L}(Y)-\nabla_{VY}=-\iota(VY)\omega$ where VY is the vector field on P associated to $Y\in\mathfrak{k}$. Thus $\mu(Y)$, viewed as a $\mathrm{SO}(V)$-invariant map from P into $\mathfrak{so}(V)$, satisfies

$$\mu(Y) = -\sum_{k,\ell}\langle\omega_{k\ell}, VY\rangle E^k_\ell.$$

For the equivariant curvature we have $\Omega(Y)=\sum_{k,\ell}\Omega_{k\ell}(Y)E^k_\ell$ with

$$\Omega_{k\ell}(Y) = d\omega_{k\ell} + \sum_j \omega_{j\ell}\wedge\omega_{kj} - \langle\omega_{k\ell}, VY\rangle.$$

As usual in equivariant cohomology, formulae for equivariant classes are obtained from the non equivariant case by replacing the curvature Ω by the equivariant curvature.

Proposition 4.29 • *The transgression form* $\beta_{\mathcal{V},\omega}(Y)$ *is defined by the following formula : for* $Y\in\mathfrak{k}$ *we have*

$$\beta_{\mathcal{V},\omega}(Y) = -\int_0^\infty \mathrm{T}\left((\sum x_k e_k)\,\mathrm{e}^{-t^2\|x\|^2+t\sum_i \eta_i e_i+\frac{1}{2}\sum_{k<\ell}\Omega_{k,\ell}(Y)e_k\wedge e_\ell}\right).$$

More explicitly,

$$\beta_{\mathcal{V},\omega}(Y) = \sum_{k,I,J}\gamma_{(k,I,J)}\,\mathrm{Pf}\left(\tfrac{\Omega_I(Y)}{2}\right)\frac{x_k\eta_J}{\|x\|^{|J|+1}}.$$

• *Let χ be a* $\mathrm{SO}(V)$ *invariant function on* V, *identically equal to* 1 *in a neighborhood of* 0. *We have thus*

$$\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V},\omega)(Y) = \frac{1}{\epsilon_d}\left(\chi\,\mathrm{Pf}(\tfrac{\Omega(Y)}{2}) + d\chi\beta_{\mathcal{V},\omega}(Y)\right), \quad Y\in\mathfrak{k}.$$

• *We have*

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y) = \frac{1}{\epsilon_d}\,\mathrm{T}\left(\mathrm{e}^{-\|x\|^2+\sum_k \eta_k e_k+\frac{1}{2}\sum_{k<\ell}\Omega_{k\ell}(Y)e_k\wedge e_\ell}\right), \quad Y\in\mathfrak{k}.$$

More explicitly

$$\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V},\omega)(Y) = \frac{1}{(\pi)^{d/2}}\,\mathrm{e}^{-\|x\|^2}\sum_{I\ \text{even}}(-1)^{\frac{|I|}{2}}\epsilon(I,I')\,\mathrm{Pf}(\tfrac{\Omega_I(Y)}{2})\,\eta_{I'}, \quad Y\in\mathfrak{k}.$$

We obtain similarly:

Theorem 4.30 • *The map* $\mathrm{p}_M : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k},\mathcal{V},\mathcal{V}\setminus M)\to\mathcal{H}^{\mathrm{pol}}_M(\mathfrak{k},\mathcal{V})$ *is an isomorphism.*

• *If* M *is compact, the map* $\mathrm{p}_c : \mathcal{H}^{\mathrm{pol}}(\mathfrak{k},\mathcal{V},\mathcal{V}\setminus M)\to\mathcal{H}^{\mathrm{pol}}_c(\mathfrak{k},\mathcal{V})$ *is an isomorphism*

5 The relative Chern character

Let N be a manifold equipped with an action of a compact Lie group K.

5.1 Quillen's Chern form of a super-connection

For an introduction to the Quillen's notion of super-connection, see [4].

Let $\mathcal{F}$ be a complex vector bundle on N. Let ∇ be a connection on $\mathcal{F}$. The curvature ∇^2 of ∇ is a $\mathrm{End}(\mathcal{F})$-valued two-form on N. Recall that the (non-normalized) Chern character of $\mathcal{F}$ is the de Rham cohomology class $\mathrm{Ch}(\mathcal{F})\in\mathcal{H}(N)^+$ of the closed differential form $\mathrm{Ch}(\nabla) := \mathrm{Tr}(\mathrm{e}^{\nabla^2})$.

Suppose now that $\mathcal{F}$ is a K-equivariant vector bundle, and suppose that ∇ is K-invariant. For any $X\in\mathfrak{k}$, we consider $\mu^\nabla(X) = \mathcal{L}(X) - \nabla_{VX}$ which is an $\mathrm{End}(\mathcal{F})$-valued function on N: here $\mathcal{L}(X)$ is the Lie derivative of X, acting on $\mathcal{A}(N,\mathcal{F})$, and ∇_{VX} is equal to $\iota(VX)\nabla$. Then $\mathrm{Ch}(\nabla)(X) = \mathrm{Tr}(\mathrm{e}^{\nabla^2+\mu^\nabla(X)})$ is a closed equivariant form on N: its class $\mathrm{Ch}(\mathcal{F})\in\mathcal{H}^\infty(\mathfrak{k},N)$ is the equivariant Chern character of $\mathcal{F}$.

More generally, let $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ be an equivariant $\mathbb{Z}_2$-graded complex vector bundle on N. We denote by $\mathcal{A}(N,\mathrm{End}(\mathcal{E}))$ the algebra of $\mathrm{End}(\mathcal{E})$-valued differential forms on N. Taking in account the $\mathbb{Z}_2$-grading of $\mathrm{End}(\mathcal{E})$, the algebra $\mathcal{A}(N,\mathrm{End}(\mathcal{E}))$ is a $\mathbb{Z}_2$-graded algebra. The super-trace on $\mathrm{End}(\mathcal{E})$ extends to a map $\mathrm{Str} : \mathcal{A}(N,\mathrm{End}(\mathcal{E}))\to\mathcal{A}(N)$.

Let $\mathbb{A}$ be a K-invariant super-connection on $\mathcal{E}$, and $\mathbf{F} = \mathbb{A}^2$ its curvature, an element of $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$. Recall that, for $X \in \mathfrak{k}$, the *moment* of $\mathbb{A}$ is the equivariant map

$$\mu^{\mathbb{A}} : \mathfrak{k} \longrightarrow \mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+ \tag{46}$$

defined by the relation $\mu^{\mathbb{A}}(X) = \mathcal{L}(X) - [\iota(VX), \mathbb{A}]$. We define the *equivariant curvature* of $\mathbb{A}$ by

$$\mathbf{F}(X) = \mathbb{A}^2 + \mu^{\mathbb{A}}(X), \quad X \in \mathfrak{k}. \tag{47}$$

We usually denote simply by $\mathbf{F}$ the equivariant curvature, keeping in mind that in the equivariant case, $\mathbf{F}$ is a function from $\mathfrak{k}$ to $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$.

Definition 5.1 *(Quillen) The equivariant Chern character of $(\mathcal{E}, \mathbb{A})$ is the equivariant differential form on N defined by $\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}}\right)$ (e.g. $\mathrm{Ch}(\mathbb{A})(X) = \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(X)}\right)$).*

The form $\mathrm{Ch}(\mathbb{A})$ is equivariantly closed. We will use the following transgression formulas:

Proposition 5.2 • *Let $\mathbb{A}_t$, for $t \in \mathbb{R}$, be a one-parameter family of K-invariant super-connections on $\mathcal{E}$, and let $\frac{d}{dt}\mathbb{A}_t \in \mathcal{A}(N, \mathrm{End}(\mathcal{E}))^-$. Let $\mathbf{F}_t$ be the equivariant curvature of $\mathbb{A}_t$. Then one has*

$$\frac{d}{dt} \mathrm{Ch}(\mathbb{A}_t) = D\left(\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}_t)\, \mathrm{e}^{\mathbf{F}_t}\right)\right). \tag{48}$$

• *Let $\mathbb{A}(s,t)$ be a two-parameters family of K-invariant super-connections. Here $s, t \in \mathbb{R}$. We denote by $\mathbf{F}(s,t)$ the equivariant curvature of $\mathbb{A}(s,t)$. Then:*

$$\frac{d}{ds} \mathrm{Str}\left((\frac{d}{dt}\mathbb{A}(s,t))\ \mathrm{e}^{\mathbf{F}(s,t)}\right) - \frac{d}{dt} \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\ \mathrm{e}^{\mathbf{F}(s,t)}\right)$$
$$= D\left(\int_0^1 \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))e^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))\ \mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right)du\right).$$

These formulae are the consequences of the two following identities. See [4], chapter 7.

• If we denote by $\mathbb{A}_{\mathfrak{k}}(X)$ the operator $\mathbb{A} - \iota(VX)$ on $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$, then we have the relation:

$$\mathbf{F}(X) = \mathbb{A}_{\mathfrak{k}}(X)^2 + \mathcal{L}(X).$$

• If α is an equivariant map from $\mathfrak{k}$ to $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$, then one has

$$D\left(\mathrm{Str}(\alpha(X))\right) = \mathrm{Str}[\mathbb{A}_{\mathfrak{k}}(X), \alpha(X)].$$

Then, using the invariance by K of all terms involved, the proof of Proposition 5.2 is entirely similar to the proof of Proposition 3.1 in [12].

In particular, the cohomology class defined by $\mathrm{Ch}(\mathbb{A})$ in $\mathcal{H}^\infty(\mathfrak{k}, N)$ is independent of the choice of the super-connection $\mathbb{A}$ on $\mathcal{E}$. By definition, this is the equivariant Chern character $\mathrm{Ch}(\mathcal{E})$ of $\mathcal{E}$. By choosing $\mathbb{A} = \nabla^+ \oplus \nabla^-$ where $\nabla^\pm$ are connections on $\mathcal{E}^\pm$, this class is just $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$. However, different choices of $\mathbb{A}$ define very different looking representatives of $\mathrm{Ch}(\mathcal{E})$.

5.2 The relative Chern character of a morphism

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be an equivariant $\mathbb{Z}_2$-graded complex vector bundle on N and $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a smooth morphism which commutes with the action of K. At each point $n \in N$, $\sigma(n) : \mathcal{E}^+_n \to \mathcal{E}^-_n$ is a linear map. The *support* of σ is the K-invariant closed subset

$$\mathrm{Supp}(\sigma) = \{n \in N \mid \sigma(n) \text{ is not invertible}\}.$$

Definition 5.3 *The morphism σ is elliptic if* $\mathrm{Supp}(\sigma)$ *is compact.*

Recall that the data $(\mathcal{E}^+, \mathcal{E}^-, \sigma)$ defines an element of the equivariant K-theory $\mathbf{K}^0_K(N)$ of N when σ is elliptic. In the following, **we do not** assume σ elliptic. Inspired by Quillen [13], we construct a cohomology class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$.

The definition will involve several choices. We choose invariant Hermitian structures on $\mathcal{E}^\pm$ and an invariant super-connection $\mathbb{A}$ on $\mathcal{E}$ **without** 0 **exterior degree term**. This means that $\mathbb{A} = \sum_{j\geq 1} \mathbb{A}_{[j]}$, where $\mathbb{A}_{[1]}$ is a connection on the bundle $\mathcal{E}$ which preserves the grading, and for $j \geq 2$, the operator $\mathbb{A}_{[j]}$ is given by the action of a differential form $\omega_{[j]} \in \mathcal{A}^j(N, \mathrm{End}(\mathcal{E}))$ on $\mathcal{A}(N, \mathcal{E})$. Furthermore, $\omega_{[j]}$ lies in $\mathcal{A}^j(N, \mathrm{End}(\mathcal{E})^-)$ if j is even, and in $\mathcal{A}^j(N, \mathrm{End}(\mathcal{E})^+)$ if j is odd.

