# Equivariant Chern characters with generalized coefficients

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# 1 Introduction

These notes form the next episode in a series of articles dedicated to a detailed proof of a cohomological index formula for transversally elliptic pseudo-differential operators and applications. The complete notes will be published as a monograph. The first two chapters are already available as [18] and [19]. We tried our best in order that each chapter is relatively self-contained, at the expense of repeating definitions. In this episode, we construct the relative equivariant Chern character of a morphism of vector bundles, *localized* by a 1-form $\lambda$ and we prove a multiplicativity property of this generalized Chern character.

Let us first give motivations for the construction of the "localized Chern character" of a morphism of vector bundles. Let $M$ be a compact manifold. The Atiyah-Singer formula for the index of an *elliptic* pseudo-differential operator $P$ on $M$ with *elliptic* symbol $\sigma$ on $\mathbf{T}^*M$ involves integration over the non compact manifold $\mathbf{T}^*M$ of the Chern character $\mathrm{Ch_c}(\sigma)$ of $\sigma$ multiplied by the Todd class of $\mathbf{T}^*M$:

$$\mathrm{index}(P) = \int_{\mathbf{T}^*M} (2i\pi)^{-\dim M}\, \mathrm{Ch_c}(\sigma)\mathrm{Todd}(\mathbf{T}^*M).$$

Here $\sigma$, the symbol of $P$, is a morphism of vector bundles on $\mathbf{T}^*M$ invertible outside the zero section of $\mathbf{T}^*M$ and the Chern character $\mathrm{Ch_c}(\sigma)$ is supported on a small neighborhood of $M$ embedded in $\mathbf{T}^*M$ as the zero section. It is important that the representative of the Chern character $\mathrm{Ch_c}(\sigma)$ is compactly supported to perform integration.

Assume that a compact Lie group $K$ acts on $M$. Let $\mathfrak{k}$ be the Lie algebra of $K$. If $X \in \mathfrak{k}$, we denote by $VX$ the infinitesimal vector field generated by $X$.

If the elliptic operator $P$ is $K$-invariant, then $\mathrm{index}(P)$ is a smooth function on $K$. The equivariant index of $P$ can be expressed similarly as the integral of the equivariant Chern character $\mathrm{Ch_c}(\sigma)$ of $\sigma$ multiplied by the equivariant Todd class of $\mathbf{T}^*M$: for $X \in \mathfrak{k}$ small enough,

$$\mathrm{index}(P)(\exp X) = \int_{\mathbf{T}^*M} (2i\pi)^{-\dim M}\, \mathrm{Ch_c}(\sigma)(X)\mathrm{Todd}(\mathbf{T}^*M)(X).$$

Here $\mathrm{Ch_c}(\sigma)(X)$ is a compactly supported closed equivariant differential form, that is a differential form on $\mathbf{T}^*M$ depending smoothly of $X \in \mathfrak{k}$, and closed for the equivariant differential $D$. The result of the integration determines a smooth function on a neighborhood of 1 in $K$ and similar formulae can be given near any point of $K$.

The motivation for this article is to extend the construction of the compactly supported class $\mathrm{Ch}_c(\sigma)$ to the case where $P$ is not necessarily elliptic, but still *transversally elliptic* relatively to the action of $K$. An equivariant pseudo-differential operator $P$ with symbol $\sigma(x,\xi)$ on $\mathbf{T}^*M$ is called transversally elliptic, if it is elliptic in the directions transversal to $K$-orbits. More precisely, let $\mathbf{T}^*_K M$ be the set of co-vectors that are orthogonal to the $K$-orbits. Then $\sigma(x,\xi)$, when restricted to $\mathbf{T}^*_K M$, is invertible when $\xi \neq 0$. In this case, the operator $P$ has again an index which is a generalized function on $K$. It is thus natural to look for a Chern character with coefficients generalized functions as well. Thus we will construct a closed compactly supported equivariant form $\mathrm{Ch_c}(\sigma,\omega)(X)$ with coefficients generalized functions on $\mathfrak{k}$ associated to such a symbol. In the next chapter, we will prove that again

$$\mathrm{index}(\sigma)(\exp X) = \int_{\mathbf{T}^*M} (2i\pi)^{-\dim M}\, \mathrm{Ch_c}(\sigma,\omega)(X)\mathrm{Todd}(\mathbf{T}^*M)(X).$$

As the differential form $\mathrm{Ch_c}(\sigma,\omega)(X)$ is compactly supported, the integration can be performed and in this case the result determines a generalized function on a neighborhood of 1 in $K$. Similar formulae can be given near any point of $K$.

Here $\omega$ is the Liouville 1-form on $\mathbf{T}^*M$ which defines a map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ by the formula

$$\langle f_\omega(n), X\rangle = \langle \omega(n), V_n X\rangle$$

and $\mathbf{T}^*_K M$ is precisely the set $f_\omega^{-1}(0)$. Our construction of the Chern character $\mathrm{Ch_c}(\sigma,\omega)$ is based on this fact.

Thus given a $K$-manifold $N$ and a real invariant one form $\lambda$ on $N$, we consider the map $f_\lambda : N \to \mathfrak{k}^*$ defined by $\langle f_\lambda, X\rangle = \langle \lambda, VX\rangle$. Define the "critical set" $C_\lambda$ by $C_\lambda = f_\lambda^{-1}(0)$. In other words, the point $n \in N$ is in $C_\lambda$ if the co-vector $\lambda(n)$ is orthogonal to the vectors tangent to the orbit $K\cdot n$. We will associate to any equivariant morphism of vector bundles $\sigma$ on $N$, invertible outside a closed invariant set $F$, an equivariant (relative) Chern character $\mathrm{Ch_{rel}}(\sigma,\lambda)$, with $C^{-\infty}$-coefficients, with support on the set $C_\lambda \cap F$. In particular, if this set is compact, our equivariant relative Chern character leads to a compactly supported Chern character $\mathrm{Ch_c}(\sigma,\lambda)$ which is supported in a neighborhood of $C_\lambda \cap F$. Then we will prove a certain number of functorial properties of the Chern character $\mathrm{Ch_{rel}}(\sigma,\lambda)$. One of the most important property is its multiplicativity.

The main ideas of our construction come from two sources:

• We use the construction basically due to Quillen of the *equivariant relative Chern character* $\mathrm{Ch_{rel}}(\sigma)$ already explained in [18], [19].

• We use a localization argument on the "critical" set $f_\lambda^{-1}(0)$ originated in Witten [23] and systematized in Paradan [14, 15]. Indeed, the fundamental remark inspired by the "non abelian localization formula of Witten" is that

$$1 = 0$$

in equivariant cohomology on the complement of the critical set $C_\lambda$.

The idea to use the Liouville one-form $\omega$ to construct a Chern character for symbols of transversally elliptic operators was already present in the preceding construction of Berline-Vergne [8]. In this work, a Chern character on

$\mathbf{T}^*M$ was constructed with gaussian look in transverse directions to the $K$-action and oscillatory behavior in parallel directions of the $K$-action. Thus this Chern character was integrable in the generalized sense. Our new construction gives directly a (relative) Chern character with compact support equal to the intersection of the support of the morphism $\sigma$ with the critical set $\mathbf{T}^*_K M$.

Let us now explain in more details the content of this article.

In Section 2, we define the equivariant differential $D$, the equivariant relative cohomologies $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$ with $C^\infty$ coefficients as well as the relative cohomology $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$ with $C^{-\infty}$ coefficients and the product in relative cohomology $\diamond : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F_1)\times\mathcal{H}^{\infty}(\mathfrak{k}, N, N\setminus F_2) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus(F_1\cap F_2))$.

In Section 3, we start by recalling Quillen's construction of the relative Chern character that we studied already in [18], [19]. Let us consider a $K$-equivariant morphism $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ between two $K$-vector bundles over $N$: the symbol $\sigma$ is invertible outside a (possibly non-compact) subset $F \subset N$. Following an idea of Quillen [20], we defined the *equivariant relative Chern character* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ as the class defined by a couple $(\alpha, \beta(\sigma))$ of equivariant forms:

- $\alpha := \mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$ is a closed equivariant form on $N$.
- $\beta(\sigma)$ is an equivariant form on $N \setminus F$ which is constructed with the help of the invariant super-connections $\mathbb{A}^\sigma(t) = \nabla + it(\sigma \oplus \sigma^*),\ t \in \mathbb{R}$. Here $\nabla$ is an invariant connection on $\mathcal{E}^+ \oplus \mathcal{E}^-$ preserving the grading.
- we have on $N \setminus F$, the equality of equivariant forms

$$\alpha|_{N\setminus F} = D(\beta(\sigma)). \tag{1}$$

In this construction, the equivariant forms $X \mapsto \alpha(X)$ and $X \mapsto \beta(\sigma)(X)$ have a **smooth** dependance relatively to the parameter $X \in \mathfrak{k}$. Thus $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ is an element of the relative cohomology group $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$.

In Subsection 3, we deform the relative Chern character of $\sigma$ by using as a further tool of deformation an invariant real 1-form $\lambda$ on $N$. Using the family of invariant super-connections

$$\mathbb{A}^{\sigma,\lambda}(t) := \nabla + it(\sigma \oplus \sigma^*) + it\lambda,\ t \in \mathbb{R}$$

we construct an equivariant form $\beta(\sigma, \lambda)$ on $N \setminus (F \cap C_\lambda)$ such that

$$\alpha|_{N\setminus(F\cap C_\lambda)} = D(\beta(\sigma, \lambda)) \tag{2}$$

holds on $N \setminus (F \cap C_\lambda)$. There are two kinds of difference between (1) and (2).

- the subset $N \setminus (F \cap C_\lambda)$ contains $N \setminus F$ so Equation (2) which holds on $N \setminus (F \cap C_\lambda)$ is in some sense "stronger" than Equation (1) which holds on $N \setminus F$.
- the equivariant form $X \mapsto \beta(\sigma, \lambda)(X)$ has a $\mathcal{C}^{-\infty}$ dependance relatively to the parameter $X \in \mathfrak{k}$.

We define the relative Chern character of $\sigma$ deformed by the 1-form $\lambda$ as the class defined by the couple $(\alpha, \beta(\sigma, \lambda))$ : we denote it by $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)$. It is an element of the relative cohomology group $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus (F \cap C_\lambda))$.

One case of interest is when $F \cap C_\lambda$ is a compact subset of $N$. Using an invariant function $\chi$ on $N$, identically equal to 1 in a neighborhood of $F \cap C_\lambda$ and with compact support, the equivariant form

$$\mathrm{p}(\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)) := \chi\alpha + d\chi\beta(\sigma, \lambda)$$

is equivariantly closed, with compact support on $N$, and has a $\mathcal{C}^{-\infty}$ dependance relatively to the parameter $X \in \mathfrak{k}$: its equivariant class is denoted $\mathrm{Ch}_{\mathrm{c}}(\sigma, \lambda)$.

An important case of $K$-equivariant bundle is the trivial bundle represented by the trivial symbol $[0] : N \times \mathbb{C} \to N \times \{0\}$. In this case, we denote the relative class $\mathrm{Ch}_{\mathrm{rel}}([0], \lambda)$ by $\mathrm{P}_{\mathrm{rel}}(\lambda)$. The class $\mathrm{P}_{\mathrm{rel}}(\lambda)$ is defined by a couple $(1, \beta(\lambda))$ where $\beta(\lambda)$ is a generalized equivariant form on $N \setminus C_\lambda$ satisfying

$$1 = D(\beta(\lambda)).$$

This equation $1 = 0$ on $N \setminus C_\lambda$, together with explicit description of $\beta(\lambda)$, is the principle explaining Witten "non abelian localisation theorem".

In Subsection 3 we study the functorial properties of the classes $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)$ and $\mathrm{Ch}_{\mathrm{c}}(\sigma, \lambda)$. We prove in particular that these classes behave nicely under the product. When $\sigma_1, \sigma_2$ are two equivariant symbols on $N$, we can take their product $\sigma_1 \odot \sigma_2$ which is a symbol on $N$ such that $\mathrm{Supp}(\sigma_1 \odot \sigma_2) = \mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$. We prove then that

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \lambda). \tag{3}$$

We have then a factorized expression for the class $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)$ by taking $\sigma_1 = [0]$ in (3): we have

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda) = \mathrm{P}_{\mathrm{rel}}(\lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma). \tag{4}$$

The first class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ is supported on $\mathrm{Supp}(\sigma)$ while the second class $\mathrm{P}_{\mathrm{rel}}(\lambda)$ (equivalent to 1) is supported on $C_\lambda$.

When $K$ is a torus, and $\lambda$ the invariant one form associated to a generic vector field $VX$ via a metric on $\mathbf{T}M$, the set $C_\lambda$ coincide with the set $N^K$ of fixed points for the action of the torus $K$. Then $\mathrm{P}_{\mathrm{rel}}(\lambda)(X)$ can be represented as a differential form with coefficients boundary values of rational functions of $X \in \mathfrak{k}$. In this special case, Equation (4) is strongly related to Segal's localization theorem on the fixed point set for equivariant $\mathbf{K}$-theory.

In Subsection 3.6, we study the multiplicativity properties of our Chern characters when a product of groups acts on $N$. If $\lambda, \mu$ are 1-forms invariant by $K_1 \times K_2$, we define $C^1_\lambda$ to be the critical set of $\lambda$ with respect to $K_1$ and $C^2_\mu$ the critical set of $\mu$ with respects to $K_2$. Let $\sigma$ and $\tau$ be two $K_1 \times K_2$-equivariant morphisms on $N$ which are invertible respectively on $N \setminus F_1$ and $N \setminus F_2$. Then the relative Chern character $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda)$ is defined as a class in $\mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (C^1_\lambda \cap F_1))$ while $\mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu)$ is defined as a class in

$\mathcal{H}^{\infty,-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (C^2_\mu \cap F_2))$. As suggested by the notation, an element in $\mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus F)$ can be represented as a couple $(\alpha(X,Y), \beta(X,Y))$ of differential forms with smooth dependence in $Y$, and a generalized function in $X$. Then we can multiply the classes $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma,\lambda)$ and $\mathrm{Ch}^2_{\mathrm{rel}}(\tau,\mu)$. One main theorem, which will be crucial for the functoriality properties of the equivariant index, is Theorem 3.32

$$\mathrm{Ch}^1_{\mathrm{rel}}(\sigma,\lambda) \diamond \mathrm{Ch}^2_{\mathrm{rel}}(\tau,\mu) = \mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau, \lambda + \mu).$$

In Section 4, we consider the case where the $K$-manifold $N$ is the cotangent bundle $\mathbf{T}^*M$ of a $K$-manifold $M$. Let $\mathbf{T}^*_K M$ be the set of co-vectors that are orthogonal to the $K$-orbits. If $\sigma$ is invertible outside a closed invariant set $F \subset \mathbf{T}^*M$, $\sigma$ is a transversally elliptic morphism if $F \cap \mathbf{T}^*_K M$ is compact. We work here with the Liouville one form $\omega$ on $\mathbf{T}^*M$. It is easy to see that the set $C_\omega$ coincides with $\mathbf{T}^*_K M$. So, for a transversally elliptic morphism $\sigma$ we define its Chern class with compact support as the class $\mathrm{Ch_c}(\sigma,\omega)$.

We finally compare our construction with the Berline-Vergne construction.

# 2 Equivariant cohomologies with $\mathcal{C}^{-\infty}$ coefficients

Let $N$ be a manifold, and let $\mathcal{A}(N)$ be the algebra of differential forms on $N$. We denote by $\mathcal{A}_c(N)$ the subalgebra of compactly supported differential forms. We will consider on $\mathcal{A}(N)$ and $\mathcal{A}_c(N)$ the $\mathbb{Z}_2$-grading in even or odd differential forms.

Let $K$ be a compact Lie group with Lie algebra $\mathfrak{k}$. We suppose that the manifold $N$ is provided with an action of $K$. We denote $X \mapsto VX$ the corresponding morphism from $\mathfrak{k}$ into the Lie algebra of vectors fields on $N$: for $n \in N$,

$$V_n X := \frac{d}{d\epsilon} \exp(-\epsilon X) \cdot n|_{\epsilon=0}.$$

Let $\mathcal{A}^\infty(\mathfrak{k}, N)$ be the $\mathbb{Z}_2$-graded algebra of equivariant smooth functions $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. Its $\mathbb{Z}_2$-grading is the grading induced by the exterior degree. Let $D = d - \iota(VX)$ be the equivariant differential: $(D\alpha)(X) = d(\alpha(X)) - \iota(VX)\alpha(X)$. Here the operator $\iota(VX)$ is the contraction of a differential form by the vector field $VX$. Let $\mathcal{H}^\infty(\mathfrak{k}, N) := \mathrm{Ker}D/\mathrm{Im}D$ be the equivariant cohomology algebra with $C^\infty$-coefficients. It is a module over the algebra $\mathcal{C}^\infty(\mathfrak{k})^K$ of $K$-invariant $C^\infty$-functions on $\mathfrak{k}$.

The sub-algebra $\mathcal{A}^\infty_c(\mathfrak{k}, N) \subset \mathcal{A}^\infty(\mathfrak{k}, N)$ of equivariant differential forms with compact support is defined as follows : $\alpha \in \mathcal{A}^\infty_c(\mathfrak{k}, N)$ if there exists a compact subset $\mathcal{K}_\alpha \subset N$ such that the differential form $\alpha(X) \in \mathcal{A}(N)$ is supported on $\mathcal{K}_\alpha$ for any $X \in \mathfrak{k}$. We denote $\mathcal{H}^\infty_c(\mathfrak{k}, N)$ the corresponding algebra of cohomology: it is a $\mathbb{Z}_2$-graded algebra.

Kumar and Vergne [12] have defined generalized equivariant cohomology spaces obtained by considering equivariant differential forms with $\mathcal{C}^{-\infty}$ coefficients. Let us recall the definition.

Let $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ be the space of *generalized* equivariant differential forms. An element $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is, by definition, a $\mathcal{C}^{-\infty}$-equivariant map $\alpha : \mathfrak{k} \to \mathcal{A}(N)$. The value taken by $\alpha$ on a smooth compactly supported density $Q(X)dX$ on $\mathfrak{k}$ is denoted by $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$. We have $\mathcal{A}^{\infty}(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ and we can extend the differential $D$ to $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ [12]. We denote by $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ the corresponding cohomology space. Note that $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{A}^{\infty}(\mathfrak{k}, N)$ under the wedge product, hence the cohomology space $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ is a module over $\mathcal{H}^{\infty}(\mathfrak{k}, N)$.

The sub-space $\mathcal{A}_c^{-\infty}(\mathfrak{k}, N) \subset \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ of generalized equivariant differential forms with compact support is defined as follows : $\alpha \in \mathcal{A}_c^{-\infty}(\mathfrak{k}, N)$ if there exits a compact subset $\mathcal{K}_\alpha \subset N$ such that the differential form $\int_{\mathfrak{k}} \alpha(X)Q(X)dX \in \mathcal{A}(N)$ is supported on $\mathcal{K}_\alpha$ for any compactly supported density $Q(X)dX$. We denote $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$ the corresponding space of cohomology. The $\mathbb{Z}_2$-grading on $\mathcal{A}(N)$ induces a $\mathbb{Z}_2$-grading on the cohomology spaces $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ and $\mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$.

Let us stress here that a generalized equivariant form $\alpha(X, n, dn)$ on $N$ is **smooth** with respect to the variable $n \in N$. Thus we can restrict generalized forms to $K$-equivariant submanifolds of $N$. However, in general, if $G$ is a subgroup of $K$, a $K$-equivariant generalized form on $N$ do not restrict to a $G$-equivariant generalized form.

More generally, if $g : M \to N$ is a $K$-equivariant map from the $K$-manifold $M$ to the $K$-manifold $N$, then we obtain a map $g^* : \mathcal{A}^{-\infty}(\mathfrak{k}, N) \to \mathcal{A}^{-\infty}(\mathfrak{k}, M)$, which induces a map $g^*$ in cohomology. When $U$ is an open invariant subset of $N$, we denote by $\alpha \to \alpha|_U$ the restriction of $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ to $U$.

There is a natural map $\mathcal{H}^{\infty}(\mathfrak{k}, N) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N)$. This map is not injective in general. Let us give a simple example. Let $U(1)$ acting on $N = \mathbb{R}^2 \setminus \{0\}$ by rotations. Let $\mathfrak{k} \sim \mathbb{R}$ be the Lie algebra of $U(1)$. The vector $X \in \mathfrak{k}$ produces the infinitesimal vector field $VX = X(y\partial_x - x\partial_y)$ on $\mathbb{R}^2$. Denote by $\frac{1}{X}$ any generalized function of $X \in \mathfrak{k}$ such that $X(\frac{1}{X}) = 1$. Let $\lambda = \frac{xdy - ydx}{x^2+y^2}$. Then

$$D(\frac{1}{X}\lambda) = (d - \iota(VX))(\frac{1}{X}\lambda) = 1.$$

Thus the image of 1 is exact in $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$, so that the image of $\mathcal{H}^{\infty}(\mathfrak{k}, N)$ is 0 in $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$.

## 2.1 Examples of generalized equivariant forms

In this article, equivariant forms with generalized coefficients appear in the following situation. Let $t \mapsto \eta_t(X)$ be a smooth map from $\mathbb{R}$ into $\mathcal{A}^{\infty}(\mathfrak{k}, N)$. For any $t \geq 0$, the integral

$$\beta_t(X) = \int_0^t \eta_s(X)ds$$

defines an element of $\mathcal{A}^{\infty}(\mathfrak{k}, N)$. One may ask if the "limit" of $\beta_t(X)$ when $t$ goes to infinity exists.

Let $X_1, \ldots, X_{\dim K}$ be a base of $\mathfrak{k}$. For any $\nu := (\nu_1, \ldots, \nu_{\dim K}) \in \mathbb{N}^{\dim K}$, we denote $\frac{\partial}{\partial X^\nu}$ the differential operator $\prod_i \left(\frac{\partial}{\partial X_i}\right)^{\nu_i}$ of degree $|\nu| := \sum_i \nu_i$.

**Definition 2.1** *For a compact subset $\mathcal{K}$ of $\mathfrak{k}$ and $r \in \mathbb{N}$, we denote $\| - \|_{\mathcal{K},r}$ the semi-norm on $\mathcal{C}^\infty(\mathfrak{k})$ defined by $\|Q\|_{\mathcal{K},r} = \sup_{X \in \mathcal{K}, |\nu| \le r} \left| \frac{\partial}{\partial X^\nu} Q(X) \right|$.*

We make the following assumption on $\eta_t(X)$. For every compact subset $\mathcal{K} \times \mathcal{K}' \subset \mathfrak{k} \times N$ and for any integer $r \in \mathbb{N}$, there exists $\mathrm{cst} > 0$ and $r' \in \mathbb{N}$ such that the following estimate

$$(5) \qquad \left\| \int_{\mathfrak{k}} \eta_t(X) Q(X) dX \right\|(n) \le \mathrm{cst} \frac{\|Q\|_{\mathcal{K},r'}}{(1+t)^r}, \quad n \in \mathcal{K}', \quad t \ge 0,$$

holds for every function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ supported in $\mathcal{K}$. Here the norm $\| - \|$ on the differential forms on $N$ is defined via the choice of a Riemannian metric on $N$.

Under estimates (5), we can define the equivariant form $\beta \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ as the limit of the equivariant forms $\beta_t \in \mathcal{A}^\infty(\mathfrak{k}, N)$ when $t$ goes to infinity. More precisely, for every $Q \in \mathcal{C}_c^\infty(\mathfrak{k})$, we have

$$(6) \qquad \int_{\mathfrak{k}} \beta(X) Q(X) dX := \int_0^\infty \left( \int_{\mathfrak{k}} \eta_t(X) Q(X) dX \right) dt.$$

In fact, in order to insure that the right hand side of (6) defines a *smooth* form on $N$, we need the following strongest version of the estimate (5) : we have

$$(7) \qquad \left\| D(\partial) \cdot \int_{\mathfrak{k}} \eta_t(X) Q(X) dX \right\|(n) \le \mathrm{cst} \frac{\|Q\|_{\mathcal{K},r'}}{(1+t)^r}, \quad n \in \mathcal{K}', \quad t \ge 0,$$

for any differential operator $D(\partial)$ acting on $\mathcal{A}(N)$. Under (7), the generalized equivariant form $\beta(X) := \int_0^\infty \eta_t(X) dt$ satisfies

$$D(\beta)(X) := \int_0^\infty D(\eta_t)(X) dt.$$

Let us consider the following basic case which appears in [14, 15]. Let $f : N \to \mathfrak{k}^*$ be an equivariant map, and let $\gamma_t(X)$ be an equivariant form on $N$ which depends polynomially on both variables $t$ and $X$. We consider the family

$$\eta_t(X) := \gamma_t(X)\, \mathrm{e}^{i\,t\,\langle f, X\rangle}, \quad t \in \mathbb{R}.$$

Then, for any $Q \in \mathcal{C}_c^\infty(\mathfrak{k})$, we have

$$\int_{\mathfrak{k}} \gamma_t(X)\, \mathrm{e}^{i\,t\,\langle f, X\rangle}\, Q(X) dX := \widehat{\gamma_t Q}(t\, f)$$

where $\widehat{\ }$ is the Fourier transform. Since the Fourier transform of a compactly supported function is rapidly decreasing, one sees that the estimates (5) and (7) holds in this case on the open subset $\{f \neq 0\}$ : the integral

$$\int_0^\infty \gamma_t(X)\, \mathrm{e}^{i\,t\,\langle f, X\rangle}\, dt$$

defines an equivariant form with generalized coefficients on $\{f \neq 0\} \subset N$.

## 2.2 Relative equivariant cohomology : the $\mathcal{C}^{-\infty}$ case

Let $F$ be a closed $K$-invariant subset of $N$. We have a restriction operation $r : \alpha \mapsto \alpha|_{N\setminus F}$ from $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ into $\mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$. To an equivariant cohomology class on $N$ vanishing on $N\setminus F$, we associate a relative equivariant cohomology class. Let us explain the construction : see [10, 18] for the non-equivariant case, and [19] for the equivariant case with $\mathcal{C}^\infty$ coefficients. Consider the complex $\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ with

$$\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F) := \mathcal{A}^{-\infty}(\mathfrak{k}, N) \oplus \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$$

and differential $D_{\mathrm{rel}}(\alpha,\beta) = \big(D\alpha, \alpha|_{N\setminus F} - D\beta\big)$. Let $\mathcal{A}^{\infty}(\mathfrak{k}, N, N\setminus F)$ be the sub-complex of $\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ formed by couples of equivariant forms with smooth coefficients : this sub-complex is stable under $D$.

**Definition 2.2** *The cohomology of the complexes $(\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F), D_{\mathrm{rel}})$ and $(\mathcal{A}^{\infty}(\mathfrak{k}, N, N\setminus F), D_{\mathrm{rel}})$ are the relative equivariant cohomology spaces $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ and $\mathcal{H}^{\infty}(\mathfrak{k}, N, N\setminus F)$.*

The class defined by a $D_{\mathrm{rel}}$-closed element $(\alpha,\beta) \in \mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ will be denoted $[\alpha,\beta]$.

The complex $\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ is $\mathbb{Z}_2$-graded : for $\epsilon \in \mathbb{Z}_2$, we take $[\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)]^{\epsilon} = [\mathcal{A}^{-\infty}(\mathfrak{k}, N)]^{\epsilon} \oplus [\mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)]^{\epsilon+1}$. Since $D_{\mathrm{rel}}$ sends $[\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)]^{\epsilon}$ into $[\mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)]^{\epsilon+1}$, the $\mathbb{Z}_2$-grading descends to the relative cohomology spaces $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F)$.

We review the basic facts concerning the relative cohomology groups. We consider now the following maps.

- The projection $j : \mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ is the degree 0 map defined by $j(\alpha,\beta) = \alpha$.
- The inclusion $i : \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F) \to \mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F)$ is the degree $+1$ map defined by $i(\beta) = (0,\beta)$.
- The restriction $r : \mathcal{A}^{-\infty}(\mathfrak{k}, N) \to \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$ is the degree 0 map defined by $r(\alpha) = \alpha|_{N\setminus F}$.

It is easy to see that $i, j, r$ induce maps in cohomology that we still denote by $i, j, r$.

**Lemma 2.3** • *We have an exact triangle*

$$\begin{tikzcd} & \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) \arrow[dr, "j"] & \\ \mathcal{H}^{-\infty}(\mathfrak{k}, N\setminus F) \arrow[ur, "i"] & & \mathcal{H}^{-\infty}(\mathfrak{k}, N). \arrow[ll, "r"] \end{tikzcd}$$

• *If $F \subset F'$ are closed $K$-invariant subsets of $N$, the restriction map $(\alpha, \beta) \mapsto (\alpha, \beta|_{N \setminus F'})$ induces a map*

$$\mathbf{r}_{F',F} : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F'). \tag{8}$$

• *The inclusion $\mathcal{A}^{\infty}(\mathfrak{k}, N, N \setminus F) \hookrightarrow \mathcal{A}^{-\infty}(\mathfrak{k}, N, N \setminus F)$ induces a map $\mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$.*

## 2.3 Product in relative equivariant cohomology

Let $F_1$ and $F_2$ be two closed $K$-invariant subsets of $N$. In [19], we have define a product $\diamond : \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2) \to \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$. Let us check that this product is still defined when one equivariant form has generalized coefficients.

Let $U_1 := N \setminus F_1$, $U_2 := N \setminus F_2$ so that $U := N \setminus (F_1 \cap F_2) = U_1 \cup U_2$. Let $\Phi := (\Phi_1, \Phi_2)$ be a partition of unity subordinate to the covering $U_1 \cup U_2$ of $U$. By averaging by $K$, we may suppose that the functions $\Phi_k$ are invariant.

Since $\Phi_k \in \mathcal{C}^{\infty}(U)^K$ is supported in $U_k$, the product $\gamma \mapsto \Phi_k \gamma$ defines maps $\mathcal{A}^{\infty}(\mathfrak{k}, U_k) \to \mathcal{A}^{\infty}(\mathfrak{k}, U)$ and $\mathcal{A}^{-\infty}(\mathfrak{k}, U_k) \to \mathcal{A}^{-\infty}(\mathfrak{k}, U)$. Since $d\Phi_1 = -d\Phi_2 \in \mathcal{A}(U)^K$ is supported in $U_1 \cap U_2$, the product $\gamma \mapsto d\Phi_1 \wedge \gamma$ defines a map $\mathcal{A}^{-\infty}(\mathfrak{k}, U_1 \cap U_2) \to \mathcal{A}^{-\infty}(\mathfrak{k}, U)$.

With the help of $\Phi$, we define a bilinear map $\diamond_{\Phi} : \mathcal{A}^{-\infty}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{A}^{\infty}(\mathfrak{k}, N, N \setminus F_2) \to \mathcal{A}^{-\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$ as follows.