We define, with the help of the invariant Hermitian metric on $\mathcal{E}^\pm$, the dual of the morphism σ as an equivariant morphism $\sigma^* : \mathcal{E}^- \to \mathcal{E}^+$. Introduce the odd Hermitian endomorphism of $\mathcal{E}$ defined by

$$v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}. \tag{49}$$

Then $v_\sigma^2 = \begin{pmatrix} \sigma^*\sigma & 0 \\ 0 & \sigma\sigma^* \end{pmatrix}$ is a non negative even Hermitian endomorphism of $\mathcal{E}$. The support of σ coincides with the set of elements $n \in N$ where the spectrum of $v_\sigma^2(n)$ contains 0.

Definition 5.4 *We denote by $h_\sigma(n) \geq 0$ the smallest eigenvalue of $v_\sigma^2(n)$.*

Consider the family of super-connections $\mathbb{A}^\sigma(t) = \mathbb{A} + it\, v_\sigma,\ t \in \mathbb{R}$ on $\mathcal{E}$. The equivariant curvature of $\mathbb{A}^\sigma(t)$ is thus the map

$$\mathbf{F}(\sigma, \mathbb{A}, t)(X) = -t^2 v_\sigma^2 + it[\mathbb{A}, v_\sigma] + \mathbb{A}^2 + \mu^{\mathbb{A}}(X), \quad X \in \mathfrak{k} \tag{50}$$

with 0 exterior degree term equal to $-t^2 v_\sigma^2 + \mu^{\mathbb{A}}_{[0]}(X)$. As the super-connection $\mathbb{A}$ do not have 0 exterior degree term, both elements $it[\mathbb{A}, v_\sigma]$ and $\mathbb{A}^2$ are sums of terms with strictly positive exterior degrees. For example, if $\mathbb{A} = \nabla^+ \oplus \nabla^-$ is a direct sum of connections, then $it[\mathbb{A}, v_\sigma] \in \mathcal{A}^1(N, \mathrm{End}(\mathcal{E})^-)$ and $\mathbb{A}^2 \in \mathcal{A}^2(N, \mathrm{End}(\mathcal{E})^+)$.

Consider the equivariant closed form $\mathrm{Ch}(\sigma, \mathbb{A}, t)(X) := \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma, \mathbb{A}, t)(X)}\right)$.

Definition 5.5 *The Quillen Chern character form* $\mathrm{Ch}_Q(\sigma)$ *attached to the connection* $\mathbb{A}$ *is the closed equivariant form*

$$\mathrm{Ch}(\sigma, \mathbb{A}, 1)(X) := \mathrm{Str}\left(\mathrm{e}^{-v_\sigma^2 + i[\mathbb{A}, v_\sigma] + \mathbb{A}^2 + \mu^{\mathbb{A}}(X)}\right).$$

Consider the *transgression form*

$$\eta(\sigma, \mathbb{A}, t)(X) := -\,\mathrm{Str}\left(iv_\sigma\ \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right). \tag{51}$$

As $iv_\sigma = \frac{d}{dt}\mathbb{A}^\sigma(t)$, we have $\frac{d}{dt}\mathrm{Ch}(\sigma, \mathbb{A}, t) = -D(\eta(\sigma, \mathbb{A}, t))$. After integration, it gives the following equality of equivariant differential forms on N

$$\mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\sigma, \mathbb{A}, t) = D\left(\int_0^t \eta(\sigma, \mathbb{A}, s)ds\right), \tag{52}$$

since $\mathrm{Ch}(\mathbb{A}) = \mathrm{Ch}(\sigma, \mathbb{A}, 0)$.

We choose a metric on the tangent bundle to N. Thus we obtain a norm $\| - \|$ on $\wedge \mathbf{T}_n^* N \otimes \mathrm{End}(\mathcal{E}_n)$ which varies smoothly with $n \in N$.

Proposition 5.6 *Let* $\mathcal{K}_1 \times \mathcal{K}_2$ *be a compact subset of* $N \times \mathfrak{k}$.

- *There exists* $\mathrm{cst} > 0$ *such that, if* $(n, X) \in \mathcal{K}_1 \times \mathcal{K}_2$,

$$\left\| \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)} \right\|(n) \leq \mathrm{cst}\,(1+t)^{\dim N}\, \mathrm{e}^{-h_\sigma(n)t^2}, \quad \text{for all } t \geq 0. \tag{53}$$

- *The differential forms* $\mathrm{Ch}(\sigma, \mathbb{A}, t)(X)$ *and* $\eta(\sigma, \mathbb{A}, t)(X)$ *(and all their partial derivatives) tends to* 0 *exponentially fast when* $t \to \infty$ *uniformly on compact subsets of* $(N \setminus \mathrm{Supp}(\sigma)) \times \mathfrak{k}$.

Proof. To estimate $\| \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)} \|$, we employ Proposition 7.3 of the Appendix, with the variable $x = (n, X)$ and the maps $R(n, X) = v_\sigma^2(n)$, $S(n, X) = \mu^{\mathbb{A}}(n)(X)$, and $T(t, n, X) = it[\mathbb{A}, v_\sigma](n) + \mathbb{A}^2(n)$. The same estimate holds for $\|D(\partial) \cdot \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)} \|$, when $D(\partial)$ is differential operator on $N \times \mathfrak{k}$. Hence the second point follows from the fact that $\inf_{n \in \mathcal{K}_1} h_\sigma(n) > 0$ when the compact subset $\mathcal{K}_1$ lies inside $N \setminus \mathrm{Supp}(\sigma)$.

The former estimates allows us to take the limit $t \to \infty$ in (52) on the open subset $N \setminus \mathrm{Supp}(\sigma)$. We get the following important lemma (see [13, 12] for the non-equivariant case).

Lemma 5.7 *We can define on* $N \setminus \mathrm{Supp}(\sigma)$ *the equivariant differential form with smooth coefficients*

$$\beta(\sigma, \mathbb{A})(X) = \int_0^\infty \eta(\sigma, \mathbb{A}, t)(X)dt, \quad X \in \mathfrak{k}. \tag{54}$$

We have $\mathrm{Ch}(\mathbb{A})|_{N \setminus \mathrm{Supp}(\sigma)} = D\left(\beta(\sigma, \mathbb{A})\right)$.

We are in the situation of Section 3. The closed equivariant form $\mathrm{Ch}(\mathbb{A})$ on N and the equivariant form $\beta(\sigma, \mathbb{A})$ on $N \setminus \mathrm{Supp}(\sigma)$ define an even relative cohomology class $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})]$ in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$.

Proposition 5.8 • *The class* $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})] \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$ *does not depend of the choice of* $\mathbb{A}$*, nor on the Hermitian structure on* $\mathcal{E}$*. We denote it by* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *and we call it Quillen's* **relative** *equivariant Chern character.*

• *Let F be an invariant closed subset of N. For $s \in [0, 1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of equivariant smooth morphisms such that* $\mathrm{Supp}(\sigma_s) \subset F$*. Then all classes* $\mathrm{Ch}_{\mathrm{rel}}(\sigma_s)$ *coincide in* $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$.

Proof. The proof is identical to the proof of Proposition 3.8 in [12] for the non equivariant case.

5.3 Tensor product

Let $\mathcal{E}_1, \mathcal{E}_2$ be two equivariant $\mathbb{Z}_2$-graded vector bundles on N. The space $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a $\mathbb{Z}_2$-graded vector bundle with even part $\mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^-$ and odd part $\mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$.

Remark 5.9 *If E_1 and E_2 are super vector spaces, the super-algebra* $\mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$ *is identified with the super-algebra* $\mathrm{End}(E_1 \otimes E_2)$ *via the following rule. For* $v_1 \in E_1, v_2 \in E_2, A \in \mathrm{End}(E_1), B \in \mathrm{End}(E_2)$ *homogeneous*

$$(A \otimes B)(v_1 \otimes v_2) = (-1)^{|B||v_1|} Av_1 \otimes Bv_2.$$

The super-algebra $\mathcal{A}(N, \mathrm{End}(\mathcal{E}_1 \otimes \mathcal{E}_2))$ can be identified with $\mathcal{A}(N, \mathrm{End}(\mathcal{E}_1)) \otimes \mathcal{A}(N, \mathrm{End}(\mathcal{E}_2))$ where the tensor is taken in the sense of super-algebras. Then, if $A \in \mathcal{A}^0(N, \mathrm{End}(\mathcal{E}_1)^-)$ and $B \in \mathcal{A}^0(N, \mathrm{End}(\mathcal{E}_2)^-)$ are odd endomorphisms, we have $(A \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes B)^2 = A^2 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes B^2$.

Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth equivariant morphisms. With the help of K-invariant Hermitian structures, we define the morphism

$$\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \longrightarrow (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$$

by $\sigma_1 \odot \sigma_2 := \sigma_1 \otimes \mathrm{Id}_{\mathcal{E}_2^+} + \mathrm{Id}_{\mathcal{E}_1^+} \otimes \sigma_2 + \mathrm{Id}_{\mathcal{E}_1^-} \otimes \sigma_2^* + \sigma_1^* \otimes \mathrm{Id}_{\mathcal{E}_2^-}$.

Let v_{σ_1} and v_{σ_2} be the odd Hermitian endomorphisms of $\mathcal{E}_1, \mathcal{E}_2$ associated to σ_1 and σ_2 (see (49)). Then $v_{\sigma_1 \odot \sigma_2} = v_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v_{\sigma_2}$ and $v^2_{\sigma_1 \odot \sigma_2} = v^2_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v^2_{\sigma_2}$. Thus the square $v^2_{\sigma_1 \odot \sigma_2}$ is the sum of two commuting non negative Hermitian endomorphisms $v^2_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2}$ and $\mathrm{Id}_{\mathcal{E}_1} \otimes v^2_{\sigma_2}$. It follows that

$$\mathrm{Supp}(\sigma_1 \odot \sigma_2) = \mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2).$$

We can now state the main result of this section.

Theorem 5.10 (The relative Chern character is multiplicative) *Let σ_1, σ_2 be two equivariant morphisms over N. The relative equivariant cohomology classes*

- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k) \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma_k))$, $k = 1, 2$,
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$

satisfy the following equality

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$$

in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$. Here $\diamond$ is the product of relative classes (see (11)).

Proof. The proof is identical to the proof of Theorem 4.3 in [12] for the non equivariant case.

5.4 The equivariant Chern character of a morphism

Let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be an equivariant morphism on N. Following Subsection 3.5, we consider the image of $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ through the map $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma)) \to \mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, N)$: the following theorem summarizes the construction of the image.

Theorem 5.11 • *For any invariant neighborhood U of $\mathrm{Supp}(\sigma)$, take $\chi \in \mathcal{C}^\infty(N)^K$ which is equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma)$ and with support contained in U. The equivariant differential form*

$$c(\sigma, \mathbb{A}, \chi) = \chi\ \mathrm{Ch}(\mathbb{A}) + d\chi\, \beta(\sigma, \mathbb{A}) \tag{55}$$

is equivariantly closed *and supported in U. Its cohomology class $c_U(\sigma) \in \mathcal{H}^\infty_U(\mathfrak{k}, N)$ does not depend of the choice of $(\mathbb{A}, \chi)$ and the Hermitian structures on $\mathcal{E}^\pm$. Furthermore, the inverse family $c_U(\sigma)$ when U runs over the neighborhoods of $\mathrm{Supp}(\sigma)$ defines a class*

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma) \in \mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, N).$$

The image of this class in $\mathcal{H}^\infty(\mathfrak{k}, N)$ is the Chern character $\mathrm{Ch}(\mathcal{E})$ of $\mathcal{E}$.

• *Let F be an invariant closed subset of N. For $s \in [0, 1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of smooth morphisms such that $\mathrm{Supp}(\sigma_s) \subset F$. Then all classes $\mathrm{Ch}_{\mathrm{sup}}(\sigma_s)$ coincide in $\mathcal{H}^\infty_F(\mathfrak{k}, N)$.*

Definition 5.12 *When σ is elliptic, we denote by*

$$\mathrm{Ch}_c(\sigma) \in \mathcal{H}^\infty_c(\mathfrak{k}, N) \tag{56}$$

the cohomology class with compact support which is the image of $\mathrm{Ch}_{\mathrm{sup}}(\sigma) \in \mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, N)$ through the canonical map $\mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, N) \to \mathcal{H}^\infty_c(\mathfrak{k}, N)$.

A representative of $\mathrm{Ch}_c(\sigma)$ is given by $c(\sigma, \mathbb{A}, \chi)$, where $\chi \in \mathcal{C}^\infty(N)^K$ is chosen with compact support, and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma)$ and $c(\sigma, \mathbb{A}, \chi)$ is given by Formula (55).

We will now rewrite Theorem 5.10 for the equivariant Chern characters $\mathrm{Ch}_{\mathrm{sup}}$ and Ch_c. Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth equivariant morphisms on N. Let $\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \to (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$ be their product.

Following (19), the product of the elements $\mathrm{Ch}_{\mathrm{sup}}(\sigma_k) \in \mathcal{H}^\infty_{\mathrm{Supp}(\sigma_k)}(\mathfrak{k}, N)$ for $k = 1, 2$ belongs to $\mathcal{H}^\infty_{\mathrm{Supp}(\sigma_1)\cap\mathrm{Supp}(\sigma_2)}(\mathfrak{k}, N) = \mathcal{H}^\infty_{\mathrm{Supp}(\sigma_1\odot\sigma_2)}(\mathfrak{k}, N)$.

Theorem 5.13 • *We have the equality*

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma_1) \wedge \mathrm{Ch}_{\mathrm{sup}}(\sigma_2) = \mathrm{Ch}_{\mathrm{sup}}(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}^\infty_{\mathrm{Supp}(\sigma_1\odot\sigma_2)}(\mathfrak{k}, N).$$

• *If the morphisms σ_1, σ_2 are elliptic, we have*

$$\mathrm{Ch}_c(\sigma_1) \wedge \mathrm{Ch}_c(\sigma_2) = \mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}^\infty_c(\mathfrak{k}, N).$$

Proof. This follows from Theorem 5.10 and the diagram (24).

The second point of Theorem 5.13 has the following interesting refinement. Let σ_1, σ_2 be two equivariant morphisms on N which are **not elliptic**, and assume that the product $\sigma_1 \odot \sigma_2$ is **elliptic**. Since $\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$ is compact, we consider equivariant neighborhoods U_k of $\mathrm{Supp}(\sigma_k)$ such that $\overline{U_1} \cap \overline{U_2}$ is compact. Choose $\chi_k \in \mathcal{C}^\infty(N)^K$ supported on U_k and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma_k)$. Then, the equivariant differential form $c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2)$ is compactly supported on N, and we have

$$\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) = c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2) \quad \text{in} \quad \mathcal{H}^\infty_c(\mathfrak{k}, N).$$

Note that the equivariant differential forms $c(\sigma_k, \mathbb{A}_k, \chi_k)$ are not compactly supported.

5.5 Retarded construction

We have defined a representative of the Chern characters $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ and $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ using the one-parameter family $\mathbb{A}^\sigma(t)$ of super-connections, for t varying between 0 and ∞. Quillen's Chern character $\mathrm{Ch}_Q(\sigma) = \mathrm{Ch}(\sigma, \mathbb{A}, 1)$ is another representative. We will compare them in appropriate cohomology spaces in the next section.

Consider any $T \in \mathbb{R}$. We have $\mathrm{Ch}(\sigma, \mathbb{A}, T) = D(\beta(\sigma, \mathbb{A}, T))$ with $\beta(\sigma, \mathbb{A}, T) = \int_T^\infty \eta(\sigma, \mathbb{A}, t)dt$. It is easy to check that the following equality

$$\Big(\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})\Big) - \Big(\mathrm{Ch}(\sigma, \mathbb{A}, T), \beta(\sigma, \mathbb{A}, T)\Big) = D_{\mathrm{rel}}\left(\int_0^T \eta(\sigma, \mathbb{A}, t)dt\,,\, 0\right) \tag{57}$$

holds in $\mathcal{A}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$. Hence we get the following

Lemma 5.14 *For any $T \in \mathbb{R}$, the relative Chern character* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *satisfies*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma) = \Big[\mathrm{Ch}(\sigma, \mathbb{A}, T), \beta(\sigma, \mathbb{A}, T)\Big] \quad \text{in} \quad \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma)).$$

Using Lemma 5.14, we get

Lemma 5.15 *For any $T \geq 0$, the class* $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$ *can be defined with the forms* $c(\sigma, \mathbb{A}, \chi, T) = \chi\ \mathrm{Ch}(\sigma, \mathbb{A}, T) + d\chi\, \beta(\sigma, \mathbb{A}, T)$.

Proof. It is due to the following transgression

$$c(\sigma, \mathbb{A}, \chi) - c(\sigma, \mathbb{A}, \chi, T) = D(\chi \int_0^T \eta(\sigma, \mathbb{A}, t)dt), \tag{58}$$

which follows from (57).

In some situations the Quillen's Chern form $\mathrm{Ch}_Q(\sigma) = \mathrm{Ch}(\sigma, \mathbb{A}, 1)$ enjoys good properties relative to the integration. So it is natural to compare the equivariant differential form $c(\sigma, \mathbb{A}, \chi)$ and $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$.

Lemma 5.16 *We have*

$$c(\sigma, \mathbb{A}, \chi) - \mathrm{Ch}_Q(\sigma) = D\left(\chi \int_0^1 \eta(\sigma, \mathbb{A}, s)ds\right) + D\Big((\chi - 1)\beta(\sigma, \mathbb{A}, 1)\Big).$$

Proof. This follows immediately from the transgressions (52) and (58).

5.6 Quillen Chern character with Gaussian look

As we have seen, Mathai-Quillen gives an explicit representative with "Gaussian look" of the Thom class of a Euclidean vector bundle $\mathcal{V} \to M$. Similarly, they give an explicit representative with "Gaussian look" of the Bott class of a complex equivariant vector bundle $\mathcal{V} \to M$. The purpose of this paragraph is to compare the Mathai-Quillen construction of Chern characters with "Gaussian look" and the relative construction.

Let $\mathcal{V}$ be a real K-equivariant vector bundle over a manifold M. We denote by $p : \mathcal{V} \to M$ the projection. We denote by (x, ξ) a point of $\mathcal{V}$ with $x \in M$ and $\xi \in \mathcal{V}_x$. Let $\mathcal{E}^{\pm} \to M$ be two K-equivariant Hermitian vector bundles. We consider a K-equivariant morphism $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ on $\mathcal{V}$.

We choose a metric on the fibers of the fibration $\mathcal{V} \to M$. We work under the following assumption on σ.

Assumption 5.17 *The morphism $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ and all its partial derivatives have at most a polynomial growth along the fibers of $\mathcal{V} \to M$. Moreover we assume that, for any compact subset $\mathcal{K}$ of M, there exist $R \geq 0$ and $c > 0$ such that*[1] *$v_\sigma^2(x, \xi) \geq c\|\xi\|^2$ when $\|\xi\| \geq R$ and $x \in \mathcal{K}$.*

[1] This inequality means that $\|\sigma(x, \xi)w\|^2 \geq c\|\xi\|^2\|w\|^2$ for any $w \in \mathcal{E}_x$.

Let $\nabla = \nabla^+ \oplus \nabla^-$ be a K-invariant connection on $\mathcal{E} \to M$, and consider the super-connection $\mathbb{A} = p^*\nabla$ so that $\mathbb{A}^\sigma(t) = p^*\nabla + itv_\sigma$. Then, the form $\mathrm{Ch}(\sigma, \mathbb{A}, 1)(X)$ has a "Gaussian" look.

Lemma 5.18 *The equivariant differential forms* $\mathrm{Ch}(\sigma, \mathbb{A}, 1)(X)$ *and* $\beta(\sigma, \mathbb{A}, 1)(X)$ *are rapidly decreasing along the fibers.*

Proof. The equivariant curvature of $\mathbb{A}^\sigma(t)$ is

$$\mathbf{F}(t)(X) = p^*\mathbf{F}(X) - t^2 v_\sigma^2 + it[p^*\nabla, v_\sigma].$$

Here $\mathbf{F} \in \mathcal{A}^2(M, \mathrm{End}(\mathcal{E}))$ is the equivariant curvature of ∇.

To estimate $\mathrm{e}^{\mathbf{F}(t)}(X)$, we apply Lemma 7.2 of the Appendix, with $H = t^2 v_\sigma^2$, and $R = -p^*\mathbf{F} - it[p^*\nabla, v_\sigma]$, $S = \mu^{\mathbb{A}}(X)$. The proof is very similar to the proof of Lemma 5.17 in [12] for the non equivariant case and we skip it.

Theorem 5.19 *Quillen's Chern character form* $\mathrm{Ch}_Q(\sigma) \in \mathcal{A}^\infty_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V})$ *represents the image of the class* $\mathrm{Ch}_{\sup}(\sigma) \in \mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, \mathcal{V})$ *in* $\mathcal{H}^\infty_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V})$.

Proof. The proof is entirely analogous to Proposition 5.18 in [12] and we skip it.

5.7 A simple example

Thus there are three useful representations of the equivariant Chern character of a morphism σ: the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma) \in \mathcal{H}^\infty(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus \mathrm{Supp}(\sigma))$, the Chern character with support $\mathrm{Ch}_{\sup}(\sigma) \in \mathcal{H}^\infty_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, \mathcal{V})$ and (in the case of vector bundles) the Chern character with Gaussian look. We will describe these explicit representatives in the three cohomology spaces in a very simple example.

Recall the following convention. Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}_2$-graded finite dimensional complex vector space and $\mathcal{A}$ a super-commutative ring (the ring of differential forms on a manifold for example). Consider the ring $\mathrm{End}(V) \otimes \mathcal{A}$. Let $\{e_i\}_{i=1}^{\dim V^+}$ be a basis of V^+ and $\{f_j\}_{j=1}^{\dim V^-}$ a basis of V^-. Consider the odd endomorphism $M^i_j : V \to V$ such that $M^i_j(e_i) = f_j$, and sending all other basis elements of V to 0. Similarly, consider the odd endomorphism $R^j_i : V \to V$ such that $R^j_i(f_j) = e_i$, and sending all other basis elements of V to 0.

Convention 5.20 *Let* $a \in \mathcal{A}$. *The matrix written with* a *in column* i *and row* j, *and* 0 *for all other entries, represents* $M^i_j \otimes a$ *in the ring End*$(V) \otimes \mathcal{A}$. *The matrix written with* a *in column* $\dim V^+ + j$ *and row* i, *and* 0 *for all other entries, represents* $R^j_i \otimes a$ *in the ring End*$(V) \otimes \mathcal{A}$.