**Definition 2.4** *For an equivariant form $a_1 := (\alpha_1, \beta_1) \in \mathcal{A}^{-\infty}(\mathfrak{k}, N, N \setminus F_1)$ with* generalized coefficients *and an equivariant form $a_2 := (\alpha_2, \beta_2) \in \mathcal{A}^{\infty}(\mathfrak{k}, N, N \setminus F_2)$ with* smooth coefficients*, we define*

$$a_1 \diamond_{\Phi} a_2 := (\alpha_1 \wedge \alpha_2, \beta(a_1, a_2)) \quad \text{with}$$

$$\beta(a_1, a_2) = \Phi_1 \beta_1 \wedge \alpha_2 + (-1)^{|a_1|} \alpha_1 \wedge \Phi_2 \beta_2 - (-1)^{|a_1|} d\Phi_1 \wedge \beta_1 \wedge \beta_2.$$

Remark that $\Phi_1 \beta_1 \wedge \alpha_2$, $\alpha_1 \wedge \Phi_2 \beta_2$ and $d\Phi_1 \wedge \beta_1 \wedge \beta_2$ are well defined equivariant forms with generalized coefficients on $U_1 \cup U_2$. So $a_1 \diamond_{\Phi} a_2 \in \mathcal{A}^{-\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$. Note also that $a_1 \diamond_{\Phi} a_2 \in \mathcal{A}^{\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2))$, if $\alpha_1, \beta_1$ have smooth coefficients.

A small computation shows that $D_{\mathrm{rel}}(a_1 \diamond_{\Phi} a_2)$ is equal to $(D_{\mathrm{rel}} a_1) \diamond_{\Phi} a_2 + (-1)^{|a_1|} a_1 \diamond_{\Phi} (D_{\mathrm{rel}} a_2)$. Thus $\diamond_{\Phi}$ defines bilinear maps

$$\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2) \xrightarrow{\diamond_{\Phi}} \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)), \tag{9}$$

and

$$\mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_1) \times \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2) \xrightarrow{\diamond_{\Phi}} \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)). \tag{10}$$

Let us see that this product do not depend of the choice of the partition of unity. If we have another partition $\Phi' = (\Phi'_1, \Phi'_2)$, then $\Phi_1 - \Phi'_1 = -(\Phi_2 - \Phi'_2)$.

It is immediate to verify that, if $D_{\rm rel}(a_1) = 0$ and $D_{\rm rel}(a_2) = 0$, one has $a_1 \diamond_\Phi a_2 - a_1 \diamond_{\Phi'} a_2 = D_{\rm rel}\big(0, (-1)^{|a_1|}(\Phi_1 - \Phi_1')\beta_1 \wedge \beta_2\big)$.

So the products (9) and (10) will be denoted by $\diamond$.

**Lemma 2.5** • *The relative product is compatible with restrictions: if $F_1 \subset F_1'$ and $F_2 \subset F_2'$ are closed invariant subsets of $N$, then the diagram*

(11)

$$\begin{tikzcd}
\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F_1) \arrow[d, "\mathbf{r}_1"] & \times & \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2) \arrow[r, "\diamond"] \arrow[d, "\mathbf{r}_2"] & \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus (F_1 \cap F_2)) \arrow[d, "\mathbf{r}_{12}"] \\
\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F_1') & \times & \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_2') \arrow[r, "\diamond"] & \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus (F_1' \cap F_2'))
\end{tikzcd}$$

*is commutative. Here the $\mathbf{r}_i$ are the restrictions maps defined in (8).*

• *The relative product is associative. More precisely, let $F_1, F_2, F_3$ be three closed invariant subsets of $N$ and take $F = F_1 \cap F_2 \cap F_3$. For any relative classes $a_1 \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F_1)$ and $a_i \in \mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus F_i)$ for $i = 2, 3$, we have $(a_1 \diamond a_2) \diamond a_3 = a_1 \diamond (a_2 \diamond a_3)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$.*

**Proof.** The proof is identical to the one done in Section 3.3 of [19].

## 2.4 Inverse limit of equivariant cohomology with support

Let $U$ be an invariant open subset of $N$.

**Definition 2.6** *A generalized equivariant form $\alpha$ on $N$ belongs to $\mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ if there exists a closed invariant subset $\mathcal{C}_\alpha \subset U$ such that the differential form $\int_{\mathfrak{k}} \alpha(X)Q(X)dX$ is supported in $\mathcal{C}_\alpha$, for any compactly supported density $Q(X)dX$ on $\mathfrak{k}$.*

Note that the vector space $\mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ is naturally a module over $\mathcal{A}^{\infty}(\mathfrak{k}, N)$. An element of $\mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ will be called an equivariant form with support in $U$. Let $\mathcal{A}^{\infty}_U(\mathfrak{k}, N)$ be the intersection of $\mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ with $\mathcal{A}^{\infty}(\mathfrak{k}, N)$ : $\alpha \in \mathcal{A}^{\infty}_U(\mathfrak{k}, N)$ if there exist a closed set $\mathcal{C}_\alpha \subset U$ such that $\alpha(X)|_n = 0$ for all $X \in \mathfrak{k}$ and all $n \in U \setminus \mathcal{C}_\alpha$.

The spaces $\mathcal{A}^{\infty}_U(\mathfrak{k}, N)$ and $\mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ are stable under the differential $D$, and we denote $\mathcal{H}^{\infty}_U(\mathfrak{k}, N)$ and $\mathcal{H}^{-\infty}_U(\mathfrak{k}, N)$ the corresponding cohomology spaces: both are modules over $\mathcal{H}^{\infty}(\mathfrak{k}, N)$.

Let $U, V$ be two invariants open subsets of $N$. The wedge product gives a natural bilinear map

$$\mathcal{H}^{-\infty}_U(\mathfrak{k}, N) \times \mathcal{H}^{\infty}_V(\mathfrak{k}, N) \xrightarrow{\wedge} \mathcal{H}^{-\infty}_{U\cap V}(\mathfrak{k}, N) \tag{12}$$

of $\mathcal{H}^{\infty}(\mathfrak{k}, N)$-modules.

Let $F$ be a closed $K$-invariant subset of $N$. We consider the set $\mathcal{F}_F$ of all open invariant neighborhoods $U$ of $F$ which is ordered by the relation $U \leq V$ if and only if $V \subset U$. If $U \leq V$, we have then the inclusion maps $\mathcal{A}^{\infty}_V(\mathfrak{k}, N) \hookrightarrow \mathcal{A}^{\infty}_U(\mathfrak{k}, N)$ and $\mathcal{A}^{-\infty}_V(\mathfrak{k}, N) \hookrightarrow \mathcal{A}^{-\infty}_U(\mathfrak{k}, N)$ which gives rise to the maps $\mathcal{H}^{\infty}_V(\mathfrak{k}, N) \to \mathcal{H}^{\infty}_U(\mathfrak{k}, N)$ and $\mathcal{H}^{-\infty}_V(\mathfrak{k}, N) \to \mathcal{H}^{-\infty}_U(\mathfrak{k}, N)$ both denoted $f_{U,V}$.

**Definition 2.7** • *We denote by* $\mathcal{H}_F^\infty(\mathfrak{k}, N)$ *the inverse limit of the inverse system* $(\mathcal{H}_U^\infty(\mathfrak{k}, N), f_{U,V}; U, V \in \mathcal{F}_F)$.

• *We denote by* $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$ *be the inverse limit of the inverse system* $(\mathcal{H}_U^{-\infty}(\mathfrak{k}, N), f_{U,V}; U, V \in \mathcal{F}_F)$.

We will call $\mathcal{H}_F^\infty(\mathfrak{k}, N)$ and $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$ the equivariant cohomology of $N$ supported on $F$ (with smooth or generalized coefficients) : both are module over $\mathcal{H}^\infty(\mathfrak{k}, N)$.

Let us give the following basic properties of the equivariant cohomology spaces with support.

**Lemma 2.8** • $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N) = \{0\}$ *if* $F = \emptyset$.

• *There is a natural map* $\mathcal{H}_F^\infty(\mathfrak{k}, N) \to \mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$.

• *There is a natural map* $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N)$. *If* $F$ *is compact, this map factors through* $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N) \to \mathcal{H}_c^{-\infty}(\mathfrak{k}, N)$.

• *If* $F \subset F'$ *are closed* $K$*-invariant subsets, there is a restriction morphism*

$$\mathbf{r}^{F',F} : \mathcal{H}_F^{-\infty}(\mathfrak{k}, N) \to \mathcal{H}_{F'}^{-\infty}(\mathfrak{k}, N). \tag{13}$$

• *If* $F_1$ *and* $F_2$ *are two closed* $K$*-invariant subsets of* $N$*, the wedge product of forms defines a natural product*

$$\mathcal{H}_{F_1}^{-\infty}(\mathfrak{k}, N) \times \mathcal{H}_{F_2}^\infty(\mathfrak{k}, N) \xrightarrow{\wedge} \mathcal{H}_{F_1 \cap F_2}^{-\infty}(\mathfrak{k}, N). \tag{14}$$

• *If* $F_1 \subset F_1'$ *and* $F_2 \subset F_2'$ *are closed* $K$*-invariant subsets, then the diagram*

$$\begin{tikzcd}
\mathcal{H}_{F_1}^{-\infty}(\mathfrak{k}, N) \arrow[d, "\mathbf{r}^1"] & \times & \mathcal{H}_{F_2}^\infty(\mathfrak{k}, N) \arrow[d, "\mathbf{r}^2"] \arrow[r, "\wedge"] & \mathcal{H}_{F_1 \cap F_2}^{-\infty}(\mathfrak{k}, N) \arrow[d, "\mathbf{r}^{12}"] \\
\mathcal{H}_{F_1'}^{-\infty}(\mathfrak{k}, N) & \times & \mathcal{H}_{F_2'}^\infty(\mathfrak{k}, N) \arrow[r, "\wedge"] & \mathcal{H}_{F_1' \cap F_2'}^{-\infty}(\mathfrak{k}, N)
\end{tikzcd} \tag{15}$$

*is commutative. Here the* $\mathbf{r}^i$ *are the restriction morphisms defined in (13).*

**Proof.** The proof of these properties are left to the reader. Note that the product (14) follows from (12).

## 2.5 Morphism $\mathrm{p}_F : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F) \to \mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$

Now we define a natural map from $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$ into $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$.

Let $\beta \in \mathcal{A}^{-\infty}(\mathfrak{k}, N \setminus F)$. If $\chi$ is a $K$-invariant function on $N$ which is identically 1 on a neighborhood of $F$, note that $d\chi\beta$ defines an equivariant form on $N$, since $d\chi$ is equal to 0 in a neighborhood of $F$.

**Proposition 2.9** *For any open invariant neighborhood* $U$ *of* $F$*, we choose* $\chi \in \mathcal{C}^\infty(N)^K$ *with support in* $U$ *and equal to* 1 *in a neighborhood of* $F$.

- *The map*

$$\mathrm{p}^{\chi}(\alpha,\beta) = \chi\alpha + d\chi\beta \tag{16}$$

*defines a homomorphism of complexes* $\mathrm{p}^{\chi} : \mathcal{A}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{A}_U^{-\infty}(\mathfrak{k}, N)$.

*In consequence, let* $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ *be a* closed *equivariant form and* $\beta \in \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$ *such that* $\alpha|_{N\setminus F} = D\beta$*, then* $\mathrm{p}^{\chi}(\alpha,\beta)$ *is a* closed *equivariant form supported in* $U$*.*

- *The cohomology class of* $\mathrm{p}^{\chi}(\alpha,\beta)$ *in* $\mathcal{H}_U^{-\infty}(\mathfrak{k}, N)$ *does not depend of* $\chi$*. We denote this class by* $\mathrm{p}_U(\alpha,\beta) \in \mathcal{H}_U^{-\infty}(\mathfrak{k}, N)$*.*
- *For any neighborhoods* $V \subset U$ *of* $F$*, we have* $f_{U,V} \circ \mathrm{p}_V = \mathrm{p}_U$*.*

**Proof.** The proof is similar to the proof of Proposition 2.3 in [18]. We repeat the main arguments. The equation $\mathrm{p}^{\chi} \circ D_{\mathrm{rel}} = D \circ \mathrm{p}^{\chi}$ is immediate to check. In particular $\mathrm{p}^{\chi}(\alpha,\beta)$ is closed, if $D_{\mathrm{rel}}(\alpha,\beta) = 0$. For two different choices $\chi$ and $\chi'$, we have $\mathrm{p}^{\chi}(\alpha,\beta) - \mathrm{p}^{\chi'}(\alpha,\beta) = D\left((\chi-\chi')\beta\right)$. Since $\chi - \chi' = 0$ in a neighborhood of $F$, the equivariant form $(\chi-\chi')\beta$ is well defined on $N$ and with support in $U$. This proves the second point. Finally, the last point is immediate, since $\mathrm{p}_U(\alpha,\beta) = \mathrm{p}_V(\alpha,\beta) = \mathrm{p}^{\chi}(\alpha,\beta)$ for $\chi \in \mathcal{C}^{\infty}(N)^K$ with support in $V \subset U$.

**Definition 2.10** *Let* $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ *be a* closed *equivariant form and* $\beta \in \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$ *such that* $\alpha|_{N\setminus F} = D\beta$*. We denote by* $\mathrm{p}_F(\alpha,\beta) \in \mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$ *the element defined by the sequence* $\mathrm{p}_U(\alpha,\beta) \in \mathcal{H}_U^{-\infty}(\mathfrak{k}, N)$*,* $U \in \mathcal{F}_F$*. We have then a morphism*

$$\mathrm{p}_F : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{H}_F^{-\infty}(\mathfrak{k}, N). \tag{17}$$

*We will say that the element* $\mathrm{p}_U(\alpha,\beta) \in \mathcal{H}_U^{-\infty}(\mathfrak{k}, N)$ is the $U$-component *of* $p_F(\alpha,\beta)$*.*

In [19], we made the same construction for the equivariant forms with smooth coefficients: for any closed invariant subset $F$ we have a morphism

$$\mathrm{p}_F : \mathcal{H}^{\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{H}_F^{\infty}(\mathfrak{k}, N). \tag{18}$$

The following proposition summarizes the functorial properties of p.

**Proposition 2.11** - *If* $F \subset F'$ *are closed invariant subsets of* $N$*, then the diagram*

$$\begin{array}{ccc} \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) & \xrightarrow{\mathrm{p}_F} & \mathcal{H}_F^{-\infty}(\mathfrak{k}, N) \\ \Big\downarrow \mathbf{r}_1 & & \Big\downarrow \mathbf{r}^1 \\ \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F') & \xrightarrow{\mathrm{p}_{F'}} & \mathcal{H}_{F'}^{-\infty}(\mathfrak{k}, N) \end{array} \tag{19}$$

*is commutative. Here* $\mathbf{r}_1$ *and* $\mathbf{r}^1$ *are the restriction morphisms (see (8) and (13)).*

• *If* $F_1, F_2$ *are closed invariant subsets of* $N$*, then the diagram*

(20)

$$\begin{array}{ccccc} \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F_1) & \times & \mathcal{H}^{\infty}(\mathfrak{k}, N, N\setminus F_2) & \xrightarrow{\diamond} & \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus (F_1\cap F_2)) \\ \downarrow{\scriptstyle \mathrm{p}_{F_1}} & & \downarrow{\scriptstyle \mathrm{p}_{F_2}} & & \downarrow{\scriptstyle \mathrm{p}_{F_1\cap F_2}} \\ \mathcal{H}^{-\infty}_{F_1}(\mathfrak{k}, N) & \times & \mathcal{H}^{\infty}_{F_2}(\mathfrak{k}, N) & \xrightarrow{\wedge} & \mathcal{H}^{-\infty}_{F_1\cap F_2}(\mathfrak{k}, N) \end{array}$$

*is commutative.*

**Proof.** The proof is entirely similar to the proof of Proposition 3.16 in [19] where one considers the case of smooth coefficients.

If we take $F' = N$ in (19), we see that the map $\mathrm{p}_F : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{H}^{-\infty}_F(\mathfrak{k}, N)$ factors the natural map $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{H}^{-\infty}(\mathfrak{k}, N)$.

If $F$ is compact, we can choose a function $\chi$ with compact support and identically equal to 1 in a neighborhood of $F$.

**Definition 2.12** *Let* $F$ *be a* compact $K$*-invariant subset of* $N$*. Choose* $\chi \in \mathcal{C}^\infty(N)^K$ *with compact support and equal to* 1 *in a neighborhood of* $F$*. Let* $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{k}, N)$ *be a* closed *equivariant form and* $\beta \in \mathcal{A}^{-\infty}(\mathfrak{k}, N\setminus F)$ *such that* $\alpha|_{N\setminus F} = D\beta$*. We denote by* $\mathrm{p}_c(\alpha,\beta) \in \mathcal{H}^{-\infty}_c(\mathfrak{k}, N)$ *the class of* $p^\chi(\alpha,\beta) = \chi\alpha + d\chi\beta$ *in* $\mathcal{H}^{-\infty}_c(\mathfrak{k}, N)$*. We have then a morphism*

$$\mathrm{p}_c : \mathcal{H}^{-\infty}(\mathfrak{k}, N, N\setminus F) \to \mathcal{H}^{-\infty}_c(\mathfrak{k}, N). \tag{21}$$

# 3 The relative Chern character: the $\mathcal{C}^{-\infty}$ case

Let $N$ be a manifold equipped with an action of a compact Lie group $K$. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be an equivariant $\mathbb{Z}_2$-graded complex vector bundle on $N$. We recall the construction of the equivariant Chern character of $\mathcal{E}$ that uses Quillen's notion of super-connection (see [6]).

We denote by $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$ the algebra of $\mathrm{End}(\mathcal{E})$-valued differential forms on $N$. Taking in account the $\mathbb{Z}_2$-grading of $\mathrm{End}(\mathcal{E})$, the algebra $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$ is a $\mathbb{Z}_2$-graded algebra. The super-trace on $\mathrm{End}(\mathcal{E})$ extends to a map $\mathrm{Str} : \mathcal{A}(N, \mathrm{End}(\mathcal{E})) \to \mathcal{A}(N)$.

Let $\mathbb{A}$ be a $K$-invariant super-connection on $\mathcal{E}$, and $\mathbf{F} = \mathbb{A}^2$ its curvature, an element of $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$. Recall that, for $X \in \mathfrak{k}$, the *moment* of $\mathbb{A}$ is the equivariant map $\mu^{\mathbb{A}} : \mathfrak{k} \longrightarrow \mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$ defined by the relation $\mu^{\mathbb{A}}(X) = \mathcal{L}(X) - [\iota(VX), \mathbb{A}]$. We define the *equivariant curvature* of $\mathbb{A}$ by

$$\mathbf{F}(X) = \mathbb{A}^2 + \mu^{\mathbb{A}}(X), \quad X \in \mathfrak{k}. \tag{22}$$

We usually denote simply by $\mathbf{F}$ the equivariant curvature, keeping in mind that in the equivariant case, $\mathbf{F}$ is a function from $\mathfrak{k}$ to $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$.

**Definition 3.1** *The equivariant Chern character of* $(\mathcal{E}, \mathbb{A})$ *is the equivariant differential form on* $N$ *defined by* $\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}(\mathrm{e}^{\mathbf{F}})$ *(e.g.* $\mathrm{Ch}(\mathbb{A})(X) = \mathrm{Str}(\mathrm{e}^{\mathbf{F}(X)})$*).*

The form $\mathrm{Ch}(\mathbb{A})$ is equivariantly closed. We will use the following transgression formulas (see [6], chapter 7, [18]).

**Proposition 3.2** • *Let* $\mathbb{A}_t$*, for* $t \in \mathbb{R}$*, be a one-parameter family of* $K$*-invariant super-connections on* $\mathcal{E}$*, and let* $\frac{d}{dt}\mathbb{A}_t \in \mathcal{A}(N, \mathrm{End}(\mathcal{E}))^-$*. Let* $\mathbf{F}_t$ *be the equivariant curvature of* $\mathbb{A}_t$*. Then one has*

$$\frac{d}{dt}\,\mathrm{Ch}(\mathbb{A}_t) = D\left(\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}_t)\,\mathrm{e}^{\mathbf{F}_t}\right)\right). \tag{23}$$

• *Let* $\mathbb{A}(s,t)$ *be a two-parameter family of* $K$*-invariant super-connections. Here* $s,t \in \mathbb{R}$*. We denote by* $\mathbf{F}(s,t)$ *the equivariant curvature of* $\mathbb{A}(s,t)$*. Then:*

$$\begin{aligned}\frac{d}{ds}\,\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)}\right) - \frac{d}{dt}\,\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)}\right)\\ = D\left(\int_0^1 \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,\mathrm{e}^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))\,\mathrm{e}^{(1-u)\mathbf{F}(s,t)}\right)du\right).\end{aligned}$$

In particular, the cohomology class defined by $\mathrm{Ch}(\mathbb{A})$ in $\mathcal{H}^\infty(\mathfrak{k}, N)$ is independent of the choice of the invariant super-connection $\mathbb{A}$ on $\mathcal{E}$. By definition, this is the equivariant Chern character $\mathrm{Ch}(\mathcal{E})$ of $\mathcal{E}$. By choosing $\mathbb{A} = \nabla^+ \oplus \nabla^-$ where $\nabla^\pm$ are connections on $\mathcal{E}^\pm$, this class is just $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$. However, different choices of $\mathbb{A}$ define very different looking representatives of $\mathrm{Ch}(\mathcal{E})$.

## 3.1 The relative Chern character of a morphism

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be an equivariant $\mathbb{Z}_2$-graded complex vector bundle on $N$ and $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a smooth morphism which commutes with the action of $K$. At each point $n \in N$, $\sigma(n) : \mathcal{E}^+_n \to \mathcal{E}^-_n$ is a linear map. The *support* of $\sigma$ is the $K$-invariant closed subset

$$\mathrm{Supp}(\sigma) = \{n \in N \mid \sigma(n) \text{ is not invertible}\}.$$

Let us recall the construction carried in [19] of the relative cohomology class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$. The definition will involve several choices. We choose invariant Hermitianstructures on $\mathcal{E}^\pm$ and an invariant super-connection $\mathbb{A}$ on $\mathcal{E}$ **without** $0$ **exterior degree term**.

Introduce the odd Hermitian endomorphism of $\mathcal{E}$ defined by

$$v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}. \tag{24}$$

Then $v_\sigma^2 = \begin{pmatrix} \sigma^*\sigma & 0 \\ 0 & \sigma\sigma^* \end{pmatrix}$ is a non negative even Hermitian endomorphism of $\mathcal{E}$ which is positive definite on $N \setminus \mathrm{Supp}(\sigma)$.

Consider the family of invariant super-connections $\mathbb{A}^\sigma(t) = \mathbb{A} + it\, v_\sigma,\ t \in \mathbb{R}$ on $\mathcal{E}$. The equivariant curvature of $\mathbb{A}^\sigma(t)$ is thus the map

$$\mathbf{F}(\sigma, \mathbb{A}, t)(X) = -t^2 v_\sigma^2 + it[\mathbb{A}, v_\sigma] + \mathbb{A}^2 + \mu^{\mathbb{A}}(X). \tag{25}$$

Consider the equivariant closed form $\mathrm{Ch}(\sigma, \mathbb{A}, t)(X) := \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right)$ with the *transgression form*

$$\eta(\sigma, \mathbb{A}, t)(X) := -\,\mathrm{Str}\left(iv_\sigma\, \mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right). \tag{26}$$

In [19], we prove the following basic fact.

**Proposition 3.3** *The differential forms* $\mathrm{Ch}(\sigma, \mathbb{A}, t)(X)$ *and* $\eta(\sigma, \mathbb{A}, t)(X)$ *(and all their partial derivatives) tends to* $0$ *exponentially fast when* $t \to \infty$ *uniformly on compact subsets of* $(N \setminus \mathrm{Supp}(\sigma)) \times \mathfrak{k}$*.*

As $iv_\sigma = \frac{d}{dt}\mathbb{A}^\sigma(t)$, we have $\frac{d}{dt}\mathrm{Ch}(\sigma, \mathbb{A}, t) = -D(\eta(\sigma, \mathbb{A}, t))$. After integration, it gives the following equality of equivariant differential forms on $N$

$$\mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\sigma, \mathbb{A}, t) = D\left(\int_0^t \eta(\sigma, \mathbb{A}, s)ds\right), \tag{27}$$

since $\mathrm{Ch}(\mathbb{A}) = \mathrm{Ch}(\sigma, \mathbb{A}, 0)$. Proposition 3.3 allows us to take the limit $t \to \infty$ in (27) on the open subset $N \setminus \mathrm{Supp}(\sigma)$. We get the following important lemma (see [20, 18] for the non-equivariant case).

**Lemma 3.4** *We can define on* $N \setminus \mathrm{Supp}(\sigma)$ *the equivariant differential form with smooth coefficients*

$$\beta(\sigma, \mathbb{A})(X) = \int_0^\infty \eta(\sigma, \mathbb{A}, t)(X)dt, \quad X \in \mathfrak{k}. \tag{28}$$

*We have* $\mathrm{Ch}(\mathbb{A})|_{N\setminus\mathrm{Supp}(\sigma)} = D\left(\beta(\sigma, \mathbb{A})\right)$*.*

We are in the situation of Subsection 2.2. The closed equivariant form $\mathrm{Ch}(\mathbb{A})$ on $N$ and the equivariant form $\beta(\sigma, \mathbb{A})$ on $N \setminus \mathrm{Supp}(\sigma)$ define an even relative cohomology class $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})]$ in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$. We have the following

**Proposition 3.5 ([19])** • *The class* $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})] \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$ *does not depend of the choice of* $\mathbb{A}$*, nor on the invariant Hermitian structure on* $\mathcal{E}$*. We denote it by* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$*.*

• *Let* $F$ *be an invariant closed subset of* $N$*. For* $s \in [0, 1]$*, let* $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ *be a family of equivariant smooth morphisms such that* $\mathrm{Supp}(\sigma_s) \subset F$*. Then all classes* $\mathrm{Ch}_{\mathrm{rel}}(\sigma_s)$ *coincide in* $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus F)$*.*

## 3.2 The relative Chern character deformed by a one-form

If we allow equivariant cohomology with $C^{-\infty}$ coefficients, we can restrict further the support of the equivariant Chern character of a bundle by modifying the term of exterior degree 1 of the super-connection. We use an idea originally due to Witten [23] and systematized in Paradan [14, 15], see also [22]. The idea is to use as further tool of deformation a $K$-invariant **real-valued** one-form $\lambda$ on $N$, or equivalently a $K$-invariant vector field on $N$. This tool was also used earlier in **K**-theory by Atiyah-Segal and Atiyah-Singer for deforming the zero section of $\mathbf{T}^*M$ (see [2, 1]).

We will need some definitions. Let $N$ be a $K$-manifold. Let $\lambda$ be a $K$-invariant **real-valued** one-form on $N$. At each point $n \in N$, $\lambda(n) \in \mathbf{T}_n^*N$.

**Definition 3.6** *The one-form $\lambda$ defines an equivariant map*

$$f_\lambda : N \to \mathfrak{k}^* \tag{29}$$

*by $\langle f_\lambda(n), X\rangle = \langle \lambda(n), V_n X\rangle$.*

**Definition 3.7** • *We define the invariant closed subset of $N$:*

$$C_\lambda = \{f_\lambda = 0\}.$$

• *For any $K$-equivariant smooth morphism $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ on $N$, we define the invariant closed subset*

$$C_{\lambda,\sigma} = C_\lambda \cap \mathrm{Supp}(\sigma).$$

In Section 3.1, we have associated to a $K$-equivariant smooth morphism $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ the relative class $\mathrm{Ch}_{\mathrm{rel}}(\sigma) \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$. Here we consider the cohomology space $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma})$. We have the diagram

$$\begin{tikzcd}
\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma}) \arrow[dr, "\mathbf{r}"] & \\
\mathcal{H}^{\infty}(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma)) \arrow[r, "e"'] & \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma)).
\end{tikzcd}$$

where $\mathbf{r}$ is the restriction morphism, and $e$ is the morphism of extension of coefficients.

The goal of this section is to construct a class $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda) \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma})$ which is equal to $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma))$.

We choose $K$-invariant Hermitian structures on $\mathcal{E}^\pm$ and a $K$-invariant super-connection $\mathbb{A}$ on $\mathcal{E}$ without 0 exterior degree term. Now we will modify $\mathbb{A}$ by introducing a 0 exterior degree term and **we will also modify** its term of exterior degree 1.

Introduce the odd Hermitian endomorphism of $\mathcal{E}$ defined by

$$\begin{pmatrix} \lambda & \sigma^* \\ \sigma & \lambda \end{pmatrix} = \lambda\,\mathrm{Id}_{\mathcal{E}} + v_\sigma.$$

To simplify notations, we may write $\lambda$ instead of $\lambda\,\mathrm{Id}_{\mathcal{E}}$. We consider the family of invariant super-connections

$$\mathbb{A}^{\sigma,\lambda}(t) = \mathbb{A} + it\,(\lambda + v_\sigma), \quad t \in \mathbb{R}.$$

The equivariant curvature of $\mathbb{A}^{\sigma,\lambda}(t)$ is $\mathbf{F}(\sigma,\lambda,\mathbb{A},t) = \mathbf{F}(\sigma,\mathbb{A},t) + itD\lambda$, where $\mathbf{F}(\sigma,\mathbb{A},t)$ is the equivariant curvature of $\mathbb{A}^\sigma(t)$. More explicitly,

$$\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X) = -t^2 v_\sigma^2 - it\langle f_\lambda, X\rangle + \mu^{\mathbb{A}}(X) + it[\mathbb{A}, v_\sigma] + \mathbb{A}^2 + itd\lambda.$$

In particular, the term of 0 exterior degree of $\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X)$ is the section of $\mathrm{End}(\mathcal{E})$ given by $-t^2 v_\sigma^2 - it\langle f_\lambda, X\rangle + \mu^{\mathbb{A}}_{[0]}(X)$. We are interested by the equivariant differential form

$$\mathrm{Ch}(\sigma,\lambda,\mathbb{A},t) := \mathrm{Ch}(\mathbb{A}^{\sigma,\lambda}(t)) = \mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)}\right).$$

Then $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},t) = \mathrm{e}^{itD\lambda}\,\mathrm{Ch}(\sigma,\mathbb{A},t)$ and the transgression forms are:

$$\begin{aligned} \eta(\sigma,\lambda,\mathbb{A},t)(X) &= -\,\mathrm{Str}\left(i(v_\sigma + \lambda)\,\mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X)}\right) \\ &= -\,\mathrm{e}^{itD\lambda(X)}\,\mathrm{Str}\left(i(v_\sigma + \lambda)\,\mathrm{e}^{\mathbf{F}(\sigma,\mathbb{A},t)(X)}\right). \end{aligned}$$

We repeat the argument of Section 3.1. The relation $\frac{d}{dt}\mathrm{Ch}(\sigma,\lambda,\mathbb{A},t) = -D(\eta(\sigma,\lambda,\mathbb{A},t))$ gives after integration the following equality of equivariant differential forms on $N$

$$\mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\sigma,\lambda,\mathbb{A},t) = D\left(\int_0^t \eta(\sigma,\lambda,\mathbb{A},s)ds\right). \tag{30}$$

Now, we will show that we can take the limit of (30) when $t$ goes to $\infty$ on the open subset $N \setminus C_{\lambda,\sigma}$.