Let $U(1) = K$ be the circle group. We identify the Lie algebra $\mathfrak{u}(1)$ of $U(1)$ with $\mathbb{R}$ so that the exponential map is $\theta \mapsto \mathrm{e}^{i\theta}$.

Consider the case where $\mathcal{V} = \mathbb{R}^2 \simeq \mathbb{C}$ and $K = U(1)$ acts by rotation: $t \cdot z = tz$ for $z \in \mathbb{C}$ and $t \in U(1)$. Take $E^+ = \mathcal{V} \times \wedge^0 \mathcal{V}$ and $E^- = \mathcal{V} \times \wedge^1 \mathcal{V}$. The action of $U(1)$ on $E^+ \equiv \mathbb{C} \times \mathbb{C}$ is $t \cdot [z, u] = [tz, u]$. The action of $U(1)$ on $E^- \equiv \mathbb{C} \times \mathbb{C}$ is $t \cdot [z, u] = [tz, tu]$.

We consider the Bott symbol $\sigma_b(z) = z$ which produces the map $\sigma_b([z, u]) = [z, zu]$ from E^+ to E^-. Then, the bundle map σ_b commutes with the action of $U(1)$ and defines an element of $\mathbf{K}^0_{U(1)}(\mathbb{R}^2)$. Recall that the Bott isomorphism tell us that $\mathbf{K}^0_{U(1)}(\mathbb{R}^2)$ is a free module over $\mathbf{K}^0_{U(1)}(\mathrm{pt}) = R(U(1))$ with base σ_b.

We choose on $E^\pm$ the trivial connections $\nabla^+ = \nabla^- = d$. Let $\mathbb{A} = \nabla^+ \oplus \nabla^-$. The moment of $\mathbb{A}$ is the map

$$\mu^{\mathbb{A}}(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & i\theta \end{pmatrix}.$$

The equivariant curvature $\mathbf{F}(\theta)$ of $\mathbb{A}$ is equal to $\mu^{\mathbb{A}}(\theta)$, thus we have the following formula for the equivariant Chern character

$$\mathrm{Ch}(\mathbb{A})(\theta) = 1 - \mathrm{e}^{i\theta}\,. \tag{59}$$

With the conventions of 5.20, the equivariant curvature $\mathbf{F}(t)$ of the superconnection $\mathbb{A}^{\sigma_b}(t) := \mathbb{A} + itv_{\sigma_b}$ is written in matrix form as

$$\mathbf{F}(t)(\theta) = \begin{pmatrix} -t^2|z|^2 & 0 \\ 0 & -t^2|z|^2 \end{pmatrix} + \begin{pmatrix} 0 & -itd\overline{z} \\ -itdz & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & i\theta \end{pmatrix},$$

for $\theta \in \mathfrak{u}(1) \simeq \mathbb{R}$. We compute $\mathrm{e}^{\mathbf{F}(t)(\theta)}$ using Volterra's formula. We obtain

$$\mathrm{e}^{\mathbf{F}(t)(\theta)} = \mathrm{e}^{-t^2|z|^2} \begin{pmatrix} 1 + (g'(i\theta) - g(i\theta))t^2 dz d\overline{z} & itg(i\theta)d\overline{z} \\ itg(i\theta)dz & \mathrm{e}^{i\theta} + g'(i\theta)t^2 dz d\overline{z} \end{pmatrix}$$

where $g(i\theta) = \frac{\mathrm{e}^{i\theta} - 1}{i\theta}$. Hence $\mathrm{Str}(\mathrm{e}^{\mathbf{F}(t)(\theta)}) = -g(i\theta)(i\theta + t^2 dz d\overline{z})\,\mathrm{e}^{-t^2|z|^2}$. Here

$$\begin{aligned} \eta(\sigma_b, \mathbb{A}, t)(\theta) &= -i\,\mathrm{Str}\left(\begin{pmatrix} 0 & \overline{z} \\ z & 0 \end{pmatrix} \mathrm{e}^{\mathbf{F}(t)(\theta)} \right) \\ &= g(i\theta)(z d\overline{z} - \overline{z} dz)\, t\, \mathrm{e}^{-t^2|z|^2}\,. \end{aligned}$$

For $z \neq 0$, we can integrate $t \mapsto \eta(\sigma_b, \mathbb{A}, t)(\theta)$ from 0 to ∞ and we obtain

$$\beta(\sigma_b, \mathbb{A})(\theta) = g(i\theta) \frac{z\, d\overline{z} - \overline{z}\, dz}{2|z|^2}.$$

Take $f \in \mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. Let $\chi(z) := f(|z|^2)$.

Similarly to the Thom form, we can give formulae for the three different representatives of the Chern character.

Proposition 5.21 • *The class* $\mathrm{Ch}_{\mathrm{rel}}(\sigma_b) \in \mathcal{H}^\infty(\mathfrak{u}(1), \mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$ *is represented by the couple of equivariant differential forms:*

$$\left(1 - \mathrm{e}^{i\theta}, \frac{\mathrm{e}^{i\theta} - 1}{i\theta} \frac{z\, d\overline{z} - \overline{z}\, dz}{2|z|^2}\right).$$

• *The Chern character with compact support is represented by the equivariant differential form*

$$\begin{aligned} c(\sigma_b, \mathbb{A}, \chi) &= \chi(1 - \mathrm{e}^{i\theta}) + d\chi\beta(\sigma_b, \mathbb{A}) \\ &= \frac{\mathrm{e}^{i\theta} - 1}{i\theta} \left(-f(|z|^2) i\theta + f'(|z|^2) dz \wedge d\overline{z}\right). \end{aligned}$$

• *Quillen's Chern character with Gaussian look* $\mathrm{Ch}_Q(\sigma_b)$ *is represented by the equivariant differential form*

$$\frac{\mathrm{e}^{i\theta} - 1}{i\theta} \mathrm{e}^{-\|z\|^2} (i\theta + dz \wedge d\overline{z}).$$

Comparing with Example 4.12, we see that the Chern character form in all of these different versions is proportional to the Thom form

$$\mathrm{Ch}(\sigma_b)(\theta) = (2i\pi) \frac{\mathrm{e}^{i\theta} - 1}{i\theta} \mathrm{Th}(V)(\theta).$$

We will see in the next section that this identity generalizes to any Euclidean vector spaces.

6 Comparison between relative Thom classes and Bott classes:Riemann Roch formula

Let $p : \mathcal{V} \to M$ be a K-equivariant Euclidean vector bundle over M of *even rank*. Here we compare the relative Chern character of the Bott symbol and the relative Thom class. Both classes live in the relative equivariant cohomology space $\mathcal{H}^\infty(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$. The formulae relating them is an important step in the proof of the Grothendieck-Riemann-Roch relative theorem [5], as well as the Atiyah-Singer theorem [1, 2]. As usual, the relation is deduced from an explicit computation in equivariant cohomology of a vector space.

6.1 The relative equivariant Bott class of a vector space

We consider first the case of an oriented Euclidean vector space V of dimension $d = 2n$. Let

$$\mathbf{c} : \mathrm{Cl}(V) \to \mathrm{End}_{\mathbb{C}}(\mathcal{S})$$

be the spinor representation of the Clifford algebra of V. We use the conventions of [4][chapter 3] for the spinor representation. In particular, as a vector space, the Clifford algebra is identified with the real exterior algebra of V. The orientation of V gives a decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ which is is stable under the action of the group $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)$.

We consider the $\mathrm{Spin}(V)$-equivariant vector bundles $\mathcal{S}_V^{\pm} := V \times \mathcal{S}^{\pm}$ over V : recall that the action of $\mathrm{Spin}(V)$ on the base V is through the twofold covering $\tau : \mathrm{Spin}(V) \to \mathrm{SO}(V)$.

The Clifford module $\mathcal{S}$ is provided with an Hermitian inner product such that $\mathbf{c}(x)^* = -\mathbf{c}(x)$, for $x \in V$. We work with the equivariant morphism $\sigma_V : \mathcal{S}_V^+ \to \mathcal{S}_V^-$ defined by: for $x \in V$,

$$\sigma_V(x) := -i\mathbf{c}(x) : \mathcal{S}^+ \longrightarrow \mathcal{S}^-.$$

Then the odd linear map $v_\sigma(x) : \mathcal{S} \to \mathcal{S}$ is equal to $-i\mathbf{c}(x)$. We choose on $\mathcal{S}_V^{\pm}$ the trivial connections $\nabla^+ = \nabla^- = d$. Thus the super-connection $A_t := \mathbb{A} + itv_\sigma$ is

$$\mathbb{A}_t = d + t\mathbf{c}(x).$$

The Lie algebra $\mathfrak{spin}(V)$ of $\mathrm{Spin}(V)$ is identified with the Lie sub-algebra $\wedge^2 V$ of $\mathrm{Cl}(V)$ and the exponential map is the exponential inside the Clifford algebra. The differential of the action of $\mathrm{Spin}(V)$ in $\mathcal{S}$ is $Y \mapsto \mathbf{c}(Y)$. We denote by $Y \in \mathfrak{spin}(V) \mapsto Y^\tau \in \mathfrak{so}(V)$ the differential of the homomorphism τ (it is an isomorphism). We need the function $\mathrm{j}_V^{1/2} : \mathfrak{so}(V) \to \mathbb{R}$ defined by

$$\mathrm{j}_V^{1/2}(X) = \det{}^{1/2}\left(\frac{\mathrm{e}^{X/2} - \mathrm{e}^{-X/2}}{X/2}\right).$$

Then $\mathrm{j}_V^{1/2}(X)$ is invertible near $X = 0$ as $\mathrm{j}_V^{1/2}(0) = 1$.

For $Y \in \mathfrak{spin}(V)$, the moment $\mu(Y) = \mathcal{L}(Y) - [\mathbb{A}_t, \iota(VY)]$ is equal to $\mathbf{c}(Y)$. Hence the equivariant curvature of the super-connection $\mathbb{A}_t$ is the function $\mathbf{F}_t : \mathfrak{spin}(V) \to \mathcal{A}(V, \mathrm{End}_{\mathbb{C}}(\mathcal{S}))$, given by

$$\mathbf{F}_t(Y) = -t^2\|x\|^2 + t\sum_k dx_k \mathbf{c}_k + \sum_{k<l} Y_{kl}\mathbf{c}_k\mathbf{c}_l. \tag{60}$$

Here $Y = \sum_{k<l} Y_{kl} e_k \wedge e_l \in \mathfrak{spin}(V)$ and $\mathbf{c}(Y) = \sum_{k<l} Y_{kl}\mathbf{c}_k\mathbf{c}_l$: we have denoted by $\mathbf{c}_k$ the odd endomorphism of $\mathcal{S}$ produced by $e_k \in V \subset \mathrm{Cl}(V)$.

This formula is very similar to the form f_t that we used to construct the Thom form, see Subsection 4.1. For $Y \in \mathfrak{spin}(V)$, we write

$$\begin{aligned} f_t(Y^\tau) &= -t^2\|x\|^2 + t\sum_k dx_k e_k + \frac{1}{2}\sum_{k<l} Y_{kl}^\tau\ e_k \wedge e_l \\ &= -t^2\|x\|^2 + t\sum_k dx_k e_k + \sum_{k<l} Y_{kl}\ e_k \wedge e_l. \end{aligned}$$

In order to compute the relative Chern character of σ we follow the strategy of [4][Section 7.7]. However, our convention for the Chern character $\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}(\mathrm{e}^{\mathbb{A}^2})$ is different from the one of [4], which decided that $\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}(\mathrm{e}^{-\mathbb{A}^2})$. Thus we carefully check signs in our formulae.