In the following proposition, $h^\sigma(n) \geq 0$ denotes the smallest eigenvalue of $v_\sigma^2(n)$. We choose a metric on the tangent bundle to $N$. Thus we obtain a norm $\|-\|$ on $\wedge\mathbf{T}_n^* N \otimes \mathrm{End}(\mathcal{E}_n)$ which varies smoothly with $n \in N$.

**Proposition 3.8** *Let $\mathcal{K}_1 \times \mathcal{K}_2$ be a compact subset of $N \times \mathfrak{k}$. Let $r$ be any positive integer. There exists a constant* cst *(depending of $\mathcal{K}_1, \mathcal{K}_2$, and $r$) such that for any smooth function $Q$ on $\mathfrak{k}$ supported in $\mathcal{K}_2$ we have*

$$\left\| \int_{\mathfrak{k}} \mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X)}\, Q(X)dX \right\|(n) \leq \mathrm{cst}\,\frac{(1+t)^{\dim N}}{(1 + \|tf_\lambda(n)\|^2)^r}\,\|Q\|_{\mathcal{K}_2,2r}\,\mathrm{e}^{-h_\sigma(n)t^2} \tag{31}$$

*for all $t \geq 0$ and $n \in \mathcal{K}_1$.*

**Proof.** We use the first estimate of Proposition 5.6 of the Appendix to estimate the integral

$$\int_{\mathfrak{k}} \mathrm{e}^{-it\langle f_\lambda, X\rangle}\, \mathrm{e}^{-t^2 R(n)+S(n,X)+T(t,n)}\, Q(X)dX$$

with $R(n) = v_\sigma^2(n)$, $S(n,X) = \mu^{\mathbb{A}}(n)(X)$, $T(t,n) = it[\mathbb{A}, v_\sigma](n) + itd\lambda(n) + \mathbb{A}^2(n)$. We obtain the estimate (31) on $\mathcal{K}_1$.

The estimate (31) on the open subset

$$N \setminus C_{\lambda,\sigma} = \{n \in N \mid h_\sigma(n) > 0 \text{ or } \|f_\lambda(n)\| > 0\}.$$

gives the following

**Corollary 3.9** • *For a function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ with compact support, the element of $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))^+$ defined by $\mathcal{I}_Q(t) := \int_{\mathfrak{k}} \mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X)}\, Q(X)dX$ tends rapidly to $0$ when $t \to \infty$, when restricted to the open subset $N \setminus C_{\lambda,\sigma}$.*

• *The integral $\int_0^\infty \mathcal{I}_Q(t)dt$ defines a smooth form on $N \setminus C_{\lambda,\sigma}$ with values in $\mathrm{End}(\mathcal{E})$.*

• *The equivariant Chern form $\mathrm{Ch}(\sigma, \lambda, \mathbb{A}, t)$, when restricted to $N \setminus C_{\lambda,\sigma}$, tends to $0$ as $t$ goes to $\infty$ in the space $\mathcal{A}^{-\infty}(\mathfrak{k}, N \setminus C_{\lambda,\sigma})$.*

• *The family of smooth equivariant forms, $\mathrm{J}_T := \int_0^T \eta(\sigma, \lambda, \mathbb{A}, t)dt$, when restricted to $N \setminus C_{\lambda,\sigma}$, admits a limit in $\mathcal{A}^{-\infty}(\mathfrak{k}, N \setminus C_{\lambda,\sigma})$ as $T$ goes to infinity.*

**Proof.** We consider the estimates (31) when the compact subset $\mathcal{K}_1$ is included in $N \setminus C_{\lambda,\sigma}$. We can choose $c > 0$ such that either $h_\sigma(n) \geq c$ or $\|f_\lambda(n)\|^2 \geq c$, for $n \in \mathcal{K}_1$. Then (31) gives for $t \geq 0$ and $n \in \mathcal{K}_1$ :

$$\text{(32)} \qquad \Big\| \mathcal{I}_Q(t) \Big\|(n) \leq \mathrm{cst}\|Q\|_{\mathcal{K}_2, 2r}(1+t)^{\dim N} \sup\Big(\frac{1}{(1+t^2c)^r}, \mathrm{e}^{-ct^2}\Big).$$

Since $r$ can be chosen large enough, (32) proves the first point: the integral $\int_0^\infty \mathcal{I}_Q(t)dt$ converge on $N \setminus C_{\lambda,\sigma}$. We have to check that it defines a *smooth* form with values in $\mathrm{End}(\mathcal{E})$. If $D(\partial_n)$ is any differential operator acting on $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$, we have to show that, outside $C_{\lambda,\sigma}$, the element of $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$ defined by $\mathcal{I}_Q^D(t) := \int_{\mathfrak{k}} D(\partial_n) \cdot \mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X)}\, Q(X)dX$ tends rapidly to $0$ when $t \to \infty$. This fact follows from the estimate (83) of Proposition 5.6. Then $\int_0^\infty \mathcal{I}_Q(t)dt$ is smooth and we have $D(\partial_n) \cdot \int_0^\infty \mathcal{I}_Q(t)dt = \int_0^\infty \mathcal{I}_Q^D(t)$.

Since we have the relations $\int_{\mathfrak{k}} \mathrm{Ch}(\sigma, \lambda, \mathbb{A}, t)(X)Q(X)dX = \mathrm{Str}\,(\mathcal{I}_Q(t))$ and $\int_{\mathfrak{k}} \mathrm{J}_T(X)Q(X)dX = -i\,\mathrm{Str}\Big((v_\sigma + \lambda)\int_0^T \mathcal{I}_Q(t)dt\Big)$, the last points follow from the first one.

**Remark 3.10** *The estimate (31) still holds when $Q$ is a smooth map from $\mathfrak{k}$ into $\mathcal{A}(N)$ (or $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$). See Remark 5.7.*

We can then define on $N \setminus C_{\lambda,\sigma}$ the equivariant differential odd form with $C^{-\infty}$ coefficients

$$\beta(\sigma, \lambda, \mathbb{A}) = \int_0^\infty \eta(\sigma, \lambda, \mathbb{A}, t)dt. \tag{33}$$

If we take the limit of (30) when $t$ goes to $\infty$ on the open subset $N \setminus C_{\lambda,\sigma}$, we get

$$\mathrm{Ch}(\mathbb{A})|_{N\setminus C_{\lambda,\sigma}} = D\left(\beta(\sigma, \lambda, \mathbb{A})\right) \quad \text{in} \quad \mathcal{A}^{-\infty}(\mathfrak{k}, N \setminus C_{\lambda,\sigma}). \tag{34}$$

**Theorem 3.11** • *The class*

$$[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \lambda, \mathbb{A})] \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma})$$

*does not depend of the choice of* $\mathbb{A}$*, nor on the invariant Hermitian structure on* $\mathcal{E}$*. We denote it by* $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)$*.*

• *Let $F$ be a closed $K$-invariant subset of $N$. For $s \in [0, 1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of smooth $K$-equivariant morphisms and $\lambda_s$ a family of $K$-invariant one-forms such that $C_{\lambda_s,\sigma_s} \subset F$. Then all classes $\mathrm{Ch}_{\mathrm{rel}}(\sigma_s, \lambda_s)$ coincide in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$.*

**Proof.** Let us prove the first point. Let $\mathbb{A}_s, s \in [0, 1]$, be a smooth one-parameter family of invariant super-connections on $\mathcal{E}$ without 0 exterior degree terms. Let $\mathbb{A}(s, t) = \mathbb{A}_s + it(v_\sigma + \lambda)$. Thus $\frac{d}{ds}\mathbb{A}(s, t) = \frac{d}{ds}\mathbb{A}_s$ and $\frac{d}{dt}\mathbb{A}(s, t) = i(v_\sigma + \lambda)$. Let $\mathbf{F}(s, t)$ be the equivariant curvature of $\mathbb{A}(s, t)$. We have

$$\frac{d}{ds}\mathrm{Ch}(\mathbb{A}_s) = D(\gamma_s), \quad \text{with} \quad \gamma_s = \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}_s)\, \mathrm{e}^{\mathbf{F}(s,0)}\right). \tag{35}$$

We have $\eta(\sigma, \lambda, \mathbb{A}_s, t) = -\,\mathrm{Str}(i(v_\sigma + \lambda)\, \mathrm{e}^{\mathbf{F}(s,t)})$. We apply the double transgression formula of Proposition 3.2, and we obtain

$$\frac{d}{ds}\eta(\sigma, \lambda, \mathbb{A}_s, t) = -\frac{d}{dt}\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}_s)\, \mathrm{e}^{\mathbf{F}(s,t)}\right) - D(\nu(s, t)) \tag{36}$$

with $\nu(s, t)(X) = \int_0^1 i\,\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}_s)\, \mathrm{e}^{u\mathbf{F}(s,t)(X)}(v_\sigma + \lambda)\, \mathrm{e}^{(1-u)\mathbf{F}(s,t)(X)}\right) du$.

Let $Q(X)$ be a smooth and compactly supported function on $\mathfrak{k}$. We consider the element of $\mathcal{A}(N, \mathrm{End}(\mathcal{E}))$ defined by

$$I_Q(u, s, t) = \int_{\mathfrak{k}} i(\frac{d}{ds}\mathbb{A}_s)\, \mathrm{e}^{u\mathbf{F}(s,t)(X)}(v_\sigma + \lambda)\, \mathrm{e}^{(1-u)\mathbf{F}(s,t)(X)}\, Q(X)dX,$$

where $u, s \in [0, 1]$ and $t \geq 0$. Now

$$\mathbf{F}(s, t)(X) = -it\langle f_\lambda, X\rangle - t^2 v_\sigma^2 + \mu^{\mathbb{A}_s}(X) + \mathbb{A}_s^2 + t[\mathbb{A}_s, v_\sigma] + itd\lambda.$$

If we write $R = v_\sigma^2$, $S(X) = \mu^{\mathbb{A}_s}(X)$ and $T(t) = \mathbb{A}_s^2 + t[\mathbb{A}_s, v_\sigma] + itd\lambda$, our integral $I_Q(u,s,t)$ is equal to

$$\int_{\mathfrak{k}} i\, \mathrm{e}^{-it\langle f_\lambda, X\rangle} (\frac{d}{ds}\mathbb{A}_s)\, \mathrm{e}^{u(-t^2R+S(X)+T(t))}(v_\sigma{+}\lambda)\, \mathrm{e}^{(1-u)(-t^2R+S(X)+T(t))}\, Q(X)dX.$$

We apply Proposition 5.8 of the Appendix. Let $\mathcal{K}_1\times\mathcal{K}_2$ be a compact subset of $N\times\mathfrak{k}$. Let $r$ be any integer. There exists a constant $\mathrm{cst}>0$, such that: for any $Q\in\mathcal{C}^\infty(\mathfrak{k})$ which is supported in $\mathcal{K}_2$, we have

$$\Big\| I_Q(u,s,t)\Big\|(n) \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}_2,2r}\, \frac{(1+t)^{\dim N}}{(1+t^2\|f_\lambda(n)\|^2)^r}\, \mathrm{e}^{-t^2 h_\sigma(n)}$$

for all $n\in\mathcal{K}_1$, $t\geq 0$, and $(u,s)\in[0,1]^2$.

If the compact subset $\mathcal{K}_1$ is included in $N\setminus C_{\lambda,\sigma}$, we can choose $c>0$ such that either $h_\sigma(n)\geq c$ or $\|f_\lambda(n)\|^2\geq c$, for $n\in\mathcal{K}_1$. Then we have

$$\Big\| I_Q(u,s,t)\Big\|(n) \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}_2,2r}\, (1+t)^{\dim N} \sup\Big(\frac{1}{(1+t^2c)^r}, \mathrm{e}^{-ct^2}\Big),$$

for $n\in\mathcal{K}_1$, $t\geq 0$, and $(u,s)\in[0,1]^2$.

Since $r$ can be chosen large enough, we have proved that $I_Q(u,s,t)\in\mathcal{A}(N,\mathrm{End}(\mathcal{E}))$, when restricted to the open subset $N\setminus C_{\lambda,\sigma}$, is rapidly decreasing in $t$ (uniformly in $(u,s)\in[0,1]^2$). Thanks to Proposition 5.8 of the Appendix, the same holds for any partial derivative $D(\partial_n)I_Q(u,s,t)$. Since

$$\int_{\mathfrak{k}} \nu(s,t)(X)Q(X)dX = \mathrm{Str}\Big(\int_0^1 I_Q(u,s,t)du\Big),$$

the integral $\epsilon_s = \int_0^\infty \nu(s,t)dt$ defines for any $s\in[0,1]$ a generalized equivariant differential form on $N\setminus C_{\lambda,\sigma}$.

So, on the open subset $N\setminus C_{\lambda,\sigma}$, we can integrate (36) in $t$ from 0 to $\infty$: we get

$$\frac{d}{ds}\beta(\sigma,\lambda,\mathbb{A}_s) = \gamma_s - D(\epsilon_s). \tag{37}$$

If we put together (35) and (37), we obtain

$$\begin{aligned}\frac{d}{ds}\left(\mathrm{Ch}(\mathbb{A}_s), \beta(\sigma,\lambda,\mathbb{A}_s)\right) &= (D(\gamma_s), \gamma_s - D(\epsilon_s))\\ &= D_{\mathrm{rel}}(\gamma_s,\epsilon_s).\end{aligned}$$

We have proved that the class $[\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\lambda,\mathbb{A})]\in\mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus C_{\lambda,\sigma})$ does not depend of $s$.

We now prove the second point. We consider the invariant super-connection $\mathbb{A}(s,t) = it(v_{\sigma_s}{+}\lambda_s){+}\mathbb{A}$. Thus $\frac{d}{ds}\mathbb{A}(s,t) = it\frac{d}{ds}(v_{\sigma_s}{+}\lambda_s)$ and $\frac{d}{dt}\mathbb{A}(s,t) = i(v_{\sigma_s}+$

$\lambda_s$). Let $\mathbf{F}(s,t)$ be the curvature of $\mathbb{A}(s,t)$. Let $\eta(\sigma_s,\mathbb{A},t) = -\operatorname{Str}((\frac{d}{dt}\mathbb{A}(s,t))\,\mathrm{e}^{\mathbf{F}(s,t)})$. By the double transgression formula,

$$\frac{d}{ds}\eta(\sigma_s,\lambda_s,\mathbb{A},t) = -\frac{d}{dt}\operatorname{Str}\left(it\Big(\frac{d}{ds}(v_{\sigma_s}+\lambda_s)\Big)\,\mathrm{e}^{\mathbf{F}(s,t)}\right) - D(\nu(s,t)) \tag{38}$$

where the equivariant form $\nu(s,t)(X)$ is given by

$$\nu(s,t)(X) = \int_0^1 \operatorname{Str}\left((it\frac{d}{ds}(v_{\sigma_s}+\lambda_s))\,\mathrm{e}^{u\mathbf{F}(s,t)(X)}(iv_{\sigma_s}+i\lambda_s)\,\mathrm{e}^{(1-u)\mathbf{F}(s,t)(X)}\right)du.$$

We use again Proposition 5.8 of the Appendix, and we see that for any test function $Q(X)$, the integral $\int_{\mathfrak{k}}\nu(s,t)(X)Q(X)dX$ is rapidly decreasing in $t$ on $N\setminus F$. Then the integral $\epsilon_s(X) = \int_0^\infty \nu(s,t)(X)dt$ defines for any $s\in[0,1]$ a generalized equivariant differential form on $N\setminus F$.

So, on the open subset $N\setminus F$, we can integrate (38) in $t$ from 0 to $\infty$. This gives the relation $\frac{d}{ds}\beta(\sigma_s,\lambda_s,\mathbb{A}) = -D(\epsilon_s)$ and then

$$\frac{d}{ds}\left(\operatorname{Ch}(\mathbb{A}),\beta(\sigma_s,\lambda_s,\mathbb{A})\right) = D_{\mathrm{rel}}\left(0,\epsilon_s\right).$$

The class of $(\operatorname{Ch}(\mathbb{A}),\beta(\sigma_s,\lambda_s,\mathbb{A}))$ does not depend of $s$.

With a similar proof, we see that it does not depend on the choice of invariant Hermitianstructure on $\mathcal{E}$.

In particular, we obtain the following corollary.

**Corollary 3.12** • *The classes* $\operatorname{Ch}_{\mathrm{rel}}(\sigma,\lambda)\in\mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus C_{\lambda,\sigma})$ *and* $\operatorname{Ch}_{\mathrm{rel}}(\sigma)\in\mathcal{H}^{\infty}(\mathfrak{k},N,N\setminus\operatorname{Supp}(\sigma))$ *are equal in* $\mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus\operatorname{Supp}(\sigma))$.

• *Let $\sigma$ be a $K$-invariant morphism. Let $\lambda_0$ and $\lambda_1$ be two $K$-invariant one-forms such that $\lambda_0(n)=\lambda_1(n)$ for any $n\in\operatorname{Supp}(\sigma)$. Then $C_{\lambda_0,\sigma}=C_{\lambda_1,\sigma}=F$ and*

$$\operatorname{Ch}_{\mathrm{rel}}(\sigma,\lambda_0) = \operatorname{Ch}_{\mathrm{rel}}(\sigma,\lambda_1)\ \text{in}\ \mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus F).$$

• *Let $\lambda$ be a $K$-invariant one-form. Let $\sigma_0:\mathcal{E}^+\to\mathcal{E}^-$ and $\sigma_1:\mathcal{E}^+\to\mathcal{E}^-$ be two $K$-invariant morphisms such that $\sigma_0(n)=\sigma_1(n)$ for any $n\in C_\lambda$. Then $C_{\lambda,\sigma_0}=C_{\lambda,\sigma_1}=F$ and*

$$\operatorname{Ch}_{\mathrm{rel}}(\sigma_0,\lambda) = \operatorname{Ch}_{\mathrm{rel}}(\sigma_1,\lambda)\ \text{in}\ \mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus F).$$

**Proof.** Indeed, for the first point, we consider the family $\lambda_s = s\lambda$. It is obvious that $\operatorname{Ch}_{\mathrm{rel}}(\sigma,0)=\operatorname{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus\operatorname{Supp}(\sigma))$. For the second point, we consider the family $\lambda_s = s\lambda_0+(1-s)\lambda_1$. For the third point, we consider the family $\sigma_s = s\sigma_0+(1-s)\sigma_1$, and we employ Proposition 3.11.

## 3.3 The trivial bundle and the "non abelian localization theorem"

A particularly important case is the zero morphism $[0]$ between the vector bundles $\mathcal{E}^+ = N \times \mathbb{C}$ and $\mathcal{E}^- = N \times \{0\}$ : $\mathcal{E}^+$ is equipped with the connection $d$, then the invariant real one-form $\lambda$ allows us to deform $d$ in $d + it\lambda$.

Then $c(t, \lambda) = \mathrm{e}^{itD\lambda}$ is the corresponding Chern character, with transgression form $\eta(t, \lambda) = -i\lambda\, \mathrm{e}^{itD\lambda}$. Outside $C_{\lambda,[0]} = C_\lambda$, we can define the generalized equivariant form $\beta(\lambda) = -i\lambda \int_0^\infty \mathrm{e}^{itD\lambda}\, dt$. The following formula, a particular case of Formula (34), is the principle of the Witten localization formula [23].

**Theorem 3.13** *(Non abelian localization theorem) We have*

$$1 = D(\beta(\lambda))$$

*outside* $C_\lambda$.

Morally, we have $\beta(\lambda) = \frac{\lambda}{D\lambda}$, so that $D(\beta(\lambda)) = \frac{D\lambda}{D\lambda} = 1$.

**Definition 3.14** *The class defined by* $(1, \beta(\lambda))$ *in* $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_\lambda)$ *is denoted*

$$\mathrm{P}_{\mathrm{rel}}(\lambda).$$

Let us rewrite Theorem 3.11 in this particular case.

**Theorem 3.15** *Let $F$ be a closed $K$-invariant subset of $N$. For $s \in [0, 1]$, let $\lambda_s$ be a family of $K$-invariant one-forms such that $C_{\lambda_s} \subset F$. Then all classes $\mathrm{P}_{\mathrm{rel}}(\lambda_s)$ coincide in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus F)$.*

Let us give some very simple examples.

• Let $N := \mathbb{R}^2$ with coordinates $(x, y)$. The circle group $S^1$ acts by rotations. We identify its Lie algebra $\mathrm{Lie}(S^1)$ with $\mathbb{R}$. The element $X \in \mathrm{Lie}(S^1)$ produces the vector field $VX = X(y\partial_x - x\partial_y)$. Let $\lambda = xdy - ydx$. Then $C_\lambda = \{(0, 0)\}$. We have $D\lambda(X) = 2dx \wedge dy + X(x^2 + y^2)$. Thus

$$\begin{aligned} \beta(\lambda)(X) &= -i\lambda \int_0^\infty \mathrm{e}^{it(X(x^2+y^2)+2dx\wedge dy)}\, dt \\ &= -i \tfrac{xdy-ydx}{x^2+y^2} \int_0^\infty \mathrm{e}^{itX}\, dt. \end{aligned}$$

The generalized function $X \mapsto -i \int_0^\infty \mathrm{e}^{itX}\, dt$ is equal to the boundary value, denoted by $\frac{1}{X+i0}$, of the function $1/z$. We obtain

$$\mathrm{P}_{\mathrm{rel}}(\lambda) = \left[1, \frac{1}{X + i0} \frac{xdy - ydx}{x^2 + y^2}\right]$$

in $\mathcal{H}^{-\infty}(\mathrm{Lie}(S^1), \mathbb{R}^2, \mathbb{R}^2 \setminus \{(0, 0)\})$.

• Let $N := \mathbf{T}^*S^1 = S^1 \times \mathbb{R}$. The circle group $S^1$ acts freely by rotations on $S^1$. If $(e^{i\theta}, \xi)$ is a point of $\mathbf{T}^*S^1$ with $\xi \in \mathbb{R}$, the Liouville 1-form is $\lambda := -\xi d\theta$.

The element $X \in \mathrm{Lie}(S^1)$ produces the vector field $VX = -X\partial_\theta$. The critical set $C_\lambda$ is $S^1$ embedded in $\mathbf{T}^*S^1$ as the zero section. We have $D\lambda(X) = d\theta d\xi - X\xi$. Thus

$$\begin{aligned} \beta(\lambda)(X) &= -i\lambda \int_0^\infty e^{it(-X\xi + d\theta\wedge d\xi)}\, dt \\ &= i\xi d\theta \int_0^\infty e^{-it\xi X}\, dt \end{aligned}$$

We obtain $\mathrm{P}_{\mathrm{rel}}(\lambda) = [1, \beta(\lambda)]$ in $\mathcal{H}^{-\infty}(\mathrm{Lie}(S^1), \mathbf{T}^*S^1, \mathbf{T}^*S^1 \setminus S^1)$ with

$$\begin{aligned} \beta(\lambda)(X) &= \frac{1}{X - i0} d\theta \qquad \text{if } \xi > 0, \\ \beta(\lambda)(X) &= \frac{1}{X + i0} d\theta \qquad \text{if } \xi < 0. \end{aligned}$$

## 3.4 Tensor product

Let $\mathcal{E}_1, \mathcal{E}_2$ be two equivariant $\mathbb{Z}_2$-graded vector bundles on $N$. The space $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a $\mathbb{Z}_2$-graded vector bundle with even part $\mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^-$ and odd part $\mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$. The super-algebra $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1 \otimes \mathcal{E}_2))$ can be identified with $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1)) \otimes \mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_2))$ where the tensor is taken in the sense of super-algebras.

Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth equivariant morphisms. With the help of invariant Hermitian structures, we define the morphism

$$\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \longrightarrow (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$$

by $\sigma_1 \odot \sigma_2 := \sigma_1 \otimes \mathrm{Id}_{\mathcal{E}_2^+} + \mathrm{Id}_{\mathcal{E}_1^+} \otimes \sigma_2 + \mathrm{Id}_{\mathcal{E}_1^-} \otimes \sigma_2^* + \sigma_1^* \otimes \mathrm{Id}_{\mathcal{E}_2^-}$.

Let $v_{\sigma_1}, v_{\sigma_2}$ and $v_{\sigma_1 \odot \sigma_2}$ be the odd Hermitian endomorphisms associated to $\sigma_1, \sigma_2$ and $\sigma_1 \odot \sigma_2$ (see (24)). Since $v_{\sigma_1 \odot \sigma_2}^2 = v_{\sigma_1}^2 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v_{\sigma_2}^2$, it follows that $\mathrm{Supp}(\sigma_1 \odot \sigma_2) = \mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$.

We proved in [19] that the relative Chern character is multiplicative : the equality $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ holds in $\mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma_1 \odot \sigma_2))$. This property admits the following generalization.

**Theorem 3.16 (The relative Chern character is multiplicative)** *Let $\sigma_1, \sigma_2$ be two equivariant morphisms on $N$. Let $\lambda$ be an invariant one form on $N$. The relative equivariant cohomology classes*

- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \lambda) \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma_1})$,
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_2) \in \mathcal{H}^\infty(\mathfrak{k}, N, N \setminus \mathrm{Supp}(\sigma_2))$,
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \lambda) \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma_1\odot\sigma_2})$

*satisfy the following equality*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \lambda) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$$

*in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma_1\odot\sigma_2})$. Here $\diamond$ is the product of relative classes (see (9)).*

In Subsection 3.3, we considered the zero morphism $[0] : N \times \mathbb{C} \to N \times \{0\}$. Since for any morphism $\sigma$ we have $[0] \odot \sigma = \sigma$, we get the following

**Corollary 3.17** *For any invariant one form $\lambda$, we have*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda) = \mathrm{P}_{\mathrm{rel}}(\lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma)$$

*in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma})$.*

The remaining part of this section is devoted to the proof of Theorem 3.16.

For $k = 1, 2$, we choose invariant super-connections $\mathbb{A}_k$, without 0 exterior degree terms on the $\mathbb{Z}_2$-graded vector bundles $\mathcal{E}_k$. We consider the closed equivariant forms

$$c_1(t) := \mathrm{Ch}(\sigma_1, \lambda, \mathbb{A}_1, t), \quad c_2(t) := \mathrm{Ch}(\sigma_2, \mathbb{A}_2, t)$$

and the transgression forms

$$\eta_1(t) := \eta(\sigma_1, \lambda, \mathbb{A}_1, t), \quad \eta_2(t) := \eta(\sigma_2, \mathbb{A}_2, t)$$

so that $\frac{d}{dt}(c_k(t)) = -D(\eta_k(t))$.

Let $\beta_1 = \int_0^\infty \eta_1(t)dt$ : it is an equivariant form on $U_1 := N \setminus C_{\lambda,\sigma_1}$ with generalized coefficients. Let $\beta_2 = \int_0^\infty \eta_2(t)dt$ : it is an equivariant form on $U_2 := N \setminus \mathrm{Supp}(\sigma_2)$ with smooth coefficients. The representatives of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \lambda)$ and $\mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ are respectively $(c_1(0), \beta_1)$, $(c_2(0), \beta_2)$.

For the symbol $\sigma_1 \odot \sigma_2$, we consider $\mathbb{A}(t) = \mathbb{A} + it(\lambda + v_{\sigma_1 \odot \sigma_2})$ where $\mathbb{A} = \mathbb{A}_1 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes \mathbb{A}_2$. Then $\mathrm{Ch}(\mathbb{A}) = c_1(0)c_2(0)$. Furthermore, it is easy to see that the transgression form for the family $\mathbb{A}(t)$ is

$$\eta(t) = \eta_1(t)c_2(t) + c_1(t)\eta_2(t).$$

Let $\beta_{12} = \int_0^\infty \eta(t)dt$ : it is an equivariant form on

$$\begin{aligned} U &:= N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2) \cap C_\lambda) \\ &= U_1 \bigcup U_2 \end{aligned}$$

with generalized coefficients. A representative of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \lambda)$ is $(c_1(0)c_2(0), \beta_{12})$.

We need the following lemma.

**Lemma 3.18** • *The integral*

$$\mathrm{I}_2 := \iint_{0 \le s \le t} \eta_1(s) \wedge \eta_2(t) ds\, dt$$

*defines an equivariant form with smooth coefficients on $U_2$.*

• *The integral*

$$\mathrm{I}_1 := \iint_{0\le t\le s} \eta_1(s)\wedge\eta_2(t)ds\,dt$$

*defines*[1] *an equivariant form with generalized coefficients on* $U_1$.

• *We have the relations* $D\mathrm{I}_1 = \beta_{12} - \beta_1 c_2(0)$ *on* $U_1$ *and* $D\mathrm{I}_2 = -\beta_{12} + \beta_2 c_1(0)$ *on* $U_2$.

**Proof.** Let $\mathcal{K}_2$ be a compact subset of $U_2$. Let $h_2 > 0$ such that $h_{\sigma_2}(n) \ge h_2$ for $n \in \mathcal{K}_2$. Let $\mathcal{K}$ be a compact subset of $\mathfrak{k}$. From Proposition 3.3, we know that there exists constants cst and cst′ (depending of $\mathcal{K}_2$, $\mathcal{K}$) such that: for $(X,n) \in \mathcal{K}\times\mathcal{K}_2$ we have

$$\Big\|\eta_2(t)(X)\Big\|(n) \le \mathrm{cst}\,(1+t)^{\dim N}\,\mathrm{e}^{-h_2 t^2}, \quad \text{for all } t \ge 0, \tag{39}$$

and

$$\Big\|\eta_1(s)(X)\Big\|(n) \le \mathrm{cst}'\,(1+s)^{\dim N}, \quad \text{for all } s \ge 0. \tag{40}$$

Then, when $0 \le s \le t$, we have, on $\mathcal{K}_2$: $\|\eta_1(s)\wedge\eta_2(t)\| \le \mathrm{cst}"\,(1+t)^{2\dim N}\,\mathrm{e}^{-h_2 t^2}$. So the integral $\mathrm{I}_2$ is absolutely convergent on $0 \le s \le t$. Since similar majoration holds for the partial derivative of $\eta_k$ (relatively to the variables $n \in N$ and $X \in \mathfrak{k}$), the integral $\mathrm{I}_2$ defines a smooth map from $\mathfrak{k}$ into $\mathcal{A}(U_2)$.

Let us prove the second point. Let $\mathcal{K}$ be a compact subset of $\mathfrak{k}$. For any test function $Q(X)$ on $\mathfrak{k}$ supported in $\mathcal{K}$, let us estimate the form $\gamma(s,t,Q) := \int_{\mathfrak{k}} \eta_1(s)(X)\eta_2(t)(X)Q(X)dX$ on $0 \le t \le s$ and on a compact subset $\mathcal{K}_1$ of $U_1$. We have

$$\gamma(s,t,Q) = \int_{\mathfrak{k}} \mathrm{e}^{-is\langle f_\lambda, X\rangle}\,\Upsilon(s,t,X)Q(X)dX$$

where $\Upsilon(s,t,X) = \mathrm{e}^{isd\lambda}\,\mathrm{Str}\left(-i(v_{\sigma_1}+\lambda)\,\mathrm{e}^{\mathbf{F}(\sigma_1,\mathbb{A}_1,s)(X)}\right)\wedge\eta_2(t)(X)$. Let $r$ be a positive integer. If we use the estimates of Proposition 5.6 (see also Remark 5.7), we get

$$\Big\|\gamma(s,t,Q)\Big\|(n) \le \mathrm{cst}\,\Big\|\eta_2(t)Q\Big\|_{\mathcal{K},2r}(n)\frac{(1+s)^{\dim N}}{(1+s^2\|f_\lambda(n)\|^2)^r}\,\mathrm{e}^{-h_1(n)s^2},$$

for all $t,s \ge 0$ and $n \in \mathcal{K}_1$. Here cst is a constant depending of $r, \mathcal{K}_1, \mathcal{K}$, and $h_1(n) \ge 0$ is the smallest eigenvalue of $v_{\sigma_1}^2(n)$.

The term $\|\eta_2(t)Q\|_{\mathcal{K},2r}(n)$ is smaller than $\|Q\|_{\mathcal{K},2r}\|\eta_2(t)\|_{\mathcal{K},2r}(n)$. If we use the second point of Proposition 5.3 of the Appendix, we see that

$$\Big\|\eta_2(t)\Big\|_{\mathcal{K},2r}(n) \le \mathrm{cst}'(1+t)^{\dim N}, \quad \text{for all } n \in \mathcal{K}_1,\ t \ge 0.$$

[1] The integral $\mathrm{I}_1$ is the limit when $T \to \infty$ of the family $(\iint_{0\le t\le s\le T}\eta_1(s)\wedge\eta_2(t)dsdt)_{T>0}$ of equivariant forms with smooth coefficients.

Finally, for $0 \leq t \leq s$ and $n \in \mathcal{K}_1$, we have:

$$\Big\|\gamma(s,t,Q)\Big\|(n) \leq \mathrm{cst}'' \Big\|Q\Big\|_{\mathcal{K},2r} \frac{(1+s)^{2\dim N}}{(1+s^2\|f_\lambda(n)\|^2)^r}\, \mathrm{e}^{-h_1(n)s^2}\,. \tag{41}$$

If the compact subset $\mathcal{K}_1$ is included in $N \setminus C_{\lambda,\sigma_1}$, we can choose $c > 0$ such that either $h_1(n) \geq c$ or $\|f_\lambda(n)\|^2 \geq c$, for $n \in \mathcal{K}_1$. Then we have

$$\Big\|\gamma(s,t,Q)\Big\|(n) \leq \mathrm{cst}''\, \|Q\|_{\mathcal{K},2r}\, (1+s)^{2\dim N} \sup\Big(\frac{1}{(1+s^2c)^r}, \mathrm{e}^{-cs^2}\Big),$$

for $n \in \mathcal{K}_1$ and $0 \leq t \leq s$.

Since $r$ can be chosen large enough, we have proved that the integral of the differential forms $\gamma(s,t,Q)$ on $0 \leq t \leq s$ is absolutely convergent. Since similar majoration holds for the partial derivative of $\gamma(s,t,Q)$ (relatively to the variables $n \in N$ and $X \in \mathfrak{k}$). the integral $\mathrm{I}_1(X)$ defines a $\mathcal{C}^{-\infty}$-map from $\mathfrak{k}$ to $\mathcal{A}(U_1)$ by the relation $\int_{\mathfrak{k}} \mathrm{I}_1(X)Q(X)dX := \iint_{0\leq t\leq s} \gamma(s,t,Q)dsdt$.

For the last point we compute

$$\begin{aligned} D(\mathrm{I}_1) &= D\left(\iint_{0\leq t\leq s} \eta_1(s)\eta_2(t)ds\,dt\right) \\ &= \iint_{0\leq t\leq s} \Big(D\eta_1(s)\eta_2(t) - \eta_1(s)D\eta_2(t)\Big)ds\,dt. \end{aligned}$$

Now we use $D(\eta_j(s)) = -\frac{d}{ds}c_j(s)$, so that we obtain

$$\begin{aligned} D(\mathrm{I}_1) &= \iint_{0\leq t\leq s} \left((-\frac{d}{ds}c_1(s))\eta_2(t) + \eta_1(s)(\frac{d}{dt}c_2(t))\right) ds\,dt \\ &= \Big(\int_0^\infty c_1(t)\eta_2(t)dt + \int_0^\infty \eta_1(s)c_2(s)ds\Big) - c_2(0)\beta_1 \\ &= \beta_{12} - c_2(0)\beta_1. \end{aligned}$$

Similarly, we compute $D(\mathrm{I}_2) = -\beta_{12} + c_1(0)\beta_2$.

Let $\Phi_1 + \Phi_2 = 1_U$ be a partition of unity subordinate to the decomposition $U = U_1 \cup U_2$ : the functions $\Phi_k$ are supposed $K$-invariant. We consider $\mathrm{I}_\Phi := \Phi_1\mathrm{I}_1 - \Phi_2\mathrm{I}_2$ which is an equivariant form with generalized coefficients on $U$. We now prove that

$$\Big(c_1(0),\beta_1\Big) \diamond_\Phi \Big(c_2(0),\beta_2\Big) - \Big(c_1(0)c_2(0),\beta_{12}\Big) = D_{\mathrm{rel}}\Big(0,\mathrm{I}_\Phi\Big). \tag{42}$$

Indeed the product $(c_1(0),\beta_1) \diamond_\Phi (c_2(0),\beta_2)$ is equal to $\Big(c_1(0)c_2(0), \Phi_1\beta_1c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2\Big)$, so that the first member of Equality (42) is $\Big(0, \Phi_1\beta_1c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}\Big)$. Thus we need to check that

$$-D(\mathrm{I}_\Phi) = \Phi_1\beta_1c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}. \tag{43}$$

Using the last point of Lemma 3.18, we have

$$\begin{aligned} -D(\mathrm{I}_\Phi) &= d\Phi_2 \mathrm{I}_2 - d\Phi_1 \mathrm{I}_1 + \Phi_2 D\mathrm{I}_2 - \Phi_1 D\mathrm{I}_1 \\ &= -d\Phi_1(\mathrm{I}_2 + \mathrm{I}_1) + \Phi_2(-\beta_{12} + c_1(0)\beta_2) - \Phi_1(\beta_{12} - c_2(0)\beta_1) \\ &= -d\Phi_1 \beta_1\beta_2 - \beta_{12} + \Phi_2 c_1(0)\beta_2 + \Phi_1 c_2(0)\beta_1. \end{aligned}$$

which was the equation to prove. Here we have used that $\Phi_1 + \Phi_2 = 1_U$, hence $d\Phi_2 = -d\Phi_1$.

## 3.5 The Chern character deformed by a one form

Let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be an equivariant morphism on $N$, and $\lambda$ be an invariant one form on $N$. Following Section 2.5, we consider the image of the relative class $\mathrm{Ch}_{\mathrm{rel}}(\sigma, \lambda)$ through the map

$$\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma}) \to \mathcal{H}^{-\infty}_{C_{\lambda,\sigma}}(\mathfrak{k}, N).$$

The following theorem summarizes the construction of the image.

**Theorem 3.19** • *For any invariant neighborhood $U$ of $C_{\lambda,\sigma}$, take $\chi \in \mathcal{C}^\infty(N)^K$ which is equal to 1 in a neighborhood of $C_{\lambda,\sigma}$ and with support contained in $U$. Then*

$$c(\sigma, \lambda, \mathbb{A}, \chi) = \chi\ \mathrm{Ch}(\mathbb{A}) + d\chi\, \beta(\sigma, \lambda, \mathbb{A}) \tag{44}$$

*is an equivariant closed differential form with generalized coefficients, supported in $U$. Its cohomology class $c_U(\sigma, \lambda) \in \mathcal{H}^{-\infty}_U(\mathfrak{k}, N)$ does not depend of the choice of $\mathbb{A}, \chi$ and the invariant Hermitian structures on $\mathcal{E}^\pm$. Furthermore, the inverse family $c_U(\sigma, \lambda)$ when $U$ runs over the neighborhoods of $C_{\lambda,\sigma}$ defines a class*

$$\mathrm{Ch}_{\mathrm{sup}}(\sigma, \lambda) \in \mathcal{H}^{-\infty}_{C_{\lambda,\sigma}}(\mathfrak{k}, N).$$

• *The image of $\mathrm{Ch}_{\mathrm{sup}}(\sigma, \lambda)$ in $\mathcal{H}^{-\infty}_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, N)$ is equal to $\mathrm{Ch}_{\mathrm{sup}}(\sigma)$.*

• *Let $F$ be a closed $K$-invariant subset of $N$. For $s \in [0, 1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of smooth $K$-equivariant morphisms and $\lambda_s$ a family of $K$-invariant one-forms such that $C_{\lambda_s,\sigma_s} \subset F$. Then all classes $\mathrm{Ch}_{\mathrm{sup}}(\sigma_s, \lambda_s)$ coincide in $\mathcal{H}^{-\infty}_F(\mathfrak{k}, N)$.*

**Definition 3.20** *When $C_{\lambda,\sigma}$ is a* compact *subset of $N$, we define*

$$\mathrm{Ch}_{\mathrm{c}}(\sigma, \lambda) \in \mathcal{H}^{-\infty}_c(\mathfrak{k}, N)$$

*as the image of $\mathrm{Ch}_{\mathrm{sup}}(\sigma, \lambda) \in \mathcal{H}^{-\infty}_{C_{\lambda,\sigma}}(\mathfrak{k}, N)$ in $\mathcal{H}^{-\infty}_c(\mathfrak{k}, N)$. A representative of $\mathrm{Ch}_{\mathrm{c}}(\sigma, \lambda)$ is given be the equivariant form $c(\sigma, \lambda, \mathbb{A}, \chi)$, with $\chi$ compactly supported.*

When $\sigma$ is *elliptic*, we have already a class $\mathrm{Ch}_c(\sigma) \in \mathcal{H}_c^\infty(\mathfrak{k}, N)$ with compact support. If we use the second point of Theorem 3.19, one sees that

$$\mathrm{Ch}_c(\sigma) = \mathrm{Ch}_c(\sigma, \lambda) \quad \text{in} \quad \mathcal{H}_c^{-\infty}(\mathfrak{k}, N),$$

for any invariant one-form $\lambda$. So the class with compact support $\mathrm{Ch}_c(\sigma, \lambda)$ will be of interest when $\sigma$ is *not elliptic*, but $C_{\lambda,\sigma}$ is *compact*.

In Subsection 3.3, we considered the zero morphism $[0] : N \times \mathbb{C} \to N \times \{0\}$ and its relative Chern character $\mathrm{P}_{\mathrm{rel}}(\lambda) \in \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_\lambda)$. The associated generalized equivariant class in $\mathcal{H}_{C_\lambda}^{-\infty}(\mathfrak{k}, N)$ will be denoted $\mathrm{Par}(\lambda)$ : this class was defined in [14, 15]. We repeat Theorem 3.19 for this special case.

Recall that $\beta(\lambda)(X) := -i\lambda \int_0^\infty \mathrm{e}^{itD(\lambda)(X)}\, dt$ is an equivariant form with generalized coefficients on $N \setminus C_\lambda$.

**Theorem 3.21** *[14]* • *Let $\chi \in \mathcal{C}^\infty(N)$ be a $K$-invariant function which is equal to 1 in a neighborhood of $C_\lambda$ and with support contained in $U$. The equivariant differential form* $\mathrm{Par}(\lambda, \chi) = \chi + d\chi\, \beta(\lambda)$ *is an* equivariantly closed *differential form with $C^{-\infty}$ coefficients, supported in $U$. Its cohomology class* $\mathrm{Par}_U(\lambda) \in \mathcal{H}_U^{-\infty}(\mathfrak{k}, N)$ *does not depend of the choice of $\chi$. Furthermore, the inverse family* $\mathrm{Par}_U(\lambda)$ *when $U$ runs over the neighborhoods of $C_\lambda$ defines a class*

$$\mathrm{Par}(\lambda) \in \mathcal{H}_{C_\lambda}^{-\infty}(\mathfrak{k}, N). \tag{45}$$

• *The image of this class in $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ coincides with* 1.

• *Let $F$ be a closed $K$-invariant subset of $N$. For $s \in [0, 1]$, let $\lambda_s$ be a family of $K$-invariant one-forms such that $C_{\lambda_s} \subset F$. Then all classes* $\mathrm{Par}(\lambda_s)$ *coincide in $\mathcal{H}_F^{-\infty}(\mathfrak{k}, N)$.*

We proved in Theorem 3.16 (see also Corollary 3.17) that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \lambda)$ is equal to the product $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus C_{\lambda,\sigma_1\odot\sigma_2})$. If we use the commutativity of the diagram (20) for the closed invariant subsets $F_1 := C_{\lambda,\sigma_1}$, $F_2 := \mathrm{Supp}(\sigma_2)$ and $F_1 \cap F_2 = C_{\lambda,\sigma_1\odot\sigma_2}$, we get

**Theorem 3.22** *We have the following relation in $\mathcal{H}_{C_{\lambda,\sigma_1\odot\sigma_2}}^{-\infty}(\mathfrak{k}, N)$ :*

$$\mathrm{Ch}_{\sup}(\sigma_1 \odot \sigma_2, \lambda) = \mathrm{Ch}_{\sup}(\sigma_1, \lambda) \wedge \mathrm{Ch}_{\sup}(\sigma_2). \tag{46}$$

*In particular, if $\sigma_1 = [0]$, we have*

$$\mathrm{Ch}_{\sup}(\sigma, \lambda) = \mathrm{Par}(\lambda) \wedge \mathrm{Ch}_{\sup}(\sigma) \quad \text{in} \quad \mathcal{H}_{C_{\lambda,\sigma}}^{-\infty}(\mathfrak{k}, N).$$

## 3.6 Product of groups

Let $K_1, K_2$ be two compact Lie groups, with Lie algebras $\mathfrak{k}_1, \mathfrak{k}_2$, and $N$ a $K_1 \times K_2$ manifold. We wish to multiply two elements $\alpha_1(X, Y)$ and $\alpha_2(X, Y)$ of $\mathcal{A}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. The product will be well defined if $\alpha_1(X, Y)$ depends smoothly on $Y$, while $\alpha_2(X,Y)$ depends smoothly on $X$. We introduce thus $\mathcal{A}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$ as the space of generalized equivariant forms $\alpha(X,Y)$ depending smoothly on $Y$: for any compactly supported function $Q\in\mathcal{C}^\infty(\mathfrak{k}_1)$, the integral $\alpha_Q(Y):=\int_{\mathfrak{k}_1}\alpha(X,Y)Q(X)dX$ converges and depends smoothly of $Y\in\mathfrak{k}_2$. We denote by $\mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$ the corresponding cohomology space of equivariant cohomology classes $\alpha(X,Y)$ depending smoothly on $Y\in\mathfrak{k}_2$. Similarly we define the space $\mathcal{H}_c^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$ of equivariant cohomology classes with compact support depending smoothly on $Y\in\mathfrak{k}_2$. If $F$ is a closed $K_1\times K_2$ invariant subset of $N$, then we define similarly $\mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N,N\setminus F)$ as well as $\mathcal{H}_F^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$. If $F_1,F_2$ are two closed $K_1\times K_2$-invariant subsets of $N$, the product

$$
\begin{aligned}
(47)\quad \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N,N\setminus F_1) \quad &\times \quad \mathcal{H}^{\infty,-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N,N\setminus F_2)\\
&\xrightarrow{\quad\diamond\quad} \mathcal{H}^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N,N\setminus(F_1\cap F_2))
\end{aligned}
$$

is well defined by the same formula as in Definition 2.4.

Similarly the wedge product

$$
(48)\qquad \mathcal{H}_{F_1}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)\times\mathcal{H}_{F_2}^{\infty,-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)\xrightarrow{\wedge}\mathcal{H}_{F_1\cap F_2}^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N)
$$

is well defined and the map $\mathrm{p}_F$ (Section 2.5) is compatible with the products.

### 3.6.1 The case of 1-forms

Consider a $K_1\times K_2$-manifold $N$ and a $K_1\times K_2$-invariant one form on $N$ denoted $\lambda$. We write the map $f_\lambda$ from $N$ into $\mathfrak{k}_1^*\times\mathfrak{k}_2^*$ as $f_\lambda:=(f_\lambda^1,f_\lambda^2)$. We have

$$C_\lambda=C_\lambda^1\cap C_\lambda^2$$

where $C_\lambda^i:=\{f_\lambda^i=0\}$.

Consider on $N\setminus C_\lambda$, the equivariant form with generalized coefficients

$$\beta(\lambda)(X,Y)=-i\lambda\int_0^\infty \mathrm{e}^{itD\lambda(X,Y)}\,dt\quad (X,Y)\in\mathfrak{k}_1\times\mathfrak{k}_2.$$

**Lemma 3.23** *The restriction of $\beta(\lambda)$ to the open subset $N\setminus C_\lambda^1\subset N\setminus C_\lambda$ defines a generalized function of $(X,Y)\in\mathfrak{k}_1\times\mathfrak{k}_2$ which depends smoothly of $Y$. In other words, $\beta(\lambda)|_{N\setminus C_\lambda^1}$ belongs to $\mathcal{A}^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N\setminus C_\lambda^1)$.*

**Proof.** Let us check that, for ¡any compactly supported function $Q\in\mathcal{C}^\infty(\mathfrak{k}_1)$, the integral $\mathrm{c}_Q(Y):=\int_{\mathfrak{k}_1}\beta(\lambda)(X,Y)Q(X)dX$ converges in $\mathcal{A}(N\setminus C_\lambda^1)$ and depends smoothly of $Y\in\mathfrak{k}_2$.

Consider the form on $N$ defined by $\mathrm{I}_Q(t,Y):=\int_{\mathfrak{k}_1}\mathrm{e}^{itD\lambda(X,Y)}\,Q(X)dX$. At a point $n\in N$, $\mathrm{I}_Q(t,Y)(n)$ is equal to

$$\mathrm{e}^{-it\langle f_\lambda^2(n),Y\rangle}\,\mathrm{e}^{itd\lambda(n)}\int_{\mathfrak{k}_1}\mathrm{e}^{-it\langle f_\lambda^1(n),X\rangle}\,Q(X)dX=\mathrm{e}^{-it\langle f_\lambda^2(n),Y\rangle}\,\mathrm{e}^{itd\lambda(n)}\,\widehat{Q}(t\,f_\lambda^1(n)).$$

The term $\mathrm{e}^{itd\lambda(n)}$ is a polynomial in $t$ of degree bounded by $\dim N$. Since the Fourier transform $\widehat{Q}$ is rapidly decreasing, we have, for any integer $r$, the estimate

$$\left\| \mathrm{I}_Q(t,Y)(n) \right\| \leq \mathrm{cst}\ \frac{(1+t)^{\dim N}}{(1+t^2\|f^1_\lambda(n)\|^2)^r}, \tag{49}$$

for all $t \geq 0$ and $(n,Y)$ in a compact subset of $N \times \mathfrak{k}_2$.

Let $\mathcal{K}$ be a compact subset of $N \setminus C^1_\lambda$ : one can choose $c > 0$ such that $\|f^1_\lambda\| > c$ on $\mathcal{K}$. For any integer $q$, we have then the estimate

$$\left\| \mathrm{I}_Q(t,Y)(n) \right\| \leq \frac{\mathrm{cst}}{(1+(ct)^2)^q}, \tag{50}$$

for all $t \geq 0$, $n \in \mathcal{K}$, and $Y$ in a compact subset of $\mathfrak{k}_2$. Thus the integral $\int_0^\infty \mathrm{I}_Q(t,Y)dt$ is absolutely convergent. Since the estimates (49) and (50) hold for any derivative $D(\partial_{n,Y})\mathrm{I}_Q(t,Y)$ in the variable $(n,Y)$, the integral $\mathrm{c}_Q(Y) = -i\lambda\int_0^\infty \mathrm{I}_Q(t,Y)dt$ defines a smooth map from $\mathfrak{k}_2$ into $\mathcal{A}(N \setminus C^1_\lambda)$.

**Definition 3.24** *We define* $\mathrm{P}^1_{\mathrm{rel}}(\lambda) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^1_\lambda)$ *to be the relative class*

$$\mathrm{P}^1_{\mathrm{rel}}(\lambda) = \left[1, \beta(\lambda)|_{N\setminus C^1_\lambda}\right].$$

Assume now that we have two $K_1 \times K_2$-invariant one forms $\lambda$ and $\mu$ on $N$. We write $f_\lambda := (f^1_\lambda, f^2_\lambda)$ and $f_\mu := (f^1_\mu, f^2_\mu)$. Let $C^1_\lambda := \{f^1_\lambda = 0\}$, and $C^2_\mu := \{f^2_\mu = 0\}$. Then the form $\mathrm{P}^1_{\mathrm{rel}}(\lambda)(X,Y) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^1_\lambda)$ depends smoothly on $Y \in \mathfrak{k}_2$, while the form $\mathrm{P}^2_{\mathrm{rel}}(\mu)(X,Y) \in \mathcal{H}^{\infty,-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^2_\mu)$ depends smoothly of $X \in \mathfrak{k}_1$, and one can form the product of the relative classes:

$$\mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu) \in \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (C^1_\lambda \cap C^2_\mu)).$$

We consider the invariant one form $\lambda + \mu$ and the associated map $(f^1_\lambda + f^1_\mu, f^2_\lambda + f^2_\mu) : N \longrightarrow \mathfrak{k}^*_1 \times \mathfrak{k}^*_2$ which vanishes on

$$C_{\lambda+\mu} := \left\{f^1_\lambda + f^1_\mu = 0\right\} \bigcap \left\{f^2_\lambda + f^2_\mu = 0\right\}.$$

Let $\mathrm{P}_{\mathrm{rel}}(\lambda+\mu) \in \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C_{\lambda+\mu})$ be the relative class associated to $\lambda+\mu$.

We take some invariant norms on $\mathfrak{k}^*_1, \mathfrak{k}^*_2$, and we consider the following functions on $N$ : $\|f^1_\lambda\|$, $\|f^2_\lambda\|$, $\|f^1_\mu\|$, $\|f^2_\mu\|$. In order to compare $\mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu)$ with $\mathrm{P}_{\mathrm{rel}}(\lambda+\mu)$, we introduce the following

**Definition 3.25** *We define the closed invariant subset*

$$\mathcal{C}(\lambda,\mu) := \left\{\|f^1_\lambda\| \leq \|f^1_\mu\|\right\} \bigcap \left\{\|f^2_\mu\| \leq \|f^2_\lambda\|\right\}. \tag{51}$$

Clearly the set $\mathcal{C}(\lambda,\mu)$ contains $C^1_\lambda \cap C^2_\lambda$ as well as the set $\mathcal{C}_{\lambda+\mu}$. Thus the following restriction operations are well defined:

- $\mathbf{r} : \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (C^1_\lambda \cap C^2_\mu)) \to \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus \mathcal{C}(\lambda, \mu))$,
- $\mathbf{r}' : \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C_{\lambda+\mu}) \to \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus \mathcal{C}(\lambda, \mu))$.

The aim of this section is to prove the following theorem.

**Theorem 3.26** *We have the following equality*

$$\mathbf{r}\left(\mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu)\right) = \mathbf{r}'\left(\mathrm{P}_{\mathrm{rel}}(\lambda + \mu)\right) \tag{52}$$

*in* $\mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus \mathcal{C}(\lambda, \mu))$.

**Proof.** Consider the closed invariant sets $F_1 = \left\{\|f^1_\lambda\| \leq \|f^1_\mu\|\right\}$, $F_2 := \left\{\|f^2_\mu\| \leq \|f^2_\lambda\|\right\}$. Then $C^1_\lambda \subset F_1$ and $C^2_\mu \subset F_2$. The form $\mathrm{P}^1_{\mathrm{rel}}(\lambda) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^1_\lambda)$ restricts to $\mathrm{P}^{\mathrm{F}_1}_{\mathrm{rel}}(\lambda) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus F_1)$ while $\mathrm{P}^2_{\mathrm{rel}}(\mu)$ restricts to $\mathrm{P}^{\mathrm{F}_2}_{\mathrm{rel}}(\mu) \in \mathcal{H}^{\infty,-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus F_2)$. Using the diagram (11), we see that $\mathbf{r}\left(\mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu)\right) = \mathrm{P}^{\mathrm{F}_1}_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^{\mathrm{F}_2}_{\mathrm{rel}}(\mu)$.

We thus need to compare the forms $\mathrm{P}^{\mathrm{F}_1}_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^{\mathrm{F}_2}_{\mathrm{rel}}(\mu)$ and $\mathrm{P}_{\mathrm{rel}}(\lambda+\mu)$. We work with the invariant open subsets $U_i := N \setminus F_i$ and $U = U_1 \cup U_2 = N \setminus \mathcal{C}(\lambda, \mu)$. The relative classes $\mathrm{P}^{\mathrm{F}_1}_{\mathrm{rel}}(\lambda)$, $\mathrm{P}^{\mathrm{F}_2}_{\mathrm{rel}}(\mu)$ and $\mathbf{r}'(\mathrm{P}_{\mathrm{rel}}(\lambda + \mu))$ are represented by the couples $(1, \beta_1(\lambda))$, $(1, \beta_2(\mu))$ and $(1, \beta(\lambda + \mu)|_U)$ :

- the form $\beta(\lambda)(X, Y) := -i\lambda \int_0^\infty \mathrm{e}^{itD\lambda(X,Y)}\, dt$ defines on $N \setminus C^1_\lambda$ an equivariant form with generalized coefficients depending smoothly on $Y$. We denote by $\beta_1(\lambda)(X, Y)$ the restriction of this form on $U_1$.
- the form $\beta(\mu)(X, Y) := -i\mu \int_0^\infty \mathrm{e}^{itD\mu(X,Y)}\, dt$ defines on $N \setminus C^2_\mu$ an equivariant form with generalized coefficients depending smoothly on $X$. We denote by $\beta_2(\mu)(X, Y)$ the restriction of this form on $U_2$.
- the form $\beta(\lambda + \mu)(X, Y) := -i(\lambda + \mu) \int_0^\infty \mathrm{e}^{itD(\lambda+\mu)(X,Y)}\, dt$ defines on $N \setminus C_{\lambda+\mu}$ an equivariant form with generalized coefficients. We denote by $\beta(\lambda+\mu)|_U(X, Y)$ the restriction of this form on $U$.

Let $\Phi_1 + \Phi_2 = 1$ be a partition of unity on $U = U_1 \cup U_2$ : the function $\Phi_i$ are supposed $K_1 \times K_2$-invariant. We want to prove that

$$(1, \beta_1(\lambda)) \diamond_\Phi (1, \beta_2(\mu)) - (1, \beta(\lambda + \mu)|_U)$$

is $D_{\mathrm{rel}}$-exact.

Let $\eta_1(s) = -i\lambda\, \mathrm{e}^{isD\lambda(X,Y)}$ and $\eta_2(s) = -i\mu\, \mathrm{e}^{isD\mu(X,Y)}$. We consider the family of smooth equivariant forms on $N$

$$\begin{aligned} \gamma_{(s,t)}(X, Y) &:= \eta_1(s) \wedge \eta_2(t) \\ &= -\lambda\mu\, \mathrm{e}^{isd\lambda + itd\mu}\, \mathrm{e}^{-i\langle f_{(s,t)}, (X,Y)\rangle} \end{aligned}$$

where $f_{(s,t)} : N \to \mathfrak{k}^*_1 \times \mathfrak{k}^*_2$ is equal to $(sf^1_\lambda + tf^1_\mu, sf^2_\lambda + tf^2_\mu)$.

**Lemma 3.27** • *The integral* $\mathrm{I}_1 := \iint_{0 \leq t \leq s} \gamma_{(s,t)} ds\, dt$ *defines a* $K_1 \times K_2$*-equivariant form with generalized coefficients on* $\overline{U_1}$.

• *The integral* $\mathrm{I}_2 := \iint_{0\leq s\leq t} \gamma_{(s,t)} ds\, dt$ *defines a* $K_1 \times K_2$*-equivariant form with generalized coefficients on* $U_2$.

• *We have* $D(\mathrm{I}_1) = \beta(\lambda+\mu)|_{U_1} - \beta_1(\lambda)$ *on* $U_1$ *and* $D(\mathrm{I}_2) = -\beta(\lambda+\mu)|_{U_2} + \beta_2(\mu)$ *on* $U_2$.

**Proof.** For any compactly supported function $Q(X,Y) \in \mathcal{C}^\infty(\mathfrak{k}_1 \times \mathfrak{k}_2)$ we consider the integral $\mathrm{J}_Q(s,t) := \int_{\mathfrak{k}_1\times\mathfrak{k}_2} \gamma_{(s,t)}(X,Y)Q(X,Y)dXdY$. At a point $n \in N$, $\mathrm{J}_Q(s,t)(n)$ is equal to

$$-\lambda\mu\, \mathrm{e}^{isd\lambda(n)+itd\mu(n)} \int_{\mathfrak{k}_1\times\mathfrak{k}_2} \mathrm{e}^{-i\langle f_{(s,t)}(n),(X,Y)\rangle}\, Q(X,Y)dXdY = \\ -\lambda\mu\, \mathrm{e}^{isd\lambda(n)+itd\mu(n)}\, \widehat{Q}(f_{(s,t)}(n)).$$

Since $\widehat{Q}$ is rapidly decreasing, and $\mathrm{e}^{isd\lambda(n)+itd\mu(n)}$ is a polynomial in the variable $(s,t)$, we have, for any integer $r$, the estimate

$$\Big\|\mathrm{J}_Q(s,t)(n)\Big\| \leq \frac{P(s,t)}{(1+\|f_{(s,t)}(n)\|^2)^r},$$

for all $s,t \geq 0$ and $n$ in a compact subset of $N$. Here $P(s,t)$ is a polynomial function and $\|f_{(s,t)}(n)\|^2 = \|sf^1_\lambda(n) + tf^1_\mu(n)\|^2 + \|sf^2_\lambda(n) + tf^2_\mu(n)\|^2$.