We consider in parallel the closed equivariant forms on V

$$\mathrm{Ch}(\mathbb{A}_t)(Y) = \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}_t(Y)}\right) \quad , \quad \mathrm{C}^t_\wedge(Y^\tau) := \mathrm{T}\left(\mathrm{e}^{f_t(Y^\tau)}\right), \quad Y \in \mathfrak{spin}(V).$$

In the first case, the exponential is computed in the super-algebra $\mathrm{End}(\mathcal{S}) \otimes \mathcal{A}(V)$ and in the second case, the exponential is computed in the super-algebra $\mathcal{A}(V) \otimes \wedge V$.

We also consider in parallel the equivariant forms

$$\eta^t_{\mathbf{c}}(Y) = -\,\mathrm{Str}\left((\sum_k x_k \mathbf{c}_k)\,\mathrm{e}^{\mathbf{F}_t(Y)}\right) \quad , \quad \eta^t_\wedge(Y^\tau) := -\,\mathrm{T}\left((\sum_k x_k e_k)\,\mathrm{e}^{f_t(Y^\tau)}\right).$$

Proposition 6.1 *Let $Y \in \mathfrak{spin}(V)$. We have*

$$\mathrm{Ch}(\mathbb{A}_t)(Y) = (-2i)^n\ \mathrm{j}_V^{1/2}(Y^\tau)\ \mathrm{C}^t_\wedge(Y^\tau), \quad \eta^t_{\mathbf{c}}(Y) = (-2i)^n\ \mathrm{j}_V^{1/2}(Y^\tau)\ \eta^t_\wedge(Y^\tau).$$

Proof. The first identity is proved in [4][Section 7.7] with other conventions. For clarity we perform the computations. We fix $Y \in \mathfrak{spin}(V) \simeq \wedge^2 V$ and we take an *oriented* orthonormal base (e_i) of V such that $Y = \sum_{k=1}^n \lambda_k e_{2k-1} \wedge\ e_{2k}$. Hence $\mathrm{j}_V^{1/2}(Y^\tau) = \Pi_k \frac{\sin\lambda_k}{\lambda_k}$. For $1 \leq k \leq n$, let

$$\begin{aligned} B_k &= t(dx_{2k-1}\mathbf{c}_{2k-1} + dx_{2k}\mathbf{c}_{2k}) + \lambda_k \mathbf{c}_{2k-1}\mathbf{c}_{2k} \\ b_k &= t(dx_{2k-1}e_{2k-1} + dx_{2k}e_{2k}) + \lambda_k e_{2k-1} \wedge e_{2k}. \end{aligned}$$

Since $B_iB_j = B_jB_i$ and $b_ib_j = b_jb_i$, we have $\mathrm{e}^{\mathbf{F}_t(Y)} = \mathrm{e}^{-t^2\|x\|^2}\,\Pi_k\,\mathrm{e}^{B_k}$ and $\mathrm{e}^{f_t(Y^\tau)} = \mathrm{e}^{-t^2\|x\|^2}\,\Pi_k\,\mathrm{e}^{b_k}$.

Lemma 6.2 *We have*

$$\begin{aligned} \mathrm{e}^{B_k} \quad &= \quad \cos\lambda_k + t^2\left(\frac{\sin\lambda_k - \lambda_k\cos\lambda_k}{\lambda_k^2}\right)dx_{2k-1}dx_{2k} + \\ &\quad \frac{\sin\lambda_k}{\lambda_k}(\lambda_k - t^2 dx_{2k-1}dx_{2k})\mathbf{c}_{2k-1}\mathbf{c}_{2k} + t\frac{\sin\lambda_k}{\lambda_k}(dx_{2k-1}\mathbf{c}_{2k-1} + dx_{2k}\mathbf{c}_{2k}) \end{aligned}$$

in $\mathcal{A}(V, \mathrm{End}_{\mathbb{C}}(\mathcal{S}))$, and

$$\mathrm{e}^{b_k} = 1 + (\lambda_k - t^2 dx_{2k-1}dx_{2k})e_{2k-1} \wedge e_{2k} + t(dx_{2k-1}e_{2k-1} + dx_{2k}e_{2k})$$

in $\mathcal{A}(V) \otimes \wedge V$.

Proof. The identity for e^{b_k} is obvious since $(b_k)^i = 0$ when $i > 2$. When $\lambda_k = 0$ the identity for e^{B_k} can be proved directly since $(dx_{2k-1}\mathbf{c}_{2k-1} + dx_{2k}\mathbf{c}_{2k})^i = 0$ for $i > 2$. When $\lambda_k \neq 0$ we can write

$$B_k = \lambda_k(\mathbf{c}_{2k-1} + t\lambda_k^{-1}dx_{2k})(\mathbf{c}_{2k} - t\lambda_k^{-1}dx_{2k-1}) - t^2\lambda_k^{-1}dx_{2k-1}dx_{2k}.$$

If we let $\xi_1 = \mathbf{c}_{2k-1} + t\lambda_k^{-1} dx_{2k}$ and $\xi_2 = \mathbf{c}_{2k} - t\lambda_k^{-1} dx_{2k-1}$, then $\xi_1^2 = \xi_2^2 = -1$, $\xi_1\xi_2 + \xi_2\xi_1 = 0$ and the ξ_i commute with $dx_{2k-1}dx_{2k}$. Thus, we see that

$$\begin{aligned}
\mathrm{e}^{B_k} &= \mathrm{e}^{\lambda_k \xi_1 \xi_2}\, \mathrm{e}^{-t^2\lambda_k^{-1} dx_{2k-1}dx_{2k}} \\
&= (\cos\lambda_k + \sin\lambda\xi_1\xi_2)\left(1 - t^2\lambda_k^{-1} dx_{2k-1}dx_{2k}\right) \\
&= \Big(\cos\lambda_k + \sin\lambda(\mathbf{c}_{2k-1} + t\lambda_k^{-1}dx_{2k})(\mathbf{c}_{2k} - t\lambda_k^{-1}dx_{2k-1})\Big) \times \\
&\quad \Big(1 - t^2\lambda_k^{-1}dx_{2k-1}dx_{2k}\Big).
\end{aligned}$$

Since $\mathrm{Str}(\mathbf{c}_1 \cdots \mathbf{c}_{2n}) = (-2i)^n$ and $\mathrm{Str}(\mathbf{c}_{i_1} \cdots \mathbf{c}_{i_l}) = 0$ for $l < 2n$, Lemma 6.2 gives that $\mathrm{Ch}(\mathbb{A}_t)(Y) = (-2i)^n \left(\mathrm{j}_V^{1/2} \mathrm{C}_\wedge^t\right)(Y^\tau)$ where

$$\mathrm{C}_\wedge^t(Y^\tau) = \mathrm{e}^{-t^2\|x\|^2}\, \Pi_{k=1}^n (\lambda_k - t^2 dx_{2k-1}dx_{2k}).$$

We found also that $\eta_{\mathbf{c}}^t(Y) = (-2i)^n \left(\mathrm{j}_V^{1/2}\eta_\wedge^t\right)(Y^\tau)$ with

$$\eta_\wedge^t(Y^\tau) = -t\,\mathrm{e}^{-t^2\|x\|^2} \sum_{k=1}^n \Pi_{i\neq k}(\lambda_i - t^2 dx_{2i-1}dx_{2i})(x_{2k}dx_{2k-1} - x_{2k-1}dx_{2k}).$$

The difference of the Chern character of the bundles $\mathcal{S}_V^\pm$ with trivial connection d is $\mathrm{Ch}(\mathcal{S}_V^+)(Y) - \mathrm{Ch}(\mathcal{S}_V^+)(Y) = \mathrm{Ch}(\mathbb{A}_0)(Y)$. By the preceding calculation, we obtain

$$\mathrm{Ch}(\mathbb{A}_0)(Y) = (-2i)^n\, \mathrm{j}_V^{1/2}(Y^\tau)\, \mathrm{Pf}\left(\tfrac{Y^\tau}{2}\right).$$

To compute the relative Chern character of the morphism σ_V we need the $\mathrm{Spin}(V)$-equivariant form

$$\beta_{\mathbf{c}}(Y) = \int_0^\infty \eta_{\mathbf{c}}^t(Y)dt$$

which is defined on $V \times \{0\}$. By Proposition 6.1, we obtain

Lemma 6.3 *The* $\mathrm{Spin}(V)$*-equivariant form* $\beta_{\mathbf{c}}(Y)$ *and* $\beta_\wedge(Y^\tau) = \int_0^\infty \eta_\wedge^t(Y^\tau)dt$ *are related on* $V \times \{0\}$ *by*

$$\beta_{\mathbf{c}}(Y) = (-2i)^n \mathrm{j}_V^{1/2}(Y^\tau)\beta_\wedge(Y^\tau), \quad Y \in \mathfrak{spin}(V).$$

We then obtain the following comparison between the Thom classes and the Chern characters of the symbol σ_V.

Theorem 6.4 • *We have the following equality in* $\mathcal{H}^\infty(\mathfrak{spin}(V), V, V\setminus\{0\})$

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_V)(Y) = (2i\pi)^n\ \mathrm{j}_V^{1/2}(Y^\tau)\ \mathrm{Th}_{\mathrm{rel}}(V)(Y^\tau). \tag{61}$$

• *We have the following equality in* $\mathcal{H}_c^\infty(\mathfrak{spin}(V), V)$

$$\mathrm{Ch}_{\mathrm{c}}(\sigma_V)(Y) = (2i\pi)^n\ \mathrm{j}_V^{1/2}(Y^\tau)\ \mathrm{Th}_{\mathrm{c}}(V)(Y^\tau). \tag{62}$$

• *We have the following equality in* $\mathcal{H}_{\text{dec-rap}}^\infty(\mathfrak{spin}(V), V)$

$$\mathrm{Ch}_Q(\sigma_V)(Y) = (2i\pi)^n\ \mathrm{j}_V^{1/2}(Y^\tau)\ \mathrm{Th}_{\mathrm{MQ}}(Y^\tau). \tag{63}$$

6.2 The Bott class of a vector bundle

Let M be a manifold equipped with the action of a compact Lie group K. Let $p : \mathcal{V} \to M$ be an oriented K-vector bundle of even rank $2n$ over M.

6.2.1 The Spin case

We assume that $\mathcal{V}$ has a K-equivariant spin structure. Thus $\mathcal{V}$ is associated to a K-equivariant principal bundle $P \to M$ with structure group $\tilde{G} = \mathrm{Spin}(V)$. We denote by $\tilde{\mathfrak{g}}$ the Lie algebra $\mathfrak{spin}(V)$.

Let ω be a K-invariant connection one form on P, with curvature form Ω. For $X \in \mathfrak{k}$, we denote by $\mu(X) = -\iota(VX)\omega \in \mathcal{C}^\infty(P) \otimes \tilde{\mathfrak{g}}$ the moment of X. The equivariant curvature form is $\Omega(X) = \Omega + \mu(X)$, $X \in \mathfrak{k}$. For any $\tilde{G}$-manifold Z, we consider the Chern-Weil homomorphisms

$$\phi_\omega^Z : \mathcal{A}^\infty(\tilde{\mathfrak{g}}, Z) \longrightarrow \mathcal{A}^\infty(\mathfrak{k}, \mathcal{Z})$$

where $\mathcal{Z} = P \times_{\tilde{G}} Z$.

Let $\mathcal{S}_M = P \times_{\tilde{G}} \mathcal{S}$ be the corresponding K-equivariant spinor bundle over M. Let $p^*\mathcal{S}_M \to \mathcal{V}$ be the pull-back of $\mathcal{S}_M$ to $\mathcal{V}$ by the projection $p : \mathcal{V} \to M$. We have $p^*\mathcal{S}_M = P \times_{\tilde{G}} \mathcal{S}_V$ where $\mathcal{S}_V = V \times \mathcal{S}$ is the trivial bundle over V.

Let $\nabla^{\mathcal{S}_M}$ be the connection on $\mathcal{S}_M \to M$ induced by the connection form ω : $\nabla^{\mathcal{S}_M} := d + \mathbf{c}(\omega)$ where $\mathbf{c}$ is the representation of $\mathrm{Spin}(V)$ on $\mathcal{S}$.

Let $\mathbf{x} : \mathcal{V} \to p^*\mathcal{V}$ be the canonical section. We consider now the K-equivariant morphism on $\mathcal{V}$, $\sigma_{\mathcal{V}} : p^*\mathcal{S}_M^+ \longrightarrow p^*\mathcal{S}_M^-$, defined by

$$\sigma_{\mathcal{V}} = -i\,\mathbf{c}(\mathbf{x}).$$

We consider the family of super-connections on $p^*\mathcal{S}_M$ defined by $\mathbb{A}_t^{\sigma_{\mathcal{V}}} = p^*\nabla^{\mathcal{S}_M} + tc(\mathbf{x})$. In the previous section we worked with a family $\mathbb{A}_t$ of super-connections on the trivial bundle $\mathcal{S} \times V \to V$. Let $\mathrm{Ch}(\sigma_{\mathcal{V}}, \mathbb{A}, t) \in \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V})$ and $\mathrm{Ch}(\mathbb{A}_t) \in \mathcal{A}^\infty(\tilde{\mathfrak{g}}, V)$ be the corresponding Chern forms. Let $\eta(\sigma_{\mathcal{V}}, \mathbb{A}, t) \in \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V})$ and $\eta^t \in \mathcal{A}^\infty(\tilde{\mathfrak{g}}, V)$ be the corresponding transgression forms.

Lemma 6.5 *We have the following equalities:*

$$\phi_\omega\left(\mathrm{Ch}(\mathbb{A}_t)\right) = \mathrm{Ch}(\sigma_{\mathcal{V}}, \mathbb{A}, t) \quad \text{in} \quad \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V}),$$

and

$$\phi_\omega\left(\eta^t\right) = \eta(\sigma_{\mathcal{V}}, \mathbb{A}, t) \quad \text{in} \quad \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V} \setminus M).$$

Proof. See [4], Section7.7.

Let $\nabla^{\mathcal{V}}$ be the connection on $\mathcal{V} \to M$ induced by the connection form ω : $\nabla^{\mathcal{V}} = d + \tau(\omega)$ where $\tau : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is the double cover. Let $\mathbf{F}^{\mathcal{V}}(X) = (\nabla^{\mathcal{V}} - \iota(VX))^2 + \mathcal{L}(X)$, $X \in \mathfrak{k}$ be the equivariant curvature of $\nabla^{\mathcal{V}}$.

Definition 6.6 *We associate to the K-equivariant (real) vector bundle $\mathcal{V} \to M$ the closed K-equivariant form on M defined by*

$$\mathrm{j}^{1/2}(\nabla^{\mathcal{V}})(X) := \det{}^{1/2}\left(\frac{\mathrm{e}^{\frac{\mathbf{F}^{\mathcal{V}}(X)}{2}} - \mathrm{e}^{-\frac{\mathbf{F}^{\mathcal{V}}(X)}{2}}}{\mathbf{F}^{\mathcal{V}}(X)}\right), \quad X \in \mathfrak{k}.$$

We denote $\mathrm{j}^{1/2}(\mathcal{V})(X)$ its cohomology class in $\mathcal{H}^\infty(\mathfrak{k}, M)$. It is an invertible class near $X = 0$ and its inverse $\mathrm{j}^{1/2}(\mathcal{V})(X)^{-1}$ is the equivariant $\hat{A}$-genus of $\mathcal{V}$.

It is easy to see that the image of the invariant polynomial $Y \mapsto \mathrm{j}_V^{1/2}(Y^\tau)$ by the Chern-Weil homomorphism ϕ_ω is equal to $\mathrm{j}^{1/2}(\nabla^{\mathcal{V}})$ (see [4], Section7.7).

We consider the sub-space $\mathcal{A}^\infty_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V}) \subset \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V})$ of K-equivariant forms on $\mathcal{V}$ which have a compact support in the fibers of $p : \mathcal{V} \to M$. Let $\mathcal{H}^\infty_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V})$ be the corresponding cohomology space. The Chern-Weil homomorphism $\phi_\omega : \mathcal{A}^\infty(\tilde{\mathfrak{g}}, V) \to \mathcal{A}^\infty(\mathfrak{k}, \mathcal{V})$ maps the sub-space $\mathcal{A}^\infty_c(\tilde{\mathfrak{g}}, V)$ into $\mathcal{A}^\infty_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V})$.

Consider now the equivariant morphism $\sigma_{\mathcal{V}}$ on $\mathcal{V}$. The support $\sigma_{\mathcal{V}}$ is equal to M, hence its relative Chern Character $\mathrm{Ch}_{\text{rel}}(\sigma_{\mathcal{V}})$ belongs to $\mathcal{H}^\infty(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M)$.

If we take the image of the equalities of Theorem 6.4 by the Chern-Weil homomorphism ϕ_ω we obtain the following

Proposition 6.7 *We have the equalities:*

$$\mathrm{Ch}_{\text{rel}}(\sigma_{\mathcal{V}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\right) \mathrm{Th}_{\text{rel}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M).$$

$$\mathrm{Ch}_{\text{fiber cpt}}(\sigma_{\mathcal{V}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\right) \mathrm{Th}_{\text{fiber cpt}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V}).$$

$$\mathrm{Ch}_Q(\sigma_{\mathcal{V}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\right) \mathrm{Th}_{\text{MQ}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V}).$$

6.2.2 The $\mathrm{Spin}^{\mathbb{C}}$ case

We assume here that the vector bundle $p : \mathcal{V} \to M$ has a K-equivariant $\mathrm{Spin}^{\mathrm{c}}$ structure. Thus $\mathcal{V}$ is associated to a K-equivariant principal bundle $P^{\mathrm{c}} \to M$ with structure group $G^{\mathrm{c}} := \mathrm{Spin}^{\mathrm{c}}(V)$.

Let $U(1) := \{\mathrm{e}^{i\theta}\}$ be the circle group with Lie algebra of $\mathfrak{u}(1) \sim \mathbb{R}$. The group $\mathrm{Spin}^{\mathrm{c}}(V)$ is the quotient $\mathrm{Spin}(V) \times_{\mathbb{Z}_2} U(1)$, where $\mathbb{Z}_2$ acts by $(-1, -1)$. There are two canonical group homomorphisms

$$\tau : \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V) \quad , \quad \mathrm{Det} : \mathrm{Spin}^{\mathrm{c}}(V) \to U(1)$$

such that $\tau^{\mathrm{c}} = (\tau, \mathrm{Det}) : \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V) \times U(1)$ is a double covering map.

Definition 6.8 *The K-equivariant line bundle $\mathbb{L}_{\mathcal{V}} := P^{\mathrm{c}} \times_{\mathrm{Det}} \mathbb{C}$ over M is called the determinant line bundle associated to the* $\mathrm{Spin}^{\mathrm{c}}$ *structure on $\mathcal{V}$.*

Let $\nabla^{\mathbb{L}_\mathcal{V}}$ be an invariant connection on $\mathbb{L}_\mathcal{V}$ adapted to an invariant Hermitian metric. Let $\mathbf{F}^{\mathbb{L}_\mathcal{V}}(X), X \in \mathfrak{k}$ be its equivariant curvature 2-form. Even if the line bundle $\mathbb{L}_\mathcal{V}$ does not admit a square root, we define (formally) the Chern character of the square root as follows.

Definition 6.9 *The Chern character* $\mathrm{Ch}(\mathbb{L}_\mathcal{V}^{1/2}) \in \mathcal{H}^\infty(\mathfrak{k}, M)$ *is defined by the equivariant form* $\mathrm{e}^{\frac{1}{2}\mathbf{F}^{\mathbb{L}_\mathcal{V}}(X)}$.

Since the spinor representation extends to $\mathrm{Spin}^{\mathrm{c}}(V)$, the $\mathrm{Spin}^{\mathrm{c}}$ structure on $\mathcal{V}$ induces a K-equivariant spinor bundle $\mathcal{S}_M^{\mathrm{c}} := P^{\mathrm{c}} \times_{G^{\mathrm{c}}} \mathcal{S}$ on M. Like in the Spin case, one considers the K-equivariant morphism $\sigma_\mathcal{V}^{\mathrm{c}} : p^*\mathcal{S}_M^{\mathrm{c},+} \longrightarrow p^*\mathcal{S}_M^{\mathrm{c},-}$, defined by $\sigma_\mathcal{V}^{\mathrm{c}} = -i\,\mathbf{c}(\mathbf{x})$ where $\mathbf{x} : \mathcal{V} \to p^*\mathcal{V}$ is the canonical section.

Proposition 6.10 *We have the equalities*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_\mathcal{V}^{\mathrm{c}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\,\mathrm{Ch}(\mathbb{L}_\mathcal{V}^{1/2})\right)\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M).$$

$$\mathrm{Ch}_{\text{fiber cpt}}(\sigma_\mathcal{V}^{\mathrm{c}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\,\mathrm{Ch}(\mathbb{L}_\mathcal{V}^{1/2})\right)\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V}).$$

$$\mathrm{Ch}_Q(\sigma_\mathcal{V}^{\mathrm{c}}) = (2i\pi)^n p^*\left(\mathrm{j}^{1/2}(\mathcal{V})\,\mathrm{Ch}(\mathbb{L}_\mathcal{V}^{1/2})\right)\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^\infty_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V}).$$

Proof. It is an easy matter to extend the proof of the Spin case. The Lie algebra of $\mathfrak{spin}^{\mathrm{c}}(V)$ of $\mathrm{Spin}^{\mathrm{c}}(V)$ is identified with $\mathfrak{spin}(V) \times \mathbb{R}$.

First one consider the the case of an oriented Euclidean vector space V of dimension $2n$ equipped with $\mathrm{Spin}^{\mathrm{c}}(V)$-equivariant vector bundles $\mathcal{S}_V^{\mathrm{c},\pm} := V \times \mathcal{S}^{\mathrm{c},\pm}$ over V. Recall that the action of $\mathrm{Spin}^{\mathrm{c}}(V)$ on the base V is through $\tau^{\mathrm{c}} : \mathrm{Spin}^{\mathrm{c}}(V) \to \mathfrak{so}(V)$. Here $\mathcal{S}^{\mathrm{c},\pm}$ are the spinor spaces $\mathcal{S}^\pm$ but with the (extended) action of $\mathrm{Spin}^{\mathrm{c}}(V)$.