Let us prove the first point. We work on a compact subset $\mathcal{K}$ of $U_1 := \{\|f^1_\mu\| < \|f^1_\lambda\|\}$. Let $0 \leq r < 1$ and $\epsilon > 0$ such that on $\mathcal{K}$ we have : $\|f^1_\mu\| \leq r\|f^1_\lambda\|$ and $\|f^1_\lambda\| \geq \epsilon$. We use then that

$$\|f_{(s,t)}(n)\|^2 \geq (1-r)^2 s^2 \|f^1_\lambda(n)\|^2 \geq cs^2,$$

for $n \in \mathcal{K}$ and $0 \leq t \leq s$ (we take $c = (1-r)^2\epsilon^2$). Finally, for $n \in \mathcal{K}$ and $0 \leq t \leq s$, we get the estimate of the form

$$\Big\|\mathrm{J}_Q(s,t)(n)\Big\| \leq \frac{\mathrm{cst}}{(1+cs^2)^q}$$

where $q$ can be taken as large as we want. For any differential operator $D(\partial)$ acting on $\mathcal{A}(N)$, we can prove by the same arguments that

$$\Big\|D(\partial)\cdot \mathrm{J}_Q(s,t)(n)\Big\| \leq \frac{\mathrm{cst}}{(1+cs^2)^q},$$

for $n \in \mathcal{K}$ and $0 \leq t \leq s$

This proves that $\mathrm{I}_1$ defines a $K_1\times K_2$-equivariant form with generalized coefficients on $U_1$ through the relation: $\int_{\mathfrak{k}_1\times\mathfrak{k}_2} \mathrm{I}_1(X,Y)Q(X,Y)dXdY = \iint_{0\leq t\leq s} \mathrm{J}_Q(s,t)dsdt$.

The proof of the second point is the same, exchanging $\lambda$ and $\mu$.

The last point follows from the computation done in the proof of Lemma 3.18. We repeat the argument. We have on $U_1$

$$\begin{aligned} D(\mathrm{I}_1) &= \iint_{0\le t\le s} D(\eta_1(s)\eta_2(t))ds\,dt \\ &= \iint_{0\le t\le s} D(\eta_1(s))\eta_2(t)ds\,dt - \iint_{0\le t\le s} \eta_1(s)D(\eta_2(t))ds\,dt \\ &= -\iint_{0\le t\le s} \frac{d}{ds}(\mathrm{e}^{isD\lambda})\eta_2(t)ds\,dt + \iint_{0\le t\le s} \eta_1(s)\frac{d}{dt}(\mathrm{e}^{itD\mu})ds\,dt \end{aligned}$$

Now $-\iint_{0\le t\le s} \frac{d}{ds}(\mathrm{e}^{isD\lambda})\eta_2(t)ds\,dt = \int_0^\infty \mathrm{e}^{itD\lambda}\,\eta_2(t)dt = -i\mu\int_0^\infty \mathrm{e}^{itD\lambda+itD\mu}\,dt$ while

$$\begin{aligned} \iint_{0\le t\le s} \eta_1(s)\frac{d}{dt}(\mathrm{e}^{itD\mu}))ds\,dt &= \int_0^\infty \eta_1(s)\,\mathrm{e}^{isD\mu}\,ds - \int_0^\infty \eta_1(s)ds \\ &= -i\lambda\int_0^\infty \mathrm{e}^{isD\lambda+isD\mu}\,ds - \beta(\lambda). \end{aligned}$$

Thus we obtain on $U_1$ the wanted equality $D(\mathrm{I}_1) = \beta(\lambda+\mu)|_{U_1} - \beta(\lambda)$. The proof of the equality $-D\mathrm{I}_2 = \beta(\lambda+\mu)|_{U_2} - \beta(\mu)$ is entirely similar.

Thanks to Lemma 3.27, we define the following equivariant form on $U$ : $\mathrm{I}_\Phi = \Phi_1\mathrm{I}_1 - \Phi_2\mathrm{I}_2$. The relation

$$\Big(1,\beta_1(\lambda)\Big)\diamond_\Phi\Big(1,\beta_2(\mu)\Big) - \Big(1,\beta(\lambda+\mu)|_U\Big) = D_{\mathrm{rel}}\Big(0,\mathrm{I}_\Phi\Big).$$

admits the same proof than the one of Equality (42).

We denote $\mathrm{Par}^1(\lambda)$ the image of $\mathrm{P}^1_{\mathrm{rel}}(\lambda)$ in $\mathcal{H}^{-\infty,\infty}_{C^1_\lambda}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$ and by $\mathrm{Par}^2(\mu)$ the image of $\mathrm{P}^2_{\mathrm{rel}}(\mu)$ in $\mathcal{H}^{\infty,-\infty}_{C^2_\mu}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$. We denote by $\mathrm{Par}(\lambda+\mu)$ the image of $\mathrm{P}_{\mathrm{rel}}(\lambda+\mu)$ in $\mathcal{H}^{-\infty}_{C_{\lambda+\mu}}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$. We use here the restriction maps

$$\begin{aligned} \mathbf{r}:\mathcal{H}^{-\infty}_{C^1_\lambda\cap C^2_\mu}(\mathfrak{k}_1\times\mathfrak{k}_2,N) &\longrightarrow \mathcal{H}^{-\infty}_{\mathcal{C}(\lambda,\mu)}(\mathfrak{k}_1\times\mathfrak{k}_2,N) \\ \mathbf{r}':\mathcal{H}^{-\infty}_{C_{\lambda+\mu}}(\mathfrak{k}_1\times\mathfrak{k}_2,N) &\longrightarrow \mathcal{H}^{-\infty}_{\mathcal{C}(\lambda,\mu)}(\mathfrak{k}_1\times\mathfrak{k}_2,N) \end{aligned}$$

The fact that the map $p_F$ is compatible with products and restrictions gives us the following corollary.

**Corollary 3.28** *We have the following equality*

$$\mathbf{r}\left(\mathrm{Par}^1(\lambda)\wedge\mathrm{Par}^2(\mu)\right) = \mathbf{r}'\left(\mathrm{Par}(\lambda+\mu)\right) \tag{53}$$

*in* $\mathcal{H}^{-\infty}_{\mathcal{C}(\lambda,\mu)}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$.

**Remark 3.29** *In later applications, it will happen that the set $\mathcal{C}(\lambda,\mu)$ is exactly equal to $C_{\lambda+\mu}$. In this case, we obtain the equality*

$$\mathbf{r}\left(\mathrm{Par}^1(\lambda)\wedge\mathrm{Par}^2(\mu)\right)=\mathrm{Par}(\lambda+\mu)$$

*in $\mathcal{H}^{-\infty}_{C_{\lambda+\mu}}(\mathfrak{k}_1\times\mathfrak{k}_2,N)$.*

### 3.6.2 The case of morphisms

We now consider the case of a $K_1\times K_2$-equivariant morphism $\sigma:\mathcal{E}^+\to\mathcal{E}^-$ over $N$. Let $\lambda$ a $K_1\times K_2$ equivariant one form on $N$. We choose an invariant super-connections $\mathbb{A}$, without 0 exterior degree terms on the $\mathbb{Z}_2$-graded vector bundle $\mathcal{E}$.

The relative Chern class $\mathrm{Ch}_{\mathrm{rel}}(\sigma,\lambda)\in\mathcal{H}^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,N,N\setminus C_{\lambda,\sigma})$ is represented by a couple $(\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\lambda,\mathbb{A}))$ where $\beta(\sigma,\lambda,\mathbb{A})$ is a $K_1\times K_2$-equivariant differential form on $N\setminus C_{\lambda,\sigma}$ with generalized coefficients.

As in Section 3.6.1 we write $C_\lambda$ as the intersection $C^1_\lambda\cap C^2_\lambda$ where $C^i_\lambda=\{f^i_\lambda=0\}$. We define the closed invariant subsets

$$C^i_{\lambda,\sigma}=\mathrm{Supp}(\sigma)\cap C^i_\lambda.$$

We will restrict equivariant forms on $N\setminus C_{\lambda,\sigma}$ to the open subsets $N\setminus C^i_{\lambda,\sigma}$.

**Lemma 3.30** *The equivariant form $\beta(\sigma,\lambda,\mathbb{A})(X,Y)$, when restricted to $N\setminus C^1_{\lambda,\sigma}$, depends smoothly on $Y\in\mathfrak{k}_2$.*

**Proof.** For any compactly supported function $Q\in\mathcal{C}^\infty(\mathfrak{k}_1)$, consider the element of $\mathcal{A}(N,\mathrm{End}(\mathcal{E}))$ defined by

$$\mathcal{I}_Q(t,Y):=\int_{\mathfrak{k}_1}\mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)(X,Y)}\,Q(X)dX.$$

At a point $n\in N$, $\mathcal{I}_Q(t,Y)(n)$ is equal to

$$\mathrm{e}^{-it\langle f^2_\lambda(n),Y\rangle}\int_{\mathfrak{k}_1}\mathrm{e}^{-it\langle f^1_\lambda(n),X\rangle}\,\mathrm{e}^{-t^2R(n)+S(n,X,Y)+T(t,n)}\,Q(X)dX$$

with $R(n)=v^2_\sigma(n)$, $S(n,X,Y)=\mu^{\mathbb{A}}(n)(X,Y)$, $T(t,n)=it[\mathbb{A},v_\sigma](n)+itd\lambda(n)+\mathbb{A}^2(n)$. If we use Proposition 5.6 of the Appendix, we have, for any integer $r$, the estimate

$$\left\|\mathcal{I}_Q(t,Y)\right\|(n)\leq\mathrm{cst}\;\frac{(1+t)^{\dim N}}{(1+t^2\|f^1_\lambda(n)\|^2)^r}\;\mathrm{e}^{-h_\sigma(n)t^2},$$

for all $t\geq 0$ and $(n,Y)$ in a compact subset of $N\times\mathfrak{k}_2$. The only change with respect to Proposition 3.8 is that we work with the map $f^1_\lambda$ instead of $f_\lambda=(f^1_\lambda,f^2_\lambda)$. As in Corollary 3.9, we see that the integral $\int_0^\infty\mathcal{I}_Q(t,Y)dt$ defines a smooth map from $\mathfrak{k}_2$ into $\mathcal{A}(N\setminus C^1_{\lambda,\sigma},\mathrm{End}(\mathcal{E}))$.

When we restrict the equivariant form $\beta(\sigma, \lambda, \mathbb{A})(X, Y)$ to the open subset $N \setminus C^1_{\lambda,\sigma}$, we get the relation

$$\int_{\mathfrak{k}_1} \beta(\sigma, \lambda, \mathbb{A})|_{N\setminus C^1_{\lambda,\sigma}}(X, Y)Q(X)dX = \operatorname{Str}\left( i(v_\sigma + \lambda) \int_0^\infty \mathcal{I}_Q(t, Y)dt \right).$$

It proves then that $Y \mapsto \beta(\sigma, \lambda, \mathbb{A})|_{N\setminus C^1_{\lambda,\sigma}}(X, Y)$ is smooth.

We can make the following definition.

**Definition 3.31** *We define* $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^1_{\lambda,\sigma})$ *to be the relative class*

$$[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \lambda, \mathbb{A})|_{N\setminus C^1_{\lambda,\sigma}}].$$

Consider now on $N$ : two $K_1 \times K_2$-equivariant morphisms $\sigma, \tau$ and two invariant one forms $\lambda, \mu$. We then consider the relative classes:

- $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^1_{\lambda,\sigma})$,
- $\mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu) \in \mathcal{H}^{\infty,-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C^2_{\mu,\tau})$,
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau, \lambda + \mu) \in \mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus C_{\lambda+\mu,\sigma\odot\tau})$.

The element $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda)$ (resp. $\mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu)$) is represented by an equivariant form with generalized coefficients which is smooth relatively to $Y \in \mathfrak{k}_2$ (resp. $X \in \mathfrak{k}_1$). Hence we can form the product $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda) \diamond \mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu)$ which belongs to $\mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (\mathrm{Supp}(\sigma \odot \tau) \cap C^1_\lambda \cap C^2_\mu))$.

We denote by $\mathrm{Ch}^1_{\mathrm{sup}}(\sigma, \lambda)$ the image of $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda)$ in $\mathcal{H}^{-\infty,\infty}_{C^1_{\lambda,\sigma}}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. It is represented by equivariant forms with generalized coefficients which are smooth relatively to $Y \in \mathfrak{k}_2$.

Similarly, we denote by $\mathrm{Ch}^2_{\mathrm{sup}}(\tau, \mu)$ the image of $\mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu)$ in $\mathcal{H}^{\infty,-\infty}_{C^2_{\mu,\tau}}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. It is represented by equivariant forms with generalized coefficients which are smooth relatively to $X \in \mathfrak{k}_1$.

As in Theorem 3.26, we look at the image of $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda) \diamond \mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu)$ and $\mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau, \lambda + \mu)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (\mathrm{Supp}(\sigma \odot \tau) \cap \mathcal{C}(\lambda, \mu)))$. We leave the natural restriction maps implicit.

**Theorem 3.32**

- *The following equality*

$$\mathrm{Ch}^1_{\mathrm{rel}}(\sigma, \lambda) \diamond \mathrm{Ch}^2_{\mathrm{rel}}(\tau, \mu) = \mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau, \lambda + \mu)$$

  *holds in* $\mathcal{H}^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, N, N \setminus (\mathrm{Supp}(\sigma \odot \tau) \cap \mathcal{C}(\lambda, \mu)))$.

- *The following equality*

$$\mathrm{Ch}^1_{\mathrm{sup}}(\sigma, \lambda) \wedge \mathrm{Ch}^2_{\mathrm{sup}}(\tau, \mu) = \mathrm{Ch}_{\mathrm{sup}}(\sigma \odot \tau, \lambda + \mu) \tag{54}$$

  *holds in* $\mathcal{H}^{-\infty}_{\mathrm{Supp}(\sigma\odot\tau)\cap\mathcal{C}(\lambda,\mu)}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$.

**Proof.** As in Corollary 3.17, it is easy to see that $\mathrm{Ch}^1_{\mathrm{rel}}(\sigma,\lambda) = \mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma)$ and $\mathrm{Ch}^2_{\mathrm{rel}}(\tau,\mu) = \mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma)$. Thus, using the associativity of the product, we have

$$\begin{aligned}
\mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu) \diamond \mathrm{Ch}_{\mathrm{rel}}(\tau) &= \mathrm{P}^1_{\mathrm{rel}}(\lambda) \diamond \mathrm{P}^2_{\mathrm{rel}}(\mu) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma) \diamond \mathrm{Ch}_{\mathrm{rel}}(\tau) \\
&= \mathrm{P}_{\mathrm{rel}}(\lambda+\mu) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau) \\
&= \mathrm{Ch}_{\mathrm{rel}}(\sigma \odot \tau, \lambda+\mu).
\end{aligned}$$

Let us recall the meaning of Equation (54). For any neighborhood $\mathcal{V}$ of $\mathrm{Supp}(\sigma \odot \tau) \cap \mathcal{C}(\lambda,\mu)$, let $c_{\mathcal{V}}(\sigma \odot \tau, \lambda+\mu)$ be the component of $\mathrm{Ch}_{\mathrm{sup}}(\sigma \odot \tau, \lambda+\mu)$ in $\mathcal{H}^{-\infty}_{\mathcal{V}}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. Then we have

$$c_{\mathcal{V}_1}(\sigma,\lambda) \wedge c_{\mathcal{V}_2}(\tau,\mu) = c_{\mathcal{V}}(\sigma \odot \tau, \lambda+\mu). \tag{55}$$

in $\mathcal{H}^{-\infty}_{\mathcal{V}}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. Here $\mathcal{V}_1$ and $\mathcal{V}_2$ are respectively any neighborhood of $\mathrm{Supp}(\sigma) \cap C^1_\lambda$ and $\mathrm{Supp}(\tau) \cap C^2_\mu$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 \subset \mathcal{V}$. The class $c_{\mathcal{V}_1}(\sigma,\lambda)$ (resp. $c_{\mathcal{V}_2}(\tau,\mu)$) is the component of $\mathrm{Ch}^1_{\mathrm{sup}}(\sigma,\lambda)$ (resp. $\mathrm{Ch}^2_{\mathrm{sup}}(\tau,\mu)$) in $\mathcal{H}^{-\infty,\infty}_{\mathcal{V}_1}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$ (resp. $\mathcal{H}^{\infty,-\infty}_{\mathcal{V}_2}(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$).

Let $\sigma$ and $\tau$ be two morphisms such that $\mathrm{Supp}(\sigma \odot \tau) \cap \mathcal{C}(\lambda,\mu)$ is **compact**, hence $\mathrm{Supp}(\sigma \odot \tau) \cap C_{\lambda+\mu}$ is compact. In this case, the Chern equivariant class with compact support $\mathrm{Ch}_c(\sigma \odot \tau, \lambda+\mu)$ is equal to the product

$$c_{\mathcal{V}_1}(\sigma,\lambda) \wedge c_{\mathcal{V}_2}(\tau,\mu)$$

in $\mathcal{H}^{-\infty}_c(\mathfrak{k}_1 \times \mathfrak{k}_2, N)$. Here $\mathcal{V}_1$ and $\mathcal{V}_2$ are respectively any neighborhood of $\mathrm{Supp}(\sigma) \cap C^1_\lambda$ and $\mathrm{Supp}(\tau) \cap C^2_\mu$ such that $\overline{\mathcal{V}_1} \cap \overline{\mathcal{V}_2}$ is compact.

In particular, we obtain the following theorem.

**Theorem 3.33** *Let $\sigma$ and $\tau$ be two equivariant morphisms such that*

- $\mathrm{Supp}(\sigma) \cap C^1_\lambda$ *is compact,*
- $\mathrm{Supp}(\tau) \cap C^2_\mu$ *is compact,*
- $\mathrm{Supp}(\sigma \odot \tau) \cap \mathcal{C}(\lambda,\mu)$ *is compact.*

*Then*

$$\mathrm{Ch}^1_c(\sigma,\lambda) \wedge \mathrm{Ch}^2_c(\tau,\mu) = \mathrm{Ch}_c(\sigma \odot \tau, \lambda+\mu).$$

## 3.7 Retarded construction

Let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a $K$-equivariant smooth morphism and let $\lambda$ be a $K$-invariant one-form.

Let $\mathbb{A}$ be a $K$-invariant super-connection without 0 exterior degree term and let $\mathbf{F}(\sigma,\lambda,\mathbb{A},t) = itD\lambda + \mathbf{F}(\sigma,\mathbb{A},t)$ be the equivariant curvature of the super-connection $\mathbb{A}^{\sigma,\lambda}(t) = \mathbb{A} + it(v_\sigma + \lambda)$. For any $T \in \mathbb{R}$, we consider the Chern character

$$\mathrm{Ch}(\sigma,\lambda,\mathbb{A},T) := \mathrm{Str}(\mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},T)}). \tag{56}$$

On $N \setminus C_{\lambda,\sigma}$, we have $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},T) = D(\beta(\sigma,\lambda,\mathbb{A},T))$ where the generalized equivariant odd form $\beta(\sigma,\lambda,\mathbb{A},T)$ is defined on $N \setminus C_{\lambda,\sigma}$ by the integral

$$\beta(\sigma,\lambda,\mathbb{A},T) = \int_T^\infty \eta(\sigma,\lambda,\mathbb{A},t)dt, \tag{57}$$

where $\eta(\sigma,\lambda,\mathbb{A},t) := -i\,\mathrm{Str}\left((v_\sigma+\lambda)\,\mathrm{e}^{\mathbf{F}(\sigma,\lambda,\mathbb{A},t)}\right)$. It is easy to check that the following equality

$$\Big(\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\lambda,\mathbb{A})\Big) - \Big(\mathrm{Ch}(\sigma,\lambda,\mathbb{A},T),\beta(\sigma,\lambda,\mathbb{A},T)\Big) = D_{\mathrm{rel}}\left(\int_0^T \eta(\sigma,\lambda,\mathbb{A},t)dt,\ 0\right) \tag{58}$$

holds in $\mathcal{A}^{-\infty}(\mathfrak{k},N,N\setminus C_{\lambda,\sigma})$. Hence we get the following

**Lemma 3.34** *For any $T\in\mathbb{R}$, the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma,\lambda)$ satisfies*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma,\lambda) = \Big[\mathrm{Ch}(\sigma,\lambda,\mathbb{A},T),\beta(\sigma,\lambda,\mathbb{A},T)\Big]$$

*in $\mathcal{H}^{-\infty}(\mathfrak{k},N,N\setminus C_{\lambda,\sigma})$.*

Using Lemma 3.34, we get

**Lemma 3.35** *For any $T\geq 0$, the class $\mathrm{Ch}_{\mathrm{sup}}(\sigma,\lambda)$ can be defined with the forms $c(\sigma,\lambda,\mathbb{A},\chi,T) := \chi\ \mathrm{Ch}(\sigma,\lambda,\mathbb{A},T) + d\chi\,\beta(\sigma,\lambda,\mathbb{A},T)$.*

**Proof.** It is due to the following transgression

$$c(\sigma,\lambda,\mathbb{A},\chi) - c(\sigma,\lambda,\mathbb{A},\chi,T) = D\left(\chi\int_0^T \eta(\sigma,\lambda,\mathbb{A},t)dt\right), \tag{59}$$

which follows from (58).

In some situations the Chern form $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)$ enjoys good properties relative to the integration. So it is natural to compare the differential form $c(\sigma,\lambda,\mathbb{A},\chi)$ and $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)$.

**Lemma 3.36** *We have*

$$c(\sigma,\lambda,\mathbb{A},\chi) - \mathrm{Ch}(\sigma,\lambda,\mathbb{A},1) = D\left(\chi\int_0^1 \eta(\sigma,\lambda,\mathbb{A},s)ds\right) + D\Big((\chi-1)\beta(\sigma,\lambda,\mathbb{A},1)\Big).$$

**Proof.** This follows immediately from the transgressions (30) and (59).

## 3.8 Example of Hamiltonian manifolds

There are several natural situations where a $K$-invariant one-form exists.

Let $(N, \Omega, \Phi)$ be a Hamiltonian $K$-manifold: here $\Omega$ is a symplectic form on $N$. The moment map $\Phi : N \to \mathfrak{k}^*$ is a $K$-equivariant map satisfying the relation

$$d\langle \Phi, X\rangle = \iota(VX)\Omega,$$

for every $X \in \mathfrak{k}$ so that the equivariant symplectic form $\Omega(X) := \langle \Phi, X\rangle + \Omega$ is a closed equivariant form.

With the help of an invariant scalar product on $\mathfrak{k}^*$, we have an identification $\mathfrak{k}^* \simeq \mathfrak{k}$ : the moment map $\Phi$ will be a map from $N$ on $\mathfrak{k}$. We consider then the Kirwan vector field $\mathbf{k}(n) = V_n(\Phi(n))$ : note that $\mathbf{k}$ is the Hamiltonian vector field of the function $\frac{1}{2}\|\Phi\|^2 : N \to \mathbb{R}$. Here we can define the invariant one-form

$$\lambda_{\mathbf{k}} := (\mathbf{k}, -)_N \tag{60}$$

where $(-,-)_N$ is any $K$-invariant Riemannian metric on $N$. It is easy to see that, for $n \in N$,

$$f_{\lambda_{\mathbf{k}}}(n) = 0 \Longleftrightarrow \lambda_{\mathbf{k}}(n) = 0 \Longleftrightarrow \mathbf{k}(n) = 0 \Longleftrightarrow d(\|\Phi\|^2)(n) = 0.$$

Hence the set $C_{\lambda_{\mathbf{k}}}$ coincides with the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of the function $\|\Phi\|^2$. In this situation, the generalized equivariant form $\mathrm{Par}(\lambda_{\mathbf{k}})$ have been studied in [14, 15].

We note that

$$\{\Phi = 0\} \subset \mathrm{Cr}(\|\Phi\|^2). \tag{61}$$

There are interesting situations where (61) is an equality.

Suppose now that the symplectic form $\Omega$ is exact: there exists a $K$-invariant one form $\omega$ on $N$ such that $\Omega = d\omega$. We can choose as associated moment map $\langle \Phi, X\rangle = -\langle \omega, VX\rangle$ and the equivariant symplectic form is exact: $\Omega(X) = D\omega(X)$. We have then two different one forms on $N$, the one form $\lambda_{\mathbf{k}}$ associated to the Kirwan vector field and the one form $\omega$.

**Lemma 3.37** *Assume $\Omega = d\omega$ and $\langle \Phi, X\rangle = -\langle \omega, VX\rangle$. We have then $f_\omega = -\Phi$ and $C_{\lambda_{\mathbf{k}}} = C_\omega = \{\Phi = 0\}$.*

**Proof.** The first equality is by definition of the moment map:

$$\langle \Phi(n), X\rangle = -\langle \omega(n), V_n X\rangle, \quad n \in N. \tag{62}$$

If one takes $X = \Phi(n)$ in (62) , it gives $\|\Phi(n)\|^2 = -\langle \omega(n), \mathbf{k}(n)\rangle$ and then

$$C_{\lambda_{\mathbf{k}}} = \mathrm{Cr}(\|\Phi\|^2) = \{\Phi = 0\} = \{f_\omega = 0\} = C_\omega.$$

It is natural to compare the elements $\mathrm{Par}(\lambda), \mathrm{Par}(-\omega) \in \mathcal{H}^{-\infty}_{\Phi^{-1}(0)}(\mathfrak{k}, N)$. We consider the following family of one-forms: $\lambda_s = s\lambda_{\mathbf{k}} - (1-s)\omega$, $s \in [0,1]$.

Since $f_{\lambda_s} = s f_{\lambda_{\mathbf{k}}} - (1-s) f_\omega$, we have $C_{\lambda_{\mathbf{k}}} \subset C_{\lambda_s}$ for any $s$. We have also: $\langle f_{\lambda_s}(n), \Phi(n)\rangle = s\|\mathbf{k}\|^2 + (1-s)\|\Phi(n)\|^2$ for any $n \in N$, which shows that $C_{\lambda_s} \subset C_{\lambda_{\mathbf{k}}}$. We have proved that $C_{\lambda_s} = C_{\lambda_{\mathbf{k}}}$ for any $s \in [0,1]$. With the help of Theorems 3.15 and 3.21, we can conclude with the following

**Proposition 3.38** *Let $N$ be a $K$-manifold, equipped with an exact symplectic two form $\Omega = d\omega$. The moment map $\Phi : N \to \mathfrak{k}^*$ is defined by (62). We have* $\mathrm{Cr}(\|\Phi\|^2) = \Phi^{-1}(0)$ *and*

$$\mathrm{P}_{\mathrm{rel}}(\lambda_{\mathbf{k}}) = \mathrm{P}_{\mathrm{rel}}(-\omega) \quad \text{in} \quad \mathcal{H}^{-\infty}(\mathfrak{k}, N, N \setminus \Phi^{-1}(0)),$$

$$\mathrm{Par}(\lambda_{\mathbf{k}}) = \mathrm{Par}(-\omega) \quad \text{in} \quad \mathcal{H}^{-\infty}_{\Phi^{-1}(0)}(\mathfrak{k}, N).$$

#### 3.8.1 The cotangent manifold

Here $N = \mathbf{T}^*M$, where $M$ is a $K$-manifold. Let $p : \mathbf{T}^*M \to M$ be the projection. We denote by $\omega$ the Liouville form on $\mathbf{T}^*M$ : $-\omega_{[x,\xi]}(w) = \langle \xi, p_* w\rangle$. Then $\Omega := d\omega$ is the canonical symplectic structure on $\mathbf{T}^*M$. The corresponding moment map for the Hamiltonian action of $K$ on $(\mathbf{T}^*M, \Omega)$ is the map $f_\omega : \mathbf{T}^*M \longrightarrow \mathfrak{k}^*$ defined by the relation

$$f_\omega(x,\xi) : X \mapsto \langle \xi, V_x X\rangle. \tag{63}$$

Here $f_\omega^{-1}(0)$ is the subset $\mathbf{T}^*_K M \subset \mathbf{T}^*M$ formed by co-vectors orthogonal to the $K$-orbits. In this situation we define a classes

$$\mathrm{P}_{\mathrm{rel}}(\omega) \in \mathcal{H}^{-\infty}(\mathfrak{k}, \mathbf{T}^*M, \mathbf{T}^*M \setminus \mathbf{T}^*_K M). \tag{64}$$

and

$$\mathrm{Par}(\omega) \in \mathcal{H}^{-\infty}_{\mathbf{T}^*_K M}(\mathfrak{k}, \mathbf{T}^*M). \tag{65}$$

This form $\mathrm{Par}(\omega)$ will be used extensively in a subsequent article to give a new cohomological formula for the index of transversally elliptic operators.

#### 3.8.2 Symplectic vector space

Let $N = V$ be a real vector space of dimension $2n$, with a non-degenerate skew-symmetric bilinear form $\Omega$ : we have $\Omega = d\omega$ where $\omega = \Omega(v, dv)$ on $V$. Let $K$ be a compact Lie group acting on $V$ by linear symplectic transformations. Then $V$ is a $K$-Hamiltonian space with moment map $\langle \Phi_K(v), X\rangle = \Omega(Xv, v)$.

Assume for the rest of this section that the moment map $\Phi_K : V \to \mathfrak{k}^*$ is *proper*. Since $\Phi_K$ is a homogeneous map, this assumption of properness is equivalent to one of the following conditions:

- $\Phi_K^{-1}(0) = 0$.
- There exists $c > 0$ such that $\|\Phi_K(v)\| \geq c\|v\|$ for all $v \in V$.

In this case, we obtain a class $\mathrm{P}_{\mathrm{rel}}(\omega) \in \mathcal{H}^{-\infty}(\mathfrak{k}, V, V \setminus \{0\})$ that we wish to compare with the relative Thom class.

We have shown in [19] that $\mathcal{H}^{\infty}(\mathfrak{k}, V, V \setminus \{0\})$ is a free module over $\mathcal{C}^{\infty}(\mathfrak{k})^K$ with basis the Thom classe $\mathrm{Th}_{\mathrm{rel}}(V)$. More precisely to any class $a = [\alpha, \beta] \in \mathcal{H}^{\infty}(\mathfrak{k}, V, V \setminus \{0\})$ we consider the integral

$$\int_V \mathrm{p}_c(a)(X) \in \mathcal{C}^{\infty}(\mathfrak{k})^K$$

where $\mathrm{p}_c(a)(X) = \chi\alpha(X) + d\chi\beta(X)$ is a $K$-equivariant class with compact support on $V$ defined with the help of a function $\chi \in \mathcal{C}^{\infty}(V)^K$ with compact support and equal to 1 in a neighborhood of 0.

We have the

**Theorem 3.39** *[19] For any class $a \in \mathcal{H}^{\infty}(\mathfrak{k}, V, V \setminus \{0\})$, we have the relation*

$$a = (\int_V \mathrm{p}_c(a)) \cdot \mathrm{Th}_{\mathrm{rel}}(V) \tag{66}$$

*in $\mathcal{H}^{\infty}(\mathfrak{k}, V, V \setminus \{0\})$.*

The same result, with same proof, holds if one work with equivariant forms with *generalized coefficients*. For any $a \in \mathcal{H}^{-\infty}(\mathfrak{k}, V, V \setminus \{0\})$ the integral $\int_V \mathrm{p}_c(a)$ defines an invariant generalized function on $\mathfrak{k}$. Since $\mathrm{Th}_{\mathrm{rel}}(V)$ has smooth coefficients, the product $(\int_V \mathrm{p}_c(a)) \cdot \mathrm{Th}_{\mathrm{rel}}(V)$ makes sense for any $a \in \mathcal{H}^{-\infty}(\mathfrak{k}, V, V \setminus \{0\})$, and Equality (66) holds in this case.

Let $dv := \frac{\Omega^n}{n!}$ be the symplectic volume form on $V$.

**Proposition 3.40** *The following relation holds in $\mathcal{H}^{-\infty}(\mathfrak{k}, V, V \setminus \{0\})$:*

$$\mathrm{P}_{\mathrm{rel}}(\omega) = \Theta \cdot \mathrm{Th}_{\mathrm{rel}}(V),$$

*where $\Theta \in \mathcal{C}^{-\infty}(\mathfrak{k})^K$ is defined by the relation*

$$\Theta(X) := (i)^n \int_V \mathrm{e}^{i\langle \Phi_K(v), X\rangle}\, dv, \quad X \in \mathfrak{k}.$$

**Proof.** Following Theorem 3.39, we have just to compute the integral $\Theta(X) := \int_V \mathrm{p}_c(\mathrm{P}_{\mathrm{rel}}(\omega))(X)$. Let $f \in \mathcal{C}^{\infty}(\mathbb{R})$ be a compactly supported function which is equal to 1 in a neighborhood of 0. We work with the invariant function $\chi(v) := f(\|v\|^2)$ on $V$, where $\| - \|$ is any $K$-invariant Euclidean norm on $V$. The equivariant form with generalized coefficient $\chi + d\chi \wedge \beta(\omega)$ which represents the cohomology class $\mathrm{p}_c(\mathrm{P}_{\mathrm{rel}}(\omega)) \in \mathcal{H}_c^{-\infty}(\mathfrak{k}, V)$ is the limit, as $T$ goes to infinity, of the equivariant forms with compact support

$$\begin{aligned}
\eta^T &= \chi + d\chi \wedge (-i\omega) \wedge \int_0^T e^{itD(\omega)} dt \\
&= \chi + D\left(\chi(-i\omega) \wedge \int_0^T e^{iD(\omega)(X)}\right) + \chi \int_0^T \frac{d}{dt} e^{itD(\omega)} dt \\
&= \chi e^{iT\, D(\omega)} + D\left(\chi(-i\omega) \wedge \int_0^T e^{iD(\omega)(X)}\right).
\end{aligned}$$

Hence $\Theta(X)$ is the limit, as $T$ goes to infinity, of the integrals

$$\int_V \eta^T(X) = \int_V \chi e^{iT\,D(\omega)(X)}.$$

Since $D(\omega)(X)|_v = \Omega_v + \langle \Phi_K(v), X\rangle$ is homogeneous of degree 2 in the variable $v$, we have $T\,D(\omega)(X) = \delta_T^*(D(\omega)(X))$ where $\delta_T(v) = \sqrt{T}v$. Then $\int_V \eta^T(X) = \int_V f(\frac{\|v\|^2}{T})e^{iD(\omega)(X)}$ tends to

$$\int_V e^{iD(\omega)(X)} = (i)^n \int_V \mathrm{e}^{i\langle \Phi_K(v), X\rangle}\, dv.$$

when $T$ goes to infinity.

## 3.9 Comparison with other constructions

### 3.9.1 Integration in mean

As stressed in the case of ordinary cohomology, one of the main purposes of constructing Chern character of an elliptic morphism $\sigma$ as a cohomology class with compact support is the fact that such classes are integrable.

In the case of equivariant cohomology, we introduce appropriate cohomology spaces for defining the integral of an equivariant differential form. Of course, if $\alpha \in \mathcal{H}^\infty(\mathfrak{k}, N)$, and the manifold $N$ is compact and oriented, the integral of $\alpha$ is the $K$-invariant $C^\infty$-function of $X \in \mathfrak{k}$ defined by $\int_N \alpha(X)$. If $N$ is non compact, we may have to define this integral in the generalized sense.

Let $\alpha$ be an equivariant form with $C^\infty$ coefficients on a vector bundle $N \to B$ over a compact basis. It may happen that although $\alpha(X)$ is not integrable on $N$, it is integrable in mean: by integrating $\alpha(X)$ against a smooth compactly supported density, we obtain a differential form $\alpha(Q) = \int_{\mathfrak{k}} \alpha(X)Q(X)dX$. If this form is rapidly decreasing over the fibers of $N \to B$, then we can integrate $\alpha(Q)$ on $N$. In other words, if for any test function $Q$ on $\mathfrak{k}$, the form $\alpha(Q)$ is rapidly decreasing over the fibers, we can define the integral $\int_N \alpha$ in the sense of generalized functions:

$$\int_{\mathfrak{k}} (\int_N \alpha)(X)Q(X)dX = \int_N \alpha(Q).$$

We define $\mathcal{A}^\infty_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ as the space of equivariant differential forms with $C^\infty$ coefficients such that, for any test function $Q$ on $\mathfrak{k}$, the form $\alpha(Q) = \int_{\mathfrak{k}} \alpha(X)Q(X)dX$ is rapidly decreasing on $N$, as well as all its derivatives.

Similarly, we define $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ as the space of equivariant differential forms with $C^{-\infty}$ coefficients such that, for any test function $Q$ on $\mathfrak{k}$, the form $\alpha(Q) = \int_{\mathfrak{k}} \alpha(X)Q(X)dX$ is rapidly decreasing on $N$, as well as all its derivatives.

Clearly $\mathcal{A}^\infty_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ is contained in $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$.

The operator $D$ is well defined on $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ and we denote the cohomology space by $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$. The inclusion $\mathcal{A}^{-\infty}_c(\mathfrak{k}, N) \hookrightarrow \mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ induces a map $\mathcal{H}^{-\infty}_c(\mathfrak{k}, N) \to \mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$.

If $\alpha$ and $\beta$ are two closed equivariant forms in $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ which defines the same class in $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$, then their integrals on $N$ define the same generalized function on $\mathfrak{k}$.

If the basis $B$ of the fibration $\pi : N \to B$ is not compact, the definition of $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ makes sense over any relatively compact open subset of the basis $B$. If the bundle $N \to B$ is oriented, then the integral over the fiber defines a map $\pi_* : \mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N) \to \mathcal{H}^{-\infty}(\mathfrak{k}, B)$.

### 3.9.2 Partial Gaussian look

Assume that $N$ is a $K$-equivariant real vector bundle over a $K$-manifold $B$: we denote by $\pi : N \to B$ the projection. We denote by $(x, \xi)$ a point of $N$ with $x \in B$ and $\xi \in N_x := \pi^{-1}(x)$. Let $\mathcal{E}^{\pm} \to B$ be two $K$-invariant Hermitian vector bundles. We consider a $K$-invariant morphism $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$. Let $\lambda$ be a $K$-invariant one-form on $N$.

We choose a metric on the fibers of the fibration $N \to B$. We work under the following assumption on $\sigma$ and $\lambda$.

**Assumption 3.41** • *The morphism $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$ and all its partial derivatives have at most a polynomial growth along the fibers of $N \to B$.*

• *The one-form $\lambda$ and all its partial derivatives have at most a polynomial growth along the fibers of $N \to B$.*

• *Moreover we assume that, on any compact subset $\mathcal{K}_1$ of $B$, there exists $R \geq 0$ and $c > 0$ such that*

$$h_\sigma(x, \xi) + \|f_\lambda(x, \xi)\|^2 \geq c\|\xi\|^2 \tag{67}$$

*when $\|\xi\| \geq R$ and $x \in \mathcal{K}_1$. Here $h_\sigma(x, \xi) \geq 0$ is the smallest eigenvalue of the positive hermitian endomorphism $v_\sigma(x, \xi)$.*

Let $U(1)$ be the circle group with Lie algebra $\mathfrak{u}(1) \simeq i\mathbb{R}$. In the following example we denote for any integer $k$ by $\mathbb{C}_{[k]}$ the vector space $\mathbb{C}$ with the action of $U(1)$ given by: $t \cdot z = t^k z$.

**Example 3.42 (Atiyah symbol 1)** *Let us consider the case of the Atiyah symbol. We consider $B = \{pt\}$ and $N = \mathbf{T}^*\mathbb{C}_{[1]} \simeq \mathbb{C}_{[1]} \times \mathbb{C}_{[1]}$. We consider the $U(1)$-equivariant symbol*

$$\begin{array}{rcl} \sigma : N \times \mathbb{C}_{[0]} & \longrightarrow & N \times \mathbb{C}_{[1]} \\ (\xi, v) & \longmapsto & (\xi, \sigma(\xi)v) \end{array}$$

*defined by $\sigma(\xi) = \xi_2 - i\xi_1$ for $\xi = (\xi_1, \xi_2)$. We take for one form $\lambda$ on $\mathbf{T}^*\mathbb{C}_{[1]}$ the Liouville one form : $\lambda = \mathrm{Re}(\xi_2 d\overline{\xi_1})$. Here we have $f_\lambda(\xi) = \mathrm{Im}(\xi_2\overline{\xi_1})$ and $h_\sigma(\xi) = |\xi_2 - i\xi_1|^2$ for $\xi \in \mathbb{C}^2$. We compute*

$$\begin{aligned} h_\sigma(\xi) + \|f_\lambda(\xi)\|^2 &= |\xi_1|^2 + |\xi_2|^2 - 2\mathrm{Im}(\xi_2\overline{\xi_1}) + \mathrm{Im}(\xi_2\overline{\xi_1})^2 \\ &\geq \frac{1}{2}\|\xi\|^2 \end{aligned}$$

*if* $\|\xi\|^2 = |\xi_1|^2 + |\xi_2|^2 \geq 2$. *Hence the Atiyah symbol satisfies Condition (67).*

If we consider the set $C_{\lambda,\sigma} := \{h_\sigma = 0\} \cap \{f_\lambda = 0\}$, Condition (67) implies that, for any compact subset $\mathcal{K}_1$ of $B$, the intersection $\pi^{-1}(\mathcal{K}_1) \cap C_{\lambda,\sigma}$ is a *compact* subset of $N$. Hence we have a natural map

$$\mathcal{H}^{-\infty}_{C_{\lambda,\sigma}}(\mathfrak{k}, N) \longrightarrow \mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N).$$

Our purpose in this section is to give a representative of $\mathrm{Ch}_{\sup}(\sigma, \lambda)$ in $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ with "partial Gaussian look". We will use the results of Section 3.7.

Let $\nabla = \nabla^+ \oplus \nabla^-$ be a connection on $\mathcal{E} \to B$, let $\mathbb{A} = \pi^*\nabla$ and consider the invariant super-connection $\mathbb{A}^{\sigma,\lambda}(t) = \mathbb{A} + it(v_\sigma + \lambda)$. Let $\mathrm{Ch}(\sigma, \lambda, \mathbb{A}, 1)$ and $\beta(\sigma, \lambda, \mathbb{A}, 1)$ be the equivariant forms defined in (56) and (57).

**Lemma 3.43** *The differential forms* $\mathrm{Ch}(\sigma, \lambda, \mathbb{A}, 1)$ *and* $\beta(\sigma, \lambda, \mathbb{A}, 1)$ *belong respectively to* $\mathcal{A}^{\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N)$ *and* $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, N \setminus C_{\lambda,\sigma})$.

Before going into the proof, let us look at the example

**Example 3.44 (Atiyah's symbol 2)** *In the case of the Atiyah symbol, we have* $C_{\lambda,\sigma} := \{|\xi_2 - i\xi_1|^2 = 0\} \cap \{\mathrm{Im}(\xi_2\overline{\xi_1}) = 0\} = \{(0,0)\}$. *We work on* $N \simeq \mathbb{C}^2$ *with the coordinates* $z_1 = \xi_2 - i\xi_1$ *and* $z_2 = \xi_2 + i\xi_1$.

*We take on the vector bundle* $N \times (\mathbb{C}_{[0]} \oplus \mathbb{C}_{[1]})$ *the connection* $\nabla = d$. *The equivariant curvature of the invariant super-connections* $\mathbb{A}^\sigma_t := d + itv_\sigma$ *is*

$$\mathbf{F}_t(i\theta) = \begin{pmatrix} -t^2|z_1|^2 & itd\overline{z_1} \\ itdz_1 & -t^2|z_1|^2 + it\theta \end{pmatrix}$$

*for* $i\theta \in \mathfrak{u}(1)$. *The Volterra expension formula gives*

$$\mathrm{e}^{\mathbf{F}_t(i\theta)} = \mathrm{e}^{-t^2|z_1|^2} \begin{pmatrix} 1 + (g'(i\theta) - g(i\theta))t^2 dz_1 d\overline{z_1} & itg(i\theta)d\overline{z} \\ itg(i\theta)dz & \mathrm{e}^{i\theta} + g'(i\theta)t^2 dz_1 d\overline{z_1} \end{pmatrix}$$

*where* $g(z) = \frac{\mathrm{e}^z - 1}{z}$. *In the coordinates* $z = (z_1, z_2)$, *we have*

$$D\lambda(i\theta) = \frac{1}{4}\left(dz_1 d\overline{z_1} + dz_2 d\overline{z_2} - \theta(|z_1|^2 - |z_2|^2)\right)$$

*Hence*

$$\begin{aligned} \mathrm{Ch}(\sigma, \lambda, \nabla, 1)(\theta) &= \mathrm{e}^{iD\lambda(i\theta)}\, \mathrm{Str}(\mathrm{e}^{\mathbf{F}_1(i\theta)}) \\ &= \alpha(\theta, z)\, \mathrm{e}^{-|z_1|^2}\, \mathrm{e}^{\frac{i\theta}{4}(|z_2|^2 - |z_1|^2)} \end{aligned}$$

*where* $\alpha(\theta, z)$ *depends polynomialy of* $z$. *For any test function* $Q$ *on* $\mathfrak{u}(1)$, *we see that the differential form* $\int_{\mathfrak{u}(1)} \mathrm{Ch}(\sigma, \lambda, \nabla, 1)(\theta) Q(\theta) d\theta$ *on* $\mathbb{C}^2$ *decomposes in forms of the type* $\eta(z)\, \mathrm{e}^{-|z_1|^2}\, h(|z_2|^2 - |z_1|^2)$ *where* $\eta$ *depends polynomialy* *of $z$, and $h$ is a rapidly decreasing function on $\mathbb{R}$ : hence* $\mathrm{Ch}(\sigma,\lambda,\nabla,1) \in \mathcal{A}^{\infty}_{\text{mean-dec-rap}}(\mathfrak{u}(1),\mathbb{C}^2)$.

*Now we consider the equivariant forms*

$$\begin{aligned} \eta(\sigma,\lambda,\nabla,t)(\theta) &= -i\, \mathrm{e}^{it\, D\lambda(i\theta)}\, \mathrm{Str}\left( \begin{pmatrix} 0 & \overline{z_1} \\ z_1 & 0 \end{pmatrix} \mathrm{e}^{\mathbf{F}_t(\phi)} \right) \\ &= g(i\theta)(z_1 d\overline{z_1} - \overline{z_1} dz_1)\, t\, \mathrm{e}^{-t^2|z_1|^2}\, \mathrm{e}^{it\, D\lambda(i\theta)} \\ &= \gamma(\theta,t,z)\, \mathrm{e}^{-t^2|z_1|^2}\, \mathrm{e}^{\frac{it\theta}{4}(|z_2|^2-|z_1|^2)} \end{aligned}$$

*where $\gamma(\theta,t,z)$ depends polynomialy of $(t,z)$. Now the integral*

$$\beta(\sigma,\lambda,\nabla,1)(\theta) = \int_1^{\infty} \eta(\sigma,\lambda,\nabla,t)(\theta)dt$$

*defines an $U(1)$-equivariant form on $\mathbb{C}^2 \setminus \{(0,0)\}$ with generalized coefficients : it decomposes in sum of generalized equivariant form of the type*

$$\alpha(\theta,z) \int_1^{\infty} t^k\, \mathrm{e}^{-t^2|z_1|^2}\, \mathrm{e}^{\frac{it\theta}{4}(|z_2|^2-|z_1|^2)}\, dt$$

*where $\alpha(\theta,z)$ is an equivariant form which depends polynomialy of $z$.*

*For any test function $Q$ on $\mathfrak{u}(1)$, we see that the differential form $\int_{\mathfrak{u}(1)} \beta(\sigma,\lambda,\nabla,1)(\theta)Q(\theta)d\theta$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ decomposes in forms of the type*

$$\gamma(z) \int_1^{\infty} t^k\, \mathrm{e}^{-t^2|z_1|^2}\, h(t(|z_2|^2 - |z_1|^2))dt$$

*where $\gamma$ depends polynomialy of $z$, and $h$ is a rapidly decreasing function on $\mathbb{R}$: hence $\beta(\sigma,\lambda,\nabla,1) \in \mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{u}(1),\mathbb{C}^2 \setminus \{(0,0)\})$.*

**Proof of Lemma 3.43.** We consider the equivariant curvature $\mathbf{F}(t) := \mathbf{F}(\sigma,\lambda,\mathbb{A},t)$ of the invariant super-connection $\mathbb{A}^{\sigma,\lambda}(t)$. We have

$$\mathbf{F}(t)(X) = -t^2 v_\sigma^2 - it\langle f_\lambda, X\rangle + \pi^*\mathbf{F}(X) + it[\pi^*\nabla, v_\sigma] + itd\lambda,$$

where $\mathbf{F}(X) \in \mathcal{A}(B,\mathrm{End}(\mathcal{E}))$ is the equivariant curvature of $\nabla$, and the terms $[\pi^*\nabla, v_\sigma] \in \mathcal{A}^1(N,\mathrm{End}(\pi^*\mathcal{E})), d\lambda \in \mathcal{A}^2(N)$, have at most a polynomial growth along the fibers of $N \to B$.

Let $Q$ be a test function on $\mathfrak{k}$ with support in a compact subset $\mathcal{K}''$ of $\mathfrak{k}$. We need to estimate the behavior on the fiber of the differential form $\int_{\mathfrak{k}} \mathrm{e}^{\mathbf{F}(t)(X)} Q(X)dX$ over $\pi^{-1}(\mathcal{K}_1)$ where $\mathcal{K}_1$ is a compact subset of $B$. More explicitly, at a point $n = (x,\xi) \in N$, we have

$$\left(\int_{\mathfrak{k}} \mathrm{e}^{\mathbf{F}(t)(X)} Q(X)dX\right)(n) = \int_{\mathfrak{k}} \mathrm{e}^{-it\langle f_\lambda, X\rangle}\, \mathrm{e}^{-t^2 R(n) + S(n,X) + T(t,n)}\, Q(X)dX,$$

with $R(n) = v_\sigma(n)^2$, $S(n,X) = \pi^*\mu^{\mathbb{A}}(X)(n)$ and $T(t,n) = itd\lambda(n) + \pi^*\nabla^2(n) + it[\pi^*\nabla, v_\sigma](n)$. The assumptions of Section 5.3 of the Appendix are satisfied:

the map $R(n)$ and $T(t,n)$ are slowly increasing along the fiber and the map $S(n,X)$ does not depend of the variable $\xi\in N_x$.

The form $\mathrm{e}^{itd\lambda}$ is a finite sum of powers of $t\,d\lambda$, so that, over $\pi^*(\mathcal{K}_1)$, it is bounded in norm by a fixed polynomial $P(t,\|\xi\|)$ (it is due to our assumption on $\lambda$).

If we use the estimate (84) of the appendix, we have, for every integer $r$, the estimate

$$\Big\|\int_{\mathfrak{k}}\mathrm{e}^{\mathbf{F}(t)(X)}\,Q(X)dX\Big\|(x,\xi)\leq \mathrm{cst}\,\|Q\|_{\mathcal{K}'',2r}\,(1+t)^{\dim N}\times$$
$$P(t,\|\xi\|)\frac{(1+\|\xi\|)^{\mu}}{(1+\|tf_\lambda(x,\xi)\|^2)^r}\ \mathrm{e}^{-t^2h_\sigma(x,\xi)}$$

for $(x,\xi)\in\pi^{-1}(\mathcal{K}_1)$ and $t\geq 0$. Here $\mu$ does not depend of the choice of $p$.

Consider the subset $S=\{(x,\xi);\|\xi\|\geq R,x\in\mathcal{K}_1\}$ of $\pi^{-1}(\mathcal{K}_1)$. Thus, on $S$, the estimate $h_\sigma(x,\xi)+\|f_\lambda(x,\xi)\|^2\geq c\|\xi\|^2$ holds. Since for any positive real $a,b$, we have $(1+a)^{-r}\,\mathrm{e}^{-b}\leq(1+a+b/r)^{-r}$, we get the following estimate

$$\Big\|\int_{\mathfrak{k}}\mathrm{e}^{\mathbf{F}(t)(X)}\,Q(X)dX\Big\|(x,\xi)\leq \mathrm{cst}\,\|Q\|_{\mathcal{K}'',2r}\,(1+t)^{\dim N}P(t,\|\xi\|)\frac{(1+\|\xi\|)^{\mu}}{(1+t^2\frac{c}{r}\|\xi\|^2)^r}$$

for $(x,\xi)\in S$ and $t\geq 0$. Combining this estimate with the fact that $r$ can be chosen large enough, we see that, for any integer $q$, we can find a constant $\mathrm{cst}(q)$ such that, on $S$, and for any $t\geq 1$,

$$\Big\|\int_{\mathfrak{k}}\mathrm{e}^{\mathbf{F}(t)(X)}\,Q(X)dX\Big\|(x,\xi)\leq\frac{\mathrm{cst}(q)}{(1+t^2\|\xi\|^2)^q}. \tag{68}$$

This implies that $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)=\mathrm{Str}\left(\mathrm{e}^{\mathbf{F}(1)}\right)$ is rapidly decreasing in mean along the fibers.

Consider now $\beta(\sigma,\lambda,\mathbb{A},1)=-i\int_1^\infty\mathrm{Str}\left(v_\sigma\,\mathrm{e}^{\mathbf{F}(t)}\right)dt$ which is defined, at least, for $\|\xi\|\geq R+1$. The estimate (68) shows also that $\beta(\sigma,\lambda,\mathbb{A},1)$ is rapidly decreasing in mean along the fibers. With the help of Proposition 5.9, we can prove in the same way that all partial derivatives of $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)$ and $\beta(\sigma,\lambda,\mathbb{A},1)$ are rapidly decreasing in mean along the fibers: hence $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)$ and $\beta(\sigma,\lambda,\mathbb{A},1)$ belong respectively to $\mathcal{A}^{\infty}_{\text{mean-dec-rap}}(\mathfrak{k},N)$ and to $\mathcal{A}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k},N\setminus C_{\lambda,\sigma})$.

Combining Lemmas 3.36 and 3.43, we obtain the following proposition.

**Proposition 3.45** *The equivariant form* $\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1)\in\mathcal{A}^{\infty}_{\text{mean-dec-rap}}(\mathfrak{k},N)$ *represents the image of* $\mathrm{Ch}_{\sup}(\sigma,\lambda)$ *in* $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k},N)$.

When the fibers of $\pi:N\to B$ are oriented, we have an integration morphism $\pi_*:\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k},N)\to\mathcal{H}^{-\infty}(\mathfrak{k},B)$.

**Corollary 3.46** *We have* $\pi_*(\mathrm{Ch}(\sigma,\lambda,\mathbb{A},1))=\pi_*(\mathrm{Ch}_{\sup}(\sigma,\lambda))$ *in* $\mathcal{H}^{-\infty}(\mathfrak{k},B)$.

# 4 The transversally elliptic case

Here $N = \mathbf{T}^*M$, where $M$ is a $K$-manifold (not necessarily compact). We denote by $\omega$ the Liouville form on $\mathbf{T}^*M$. The moment map for the action of $K$ on $(\mathbf{T}^*M, d\omega)$ is the map $f_\omega : \mathbf{T}^*M \to \mathfrak{k}^*$ defined by (63). We denote by $\mathbf{T}^*_K M \subset N$ the set of zeroes of $f_\omega$. In other words, an element $(x,\xi)$ is in $\mathbf{T}^*_K M$, if $\xi$ vanishes on all the tangent vectors at $x$ to the orbit $K \cdot x$.

In Example 3.8.1, we have defined in this situation the generalized equivariant class

$$\mathrm{Par}(\omega) \in \mathcal{H}^{-\infty}_{\mathbf{T}^*_K M}(\mathfrak{k}, \mathbf{T}^*M).$$

Let $\mathcal{E}^\pm \to M$ be Hermitian $K$-vector bundles. Let $p : \mathbf{T}^*M \to M$ be the projection. Let $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a $K$-equivariant morphism. We suppose that $\sigma$ is $K$-transversally elliptic: the subset

$$C_{\omega,\sigma} = \mathrm{Supp}(\sigma) \cap \mathbf{T}^*_K M$$

is *compact*.

Choose an invariant super-connection $\mathbb{A}$ on $p^*\mathcal{E}$, without 0 exterior degree term. We consider, as in Subsection 3.5, the family of invariant super-connections $\mathbb{A}^{\sigma,\omega}(t) = \mathbb{A} + it\,(\omega + v_\sigma), t \in \mathbb{R}$, on $\mathcal{E}$ with equivariant curvature $\mathbf{F}(\sigma,\omega,\mathbb{A},t)$. Recall the equivariant forms :

$$\begin{aligned}
\eta(\sigma,\omega,\mathbb{A},t) &= -i\,\mathrm{Str}\Big((v_\sigma + \omega)\ \mathrm{e}^{\mathbf{F}(\sigma,\omega,\mathbb{A},t)}\Big),\\
\beta(\sigma,\omega,\mathbb{A}) &= \int_0^\infty \eta(\sigma,\omega,\mathbb{A},t)dt.
\end{aligned}$$

The Chern character $\mathrm{Ch}_{\mathrm{sup}}(\sigma,\omega)$ can be constructed as a class in $\mathcal{H}^{-\infty}_{C_{\omega,\sigma}}(\mathfrak{k}, \mathbf{T}^*M)$ (see Section 3.5). Since $C_{\omega,\sigma} = \mathrm{Supp}(\sigma) \cap \mathbf{T}^*_K M$ is compact, we have a natural map $\mathcal{H}^{-\infty}_{C_{\omega,\sigma}}(\mathfrak{k}, \mathbf{T}^*M) \to \mathcal{H}^{-\infty}_c(\mathfrak{k}, \mathbf{T}^*M)$, and we define $\mathrm{Ch}_c(\sigma,\omega)$ as the image of $\mathrm{Ch}_{\mathrm{sup}}(\sigma,\omega)$ in $\mathcal{H}^{-\infty}_c(\mathfrak{k}, \mathbf{T}^*M)$ (see Definition 3.20). We gave in Theorem 3.22 another way to represent the class $\mathrm{Ch}_c(\sigma,\omega)$ as the product of $\mathrm{Par}(\omega) \in \mathcal{H}^{-\infty}_{\mathbf{T}^*_K M}(\mathfrak{k}, \mathbf{T}^*M)$, with $\mathrm{Ch}_{\mathrm{sup}}(\sigma) \in \mathcal{H}^{-\infty}_{\mathrm{Supp}(\sigma)}(\mathfrak{k}, \mathbf{T}^*M)$.

We summarize our results in the following proposition.

**Proposition 4.1** • *Let $\chi \in \mathcal{C}^\infty(\mathbf{T}^*M)$ be a $K$-invariant function, with* compact support *and equal to* 1 *in a neighborhood of* $\mathrm{Supp}(\sigma) \cap \mathbf{T}^*_K M$. *The following generalized equivariant form on* $\mathbf{T}^*M$

$$c(\sigma,\omega,\mathbb{A},\chi) = \chi\,\mathrm{Ch}(\mathbb{A})(X) + d\chi\,\beta(\sigma,\omega,\mathbb{A})(X) \tag{69}$$

*is closed, with* compact support*, and its cohomology class* $\mathrm{Ch}_c(\sigma,\omega)$ *in* $\mathcal{H}^{-\infty}_c(\mathfrak{k}, \mathbf{T}^*M)$ *does not depend of the data* $(\mathbb{A},\chi)$. *Furthermore this class depends only of the restriction of $\sigma$ to $\mathbf{T}^*_K M$.*

• *Let $\chi_1, \chi_2 \in \mathcal{C}^\infty(\mathbf{T}^*M)$ be $K$-invariant functions such that : $\chi_1$ is equal to* 1 *in a neighborhood of $\mathbf{T}^*_K M$, $\chi_2$ is equal to* 1 *in a neighborhood of* $\mathrm{Supp}(\sigma)$ *and the product $\chi_1\chi_2$ is compactly supported. Then the product*

$$\Big(\chi_1 + d\chi_1\beta(\omega)(X)\Big) \wedge \Big(\chi_2\,\mathrm{Ch}(\mathbb{A})(X) + d\chi_2\,\beta(\sigma,\mathbb{A})(X)\Big)$$

*is a closed equivariant form with generalized coefficients and with* compact support *on* $\mathbf{T}^*M$*. Its cohomology class coincides with* $\mathrm{Ch}_c(\sigma,\omega)$ *in* $\mathcal{H}_c^{-\infty}(\mathfrak{k},\mathbf{T}^*M)$*.*

## 4.1 Free action

Let $G, K$ be two compact Lie groups. Let $P$ be a compact manifold provided with an action of $G\times K$. We assume that the action of $K$ is free. Then the manifold $M := P/K$ is provided with an action of $G$ and the quotient map $q : P \to M$ is $G$-equivariant.

We consider the canonical bundle map $\mathrm{V} : P\times\mathfrak{k} \to \mathbf{T}P$ defined by the $K$-action : $\mathrm{V}(x, X) = V_x X$. Let $\theta$ be an invariant connection one form on $P$: it is a $K\times G$-equivariant bundle map

$$\theta : \mathbf{T}P \longrightarrow P\times\mathfrak{k}$$

such that $\theta\circ\mathrm{V}$ is the identity on $P\times\mathfrak{k}$. We may also think at $\theta$ as an invariant one-form on $P$ with values in $\mathfrak{k}$.

Let $j : \mathbf{T}^*P \to P\times\mathfrak{k}^*$ and $\theta^* : P\times\mathfrak{k}^* \longrightarrow \mathbf{T}^*P$ be the bundle maps which are respectively dual to the bundle maps V and $\theta$. The kernel of $j$ is equal to $\mathbf{T}^*_K P$. We obtain the direct sum decomposition

$$\mathbf{T}^*P = P\times\mathfrak{k}^* \oplus \mathbf{T}^*_K P$$

and the dual direct sum decomposition

$$\mathbf{T}P = P\times\mathfrak{k}\oplus \mathrm{Hor}.$$

Here $P\times\mathfrak{k}$ is isomorphic to the vertical tangent bundle and the bundle Hor is the bundle of horizontal tangent vectors. The projection $\mathbf{T}^*P \to \mathbf{T}^*_K P$ is defined by $\eta\mapsto \eta-\theta^*\circ j(\eta)$.