Then we consider the $\mathrm{Spin}^{\mathrm{c}}(V)$-equivariant morphism $\sigma_V^{\mathrm{c}} : \mathcal{S}_V^{\mathrm{c},+} \longrightarrow \mathcal{S}_V^{\mathrm{c},-}$, defined by $\sigma_\mathcal{V}^{\mathrm{c}} = -i\,\mathbf{c}(\mathbf{x})$. The $\mathrm{Spin}^{\mathrm{c}}(V)$-equivariant curvature of the super-connection $\mathbb{A}_t = d + t\mathbf{c}(x)$ is

$$\mathbf{F}_t(Y,\theta) = \mathbf{F}_t(Y) + i\theta, \quad (Y,\theta) \in \mathfrak{spin}^{\mathrm{c}}(V), \tag{64}$$

where $\mathbf{F}_t(Y)$ is the $\mathrm{Spin}(V)$-equivariant curvature compute in (60). Then we get the easy extension of Proposition 6.4

Lemma 6.11 *We have the following equality in* $\mathcal{A}_c^\infty(\mathfrak{spin}^{\mathrm{c}}(V), V)$

$$\mathrm{Ch}_c(\sigma_V^{\mathrm{c}})(Y,\theta) = (2i\pi)^n\ \mathrm{e}^{i\theta}\ \mathrm{j}_V^{1/2}(Y^\tau)\mathrm{Th}_c(V)(Y^\tau). \tag{65}$$

We come back to the situation of the K-equivariant vector bundle $\mathcal{V} \to M$. We consider a K-invariant connection one form ω on the $\mathrm{Spin}^{\mathrm{c}}(V)$-principal bundle $P^{\mathrm{c}} \to M$.

We prove Proposition 6.7 after taking the image of (65) by the Chern-Weil homomorphism ϕ_ω. Note that $\phi_\omega(\mathrm{e}^{i\theta}) = \mathrm{Ch}(\mathbb{L}_\mathcal{V}^{1/2})$ in $\mathcal{H}^\infty(\mathfrak{k}, M)$.

6.2.3 The complex case

In this section we treat the special case where the Spin^c structure comes from a complex structure.

We assume that $p : \mathcal{V} \to M$ is a K-equivariant complex vector bundle equipped with compatible Hermitian inner product $\langle -, - \rangle$ and connection $\nabla^{\mathcal{V}}$. We consider the super-vector bundle $\wedge_{\mathbb{C}}\mathcal{V} \to M$, where $\wedge_{\mathbb{C}}\mathcal{V} \to M$ means that we consider $\mathcal{V}$ as a complex vector bundle.

We consider now the K-equivariant morphism on $\mathcal{V}$, $\sigma_{\mathcal{V}}^{\mathbb{C}} : p^*(\wedge_{\mathbb{C}}^{+}\mathcal{V}) \longrightarrow p^*(\wedge_{\mathbb{C}}^{-}\mathcal{V})$, defined by

$$\sigma_{\mathcal{V}}^{\mathbb{C}}(v) = -i\left(\iota(v) - \varepsilon(v)\right) \quad \text{on} \quad \wedge_{\mathbb{C}}^{+}\mathcal{V}_{p(v)},$$

where $\iota(v)$ and $\varepsilon(v)$ are respectively the contraction by v ($\iota(v)(w) = \langle w, v\rangle$) and the wedge product by v. Since $\left(\iota(v) - \varepsilon(v)\right)^2 = -\|v\|^2$, we know that the support of $\sigma_{\mathcal{V}}^{\mathbb{C}}$ is the zero section of $\mathcal{V}$.

Let $\mathbf{F}^{\mathcal{V}}(X), X \in \mathfrak{k}$ be the equivariant curvature of $\nabla^{\mathcal{V}}$.

Definition 6.12 *The equivariant Todd form of* $(\mathcal{V}, \nabla^{\mathcal{V}})$ is defined by for X small*, by*

$$\mathrm{Todd}(\nabla^{\mathcal{V}})(X) := \det{}_{\mathbb{C}}\left(\frac{\mathbf{F}^{\mathcal{V}}(X)}{\mathrm{e}^{\mathbf{F}^{\mathcal{V}}(X)} - 1}\right).$$

We denote $\mathrm{Todd}(\mathcal{V})$ *its cohomology class in* $\mathcal{H}^{\infty}(\mathfrak{k}, M)$*.*

Remark that the inverse $\det_{\mathbb{C}}\left(\frac{\mathrm{e}^{\mathbf{F}^{\mathcal{V}}(X)} - 1}{\mathbf{F}^{\mathcal{V}}(X)}\right)$ of the equivariant Todd form is defined for any $X \in \mathfrak{k}$.

Proposition 6.13 *We have the equalities:*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_{\mathcal{V}}^{\mathbb{C}}) = (2i\pi)^n p^*\left(\mathrm{Todd}(\mathcal{V})^{-1}\right)\mathrm{Th}_{\mathrm{rel}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^{\infty}(\mathfrak{k}, \mathcal{V}, \mathcal{V} \setminus M),$$

$$\mathrm{Ch}_{\text{fiber cpt}}(\sigma_{\mathcal{V}}^{\mathbb{C}}) = (2i\pi)^n p^*\left(\mathrm{Todd}(\mathcal{V})^{-1}\right)\mathrm{Th}_{\text{fiber cpt}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^{\infty}_{\text{fiber cpt}}(\mathfrak{k}, \mathcal{V}),$$

$$\mathrm{Ch}_{Q}(\sigma_{\mathcal{V}}^{\mathbb{C}}) = (2i\pi)^n p^*\left(\mathrm{Todd}(\mathcal{V})^{-1}\right)\mathrm{Th}_{\mathrm{MQ}}(\mathcal{V}) \quad \text{in} \quad \mathcal{H}^{\infty}_{\text{dec-rap}}(\mathfrak{k}, \mathcal{V}).$$

Proof. The complex structure on the bundle $\mathcal{V}$ induces canonically a Spin^c structure where the bundle of spinors is $\wedge_{\mathbb{C}}\mathcal{V}$. The corresponding determinant line bundle is $\mathbb{L}_{\mathcal{V}} := \wedge_{\mathbb{C}}^{\max}\mathcal{V}$. Then one has just to check that

$$\mathrm{j}^{1/2}(\mathcal{V})\,\mathrm{Ch}(\mathbb{L}_{\mathcal{V}}^{1/2}) = \mathrm{Todd}(\mathcal{V})^{-1},$$

and we conclude with Proposition 6.10.

7 Appendix

We give proofs of the estimates used in this article. They are all based on Volterra's expansion formula: if R and S are elements in a finite dimensional associative algebra, then

$$\mathrm{e}^{(R+S)} = \mathrm{e}^{R} + \sum_{k=1}^{\infty} \int_{\Delta_k} \mathrm{e}^{s_1 R}\, S\, \mathrm{e}^{s_2 R}\, S \cdots S\, \mathrm{e}^{s_k R}\, S\, \mathrm{e}^{s_{k+1} R}\, ds_1 \cdots ds_k \tag{66}$$

where Δ_k is the simplex $\{s_i \geq 0; s_1 + s_2 + \cdots + s_k + s_{k+1} = 1\}$. We recall that the volume of Δ_k for the measure $ds_1 \cdots ds_k$ is $\frac{1}{k!}$.

Now, let $\mathcal{A} = \oplus_{i=0}^{q} \mathcal{A}_i$ be a finite dimensional graded commutative algebra with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$. We assume $\mathcal{A}_0 = \mathbb{C}$ and we denote by $\mathcal{A}_+ = \oplus_{i=1}^{q} \mathcal{A}_i$. Thus $\omega^{q+1} = 0$ for any $\omega \in \mathcal{A}_+$. Let V be a finite dimensional Hermitian vector space. Then $\mathrm{End}(V) \otimes \mathcal{A}$ is an algebra with a norm still denoted by $\|\cdot\|$. If $S \in \mathrm{End}(V)$, we denote also by S the element $S \otimes 1$ in $\mathrm{End}(V) \otimes \mathcal{A}$.

Remark 7.1 *In the rest of this section we will denote* $\mathrm{cst}(a, b, \cdots)$ *some positive constant which depends on the parameter* $a, b, \cdots$.

7.1 First estimates

We denote $\mathrm{Herm}(V) \subset \mathrm{End}(V)$ the subspace formed by the Hermitian endomorphisms. When $R \in \mathrm{Herm}(V)$, we denote $\mathrm{m}(R) \in \mathbb{R}$ the smallest eigenvalue of R : we have

$$\left\| \mathrm{e}^{-R} \right\| = \mathrm{e}^{-\mathrm{m}(R)}.$$

Lemma 7.2 *Let* $\mathcal{P}(t) = \sum_{k=0}^{q} \frac{t^k}{k!}$. *Then, for any* $S \in \mathrm{End}(V) \otimes \mathcal{A}$, $T \in \mathrm{End}(V) \otimes \mathcal{A}_+$, *and* $R \in \mathrm{Herm}(V)$, *we have*

$$\| \mathrm{e}^{-R+S+T} \| \leq \mathrm{e}^{-\mathrm{m}(R)}\, \mathrm{e}^{\|S\|}\, \mathcal{P}(\|T\|).$$

Proof. Let $c = \mathrm{m}(R)$. Then $\| \mathrm{e}^{-uR} \| = \mathrm{e}^{-uc}$ for all $u \geq 0$. Using Volterra's expansion for the couple sR, sS, we obtain $\| \mathrm{e}^{s(-R+S)} \| \leq \mathrm{e}^{-sc}\, \mathrm{e}^{s\|S\|}$. Indeed, $\mathrm{e}^{s(-R+S)} = \mathrm{e}^{-sR} + \sum_{k=1}^{\infty} I_k$ with

$$I_k = s^k \int_{\Delta_k} \mathrm{e}^{-s_1 sR}\, S \cdots S\, \mathrm{e}^{-s_k sR}\, S\, \mathrm{e}^{-s_{k+1} sR}\, ds_1 \cdots ds_k.$$

The term I_k is bounded in norm by $\frac{s^k}{k!} \|S\|^k\, \mathrm{e}^{-sc}$. Summing in k, we obtain $\| \mathrm{e}^{-s(R+S)} \| \leq \mathrm{e}^{-sc}\, \mathrm{e}^{s\|S\|}$ for $s \geq 0$. We reapply Volterra's expansion to compute $\mathrm{e}^{(-R+S)+T}$ as the sum

$$\mathrm{e}^{-R+S} + \sum_{k \geq 1}^{q} \int_{\Delta_k} \mathrm{e}^{s_1(-R+S)}\, T \cdots T\, \mathrm{e}^{s_k(-R+S)}\, T\, \mathrm{e}^{s_{k+1}(-R+S)}\, ds_1 \cdots ds_k.$$

Here the sum in k is finite and stops at $k = q$. The norm of the k^{th} term is bounded by $\frac{1}{k!}\, \mathrm{e}^{-c}\, \mathrm{e}^{\|S\|}\, \|T\|^k$. Summing up in k, we obtain our estimate.

For proving Proposition 5.6, we need to consider the following situation. Let E be a (finite dimensional) vector space. We consider the following smooth maps

- $x \mapsto S(x)$ from E to $\mathrm{End}(V) \otimes \mathcal{A}$.
- $(t, x) \mapsto t^2 R(x)$ from $\mathbb{R} \times E$ to $\mathrm{Herm}(V)$.
- $(t, x) \mapsto T(t, x) = T_0(x) + tT_1(x)$ from $\mathbb{R} \times E$ to $\mathrm{End}(V) \otimes \mathcal{A}_+$.