Note that, for each $x\in P$, we have a canonical isomorphism $\mathbf{T}^*_K P|_x \simeq \mathbf{T}^*M|_{q(x)}$ defined by $\eta\mapsto\eta\circ\mathbf{T}q|_x$.

**Definition 4.2** *The smooth map* $Q : \mathbf{T}^*P \to \mathbf{T}^*M$ *is defined as follows. For* $\eta\in\mathbf{T}^*P|_x$*, we have* $Q(x,\eta) = (q(x),\eta')$ *where* $\eta'\in\mathbf{T}^*M|_{q(x)}$ *is the image of* $\eta$ *through the projection* $\mathbf{T}^*P|_x \to \mathbf{T}^*_K P|_x$ *composed with the isomorphism* $\mathbf{T}^*_K P|_x \simeq \mathbf{T}^*M|_{q(x)}$*.*

Let $\omega_P$ and $\omega_M$ be the Liouville 1-forms on $\mathbf{T}^*P$ and $\mathbf{T}^*M$ respectively, and let $f^{K\times G}_{\omega_P} : \mathbf{T}^*P \to \mathfrak{k}^*\times\mathfrak{g}^*$ and $f^{G}_{\omega_M} : \mathbf{T}^*M \to \mathfrak{g}^*$ be the corresponding equivariant maps.

We consider the $K\times G$ invariant one form $\nu$ on $P\times\mathfrak{k}^*$ which is defined by

$$\nu(x,\xi) := \langle\theta(x),\xi\rangle. \tag{70}$$

The corresponding map $(f^K_\nu, f^G_\nu) : P\times\mathfrak{k}^* \to \mathfrak{k}^*\times\mathfrak{g}^*$ satisfies $f^K_\nu(x,\xi)=\xi$, and $f^G_\nu(x,\xi) = -\xi\circ\mu(x)$. Here $\mu : P\to \hom(\mathfrak{g},\mathfrak{k})$ is the *moment* of the connection 1-form $\theta$ : $\mu(x)(Y) = -\langle\theta(x), V_x Y\rangle$.

**Lemma 4.3** • *We have* $\omega_P = Q^*(\omega_M) + j^*(\nu)$.

• *We have*

$$f^{K\times G}_{\omega_P} = \left( f^{K}_{\nu} \circ j \, , \, f^{G}_{\omega_M} \circ Q + f^{G}_{\nu} \circ j \right). \tag{71}$$

**Proof.** We write $\theta = \sum_i \theta_i \otimes E^i$ where $(E^i)$ is a base of $\mathfrak{k}$. We denote $\langle -, E^i_P \rangle$ the smooth function on $\mathbf{T}^*P$ defined by $(x,\eta) \mapsto \langle \eta, E^i_P(x) \rangle$. First we have $j^*(\nu) = \sum_i p^*(\theta_i)\langle -, E^i_P \rangle$ where $p : \mathbf{T}^*P \to P$ is the projection. Next we have, for $(x,\eta) \in \mathbf{T}^*P$ and $v \in \mathbf{T}(\mathbf{T}^*P)|_{(x,\eta)}$, the relations :

$$\begin{aligned}
\langle Q^*(\omega_M)(x,\eta), v \rangle &= \langle \omega(Q(x,\eta)), \mathbf{T}Q|_{(x,\eta)}\xi \rangle \\
&= \langle \eta', \mathbf{T}p_1|_{Q(x,\eta)} \circ \mathbf{T}Q|_{(x,\eta)} v \rangle \\
&= \langle \eta', \mathbf{T}(q \circ p_2)|_{(x,\eta)} v \rangle \\
&= \langle \eta - \sum_i \theta_i(x) \langle \eta, E^i_P(x) \rangle, \mathbf{T}p_2(v) \rangle \\
&= \langle \omega_P(x,\eta), v \rangle + \langle j^*(\nu)(x,\eta), v \rangle.
\end{aligned}$$

Here $Q(x,\eta) = (q(x), \eta')$ and we use the relation $p' \circ Q = q \circ p$, where $p' : \mathbf{T}^*M \to M$ is the projection. The last point is a consequence of the first one.

Following Section 3.3, we associate to the invariant 1-forms $\omega_P$ and $\omega_M$ the relative equivariant classes :

- $\mathrm{P}_{\mathrm{rel}}(\omega_P) \in \mathcal{H}^{-\infty}(\mathfrak{k} \times \mathfrak{g}, \mathbf{T}^*P, \mathbf{T}^*P \setminus \mathbf{T}^*_{K\times G}P)$,
- $\mathrm{P}_{\mathrm{rel}}(\omega_M) \in \mathcal{H}^{-\infty}(\mathfrak{g}, \mathbf{T}^*M, \mathbf{T}^*M \setminus \mathbf{T}^*_G M)$.

We are in the setting of Section 3.6. We consider the manifold $N := \mathbf{T}^*P$ equipped with the actions of the group $K_1 := K$ and $K_2 := G$, and the invariant one forms $\mu = Q^*(\omega_M)$, $\lambda = j^*(\nu)$.

We first consider the $K \times G$ invariant form $\nu$ on $P \times \mathfrak{k}^*$, and the map $f_\nu = (f^K_\nu, f^G_\nu)$ from $P \times \mathfrak{k}^*$ to $\mathfrak{k}^* \times \mathfrak{g}^*$.

**Lemma 4.4** *We have*

$$C_\nu = C^K_\nu = P \times \{0\}.$$

**Proof.** As $f^{K}_{\nu}(x,\xi) = \xi$, and $f^{G}_{\nu}(x,\xi) = -\xi \circ \mu(x)$, the relation $f^{K}_{\nu}(x,\xi) = 0$ implies that $f^G_\nu = 0$, so $f_\nu = 0$.

We consider the class $\mathrm{P}^1_{\mathrm{rel}}(\nu) \in \mathcal{H}^{-\infty,\infty}(\mathfrak{k} \times \mathfrak{g}, P \times \mathfrak{k}^*, P \times (\mathfrak{k}^* \setminus \{0\}))$.

The pull-backs $Q^*(\mathrm{P}_{\mathrm{rel}}(\omega_M))$ and $j^*\left(\mathrm{P}^1_{\mathrm{rel}}(\nu)\right)$ belong respectively to $\mathcal{H}^{-\infty}(\mathfrak{g}, \mathbf{T}^*P, \mathbf{T}^*P \setminus \mathbf{T}^*_G P)$, $\mathcal{H}^{-\infty,\infty}(\mathfrak{k} \times \mathfrak{g}, \mathbf{T}^*P, \mathbf{T}^*P \setminus \mathbf{T}^*_K P)$: the relative class $Q^*(\mathrm{P}_{\mathrm{rel}}(\omega_M))(X,Y)$ does not depend of $X \in \mathfrak{k}$ and the relative class $Q^*\left(\mathrm{P}^1_{\mathrm{rel}}(\nu)\right)(X,Y)$ is smooth relatively to $Y \in \mathfrak{g}$. We can then take the product

$$Q^*(\mathrm{P}_{\mathrm{rel}}(\omega_M))(Y) \diamond j^*\left(\mathrm{P}^1_{\mathrm{rel}}(\nu)\right)(X,Y)$$

which belongs to $\mathcal{H}^{-\infty}(\mathfrak{k} \times \mathfrak{g}, \mathbf{T}^*P, \mathbf{T}^*P \setminus \mathbf{T}^*_{K\times G}P)$.

The main point of this section is the following

**Theorem 4.5** *The following equality*

$$\mathrm{P}_{\mathrm{rel}}(\omega_P) = Q^*\left(\mathrm{P}_{\mathrm{rel}}(\omega_M)\right) \diamond j^*\left(\mathrm{P}^1_{\mathrm{rel}}(\nu)\right)$$

*holds in* $\mathcal{H}^{-\infty}(\mathfrak{k}\times\mathfrak{g}, \mathbf{T}^*P, \mathbf{T}^*P \setminus \mathbf{T}^*_{K\times G}P)$.

**Proof.** This theorem follows from Theorem 3.26 and of the following description of the sets where we need to work. Indeed, let us see that we have

$$C_{Q^*\omega_M} = C^G_{Q^*\omega_M} = Q^*\mathbf{T}^*_G M, \qquad C_{j^*\nu} = C^K_{j^*\nu} = \mathbf{T}^*_K P$$

and

$$\mathcal{C}(Q^*\omega_M, j^*\nu) = C_{\omega_P} = T^*_{G\times K}P.$$

As the component of $f_{\omega_M}$ on $\mathfrak{k}^*$ is equal to 0, the first equality is clear. The second equality follows from Lemma 4.4.

To compute $\mathcal{C}(Q^*\omega_M, j^*\nu)$, we take some invariant metrics on $\mathfrak{g}^*$ and $\mathfrak{k}^*$. The set $\mathcal{C}(Q^*\omega_M, j^*\nu)$ is the set $\left\{\|Q^* f^G_{\omega_M}\| \leq \|j^* f^G_\nu\|\right\} \bigcap \left\{\|j^* f^K_\nu\| \leq 0\right\}$.

Thus, on $\mathcal{C}(Q^*\omega_M, j^*\nu)$, we have $j^* f^K_\nu = 0$. As shown by Lemma 4.4, this implies $j^* f^G_\nu = 0$, so that all maps $j^* f^K_\nu, j^* f^G_\nu, Q^* f^G_{\omega_M}$ are zero on $\mathcal{C}(Q^*\omega_M, j^*\nu)$. We obtain the last equality.

We denote by $\mathrm{Par}^1(\nu) \in \mathcal{H}^{-\infty,\infty}_P(\mathfrak{k}\times\mathfrak{g}, P\times\mathfrak{k}^*)$ the image of the class $\mathrm{P}^1_{\mathrm{rel}}(\nu)$. Then $\mathrm{Par}^1(\nu)(X,Y)$ depends smoothly on $Y$. As $P$ is compact, it defines a class still denoted by $\mathrm{Par}^1(\nu)$ in $\mathcal{H}^{-\infty,\infty}_c(\mathfrak{k}\times\mathfrak{g}, P\times\mathfrak{k}^*)$

Let $\sigma_P : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a $K\times G$-transversally elliptic morphism on $\mathbf{T}^*P$. Let $\overline{\mathcal{E}}^\pm \to M$ be the vector bundles equal to the quotient $\mathcal{E}^\pm/K$. We define the morphism $\sigma_M : p^*\overline{\mathcal{E}}^+ \to p^*\overline{\mathcal{E}}^-$ on $\mathbf{T}^*M$ by the relation

$$Q^*\sigma_M(x,\eta) = \sigma_P([x,\eta]_{\mathrm{T}^*_K P}), \quad (x,\eta)\in\mathbf{T}^*P.$$

Here $(x,\eta)\to[x,\eta]_{\mathrm{T}^*_K M}$ denotes the projection $\mathbf{T}^*P \to \mathbf{T}^*_K P$.

It is immediate to see that $\sigma_M$ is $G$-transversally elliptic, and that $Q^*\sigma_M$ defines the same class than $\sigma_P$ in $\mathbf{K}^0_{K\times G}(\mathbf{T}^*_{K\times G}P)$.

**Proposition 4.6** *We have the following equality*

$$\mathrm{Ch}_c(\sigma_P,\omega_P) = Q^*\Big(\mathrm{Ch}_c(\sigma_M,\omega_M)\Big) \wedge j^*\Big(\mathrm{Par}^1(\nu)\Big)$$

*in* $\mathcal{H}^{-\infty}_c(\mathfrak{k}\times\mathfrak{g}, \mathbf{T}^*P)$.

**Proof.** We use here the results of Section 3.6. We work on $N := \mathbf{T}^*P$ with the symbol $\sigma = Q^*\sigma_M$ and the symbol $\tau = [0]$. The Chern character with compact support $\mathrm{Ch}_c(\sigma_P,\omega_P) = \mathrm{Ch}_c(Q^*(\sigma_M), Q^*(\omega_M) + j^*(\nu))$ is equal to the product

$$c_{\mathcal{V}_1}(Q^*(\sigma_M), Q^*(\omega_M)) \wedge c_{\mathcal{V}_2}([0], j^*(\nu))$$

in $\mathcal{H}^{-\infty}_c(\mathfrak{k}_1\times\mathfrak{k}_2, N)$. Here $\mathcal{V}_1$ is any neighborhood of

$$\mathrm{Supp}(Q^*(\sigma_M)) \cap C_{Q^*(\omega_M)} = Q^{-1}\left(\mathrm{Supp}(\sigma_M) \cap \mathbf{T}^*_G M\right)$$

and $\mathcal{V}_2$ is any neighborhood of

$$\mathrm{Supp}([0]) \cap C_{j^*(\nu)} = j^{-1}(P \times \{0\}) = \mathbf{T}^*_K P$$

with the condition that $\overline{\mathcal{V}_1} \cap \overline{\mathcal{V}_2}$ is compact.

Here we take $\mathcal{V}_1$ of the form $Q^{-1}(\mathcal{U}_1)$ where $\mathcal{U}_1$ is a neighborhood of $\mathrm{Supp}(\sigma_M) \cap \mathbf{T}^*_K M$ in $\mathbf{T}^* M$ with $\overline{\mathcal{U}_1}$ compact. We take $\mathcal{V}_2$ of the form $j^{-1}(\mathcal{U}_2)$ where $\mathcal{U}_2 = \{(p, \xi) \in P \times \mathfrak{k}^* \mid \|\xi\| \leq \epsilon\}$ is defined for $\epsilon$ small enough.

Then the class $c_{\mathcal{V}_1}(Q^*(\sigma_M), Q^*(\omega_M))$ and $c_{\mathcal{V}_2}([0], j^*(\nu))$ are respectively equal to $Q_1^*(c_{\mathcal{U}_1}(\sigma_M, \omega_M))$ and to $j^*(c_{\mathcal{U}_2}([0], \nu))$. The class $c_{\mathcal{U}_1}(\sigma_M, \omega_M)$ is equal to $\mathrm{Ch}_c(\sigma_M, \omega_M)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{g}, \mathbf{T}^* M)$, and the class $c_{\mathcal{U}_2}([0], \nu)$ defines $\mathrm{Par}^1(\nu)$ in $\mathcal{H}_c^{-\infty,\infty}(\mathfrak{k} \times \mathfrak{g}, P \times \mathfrak{k}^*)$.

## 4.2 Exterior product

To define products of symbols, we will need to use "almost homogeneous symbols".

**Definition 4.7** *A morphism $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ over $\mathbf{T}^* M$ is said to be* almost homogeneous *of order $m$ if $\sigma([x, t\xi]) = t^m \sigma([x, \xi])$, for every $t \geq 1$ and for $\xi$ large enough*[2].

**Lemma 4.8** *A $K$-transversally elliptic morphism $\sigma$ is homotopic to a $K$- transversally elliptic morphism which is furthermore* almost homogeneous *of order 0.*

**Proof.** Let $c > 0$ such that $C_{\omega,\sigma} = \mathrm{Supp}(\sigma) \cap \mathbf{T}^*_K M \subset \{(x, \xi) \in \mathbf{T}^* M \,|\, \|\xi\| \leq c\}$. We consider a smooth function $\phi : \mathbb{R} \to \mathbb{R}^{\geq 0}$ satisfying: $\phi = 1$ on $[0, c]$, $\phi \geq 1$ on $[c, 2c]$, and $\phi(y) = \frac{2c}{y}$ for $y \geq 2c$.

We define now, for $s \in [0, 1]$, the morphism $\sigma^s(x, \xi) := \sigma(x, \phi(s\|\xi\|)\xi)$. We see easily that $C_{\omega,\sigma} = C_{\omega,\sigma^s}$ for all $s \in [0, 1]$. Hence $\sigma = \sigma^0$ is homotopic to $\sigma^1$ which is almost homogeneous of order 0.

Let $K_1, K_2$ be two compact Lie groups. We work with the following data:

- $M_1$ is a $K_1 \times K_2$-manifold not *necessarily compact* ,
- $M_2$ is a *compact* $K_2$-manifold ,
- $\mathcal{E}_1$ is a $K_1 \times K_2$-equivariant complex super-vector bundle on $M_1$,
- $\mathcal{E}_2$ is a $K_2$-equivariant complex super-vector bundle on $M_2$,
- $\sigma_1 : p_1^*\mathcal{E}_1^+ \to p_1^*\mathcal{E}_1^-$ is a $K_1 \times K_2$-equivariant morphism on $\mathbf{T}^* M_1$ which is $K_1$-transversally elliptic
- $\sigma_2 : p_2^*\mathcal{E}_2^+ \to p_2^*\mathcal{E}_2^-$ is a $K_2$-equivariant morphism on $\mathbf{T}^* M_2$ which is $K_2$-transversally elliptic.

[2]It means that $\|\xi\| \geq c$ for some Riemannian metric $\|\cdot\|$ and some constant $c > 0$.

We consider now the exterior product $\sigma := \sigma_1 \odot_{\rm ext} \sigma_2$ which is an equivariant morphism on $M := M_1 \times M_2$ with support equal to $\mathrm{Supp}(\sigma_1) \times \mathrm{Supp}(\sigma_2)$. Since $\mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2) \neq \mathbf{T}^*_{K_1} M_1 \times \mathbf{T}^*_{K_2} M_2$, the morphism $\sigma$ is not necessarily $K_1 \times K_2$-transversally elliptic. However, we will see that it is so when the morphism $\sigma_2$ is taken **almost homogeneous** of order 0.

For $k = 1, 2$, let $p_k : \mathbf{T}^*(M_1 \times M_2) \to \mathbf{T}^* M_k$ be the projection. The Liouville one form $\omega$ on $\mathbf{T}^*(M_1 \times M_2)$ is equal to $p_1^*\omega_1 + p_2^*\omega_2$, where $\omega_k$ is the Liouville one form on $\mathbf{T}^* M_k$.

**Lemma 4.9** *Assume that the morphism $\sigma_2$ is taken* **almost homogeneous** *of order* 0*. Then the morphism $\sigma := \sigma_1 \odot_{\rm ext} \sigma_2$ on $M := M_1 \times M_2$ is $K_1 \times K_2$-transversally elliptic.*

**Proof.** Let $f^2_{\omega_2} : \mathbf{T}^* M_2 \to \mathfrak{k}_2^*$ and $(f^1_{\omega_1}, f^2_{\omega_1}) : \mathbf{T}^* M_1 \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ be the moment maps associated to the actions of $K_2$ on $M_2$ and $K_1 \times K_2$ on $M_1$. An element $(n_1, n_2) \in \mathbf{T}^* M_1 \times \mathbf{T}^* M_2$ belongs to $\mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2)$ if and only if $f^1_{\omega_1}(n_1) = 0$ and $f^2_{\omega_1}(n_1) + f^2_{\omega_2}(n_2) = 0$.

Let $f_2$ be the restriction of the map $f^2_{\omega_2}$ to the subset $\mathrm{Supp}(\sigma_2)$. Then

$$\mathrm{Supp}(\sigma_1 \odot_{\rm ext} \sigma_2) \cap \mathbf{T}^*_{K_1\times K_2}(M_1 \times M_2) \subset C^1_{\sigma_1,\omega_1} \times f_2^{-1}(\mathcal{K})$$

where $C^1_{\sigma_1,\omega_1} = \mathrm{Supp}(\sigma_1) \cap \mathbf{T}^*_{K_1}(M_1)$ and $\mathcal{K} := -f^{K_2}_{\omega_1}(C^1_{\sigma_1,\omega_1})$ are compacts. The proof follows from the

**Lemma 4.10** *The map $f_2 : \mathrm{Supp}(\sigma_2) \to \mathfrak{k}_2^*$ is proper.*

**Proof.** Since $\sigma_2$ is $K_2$-transversally elliptic, we have

$$\mathrm{Supp}(\sigma_2) \cap \mathbf{T}^*_{K_2} M_2 \subset \{\|\xi_2\| < c\}. \tag{72}$$

Thus the function $\|f_2\|$ is strictly positive on the compact subset $\mathrm{Supp}(\sigma_2) \cap \{\|\xi_2\| = c\}$. We choose $u > 0$ such that $\|f_2\| > u$ on $\mathrm{Supp}(\sigma_2) \cap \{\|\xi_2\| = c\}$.

As $\sigma_2$ is almost homogeneous of order 0, we can choose $c$ sufficiently large such that $\sigma_2([x, t\xi]) = \sigma_2([x, \xi])$ for every $t \geq 1$ and for $\|\xi\| \geq c$, so that $S_2' = \mathrm{Supp}(\sigma_2) \cap \{\|\xi_2\| \geq c\}$ is stable by multiplication by $t \geq 1$. It is sufficient to prove that the restriction of $f_2$ to $S_2'$ is proper. By homogeneity, $f_2([x, \xi]) > uc\|\xi\|$ on $S_2'$. It follows that the restriction of $\mathrm{f}_2$ to $S_2'$ is proper.

Consider first $N_1 = \mathbf{T}^* M_1$ with the $K_1 \times K_2$ invariant form $\omega_1$. We are in the situation of Section 3.6. We write the map $f_{\omega_1} : N_1 \to \mathfrak{k}_1^* \times \mathfrak{k}_2^*$ as $(f^1_{\omega_1}, f^2_{\omega_1})$. We see that the set $C^1_{\omega_1}$ is just $\mathbf{T}^*_{K_1} M_1$. Let

$$C^1_{\sigma_1,\omega_1} = \mathrm{Supp}(\sigma_1) \cap \mathbf{T}^*_{K_1} M_1.$$

By our assumption, this is a compact subset of $\mathbf{T}^* M_1$. The relative class $\mathrm{Ch}^1_{\rm rel}(\sigma_1, \omega_1)$ belongs to $\mathcal{H}^{-\infty,\infty}(\mathfrak{k}_1^* \times \mathfrak{k}_2^*, N_1 \setminus C^1_{\sigma_1,\omega_1})$. We consider its image

$\mathrm{Ch}^1_c(\sigma_1,\omega_1)$ in $\mathcal{H}_c^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M_1)$. A representant of $\mathrm{Ch}^1_c(\sigma_1,\omega_1)$ is given by $c(\sigma_1,\omega_1,\mathbb{A}_1,\chi_1)$ (see (69)), where $\chi_1$ is an invariant compactly supported function on $\mathbf{T}^*M_1$ which is equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma_1)\cap\mathbf{T}^*_{K_1}M_1$.

From now on, we assume that the morphism $\sigma_2$ is taken **almost homogeneous** of order 0.

We consider the Chern classes with compact support associated to the transversally elliptic morphisms $\sigma_1$, $\sigma_2$ and $\sigma:=\sigma_1\odot_{\mathrm{ext}}\sigma_2$ :

- $\mathrm{Ch}^1_c(\sigma_1,\omega_1)\in\mathcal{H}_c^{-\infty,\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M_1)$,
- $\mathrm{Ch}_c(\sigma_2,\omega_2)\in\mathcal{H}_c^{-\infty}(\mathfrak{k}_2,\mathbf{T}^*M_2)$,
- $\mathrm{Ch}_c(\sigma,\omega)\in\mathcal{H}_c^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M)$.

We may then form the product of the generalized equivariant forms $p_1^*\,\mathrm{Ch}^1_c(\sigma_1,\omega_1)$ and $p_2^*\,\mathrm{Ch}_c(\sigma_2,\omega_2)$. The main result of this Section is the

**Theorem 4.11** *The following equality*

$$p_1^*\,\mathrm{Ch}^1_c(\sigma_1,\omega_1)(X,Y)\wedge p_2^*\,\mathrm{Ch}_c(\sigma_2,\omega_2)(Y)=\mathrm{Ch}_c(\sigma,\omega)(X,Y)$$

*holds in* $\mathcal{H}_c^{-\infty}(\mathfrak{k}_1\times\mathfrak{k}_2,\mathbf{T}^*M)$.

**Proof.** This theorem follows from the results proved in Section 3.6. We consider the manifold $N:=\mathbf{T}^*(M_1\times M_2)$ equipped with the actions of the groups $K_1$, $K_2$ and the invariant one forms $\lambda=p_1^*(\omega_1)$ and $\mu:=p_2^*(\omega_2)$ : $\lambda+\mu=\omega$. The morphism $\sigma$ is equal to the product $p_1^*(\sigma_1)\odot p_2^*(\sigma_2)$. As the component of $f_\mu$ on $\mathfrak{k}_1^*$ is equal to 0, the closed subset $\mathcal{C}:=\mathcal{C}(\lambda,\mu)$ of $N$ is equal to

$$\begin{aligned}\mathcal{C} &:= \left\{\|f^1_\lambda\|=0\right\}\bigcap\left\{\|f_\mu\|\leq\|f^2_\lambda\|\right\}\\ &= \mathbf{T}^*_{K_1}M\bigcap\left\{(n_1,n_2)\in\mathbf{T}^*M_1\times\mathbf{T}^*M_2\ \mid\ \|f^2_{\omega_2}(n_2)\|\leq\|f^2_{\omega_1}(n_1)\|\right\}.\end{aligned}$$

Let us check that $\mathrm{Supp}(\sigma)\cap\mathcal{C}$ is compact. Since $\mathrm{Supp}(\sigma)=\mathrm{Supp}(\sigma_1)\times\mathrm{Supp}(\sigma_2)$, we have

$$\mathrm{Supp}(\sigma)\cap\mathcal{C}=\left\{(n_1,n_2)\in C^1_{\omega_1,\sigma_1}\times\mathrm{Supp}(\sigma_2)\ \mid\ \|f^2_{\omega_2}(n_2)\|\leq\|f^2_{\omega_1}(n_1)\|\right\}$$

where $C^1_{\omega_1,\sigma_1}=\mathrm{Supp}(\sigma_1)\cap\mathbf{T}^*_{K_1}M_1$ is compact. Let $c>0$ such that $\|f^2_{\omega_1}(n_1)\|\leq c$ for all $n_1\in C^1_{\omega_1,\sigma_1}$. We have then

$$\mathrm{Supp}(\sigma)\cap\mathcal{C}\subset C^1_{\omega_1,\sigma_1}\times\Big(\mathrm{Supp}(\sigma_2)\cap\{\|f^2_{\omega_2}\|\leq c\}\Big)$$

We know from Lemma 4.10 that the map $f^2_{\omega_2}$ is proper on $\mathrm{Supp}(\sigma_2)$: the set $\mathrm{Supp}(\sigma_2)\cap\{\|f^2_{\omega_2}\|\leq c\}$ is compact and then $\mathrm{Supp}(\sigma)\cap\mathcal{C}$ is also compact.

We are exactly in the situation of Theorem 3.33. The equivariant Chern character with compact support $\mathrm{Ch}_c(\sigma,\omega)=\mathrm{Ch}_c(p_1^*(\sigma_1)\odot p_2^*(\sigma_2),\lambda+\mu)$ is equal to the product

$$c_{\mathcal{V}_1}(p_1^*(\sigma_1),\lambda)\wedge c_{\mathcal{V}_2}(p_2^*(\sigma_2),\mu)$$

in $\mathcal{H}_c^{-\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, \mathbf{T}^*M)$. Here $\mathcal{V}_1$ and $\mathcal{V}_2$ are respectively any neighborhood of $\mathrm{Supp}(p_1^*(\sigma_1)) \cap C_\lambda^1 = p_1^*(\mathrm{Supp}(\sigma_1) \cap \mathbf{T}_{K_1}^* M_1)$ and $\mathrm{Supp}(p_2^*(\sigma_2)) \cap C_\mu = p_2^*(\mathrm{Supp}(\sigma_2) \cap \mathbf{T}_{K_2}^* M_2)$ such that $\overline{\mathcal{V}_1} \cap \overline{\mathcal{V}_2}$ is compact. Here we take $\mathcal{V}_k$ of the form $p_k^{-1}(\mathcal{U}_k)$ where $\mathcal{U}_k$ is a neighborhood of $\mathrm{Supp}(\sigma_k) \cap \mathbf{T}_{K_k}^* M_k$ in $\mathbf{T}^* M_k$ with $\overline{\mathcal{U}_k}$ compact. Then the class $c_{\mathcal{V}_1}(p_1^*(\sigma_1), \lambda)$ and $c_{\mathcal{V}_2}(p_1^*(\sigma_1), \mu)$ are respectively equal to $p_1^*(c_{\mathcal{U}_1}(\sigma_1, \omega_1))$ and to $p_2^*(c_{\mathcal{U}_2}(\sigma_2, \omega_2))$. The proof is completed since each $c_{\mathcal{U}_1}(\sigma_1, \omega_1)$ is equal to $\mathrm{Ch}_c^1(\sigma_1, \omega_1)$ in $\mathcal{H}_c^{-\infty,\infty}(\mathfrak{k}_1 \times \mathfrak{k}_2, \mathbf{T}^* M_1)$, while $c_{\mathcal{U}_2}(\sigma_2, \omega_2)$ is equal to $\mathrm{Ch}_c(\sigma_2, \omega_2)$ in $\mathcal{H}_c^{-\infty}(\mathfrak{k}_2, \mathbf{T}^* M_2)$.

## 4.3 The Berline-Vergne Chern character

In this section, we compare $\mathrm{Ch}_c(\sigma, \omega)$ with the class defined by Berline-Vergne [8], with the help of transversally good symbols. We suppose in this Section that the manifold $M$ is compact.

In Berline-Vergne [8], we associated to a "transversally good elliptic symbol" $\sigma$ a class $\mathrm{Ch}_{\mathrm{BV}}(\sigma)$ which was an equivariant differential form on $\mathbf{T}^*M$ with smooth coefficients, rapidly decreasing in mean on $\mathbf{T}^*M$. If $\sigma$ is any transversally elliptic symbol, the Chern character $\mathrm{Ch}_c(\sigma, \omega)$ is compactly supported on $\mathbf{T}^*M$, so defines an element of $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, \mathbf{T}^*M)$. Our aim is to prove that the classes $\mathrm{Ch}_{\mathrm{BV}}(\sigma)$ and $\mathrm{Ch}_c(\sigma, \omega)$ coincide in $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, \mathbf{T}^*M)$.