Proposition 7.3 *Let $D(\partial)$ be a constant coefficient differential operator in $x \in E$ of degree r. Let $\mathcal{K}$ be a compact subset of E. There exists a constant* $\mathrm{cst} > 0$ *(depending on $\mathcal{K}, R(x)$, $S(x), T_0(x), T_1(x)$ and $D(\partial)$) such that*

$$\left\| D(\partial) \cdot \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} \right\| \leq \mathrm{cst}\, (1+t)^{2r+q}\, \mathrm{e}^{-t^2 \mathrm{m}(R(x))}, \tag{67}$$

for all $(x, t) \in \mathcal{K} \times \mathbb{R}^{\geq 0}$.

Here the integer q is highest degree of the graded algebra $\mathcal{A}$.

Corollary 7.4 *Let $\mathcal{U}$ be an open subset of E such that $R(x)$ is positive definite for any $x \in \mathcal{U}$, that is $\mathrm{m}(R(x)) > 0$ for all $x \in \mathcal{U}$. Then the integral*

$$\int_0^\infty \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)}\, dt$$

defines a smooth map from $\mathcal{U}$ into $\mathrm{End}(V) \otimes \mathcal{A}$.

Proof. We fix a basis $v_1, \ldots, v_p$ of E. Let us denote ∂_i the partial derivative along the vector v_i. For any sequence $I := [i_1, \ldots, i_n]$ of integers $i_k \in \{1, \ldots, p\}$, we denote ∂_I the differential operator of order $n = |I|$ defined by the product $\prod_{k=1}^n \partial_{i_k}$.

For any smooth function $g : E \to \mathrm{End}(V) \otimes \mathcal{A}$ we define the functions

$$\left\| g \right\|_n (x) := \sup_{|I| \leq n} \|\partial_I \cdot g(x)\|$$

and the semi-norms $\|g\|_{\mathcal{K},n} := \sup_{x \in \mathcal{K}} \|g\|_n(x)$ attached to a compact subset $\mathcal{K}$ of E. We will use the trivial fact that $\|g\|_n(x) \leq \|g\|_m(x)$ when $n \leq m$. Since any constant differential operator $D(\partial)$ is a finite sum $\sum_I a_I \partial_I$, it is enough to proves (67) for the ∂_I.

First, we analyze $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x)}\right)$. The Volterra expansion formula gives

$$\partial_i \cdot \left(\mathrm{e}^{-t^2 R(x)}\right) = -t^2 \int_{\Delta_1} \mathrm{e}^{-s_1 t^2 R(x)}\, \partial_i \cdot R(x)\, \mathrm{e}^{-s_2 t^2 R(x)}\, ds_1, \tag{68}$$

and then $\|\partial_i \cdot \mathrm{e}^{-t^2 R(x)}\| \leq \|R\|_1(x)\,(1+t)^2\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}$ for $(x,t) \in E \times \mathbb{R}^{\geq 0}$.

With (68), one can easily prove by induction on the degree of ∂_I that: if $|I| = n$ then

$$\left\|\partial_I \cdot \mathrm{e}^{-t^2 R(x)}\right\| \leq \mathrm{cst}(n)\left(1 + \|R\|_n(x)\right)^n (1+t)^{2n}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))} \tag{69}$$

for $(x,t) \in E \times \mathbb{R}^{\geq 0}$. Note that (69) is still true when $I = \emptyset$ with $\mathrm{cst}(0) = 1$.

Now we look at $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x) + S(x)}\right)$ for $|I| = n$. The Volterra expansion formula gives $\mathrm{e}^{-t^2 R(x)+S(x)} = \mathrm{e}^{-t^2 R(x)} + \sum_{k=1}^{\infty} \mathcal{Z}_k(x)$ with

$$\mathcal{Z}_k(x) = \int_{\Delta_k} \mathrm{e}^{-s_1(t^2 R(x))}\, S(x)\, \mathrm{e}^{-s_2(t^2 R(x))}\, S(x) \cdots S(x)\, \mathrm{e}^{-s_{k+1}(t^2 R(x))}\, ds_1 \cdots ds_k.$$

The term $\partial_I \cdot \mathcal{Z}_k(x)$ is equal to the sum, indexed by the partitions (we allow some of the I_j to be empty.) $\mathcal{P} := \{I_1, I_2, \ldots, I_{2k+1}\}$ of I, of the terms

$$\mathcal{Z}_k(\mathcal{P})(x) := \tag{70}$$

$$\int_{\Delta_k} \left(\partial_{I_1} \cdot \mathrm{e}^{-s_1(t^2 R(x))}\right) (\partial_{I_2} \cdot S(x)) \cdots (\partial_{I_{2k}} \cdot S(x)) \left(\partial_{I_{2k+1}} \cdot \mathrm{e}^{-s_{k+1}(t^2 R(x))}\right) ds_1 \cdots ds_k$$

which are, thanks to (69), smaller in norm than

$$\mathrm{cst}(\mathcal{P})\left(1 + \|R\|_{n_{\mathcal{P}}^+}(x)\right)^{n_{\mathcal{P}}^+} \frac{\left(\|S\|_{n_{\mathcal{P}}^-}(x)\right)^k}{k!} (1+t)^{2n_{\mathcal{P}}^+}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}. \tag{71}$$

The integer $n_{\mathcal{P}}^+, n_{\mathcal{P}}^-$ are respectively equal to the sums $|I_1| + |I_3| + \cdots + |I_{2k+1}|$, $|I_2| + |I_4| + \cdots + |I_{2k}|$, and then $n_{\mathcal{P}}^+ + n_{\mathcal{P}}^- = n$. The constant $\mathrm{cst}(\mathcal{P})$ is equal to the products $\mathrm{cst}(|I_1|)\mathrm{cst}(|I_3|) \cdots \mathrm{cst}(|I_{2k+1}|)$. Since the sum $\sum_{\mathcal{P}} \mathrm{cst}(\mathcal{P})$ is bounded by a constant $\mathrm{cst}'(n)$, we find that

$$\left\|\partial_I \cdot \mathrm{e}^{-t^2 R(x) + S(x)}\right\| \leq \mathrm{cst}'(n)\left(1 + \|R\|_n(x)\right)^n \mathrm{e}^{\|S\|_n(x)}(1+t)^{2n}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))} \tag{72}$$

for $(x,t) \in E \times \mathbb{R}^{\geq 0}$. Note that (72) is still true when $I = \emptyset$ with $\mathrm{cst}'(0) = 1$.

Finally we look at $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x) + S(x) + T(t,x)}\right)$ for $|I| = n$. The Volterra expansion formula gives $\mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} = \mathrm{e}^{-t^2 R(x)+S(x)} + \sum_{k=1}^{q} \mathcal{W}_k(x)$ with

$$\mathcal{W}_k(x) = \int_{\Delta_k} \mathrm{e}^{s_1(-t^2 R(x)+S(x))}\, T(t,x) \cdots T(t,x)\, \mathrm{e}^{s_{k+1}(-t^2 R(x)+S(x))}\, ds_1 \cdots ds_k.$$

Note that the term $\mathcal{W}_k(x)$ vanishes for $k > q$. If we use (72), we get for $(x,t) \in E \times \mathbb{R}^{\geq 0}$:

$$\|\partial_I \cdot \mathcal{W}_k(x)\| \leq \mathrm{cst}''(n)\Big(\|T_0\|_n(x) + \|T_1\|_n(x)\Big)^k \times$$
$$\Big(1 + \|R\|_n(x)\Big)^n \frac{(1+t)^{2n+k}}{k!}\ \mathrm{e}^{\|S\|_n(x)}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}.$$

Finally we get for $(x,t) \in E \times \mathbb{R}^{\geq 0}$:

$$\left\| \partial_I \cdot \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} \right\| \leq \mathrm{cst}''(n)\Big(1+\|R\|_n(x)\Big)^n \times \tag{73}$$

$$\mathrm{P}\Big(\|T_0\|_n(x)+\|T_1\|_n(x)\Big)\ \mathrm{e}^{\|S\|_n(x)}\ (1+t)^{2n+q}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}$$

where P is the polynomial $\mathrm{P}(z)=\sum_{k=0}^{q} \frac{z^k}{k!}$.

So (67) is proved with

$$\mathrm{cst} = \mathrm{cst}''(n) \sup_{x\in\mathcal{K}} \left\{ \Big(1+\|R\|_n(x)\Big)^n \mathrm{P}\Big(\|T_0\|_n(x)+\|T_1\|_n(x)\Big)\ \mathrm{e}^{\|S\|_n(x)} \right\}.$$

7.2 Second estimates

Consider now the case where $E = W \times \mathfrak{k}$: the variable $x \in E$ will be replaced by $(y,X) \in W \times \mathfrak{k}$. **We suppose that the maps R and T are constant relatively to the parameter $X \in \mathfrak{k}$.**

Let $\mathcal{K} = \mathcal{K}' \times \mathcal{K}''$ be a compact subset of $W \times \mathfrak{k}$. Let $D(\partial)$ be a constant coefficient differential operator in $(y,X) \in W \times \mathfrak{k}$ of degree r : let r_W be its degree relatively to the variable $y \in W$.

Proposition 7.5 *There exists a constant* cst > 0*, depending on* $\mathcal{K}, R(y), S(y,X), T_0(y), T_1(y)$ *and* $D(\partial)$*, such that*

$$\left\| D(\partial) \cdot \mathrm{e}^{-t^2 R(y)+S(y,X)+T(t,y)} \right\| \leq \mathrm{cst}\,(1+t)^{2r_W+q}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y))}, \tag{74}$$

for all $(y,X,t) \in \mathcal{K}' \times \mathcal{K}'' \times \mathbb{R}^{\geq 0}$*. Here q is the highest degree of the graded algebra $\mathcal{A}$.*

Proof. We follow the proof of Proposition 7.3. We have just to explain why we can replace in (67) the factor $(1+t)^{2r}$ by $(1+t)^{2r_W}$.

We choose some basis $v_1,\ldots,v_{\pi_1}$ of W and $X_1,\ldots,X_{\pi_2}$ of $\mathfrak{k}$. Let us denote $\partial^1_i, \partial^2_j$ the partial derivatives along the vector v_i and X_j. For any sequence

$$I := \underbrace{\{i_1,\ldots,i_{n_1}\}}_{I(1)} \cup \underbrace{\{j_1,\ldots,j_{n_2}\}}_{I(2)}$$

of integers where $i_k \in \{1,\ldots,\pi_1\}$ and $j_k \in \{1,\ldots,\pi_2\}$, we denote ∂_I the differential operator of order $|I| = n_1+n_2$ defined by the product $\prod_{k=1}^{n} \partial^1_{i_k} \prod_{l=1}^{m} \partial^2_{j_k}$.

We first notice that $\partial_I \cdot \mathrm{e}^{-t^2 R(y)} = 0$ if $I(2) \neq \emptyset$. Now we look at $\partial_I \cdot \Big(\mathrm{e}^{-t^2 R(y)+S(y,X)}\Big)$ for $I = I(1)\cup I(2)$. The term $\mathcal{Z}_k(\mathcal{P})$ of (70) vanishes when there exists a sub-sequence I_{2l+1} with $I_{2l+1}(2) \neq \emptyset$. In the other cases, the integer $n^+_{\mathcal{P}} = |I_1|+|I_3|+\cdots+|I_{2k+1}|$ appearing in (71) is smaller than $|I(1)| = n_1$. So the inequalities (72) and (73) hold with the factor $(1+t)^{2n}$ replaced by $(1+t)^{2n_1}$.

The preceding estimates hold if we work in the algebra $\mathrm{End}(\mathcal{E}) \otimes \mathcal{A}$, where $\mathcal{E}$ is a super-vector space and $\mathcal{A}$ a super-commutative algebra.

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