We recall the definition of $\mathrm{Ch}_{\mathrm{BV}}(\sigma)$. A $K$-transversally elliptic symbol $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ is "good" if it satisfies the following conditions:

• $\sigma$ and all its derivatives are slowly increasing along the fibers,

• the endomorphism $v_\sigma^2$ is "good" with respect to the moment map $f_\omega$. That is, there exists $r > 0$, $c > 0$ and $a > 0$ such that[3] for every $(x, \xi)$ :

$$\|f_\omega(x, \xi)\| \leq a\|\xi\| \text{ and } \|\xi\| \geq r \Longrightarrow h_\sigma(x, \xi) \geq c\|\xi\|^2. \tag{73}$$

Let $\mathbb{A} = p^*\nabla$, where $\nabla = \nabla^+ \oplus \nabla^-$ is a sum of connections on the bundles $\mathcal{E}^\pm \to B$. Consider the invariant super-connection $\mathbb{A}_1 = \mathbb{A} + iv_\sigma + i\omega$ with equivariant curvature

$$\mathbf{F}(\sigma, \mathbb{A}, 1)(X) = -v_\sigma^2 + i\langle f_\omega, X\rangle + i\Omega + i[\mathbb{A}, v_\sigma] + \mathbb{A}^2 + \mu^{\mathbb{A}}(X).$$

If $\sigma$ is a $K$-transversally good symbol, Berline and Vergne have shown that the smooth equivariant form $\mathrm{Ch}(\sigma, \omega, \mathbb{A}, 1) = \mathrm{e}^{iD\omega}\,\mathrm{Str}(\mathrm{e}^{F(\sigma,\mathbb{A},1)})$ is rapidly decreasing in mean: for any test function $Q$ on $\mathfrak{k}$, $\int_{\mathfrak{k}} \mathrm{Ch}(\sigma, \omega, \mathbb{A}, 1)(X)Q(X)dX$ is a differential form on $\mathbf{T}^*M$ which is rapidly decreasing along the fibers of the projection $\mathbf{T}^*M \to M$. It thus defines a class

$$\mathrm{Ch}_{\mathrm{BV}}(\sigma) \in \mathcal{H}^{\infty}_{\text{mean-dec-rap}}(\mathfrak{k}, \mathbf{T}^*M).$$

**Theorem 4.12** *If $\sigma$ is a transversally good symbol, then $\mathrm{Ch}_{\mathrm{BV}}(\sigma) = \mathrm{Ch}_c(\sigma, \omega)$ in $\mathcal{H}^{-\infty}_{\mathrm{mean-dec-rap}}(\mathfrak{k}, \mathbf{T}^*M)$. In particular, the integrals on the fibers of $\mathrm{Ch}_{\mathrm{BV}}(\sigma)$ and of $\mathrm{Ch}_c(\sigma, \omega)$ defines the same element in $\mathcal{H}^{-\infty}(\mathfrak{k}, M)$.*

[3] $h_\sigma(x, \xi) \geq 0$ is the smallest eigenvalue of the positive hermitian endomorphism $\sigma(x, \xi)^2$.

**Proof.** The condition (73) implies that

$$h_\sigma(x,\xi) + \|f_\omega(x,\xi)\|^2 \geq c'\|\xi\|^2 \quad \text{when} \quad \|\xi\| \geq r$$

where $c' = \min(a^2, c)$. So we can exploit the result of Proposition 3.45 : we have $\mathrm{Ch}(\sigma,\omega,\mathbb{A},1) = \mathrm{Ch}_{\sup}(\sigma,\omega)$ in $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k},\mathbf{T}^*M)$. The proof is finished, since $\mathrm{Ch}_{\sup}(\sigma,\omega) = \mathrm{Ch}_c(\sigma,\omega)$ in $\mathcal{H}^{-\infty}_{\text{mean-dec-rap}}(\mathfrak{k},\mathbf{T}^*M)$.

# 5 Appendix

We give proofs of the estimates used in this article. They are all based on Volterra's expansion formula: if $R$ and $S$ are elements in a finite dimensional associative algebra, then

$$\mathrm{e}^{(R+S)} = \mathrm{e}^R + \sum_{k=1}^{\infty} \int_{\Delta_k} \mathrm{e}^{s_1 R}\, S\, \mathrm{e}^{s_2 R}\, S \cdots S\, \mathrm{e}^{s_k R}\, S\, \mathrm{e}^{s_{k+1} R}\, ds_1 \cdots ds_k \tag{74}$$

where $\Delta_k$ is the simplex $\{s_i \geq 0; s_1 + s_2 + \cdots + s_k + s_{k+1} = 1\}$. We recall that the volume of $\Delta_k$ for the measure $ds_1 \cdots ds_k$ is $\frac{1}{k!}$.

Now, let $\mathcal{A} = \oplus_{i=0}^q \mathcal{A}_i$ be a finite dimensional graded commutative algebra with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$. We assume $\mathcal{A}_0 = \mathbb{C}$ and we denote by $\mathcal{A}_+ = \oplus_{i=1}^q \mathcal{A}_i$. Thus $\omega^{q+1} = 0$ for any $\omega \in \mathcal{A}_+$. Let $V$ be a finite dimensional Hermitian vector space. Then $\mathrm{End}(V) \otimes \mathcal{A}$ is an algebra with a norm still denoted by $\|\cdot\|$. If $S \in \mathrm{End}(V)$, we denote also by $S$ the element $S \otimes 1$ in $\mathrm{End}(V) \otimes \mathcal{A}$.

**Remark 5.1** *In the rest of this section we will denote* $\mathrm{cst}(a, b, \cdots)$ *some positive constant which depends on the parameter* $a, b, \cdots$.

## 5.1 First estimates

We denote $\mathrm{Herm}(V) \subset \mathrm{End}(V)$ the subspace formed by the Hermitian endomorphisms. When $R \in \mathrm{Herm}(V)$, we denote $\mathrm{m}(R) \in \mathbb{R}$ the smallest eigenvalue of $R$ : we have

$$\left\| \mathrm{e}^{-R} \right\| = \mathrm{e}^{-\mathrm{m}(R)}.$$

**Lemma 5.2** *Let* $\mathcal{P}(t) = \sum_{k=0}^q \frac{t^k}{k!}$. *Then, for any* $S \in \mathrm{End}(V) \otimes \mathcal{A}$, $T \in \mathrm{End}(V) \otimes \mathcal{A}_+$, *and* $R \in \mathrm{Herm}(V)$, *we have*

$$\|\, \mathrm{e}^{-R+S+T} \,\| \leq \mathrm{e}^{-\mathrm{m}(R)}\, \mathrm{e}^{\|S\|}\, \mathcal{P}(\|T\|).$$

**Proof.** Let $c = \mathrm{m}(R)$. Then $\|\, \mathrm{e}^{-uR} \,\| = \mathrm{e}^{-uc}$ for all $u \geq 0$. Using Volterra's expansion for the couple $sR, sS$, we obtain $\|\, \mathrm{e}^{s(-R+S)} \,\| \leq \mathrm{e}^{-sc}\, \mathrm{e}^{s\|S\|}$. Indeed, $\mathrm{e}^{s(-R+S)} = \mathrm{e}^{-sR} + \sum_{k=1}^{\infty} I_k$ with

$$I_k = s^k \int_{\Delta_k} \mathrm{e}^{-s_1 sR}\, S \cdots S\, \mathrm{e}^{-s_k sR}\, S\, \mathrm{e}^{-s_{k+1} sR}\, ds_1 \cdots ds_k.$$

The term $I_k$ is bounded in norm by $\frac{s^k}{k!}\|S\|^k \mathrm{e}^{-sc}$. Summing in $k$, we obtain $\| \mathrm{e}^{-s(R+S)} \| \leq \mathrm{e}^{-sc} \mathrm{e}^{s\|S\|}$ for $s \geq 0$. We reapply Volterra's expansion to compute $\mathrm{e}^{(-R+S)+T}$ as the sum

$$\mathrm{e}^{-R+S} + \sum_{k\geq 1}^{q} \int_{\Delta_k} \mathrm{e}^{s_1(-R+S)} T \cdots T \mathrm{e}^{s_k(-R+S)} T \mathrm{e}^{s_{k+1}(-R+S)} ds_1 \cdots ds_k.$$

Here the sum in $k$ is finite and stops at $k = q$. The norm of the $k^{th}$ term is bounded by $\frac{1}{k!} \mathrm{e}^{-c} \mathrm{e}^{\|S\|} \|T\|^k$. Summing up in $k$, we obtain our estimate.

For proving Proposition 3.8, we need to consider the following situation. Let $E$ be a (finite dimensional) vector space. We consider the following smooth maps

- $x \mapsto S(x)$ from $E$ to $\mathrm{End}(V) \otimes \mathcal{A}$.
- $(t, x) \mapsto t^2 R(x)$ from $\mathbb{R} \times E$ to $\mathrm{Herm}(V)$.
- $(t, x) \mapsto T(t, x) = T_0(x) + tT_1(x)$ from $\mathbb{R} \times E$ to $\mathrm{End}(V) \otimes \mathcal{A}_+$.

**Proposition 5.3** *Let $D(\partial)$ be a constant coefficient differential operator in $x \in E$ of degree $r$. Let $\mathcal{K}$ be a compact subset of $E$. There exists a constant* $\mathrm{cst} > 0$ *(depending on $\mathcal{K}, R(x)$, $S(x), T_0(x), T_1(x)$ and $D(\partial)$) such that*[4]

$$\left\| D(\partial) \cdot \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} \right\| \leq \mathrm{cst}\, (1 + t)^{2r+q}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}, \tag{75}$$

*for all $(x, t) \in \mathcal{K} \times \mathbb{R}^{\geq 0}$.*

**Corollary 5.4** *Let $\mathcal{U}$ be an open subset of $E$ such that $R(x)$ is positive definite for any $x \in \mathcal{U}$, that is $\mathrm{m}(R(x)) > 0$ for all $x \in \mathcal{U}$. Then the integral*

$$\int_0^\infty \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} dt$$

*defines a smooth map from $\mathcal{U}$ into* $\mathrm{End}(V) \otimes \mathcal{A}$.

**Proof.** We fix a basis $v_1, \ldots, v_p$ of $E$. Let us denote $\partial_i$ the partial derivative along the vector $v_i$. For any sequence $I := [i_1, \ldots, i_n]$ of integers $i_k \in \{1, \ldots, p\}$, we denote $\partial_I$ the differential operator of order $n = |I|$ defined by the product $\prod_{k=1}^n \partial_{i_k}$.

For any smooth function $g : E \to \mathrm{End}(V) \otimes \mathcal{A}$ we define the functions

$$\left\| g \right\|_n (x) := \sup_{|I|\leq n} \|\partial_I \cdot g(x)\|$$

[4] $q$ is highest degree of the graded algebra $\mathcal{A}$.

and the semi-norms $\|g\|_{\mathcal{K},n} := \sup_{x\in\mathcal{K}} \|g\|_n(x)$ attached to a compact subset $\mathcal{K}$ of $E$. We will use the trivial fact that $\|g\|_n(x) \leq \|g\|_m(x)$ when $n \leq m$. Since any constant differential operator $D(\partial)$ is a finite sum $\sum_I a_I \partial_I$, it is enough to proves (75) for the $\partial_I$.

First, we analyze $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x)}\right)$. The Volterra expansion formula gives

$$\partial_i \cdot \left(\mathrm{e}^{-t^2 R(x)}\right) = -t^2 \int_{\Delta_1} \mathrm{e}^{-s_1 t^2 R(x)}\, \partial_i \cdot R(x)\, \mathrm{e}^{-s_2 t^2 R(x)}\, ds_1, \tag{76}$$

and then $\|\partial_i \cdot \mathrm{e}^{-t^2 R(x)}\| \leq \|R\|_1(x)\,(1+t)^2\, \mathrm{e}^{-t^2 \mathrm{m}(R(x))}$ for $(x,t) \in E \times \mathbb{R}^{\geq 0}$.

With (76), one can easily prove by induction on the degree of $\partial_I$ that: if $|I| = n$ then

$$\left\|\partial_I \cdot \mathrm{e}^{-t^2 R(x)}\right\| \leq \mathrm{cst}(n) \left(1 + \|R\|_n(x)\right)^n (1+t)^{2n}\, \mathrm{e}^{-t^2 \mathrm{m}(R(x))} \tag{77}$$

for $(x,t) \in E \times \mathbb{R}^{\geq 0}$. Note that (77) is still true when $I = \emptyset$ with $\mathrm{cst}(0) = 1$.

Now we look at $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x) + S(x)}\right)$ for $|I| = n$. The Volterra expansion formula gives $\mathrm{e}^{-t^2 R(x) + S(x)} = \mathrm{e}^{-t^2 R(x)} + \sum_{k=1}^{\infty} \mathcal{Z}_k(x)$ with

$$\mathcal{Z}_k(x) = \int_{\Delta_k} \mathrm{e}^{-s_1 (t^2 R(x))}\, S(x)\, \mathrm{e}^{-s_2 (t^2 R(x))}\, S(x) \cdots S(x)\, \mathrm{e}^{-s_{k+1} (t^2 R(x))}\, ds_1 \cdots ds_k.$$

The term $\partial_I \cdot \mathcal{Z}_k(x)$ is equal to the sum, indexed by the partitions[5] $\mathcal{P} := \{I_1, I_2, \ldots, I_{2k+1}\}$ of $I$, of the terms

$$\mathcal{Z}_k(\mathcal{P})(x) := \tag{78}$$

$$\int_{\Delta_k} \left(\partial_{I_1} \cdot \mathrm{e}^{-s_1 (t^2 R(x))}\right) (\partial_{I_2} \cdot S(x)) \cdots (\partial_{I_{2k}} \cdot S(x)) \left(\partial_{I_{2k+1}} \cdot \mathrm{e}^{-s_{k+1} (t^2 R(x))}\right) ds_1 \cdots ds_k$$

which are, thanks to (77), smaller in norm than

$$\mathrm{cst}(\mathcal{P}) \left(1 + \|R\|_{n^+_{\mathcal{P}}}(x)\right)^{n^+_{\mathcal{P}}} \frac{\left(\|S\|_{n^-_{\mathcal{P}}}(x)\right)^k}{k!} (1+t)^{2 n^+_{\mathcal{P}}}\, \mathrm{e}^{-t^2 \mathrm{m}(R(x))}. \tag{79}$$

The integer $n^+_{\mathcal{P}}, n^-_{\mathcal{P}}$ are respectively equal to the sums $|I_1| + |I_3| + \cdots + |I_{2k+1}|$, $|I_2| + |I_4| + \cdots + |I_{2k}|$, and then $n^+_{\mathcal{P}} + n^-_{\mathcal{P}} = n$. The constant $\mathrm{cst}(\mathcal{P})$ is equal to the products $\mathrm{cst}(|I_1|)\mathrm{cst}(|I_3|) \cdots \mathrm{cst}(|I_{2k+1}|)$. Since the sum $\sum_{\mathcal{P}} \mathrm{cst}(\mathcal{P})$ is bounded by a constant $\mathrm{cst}'(n)$, we find that

$$\left\|\partial_I \cdot \mathrm{e}^{-t^2 R(x) + S(x)}\right\| \leq \mathrm{cst}'(n) \left(1 + \|R\|_n(x)\right)^n \mathrm{e}^{\|S\|_n(x)} (1+t)^{2n}\, \mathrm{e}^{-t^2 \mathrm{m}(R(x))} \tag{80}$$

for $(x,t) \in E \times \mathbb{R}^{\geq 0}$. Note that (80) is still true when $I = \emptyset$ with $\mathrm{cst}'(0) = 1$.

[5] We allow some of the $I_j$ to be empty.

Finally we look at $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)}\right)$ for $|I| = n$. The Volterra expansion formula gives $\mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)} = \mathrm{e}^{-t^2 R(x)+S(x)} + \sum_{k=1}^q \mathcal{W}_k(x)$ with

$$\mathcal{W}_k(x) = \int_{\Delta_k} \mathrm{e}^{s_1(-t^2 R(x)+S(x))}\, T(t,x) \cdots T(t,x)\, \mathrm{e}^{s_{k+1}(-t^2 R(x)+S(x))}\, ds_1 \cdots ds_k.$$

Note that the term $\mathcal{W}_k(x)$ vanishes for $k > q$. If we use (80), we get for $(x,t) \in E \times \mathbb{R}^{\geq 0}$ :

$$\|\partial_I \cdot \mathcal{W}_k(x)\| \leq \mathrm{cst}''(n)\Big(\|T_0\|_n(x) + \|T_1\|_n(x)\Big)^k \times$$
$$\Big(1 + \|R\|_n(x)\Big)^n \frac{(1+t)^{2n+k}}{k!}\ \mathrm{e}^{\|S\|_n(x)}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}.$$

Finally we get for $(x,t) \in E \times \mathbb{R}^{\geq 0}$ :

$$\left\|\partial_I \cdot \mathrm{e}^{-t^2 R(x)+S(x)+T(t,x)}\right\| \leq \mathrm{cst}''(n)\Big(1 + \|R\|_n(x)\Big)^n \times \tag{81}$$
$$\mathrm{P}\Big(\|T_0\|_n(x) + \|T_1\|_n(x)\Big)\ \mathrm{e}^{\|S\|_n(x)}\ (1+t)^{2n+q}\ \mathrm{e}^{-t^2 \mathrm{m}(R(x))}$$

where P is the polynomial $\mathrm{P}(z) = \sum_{k=0}^q \frac{z^k}{k!}$.

So (75) is proved with

$$\mathrm{cst} = \mathrm{cst}''(n) \sup_{x \in \mathcal{K}} \left\{ \Big(1 + \|R\|_n(x)\Big)^n \mathrm{P}\Big(\|T_0\|_n(x) + \|T_1\|_n(x)\Big)\ \mathrm{e}^{\|S\|_n(x)} \right\}.$$

## 5.2 Second estimates

Consider now the case where $E = W \times \mathfrak{k}$ : the variable $x \in E$ will be replaced by $(y, X) \in W \times \mathfrak{k}$. **We suppose that the maps $R$ and $T$ are constant relatively to the parameter $X \in \mathfrak{k}$.**

Let $\mathcal{K} = \mathcal{K}' \times \mathcal{K}''$ be a compact subset of $W \times \mathfrak{k}$. Let $D(\partial)$ be a constant coefficient differential operator in $(y, X) \in W \times \mathfrak{k}$ of degree $r$ : let $r_W$ be its degree relatively to the variable $y \in W$.

**Proposition 5.5** *There exists a constant* $\mathrm{cst} > 0$*, depending on* $\mathcal{K}, R(y), S(y,X), T_0(y), T_1(y)$ *and* $D(\partial)$*, such that*[6]

$$\left\|D(\partial) \cdot \mathrm{e}^{-t^2 R(y)+S(y,X)+T(t,y)}\right\| \leq \mathrm{cst}\,(1+t)^{2r_W+q}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y))}, \tag{82}$$

*for all* $(y, X, t) \in \mathcal{K}' \times \mathcal{K}'' \times \mathbb{R}^{\geq 0}$.

**Proof.** We follow the proof of Proposition 5.3. We have just to explain why we can replace in (75) the factor $(1+t)^{2r}$ by $(1+t)^{2r_W}$.

[6] $q$ is the highest degree of the graded algebra $\mathcal{A}$.

We choose some basis $v_1, \ldots, v_{p_1}$ of $W$ and $X_1, \ldots, X_{p_2}$ of $\mathfrak{k}$. Let us denote $\partial_i^1, \partial_j^2$ the partial derivatives along the vector $v_i$ and $X_j$. For any sequence

$$I := \underbrace{\{i_1, \ldots, i_{n_1}\}}_{I(1)} \cup \underbrace{\{j_1, \ldots, j_{n_2}\}}_{I(2)}$$

of integers where $i_k \in \{1, \ldots, p_1\}$ and $j_k \in \{1, \ldots, p_2\}$, we denote $\partial_I$ the differential operator of order $|I| = n_1 + n_2$ defined by the product $\prod_{k=1}^n \partial_{i_k}^1 \prod_{l=1}^m \partial_{j_k}^2$.

We first notice that $\partial_I \cdot \mathrm{e}^{-t^2 R(y)} = 0$ if $I(2) \neq \emptyset$. Now we look at $\partial_I \cdot \left(\mathrm{e}^{-t^2 R(y) + S(y,X)}\right)$ for $I = I(1) \cup I(2)$. The term $\mathcal{Z}_k(\mathcal{P})$ of (78) vanishes when there exists a subsequence $I_{2l+1}$ with $I_{2l+1}(2) \neq \emptyset$. In the other cases, the integer $n_{\mathcal{P}}^+ = |I_1| + |I_3| + \cdots + |I_{2k+1}|$ appearing in (79) is smaller than $|I(1)| = n_1$. So the inequalities (80) and (81) hold with the factor $(1+t)^{2n}$ replaced by $(1+t)^{2n_1}$.

In order to prove Proposition 3.8, we need to consider for every compactly supported function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ the integral

$$J_Q(\xi, y, t) := \int_{\mathfrak{k}} \mathrm{e}^{i\langle \xi, X\rangle}\, \mathrm{e}^{-t^2 R(y) + S(y,X) + T(t,y)}\, Q(X) dX.$$

**Proposition 5.6** *Let $\mathcal{K}' \times \mathcal{K}''$ be a compact subset of $W \times \mathfrak{k}$, and let $p$ be any positive integer.*

• *There exists a constant* $\mathrm{cst} > 0$*, such that: for any function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ with support on $\mathcal{K}''$, we have*

$$\left\| J_Q(\xi, y, t) \right\| \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}'', 2p} \, \frac{(1+t)^q}{(1 + \|\xi\|^2)^p}\, \mathrm{e}^{-t^2 \mathrm{m}(R(y))}.$$

*for all $(\xi, y, t) \in \mathfrak{k}^* \times \mathcal{K}' \times \mathbb{R}^{\geq 0}$.*

• *Let $D(\partial_y)$ be a constant differential operator on $W$ of order $r$. There exists a constant* $\mathrm{cst} > 0$*, such that: for any function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ with support on $\mathcal{K}''$, we have*

$$\left\| D(\partial_y) \cdot J_Q(\xi, y, t) \right\| \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}'', 2p} \, \frac{(1+t)^{q+2r}}{(1 + \|\xi\|^2)^p}\, \mathrm{e}^{-t^2 \mathrm{m}(R(y))} \tag{83}$$

*for all $(\xi, y, t) \in \mathfrak{k}^* \times \mathcal{K}' \times \mathbb{R}^{\geq 0}$.*

**Proof.** Let us concentrate on the first point. We have

$$\begin{aligned}
(1 + \|\xi\|^2)^p J_Q(\xi, y, t) &= \int_{\mathfrak{k}} \left( D_{2p}(\partial_X) \cdot \mathrm{e}^{i\langle \xi, X\rangle} \right) \mathrm{e}^{-t^2 R(y) + S(y,X) + T(t,y)}\, Q(X) dX \\
&= \int_{\mathfrak{k}} \mathrm{e}^{i\langle \xi, X\rangle}\, D_{2p}(\partial_X) \cdot \left( \mathrm{e}^{-t^2 R(y) + S(y,X) + T(t,y)}\, Q(X) \right) dX
\end{aligned}$$

where $D_{2p}(\partial_X) = (1 - \sum_a (\partial_{X_a})^2)^p$ is a constant coefficients differential operator in $X$ with order equal to $2p$. Now $D_{2p}(\partial_X) \cdot \left( \mathrm{e}^{-t^2 R(y) + S(y,X) + T(t,y)}\, Q(X) \right)$ is a finite sum of terms $\partial_X^\alpha \cdot \left(\mathrm{e}^{-t^2 R(y)+S(y,X)+T(t,y)}\right) (\partial_X^\beta \cdot Q(X))$ with $|\alpha|$ and $|\beta|$ less or equal than $2p$. All the derivatives $\partial_X^\beta \cdot Q(X)$ are bounded by $\|Q\|_{\mathcal{K}_2,p}$. We now employ the estimate of Proposition 5.5 for $\partial_X^\alpha \cdot \left(\mathrm{e}^{-t^2 R(y)+S(y,X)+T(t,y)}\right)$, where $\|\alpha\| \leq 2p$, and $(y,X) \in \mathcal{K}' \times \mathcal{K}''$, and we obtain our estimate.

The second point works similarly. We need to estimate $\partial_X^\alpha D(\partial_y) \cdot \left(\mathrm{e}^{-t^2 R(y)+S(y,X)+T(t,y)}\right)$. We use also the estimate of Proposition 5.5.

**Remark 5.7** *The estimate of Proposition 5.6 still holds when $Q$ is a smooth map (with compact support) from $\mathfrak{k}$ with values in* $\mathrm{End}(V) \otimes \mathcal{A}$.

In fact, we need still a slightly more general situation. The proof is identical to the preceding proof.

Let us denote $\mathbf{F}(y,X,t) := -t^2 R(y) + S(y,X) + T(t,y)$. Let $U_1(y), U_2(y)$ be two smooth maps with values in $\mathrm{End}(V) \otimes \mathcal{A}$. For any smooth function $Q$ on $\mathfrak{k}$ with compact support, we consider the integral

$$\mathcal{I}_Q(u,t,y,\xi) := \int_{\mathfrak{k}} \mathrm{e}^{i\langle \xi, X\rangle}\, U_1(y)\, \mathrm{e}^{u\mathbf{F}(y,X,t)}\, U_2(y)\, \mathrm{e}^{(1-u)\mathbf{F}(y,X,t)}\, Q(X) dX.$$

**Proposition 5.8** *Let $p$ be any positive integer. Let $\mathcal{K}' \times \mathcal{K}''$ be a compact subset of $W \times \mathfrak{k}$. Let $D(\partial_y)$ be a constant differential operator on $W$ of order $r$. There exists* $\mathrm{cst} > 0$ *such that: for any function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ with support in the compact $\mathcal{K}''$ we have*

$$\left\| D(\partial_y) \cdot \mathcal{I}_Q(u,t,y,\xi) \right\| \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}_2,2p} \frac{(1+t)^{q+2r}}{(1+\|\xi\|^2)^p}\, \mathrm{e}^{-t^2 \mathrm{m}(R(y))}$$

*for all $(t,y,\xi) \in \mathbb{R}^{\geq 0} \times \mathcal{K}' \times \mathfrak{k}^*$, and $u \in [0,1]$.*

## 5.3 Third estimates

In order to prove Theorem 4.12, we need to consider the following setting:

- The vector space $W$ decomposes as $W = W_1 \times W_2$,
- The maps $R$ and $T$ do not depend of the variable $X \in \mathfrak{k}$,
- The map $S$ does not depend of the variable $y_2 \in W_2$,
- The maps $R$, $T_0$ and $T_1$ are *slowly increasing* relatively to the variable $y_2 \in W_2$. Let us recall the definition. For any integer $n$, and any compact subset $\mathcal{K}_1$ of $W_1$, there exist some positive constants $\mathrm{cst}, \mu$ such that each function $\|R\|_n(y_1,y_2)$, $\|T_0\|_n(y_1,y_2)$ and $\|T_1\|_n(y_1,y_2)$ is bounded by $\mathrm{cst}(1+\|y_2\|)^\mu$ on $\mathcal{K}_1 \times W_2$.

Here we consider for every compactly supported function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ the integral

$$J_Q(\xi, y_1, y_2, t) := \int_{\mathfrak{k}} \mathrm{e}^{i\langle \xi, X\rangle}\, \mathrm{e}^{-t^2 R(y_1,y_2) + S(y_1,X) + T(t,y_1,y_2)}\, Q(X) dX.$$

**Proposition 5.9** *Let $\mathcal{K}_1 \times \mathcal{K}''$ be a compact subset of $W_1 \times \mathfrak{k}$. Let $D(\partial_y)$ be a constant differential operator on $W$ of order $r_W$.*

*There exists a constant $\mu \geq 0$, such that for any positive integer $p$, there exists* $\mathrm{cst} > 0$ *for which the following estimate holds for any function $Q \in \mathcal{C}^\infty(\mathfrak{k})$ with support on $\mathcal{K}''$ :*

$$\left\| D(\partial_y) \cdot J_Q(\xi, y_1, y_2, t) \right\| \leq \mathrm{cst}\, \|Q\|_{\mathcal{K}'',2p} \frac{(1 + \|y_2\|)^\mu}{(1 + \|\xi\|^2)^p} (1+t)^{q+2r_W}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y_1,y_2))} \tag{84}$$

*for $(\xi, y_1, y_2, t) \in \mathfrak{k}^* \times \mathcal{K}' \times W_2 \times \mathbb{R}^{\geq 0}$.*

**Remark 5.10** *In the estimate (84), the crucial point is that the constant $\mu$ is the same for all integer $p$.*

**Proof.** In the following the parameter $(y_1, X)$ belongs to the compact $\mathcal{K} := \mathcal{K}'_1 \times \mathcal{K}''$ and the parameter $(y_2, t)$ belongs to $W_2 \times \mathbb{R}^{\geq 0}$.

As in the proof of Proposition 5.6, we get the estimates (84) if we show that for any differential operator $D(\partial_X)$ we have the estimate:

$$\left\| D(\partial_y) \circ D(\partial_X) \cdot \mathrm{e}^{-t^2 R(y_1,y_2) + S(y_1,X) + T(t,y_1,y_2)} \right\| \leq \mathrm{cst}\, (1 + \|y_2\|)^\mu\, (1+t)^{q+2r_W}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y_1,y_2))}, \tag{85}$$

where the parameter $\mu$ in (85) does not depend of the choice of $D(\partial_X)$.

First, we consider the term $D(\partial_y) \circ D(\partial_X) \cdot \mathrm{e}^{-t^2 R(y_1,y_2)}$ : it vanishes if the order of $D(\partial_X)$ is not zero. In the other case we exploit (77) with the *slowly increasing* behavior of $R$ to get

$$\left\| D(\partial_y) \cdot \mathrm{e}^{-t^2 R(y_1,y_2)} \right\| \leq \mathrm{cst}\, (1 + \|y_2\|)^\alpha\, (1+t)^{2r_W}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y_1,y_2))},$$

where $\alpha$ depends of the order $D(\partial_y)$.

Now we consider the term $D(\partial_y) \circ D(\partial_X) \cdot \mathrm{e}^{-t^2 R(y_1,y_2) + S(y_1,X)}$. The estimate (79) gives, modulo the changes explained in the proof of Proposition 5.5, the following

$$\left\| D(\partial_y) \circ D(\partial_X) \cdot \mathrm{e}^{-t^2 R(y_1,y_2) + S(y_1,X)} \right\| \leq \mathrm{cst}\, (1 + \|y_2\|)^\delta\, (1+t)^{2r_W}\ \mathrm{e}^{-t^2 \mathrm{m}(R(y_1,y_2))} \tag{86}$$

where cst take into account the term[7] $\mathrm{e}^{\|S\|_{\mathcal{K},n}}$. Here the term $(1 + \|y_2\|)^\delta$ comes from the term $(1 + \|R\|_{n^+_\mathcal{P}}(x))^{n^+_\mathcal{P}}$ of (79). The factor $n^+_\mathcal{P}$ is bounded by the order of $D(\partial_y)$, hence it explains why $\delta$ does not depend of the order of $D(\partial_X)$.

Using Volterra expansion formula, it is now an easy matter to derive (85) from (86).

The preceding estimates hold if we work in the algebra $\mathrm{End}(\mathcal{E}) \otimes \mathcal{A}$, where $\mathcal{E}$ is a super-vector space and $\mathcal{A}$ a super-commutative algebra.

[7] $n$ is the order of $D(\partial_y) \circ D(\partial_X)$

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